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Response to Wilson's note on 'Influences of resource limitations and transmission costs on epidemic simulations and critical thresholds in scale-free networks'

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Abstract

James R. Wilson points out what he describes as flaws in our proof in 'Influences of Resource Limitations and Transmission Costs on Epidemic Simulations and Critical Thresholds in Scale-free Networks' (*Simulation* 85(3): 205–219) and offers an alternative steady-state behavior derivation based on our epidemic simulation model. In this response we will explain our definitions for the terms used in our paper and the derivation process for our analysis, then compare and contrast our mathematical model with that proposed by Wilson. We suggest that more compartmental models can be used to support our argument that increasing transmission costs or decreasing individual resources increases the critical threshold of a contagion event in a scale-free network.

Keywords

critical thresholds, effective spreading rate, complex networks, heterogeneous networks, resource limitations, transmission costs

1. Introduction

In the article entitled 'Influences of Resource Limitations and Transmission Costs on Epidemic Simulations and Critical Thresholds in Scale-Free Networks' [1] we (a) present five key characteristics of resource and transmission costs associated with social interactions and daily contact, (b) apply network-oriented modeling and population-based analytical approaches to respectively construct simulation and mathematical models to investigate the influences of resource limitations and transmission costs on epidemic dynamics and critical thresholds in scale-free networks, and (c) compare results from our mathematical analyses and simulation experiments (Figure 1). The mathematical model supporting our epidemic simulations is based on Pastor-Satorras and Vespignani's efforts.[2–6] We have modified their model by adding the term S_k, defined as the spread of an infection according to minimum values for available resources and numbers of links among active nodes.

As shown in Figure 1, our mathematical and simulation results consistently indicate that when resources and transmission costs are taken into consideration, a significant critical threshold (above which a contagious disease exceeds control and becomes epidemic and below which a contagious disease disappears) exists when a contagion event occurs in a scale-free network: in short, a non-zero critical threshold exists in scale-free networks. Our results also indicate that the appearance

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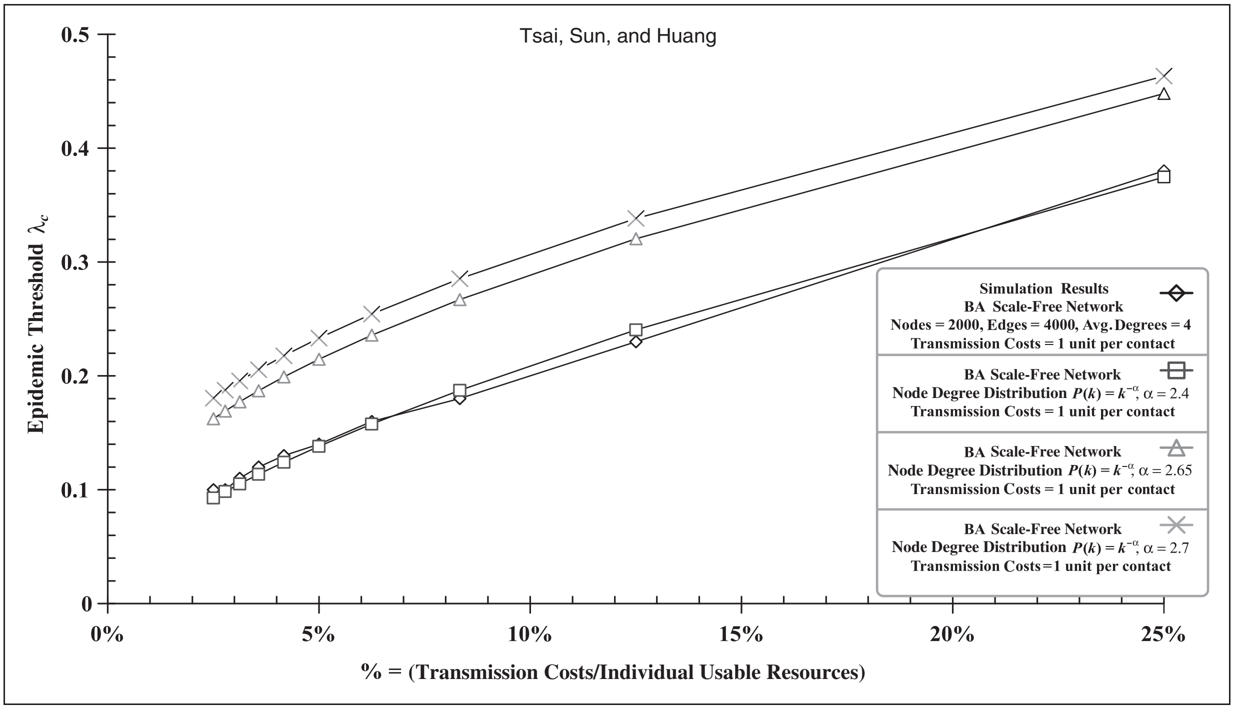

Figure 1. Critical threshold λ_c is a function of the ratio of transmission costs to individual resources (c/R) in scale-free networks. We used it to analyze results from our simulation experiments and three mathematical analyses.

of a critical threshold is tied to a ratio of transmission costs to available resources. Our findings suggest two potential intervention strategies for controlling a variety of biological and non-biological epidemic events. In the cases of computer viruses and network attacks, strategies may include (a) placing restrictions on individual uploads to and downloads from remote servers (i.e., limiting resources), or (b) charging upload/download fees when users want to exceed daily limits (i.e., increasing transmission costs). Both strategies can achieve the same end result: increasing the critical threshold value.

Wilson [7] argues that Pastor-Satorras and Vespignani's [2–6] original $\rho_k(t)$ and $\theta\left[\{\rho_k(t)\}\right]$ terms (respectively defined as the relative density of infected nodes having k connections and the probability that any given individual will establish a link with an infected individual) should be revised as follows (both dependent on a λ effective spreading rate):

1. infection rate $\rho_k(t;\ \lambda)$ is the probability that any node with k links will be infected at time t; and
2. $\theta(t;\ \lambda)$ is the probability that at time t a randomly selected link in a network will be incident on (connected to) an infected node, given that the infection spreading rate is λ.

Wilson thereby provides an alternative derivation for our proposed simulation model for depicting the influences of resource limitations and transmission costs on epidemic dynamics and critical thresholds in scale-free networks with different features. In the next section we will explain the original basis for our mathematical model and compare it to Wilson's proposed alternative.

2. Supplemental explanation

Our mathematical model for describing how resource limitations and transmission costs influence epidemic dynamics and critical thresholds in scale-free networks is expressed as

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t) + \lambda S_k\left[1 - \rho_k(t)\right]\theta\left[\{\rho_k(t)\}\right] \qquad (1)$$

where S_k is the minimum value for the ratio between an individual's resources in relation to contagion transmission costs and the individual's connectivity (k). With the exception of S_k, the symbols used here are consistent with those defined by Pastor-Satorras and Vespignani [2–6] in their discussions of spreading dynamics. However, those (and other) authors have tended to overlook two important factors: *resource limitations* and *transmission costs*. The term $\rho_k(t)$ represents the relative density of infected nodes with k connections at time t. The phrase 'relative density' refers to the proportion of infected individuals within a population who have k connections. The infection spreading rate λ is a pre-determined constant [2–6] representing the spreading capability of specific diseases, defined as a ratio between the rate that healthy individuals in a population become infected and the rate that infected individuals recover. The term $\{\rho_k(t)\}$ denotes the set containing all $\rho_k(t)$ for all positive k, as well as the alterative representation $\{\rho_1(t),\ \rho_2(t),\ \rho_3(t), \ldots\}$. Accordingly, $\theta[\{\rho_k(t)\}]$ is the probability that any given individual will be linked to an infected individual. Pastor-Satorras and Vespignani [2–6] claim that this probability is proportional to the infection rate, and therefore can be reduced to $\theta(\lambda)$. In Equation (2) (numbered Equation (6) in our original paper) we define ρ_k as the steady state of $\rho_k(t)$ by solving the stationary condition $d\rho_k(t)/dt = 0$. After substituting $\theta(\lambda)$ in that equation we obtain

$$\theta = \frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda S_k\theta}{1+\lambda S_k\theta} \tag{2}$$

As shown, one trivial solution for θ is $\theta = 0$. Next, we derive inequality (3) (inequality (8) in our original paper) based on the possibility that the right-hand side of Equation (2) has a non-singular solution:

$$\frac{d}{d\theta}\left(\frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda S_k\theta}{1+\lambda S_k\theta}\right)\Bigg|_{\theta=0} \geq 1. \tag{3}$$

Without using a concave function as an alternative proof,[8] we show that Equation (3) is a contradiction. Assuming that Equation (3) does not hold, it should therefore be expressed as

$$\frac{d}{d\theta}\left(\frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda S_k\theta}{1+\lambda S_k\theta}\right)\Bigg|_{\theta=0} < 1. \tag{4}$$

By defining

$$F(\theta) = \theta - \frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda S_k\theta}{1+\lambda S_k\theta} \tag{5}$$

we observe that a trivial solution for $F(0) = 0$ is $\theta = 0$. Next, note that the first derivative of $F(\theta)$ at 0 with respect to θ is larger than zero as follows:

$$\frac{d}{d\theta}F(\theta)\Bigg|_{\theta=0} = 1 - \frac{d}{d\theta}\left(\frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda S_k\theta}{1+\lambda S_k\theta}\right)\Bigg|_{\theta=0} > 0. \tag{6}$$

However, this conclusion implies that non-trivial solutions for $F(\theta) = 0$ do not exist for any $\theta > 0$, which contradicts inequality (4). We therefore obtained $\lambda_c = \langle k\rangle / \sum_k kP(k)S_k$ as a conclusion regarding critical thresholds (Equation (10) in our original paper). By deriving the above conclusion in advance, we were able to obtain a separate conclusion concerning the lower critical threshold boundary, $\lambda_c \geq 1/\left((R/c)^2/\langle k\rangle\right) + R/c$, which also implies that resources and transmission costs significantly affect critical threshold values.

3. Comparing and contrasting two mathematical models

Wilson's [7] proposed alternative is expressed as

$$\frac{\partial\rho_k(t;\ \lambda)}{\partial t} = -\rho_k(t;\ \lambda) + \lambda S_k[1-\rho_k(t;\ \lambda)]\theta(t;\ \lambda). \tag{7}$$

We feel there are three aspects of this model worthy of note. Firstly, Wilson defines $\rho_k(t)$, the infection rate, as the probability that any node with k links will be infected at time t, whereas we define $\rho_k(t)$ as the proportion of infected individuals in a population having k connections. Secondly, he defines the infection spreading rate λ as a variable parameter of $\rho_k(t)$ rather than as a pre-determined constant that affects $\rho_k(t)$; accordingly, he defines $\rho_k(t;\ \lambda)$ as the probability of a node with k links being infected at time t. Thirdly, based on his definition of infection rate, Wilson uses $\theta[\{\rho_k(t)\}]$ as a function that is dependent on both λ and t; in other words, $\theta(t;\ \lambda)$ is the probability that at time t any randomly selected link in a network will be incident on (connected to) an infected node, given the λ infection spreading rate.

We disagree with Wilson's perception of $\rho_k(t;\ \lambda)$ in Equation (7) as the probability that any node with k links will be infected at time t. However, we believe that Equation (7) can be considered an extension of Equation (1), using a variable rather than constant λ. In our paper we present λ as a parameter for discussing specific contagious diseases. Based on Pastor-Satorras and Vespignani's work[2], λ can be defined as

$$\lambda = \frac{\text{Rate each healthy individual is infected}}{\text{Rate each infected individual becomes healthy}}.$$

In our mathematical model, it is reasonable to consider λ as a constant rather than a variable; this is consistent with our research question (i.e., under what value of λ does a contagious disease become an epidemic). We believe that viewing λ as a variable for $\rho_k(t;\ \lambda)$ supports predictions of epidemic dynamics. However, we admit that our model is a preliminary tool for exploring epidemic dynamics and critical thresholds associated with resource limitations and transmission costs in scale-free networks. We expect that more complex models will be developed to investigate the effects of those two factors in detail.

To derive his critical threshold, Wilson established alternatives for the $\rho_k(t;\ \lambda)$ and $\theta(t;\ \lambda)$ steady states in Equation (7) as

$$\rho_k(\lambda) \equiv \lim_{t\to\infty} \rho_k(t;\ \lambda) = \frac{\lambda S_k \theta(\lambda)}{1 + \lambda S_k \theta(\lambda)}$$
$$\text{for all } k \text{ and for } \lambda \in [0,\ 1]$$

and

$$\theta(\lambda) = \lim_{t\to\infty} \theta(t;\ \lambda) \left\{ \begin{array}{ll} = 0 & \text{if} \quad \lambda \in [0,\ \lambda_c], \\ > 0 & \text{if} \quad \lambda \in [\lambda_c,\ 1], \end{array} \right\}.$$

According to these definitions, Equation (2) can be extended as

$$\theta(\lambda) = \frac{1}{\langle k \rangle} \sum_k kP(k) \frac{\lambda S_k \theta(\lambda)}{1 + \lambda S_k \theta(\lambda)}. \qquad (8)$$
$$\text{for all}\, \lambda \in [0,\ 1].$$

By claiming that $\theta(\lambda)$ is analytic (meaning that a real number $\omega \in (0,\ 1)$ exists such that for any $\lambda \in [0,\ \omega]$, the $\theta(\lambda)$) derivatives for all orders exist (or are infinitely differ-entiable) in any neighborhood at its origin. After defining critical epidemic threshold as $\lambda_c = \sup\{\lambda \in [0,\ 1] : \theta(\lambda) = 0\}$, Wilson uses derivatives of $\theta(\lambda)$ on both sides of Equation (8) according to all orders, and thus obtains the critical epidemic threshold $\lambda_c = \langle k \rangle / \sum_k kP(k)S_k$.

We believe that the use of a Taylor expansion is suitable for any analysis of $\theta(\lambda)$ as a function of a λ variable. In conclusion, the result from Wilson's proposed alternative (Equation (7)) is the same as for our proposed model (Equation (1)).

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