Examples of 2nd Order Linear ODE's, with Complex Roots.

Ex: Solve $d^2y/dt^2 + dy/dt + y = 0$

Step 1: Char. eq: $a, b, c = 1 \longrightarrow r^2 + r + 1 = 0$

Step 2: Roots: $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$

Step 3: Find u, v:

$$e^{r_1 t} = e^{-\left(\frac{1+i\sqrt{3}}{2}\right)t} = e^{-\frac{t}{2}} \cdot e^{\frac{i\sqrt{3}}{2}t} = e^{-t/2}\left(\cos\left(\tfrac{\sqrt{3}}{2}t\right) + i\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right)$$

then $u = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$ and $v = e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$

Step 4: General Solution:

$$y = c_1 u + c_2 v \Rightarrow y = c_1 e^{-t/2}\cos\left(\tfrac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2}\sin\left(\tfrac{\sqrt{3}}{2}t\right)$$

Ex: Solve $y'' + 2y' + 3y = 0$

Step 1: char eq: $r^2 + 2r + 3 = 0$

Step 2: roots: $r_{1,2} = \dfrac{-2 \pm \sqrt{4 - 4(1)(3)}}{2(1)} = \dfrac{-2 \pm \sqrt{-8}}{2} = \dfrac{-2 \pm \sqrt{4}\sqrt{-2}}{2} = -1 \pm i\sqrt{2}$

Step 3: find u, v:

$$e^{r_1 t} = e^{(-1+i\sqrt{2})t} = e^{-t} \cdot e^{i\sqrt{2}t} = e^{-t}\left(\cos(t\sqrt{2}) + i\sin(t\sqrt{2})\right)$$

then, $u = e^{-t}\cos(t\sqrt{2})$
$v = e^{-t}\sin(t\sqrt{2})$

Step 4: General Solution: $y = c_1 u + c_2 v$

$$y = c_1 e^{-t}\cos(t\sqrt{2}) + c_2 e^{-t}\sin(t\sqrt{2})$$

Not doing an IVP example since it's exactly the same as others.