Examples of first-order linear differential equations:

Ex: Find the general solution of: $(1+t^2)\,dy/dt + ty = (1+t^2)^{5/2}$

*Notice it is not in standard form. Rewrite in form: $dy/dt + a(t)y = b(t)$

$\hookrightarrow dy/dt + \left(\frac{t}{1+t^2}\right)y = (1+t^2)^{3/2}$

$a(t) = \left(\frac{t}{1+t^2}\right)$

$\mu(t) = e^{\int a(t)dt}$

$= \exp\left(\int \frac{t}{1+t^2}\,dt\right) \longrightarrow \int \frac{t}{1+t^2}\,dt$

use u-substitution:
$u = t^2$
$du = 2t\,dt$
$\frac{1}{2}du = t$

$= \frac{1}{2}\int \frac{1}{1+u}\,du$

$= \exp\left(\frac{1}{2}\ln(1+t^2)\right) \longleftarrow = \frac{1}{2}\ln(1+u) = \frac{1}{2}\ln(1+t^2)$

$\mu(t) = (1+t^2)^{1/2}$

Step 1: Muliply both sides by $\mu(t)$: $\frac{d(\mu y)}{dt} = (1+t^2)^{1/2}\cdot(1+t^2)^{3/2}$
$= (1+t^2)^2$

Step 2: Integrate both sides: $\int \frac{d(\mu y)}{dt}\,dt = \int (1+t^2)^2\,dt$

$\mu y = \int 1 + 2t^2 + t^4\,dt$

$\mu y = t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C$

Step 3: Solve for y: $y = \dfrac{t + 2/3\,t^3 + 1/5\,t^5 + C}{(1+t^2)^{1/2}}$

Ex: Solve the initial value problem: $dy/dt - 1/t\,y = t\sin t$; $t>0$; $y(\pi) = 0$.

*find $\mu(t)$: $a(t) = -1/t \rightarrow \int -t^{-1}dt = -\ln(t) \rightarrow$ then $\mu(t) = e^{-\ln t}$
$\mu(t) = t^{-1}$

Step 1: Multiply by $\mu(t)$: $d(\mu y)/dt = t\sin t\cdot\left(\frac{1}{t}\right)$
$= \sin t$

Step 2: Integrate: $\int \frac{d(\mu y)}{dt}\,dt = \int \sin t\,dt$

$\mu y = -\cos t + C$

Step 3: Solve for y: $y = (-\cos t + C)t$ This is the general solution.

$\rightarrow$ initial value: $y(\pi) = 0 \longrightarrow 0 = (-\cos(\pi) + c)\pi$
$= -\pi\cos(\pi) + \pi C$
$= -\pi(-1) + \pi c$
$= \pi + \pi c = \pi(1+c)$
$-1 = c$

Then, the unique solution is $y = -t\cos t - t$