Applied Mathematics
005 – Fourier Transform

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The message

- The Fourier transform is nothing magic
- First, it is a transform, that is, it changes the representation of the data
- Second, this transform is easily reversible, that is, one can go back and forth between the representations
- The standard representation of data is in space or time
  - You measure data as a function of position, or time
- The Fourier representation of data is in frequencies
  - How much is a certain frequency present in the data?
- It therefore is most useful for understanding periodic data, that is data that contains re-occurring patterns

Mathematical Background: Complex Numbers

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- A complex number **x** has the form:

j sometimes called i

$$x = a + jb, \text{ where } j = \sqrt{-1}$$

a: **real part**, b: **imaginary** part

- Addition is done by adding real and imaginary parts separately

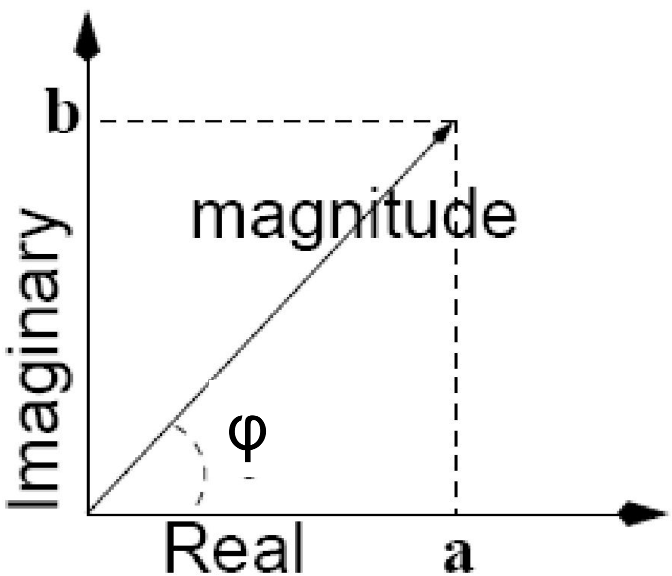
$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

- Multiplication (taking into account the fact that j*j = -1)

$$(a + jb) * (c + jd) = (ac - bd) + j(ad + bc)$$

Mathematical Background: Complex Numbers

- Magnitude-Phase (i.e., vector) representation of complex numbers for the positive, upper quadrant



Magnitude: $|x| = r = \sqrt{a^2 + b^2}$

Phase: $\phi(x) = \tan^{-1}(b/a)$

Polar notations:

$$x = r(\cos\phi + j\sin\phi)$$

$$\boxed{x = |x|e^{j\phi(x)}}$$

Mathematical Background: Complex Numbers

- Euler's formula

$$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

- Properties

$$e^{j\pi} + 1 = 0$$

$$|e^{\pm j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$\phi(e^{\pm j\theta}) = \tan^{-1}(\pm \frac{sin(\theta)}{cos(\theta)}) = \tan^{-1}(\pm tan(\theta)) = \pm\theta$$

$$sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Mathematical Background: Complex Numbers

- Multiplication using magnitude-phase representation

$$xy = |x|e^{j\phi(x)} \cdot |y|e^{j\phi(y)} = |x|\ |y|\ e^{j(\phi(x)+\phi(y))}$$

- Complex conjugate

$$x^{*} = a - jb$$

- Properties

$$|x| = |x^{*}|$$
$$\phi(x) = -\phi(x^{*})$$
$$xx^{*} = |x|^{2}$$

Mathematical Background: Sine and Cosine Functions

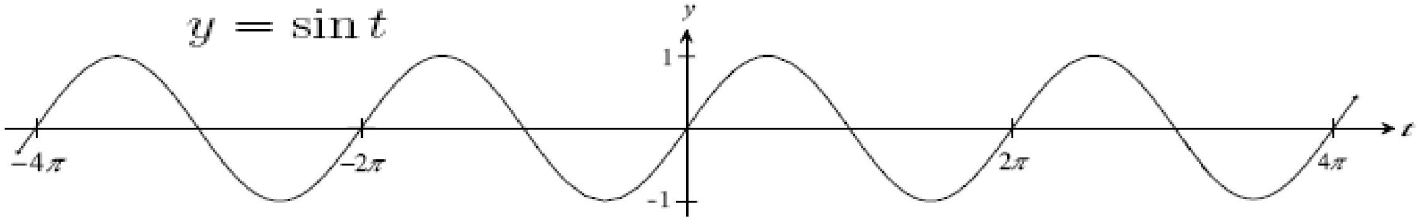
- Sine and Cosine functions are periodic
- As we have seen, the general form of sine and cosine functions are:

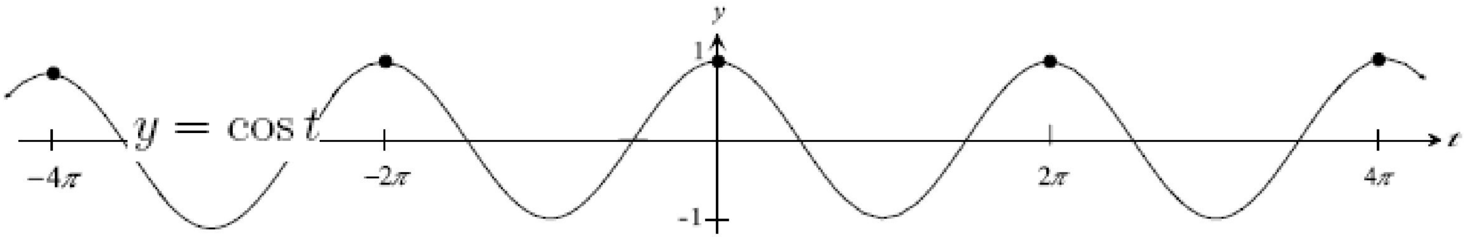
$$y(t) = A \sin[a(t+b)] \qquad y(t) = A \cos[a(t+b)]$$

$\lvert A \rvert$	amplitude
$\frac{2\pi}{\lvert a \rvert}$	period
b	phase shift

Mathematical Background: Sine and Cosine Functions

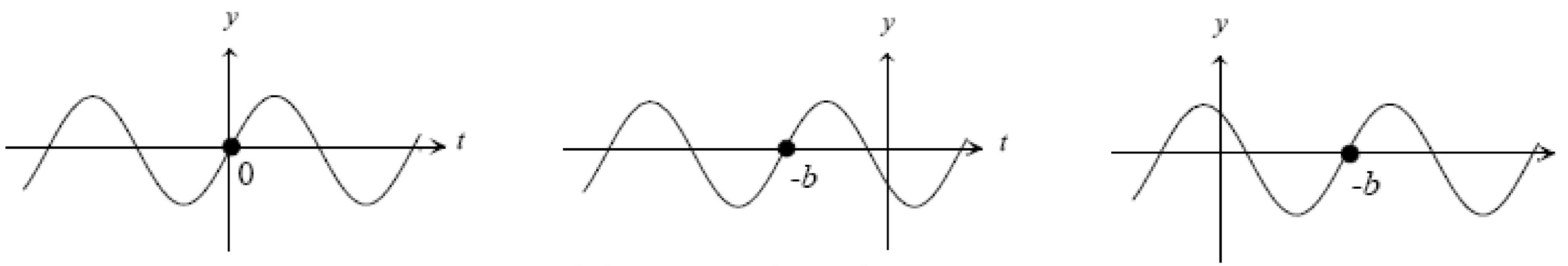
Special case: unit amplitude, no phase shift, unit period
A=1, b=0, α=1





Mathematical Background: Sine and Cosine Functions

- Shifting or translating the sine function by a constant b (a phase shift)



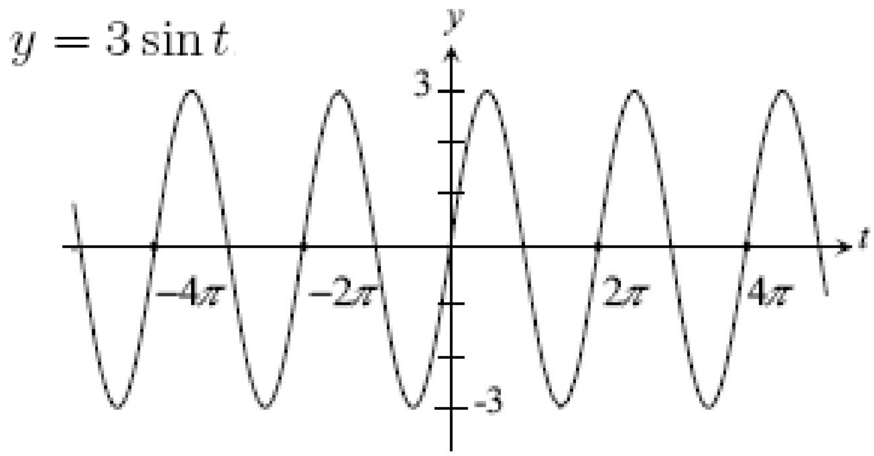
(a) $y = \sin t$ (b) $y = \sin(t + b)$, $b > 0$ (c) $y = \sin(t + b)$, $b < 0$

- We see, therefore, that the cosine is merely a shifted sine function:

$$\cos(t) = \sin(t + \frac{\pi}{2})$$

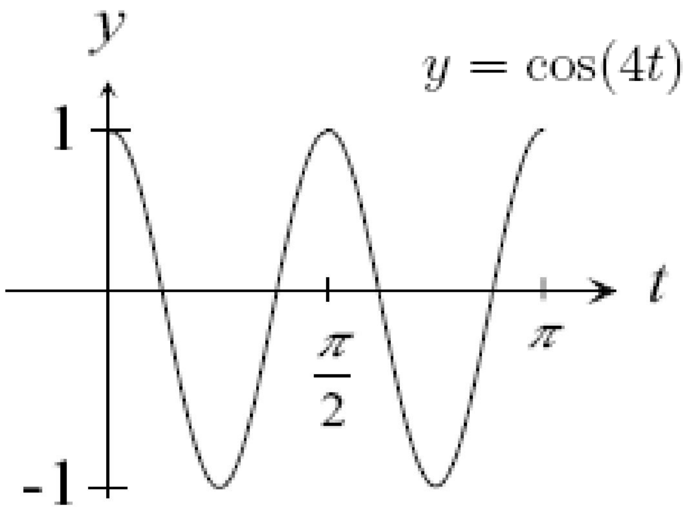
Mathematical Background: Sine and Cosine Functions

- Effect of changing the amplitude A



Mathematical Background: Sine and Cosine Functions

- Effect of changing the period T=2π/|α|, for example, for y=cos(αt)



α =4

period
2π/4=π/2

shorter period
higher frequency
(i.e., the function oscillates **faster**)

- Frequency is defined as f=1/T
- Frequency notation: sin(αt)=sin(2πt/T)=sin(2πft)

Fourier Series Theorem

- Any **periodic** signal can be expressed as a weighted (infinite) sum of sine and cosine functions of varying frequency

$$\boxed{\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]}$$

a_n and b_n, are the weights of the expansion

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, \quad n \geq 1$$

Fourier Series Theorem

- Any periodic signal can be expressed as a weighted (infinite) sum of sine and cosine functions of varying frequency (this is the compact notation):

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{jnx}$$

$$c_n = \int_{-\pi}^{\pi} f(x)\, e^{-jnx}$$ are the weights of the expansion

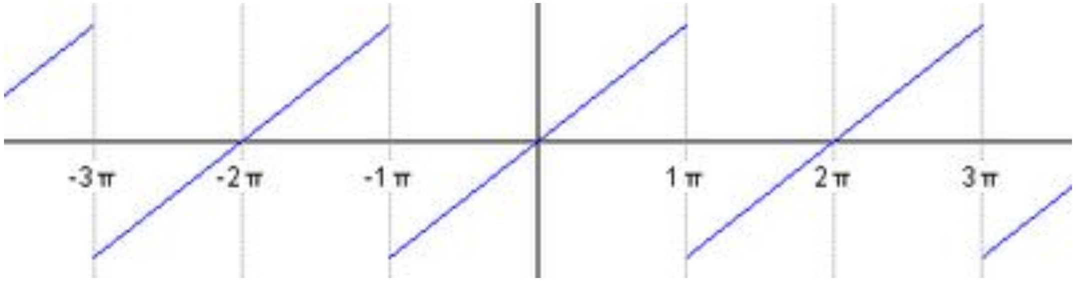
$a_n = c_n + c_{-n}$ for n=0,1,2,….

$b_n = j(c_n - c_{-n})$ for n=1,2,….

Example – the sawtooth wave

- The sawtooth wave is defined as:

$$f(x) = x, \quad \text{for } -\pi < x < \pi,$$
$$f(x + 2\pi) = f(x), \quad \text{for } -\infty < x < \infty.$$



- The coefficients are therefore:

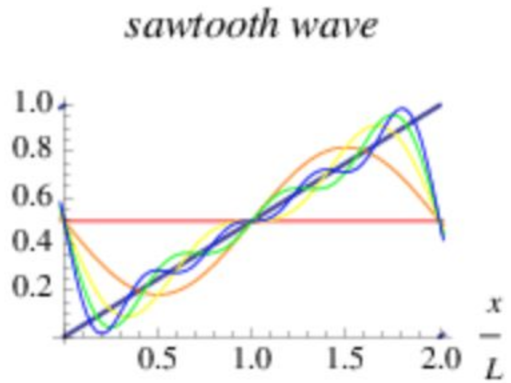
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0, \quad n \geq 0.$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = -\frac{2}{n}\cos(n\pi) + \frac{2}{n^2\pi}\sin(n\pi) = 2\,\frac{(-1)^{n+1}}{n}, \quad n \geq 1.$$

Example – the sawtooth wave

- With that the Fourier series becomes:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$
$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx), \quad \text{for} \quad x - \pi \notin 2\pi\mathbf{Z}.$$

- Here are the first five approximations of the Fourier series plotted



Fourier series – applications

- The Fourier series is mainly used for solving partial differential equations
- This is also the origin of the work of Fourier who used the Fourier series to describe possible solutions to the heat equation

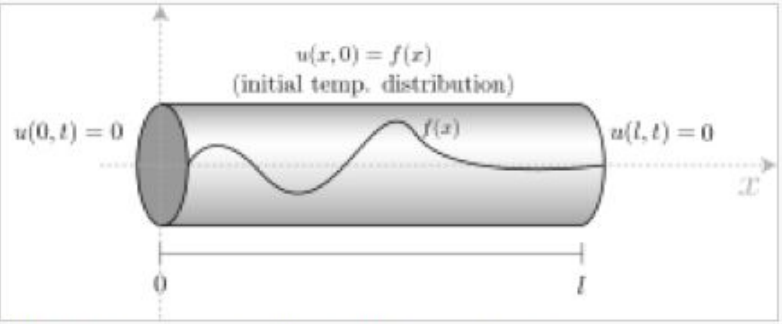
$$\frac{\partial u}{\partial t} - \alpha \Delta u = 0$$

where u is a function of x,y,z,t

The heat equation

The following solution technique for the heat equation was proposed by Joseph Fourier in his treatise *Théorie analytique de la chaleur*, published in 1822. Let us consider the heat equation for one space variable. This could be used to model heat conduction in a rod. The equation is

$$u_t = \alpha u_{xx} \qquad (1)$$



Idealized physical setting for heat conduction in a rod with homogeneous boundary conditions.

where $u = u(x, t)$ is a function of two variables x and t. Here

- x is the space variable, so $x \in [0, L]$, where L is the length of the rod.
- t is the time variable, so $t \geq 0$.

We assume the initial condition

$$u(x,0) = f(x) \quad \forall x \in [0, L] \qquad (2)$$

where the function f is given, and the boundary conditions

$$u(0,t) = 0 = u(L,t) \quad \forall t > 0. \qquad (3)$$

The heat equation

Let us attempt to find a solution of (1) which is not identically zero satisfying the boundary conditions (3) but with the following property: *u* is a product in which the dependence of *u* on *x*, *t* is separated, that is:

$$u(x,t) = X(x)T(t). \tag{4}$$

This solution technique is called separation of variables. Substituting *u* back into equation (1),

$$\frac{\dot{T}(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)}.$$

Since the right hand side depends only on *x* and the left hand side only on *t*, both sides are equal to some constant value −λ. Thus:

$$\dot{T}(t) = -\lambda\alpha T(t) \tag{5}$$

and

$$X''(x) = -\lambda X(x). \tag{6}$$

The heat equation

We will now show that nontrivial solutions for (6) for values of λ ≤ 0 cannot occur:

1. Suppose that $\lambda < 0$. Then there exist real numbers B, C such that
   $$X(x) = Be^{\sqrt{-\lambda}\,x} + Ce^{-\sqrt{-\lambda}\,x}.$$
   From (3) we get $X(0) = 0 = X(L)$ and therefore $B = 0 = C$ which implies u is identically 0.
2. Suppose that $\lambda = 0$. Then there exist real numbers B, C such that $X(x) = Bx + C$. From equation (3) we conclude in the same manner as in 1 that u is identically 0.
3. Therefore, it must be the case that $\lambda > 0$. Then there exist real numbers A, B, C such that
   $$T(t) = Ae^{-\lambda\alpha t}$$
   and
   $$X(x) = B\sin(\sqrt{\lambda}\,x) + C\cos(\sqrt{\lambda}\,x).$$
   From (3) we get $C = 0$ and that for some positive integer n,
   $$\sqrt{\lambda} = n\frac{\pi}{L}.$$

This solves the heat equation in the special case that the dependence of u has the special form (4).

The heat equation

This solves the heat equation in the special case that the dependence of u has the special form (4).

In general, the sum of solutions to (1) which satisfy the boundary conditions (3) also satisfies (1) and (3). We can show that the solution to (1), (2) and (3) is given by

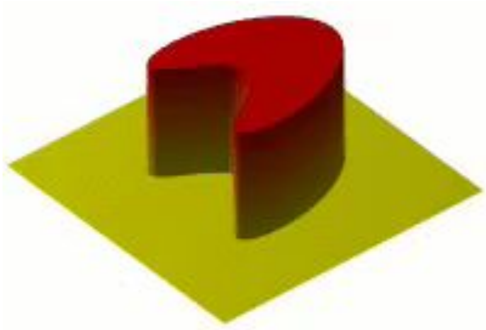
$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha t}{L^2}}$$

where

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

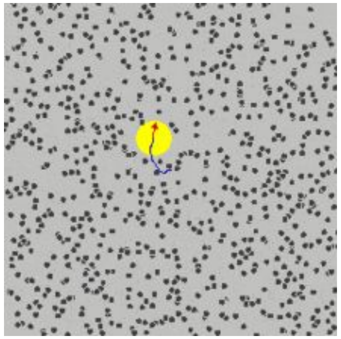
The heat equation

- Given the initial heating profile of a cool plate, the heat equation can be solved and predicts how temperature evolves over time
- Interestingly, the heat equation underlies also modelling of Brownian motion and of financial processes, such as the Black-Scholes model for option trading



Evolution of temperature on a heated plate

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2},$$



Diffusion equation for Brownian motion

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Black-Scholes equation

Continuous Fourier Transform (FT)

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- Transforms a signal (i.e., function) from the spatial domain (x) to the time/frequency domain (u or t).

Forward FT: $$F(f(x)) = F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Inverse FT: $$F^{-1}(F(u)) = f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

with $e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$

Definitions

- The Fourier Transform (FT) **F(u)** is a complex function:

  $$F(u) = R(u) + jI(u)$$

- Magnitude of FT (spectrum):

  $$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- Phase of FT:

  $$\phi(F(u)) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right)$$

- Magnitude-Phase representation:

  $$F(u) = |F(u)|e^{j\phi(u)}$$

- Power is the squared spectrum:

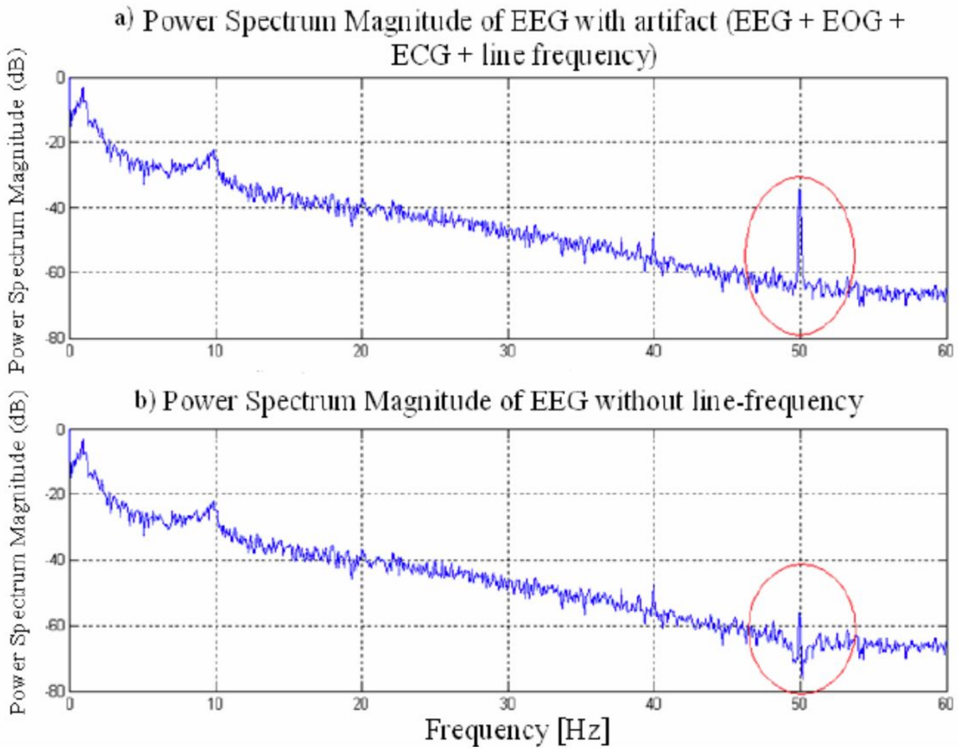
  P(u)$=|F(u)|^2= R^2(u) + I^2(u)$

Typical applications of the FT

- Analyze frequency content of signals (EEG)
- Filtering: Remove frequencies from a signal
  - High-pass filtering
  - Low-pass filtering
- Easier and faster to perform certain operations in the **frequency** domain than in the **spatial** domain
  - Convolution
- Solution of certain numerical problems easier in the frequency domain

Example: removing line noise

- Most signals have artifacts that can be reduced with filtering methods
- We need to remove specific noise frequencies
- Here is an example of the removal of the 50Hz line noise in an EEG recording



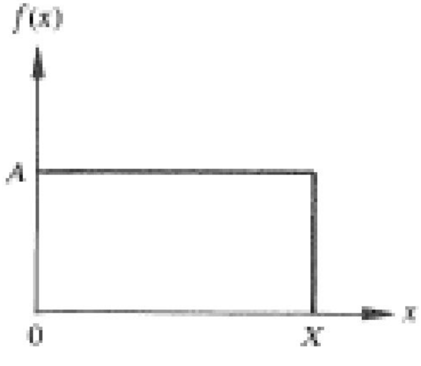
Removal of power line 50Hz artifact from an EEG recording

Frequency Filtering Steps

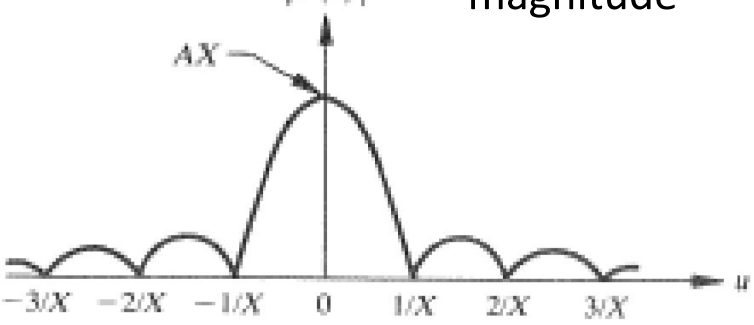
- 1. Take the FT of f(x): $F(f(x))$
- 2. Remove undesired frequencies: $D(F(f(x)))$
- 3. Convert back to a signal: $\hat{f}(x) = F^{-1}(D(F(f(x))))$

We'll talk more about this later .....

Example: rectangular pulse



(a)



(b)

magnitude

rect(x) function

sinc(x)=sin(x)/x

Extending FT to two dimensions

- Forward FT

$$F(f(x,y)) = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

- Inverse FT

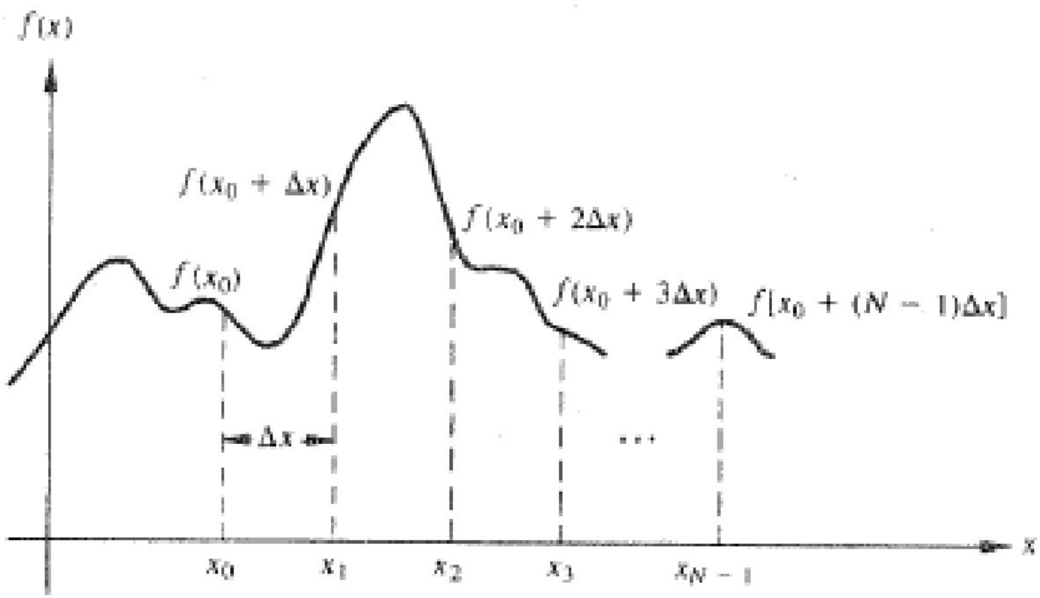
$$F^{-1}(F(u,v)) = f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+uy)}dudv$$

Intuition

- The Fourier transformation tries to explain the signal as a sum of sines and cosines
  - So the explanation of sines and cosines is trivial [one value]
  - In order to explain an infinitely localized signal in space [delta function], we need infinitely many frequencies
  - Sharp changes in the spatial signal [edges] will need many frequencies to explain them [rect -> sinc]

Discrete Fourier Transform (DFT)

- In most applications, we have discrete data, that is, data that is sampled at discrete locations
  - Digitized time series (from an Analog-Digital-Converter)
  - Images



$$f(x) = f(x_0 + x\Delta x),\ x = 0, 1, \ldots, N-1$$

Discrete Fourier Transform (DFT)

- Forward DFT

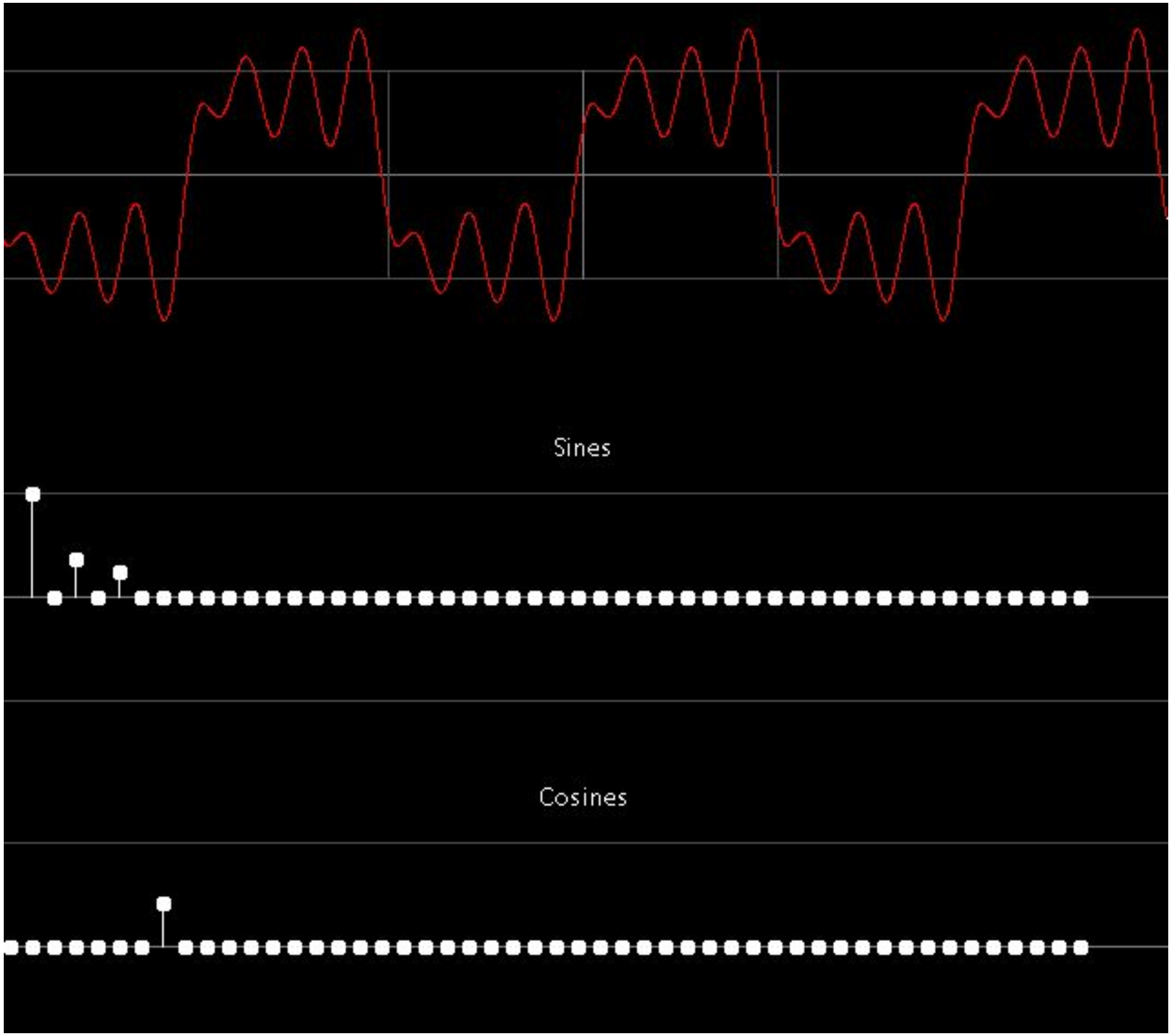
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{\frac{-j2\pi ux}{N}}, \, u = 0, 1, \ldots, N-1$$

- Inverse DFT

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{\frac{j2\pi ux}{N}}, \, x = 0, 1, \ldots, N-1$$

$F(u)$ is discrete: $F(u) = F(u\Delta u)$, $u = 0, 1, \ldots, N-1$, Δu= 1/NΔ
x

Discrete Fourier Transform (DFT)



https://www.falstad.com/fourier/

Extending DFT to 2D

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- Assume that f(x,y) is M x N image.

- Forward DFT

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

$$(u = 0, 1, \ldots, M-1, v = 0, 1, \ldots, N-1)$$

- Inverse DFT:

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

$$(x = 0, 1, \ldots, M-1, y = 0, 1, \ldots, N-1)$$

Extending DFT to 2D

- Special case: f(x,y) is N x N image.

- Forward DFT

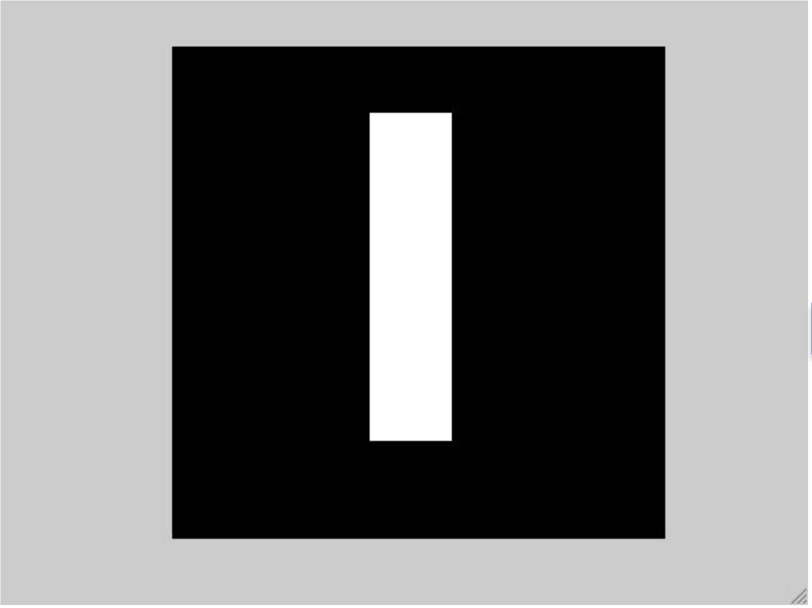
$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux+vy}{N}\right)},$$

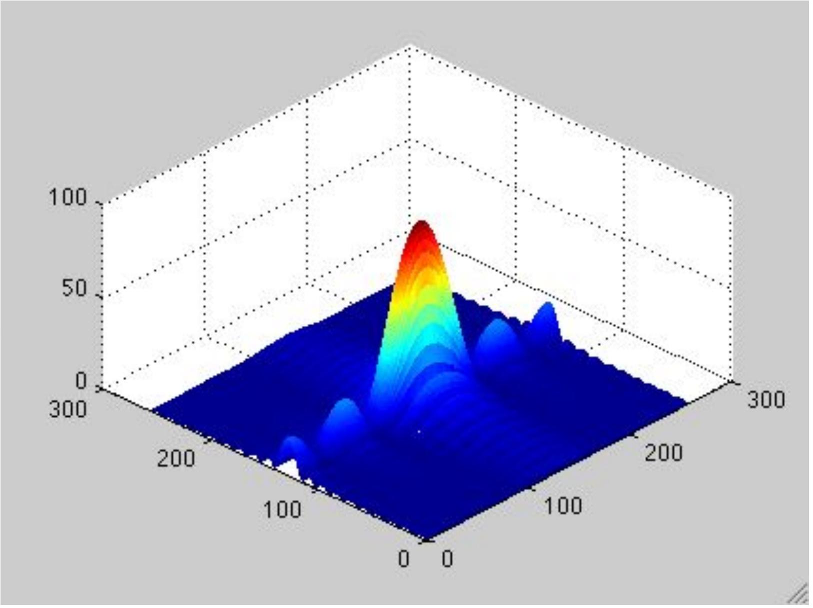
- Inverse DFT

$$f(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux+vy}{N}\right)},$$

Example: 2D rectangle function

- FT of 2D rectangle function

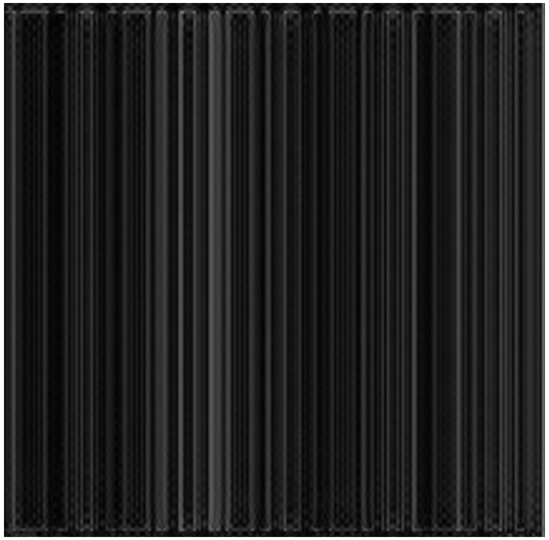




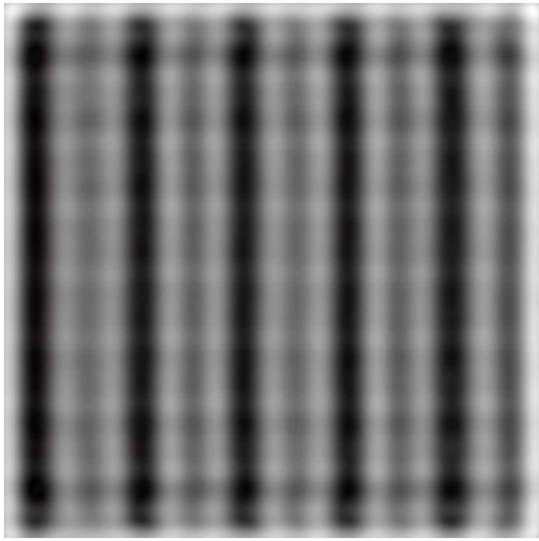
How do frequencies show up in an image?

- High frequencies correspond to quickly varying information (e.g., edges)
- Low frequencies correspond to slowly varying information (e.g., continuous surface)

Original Image

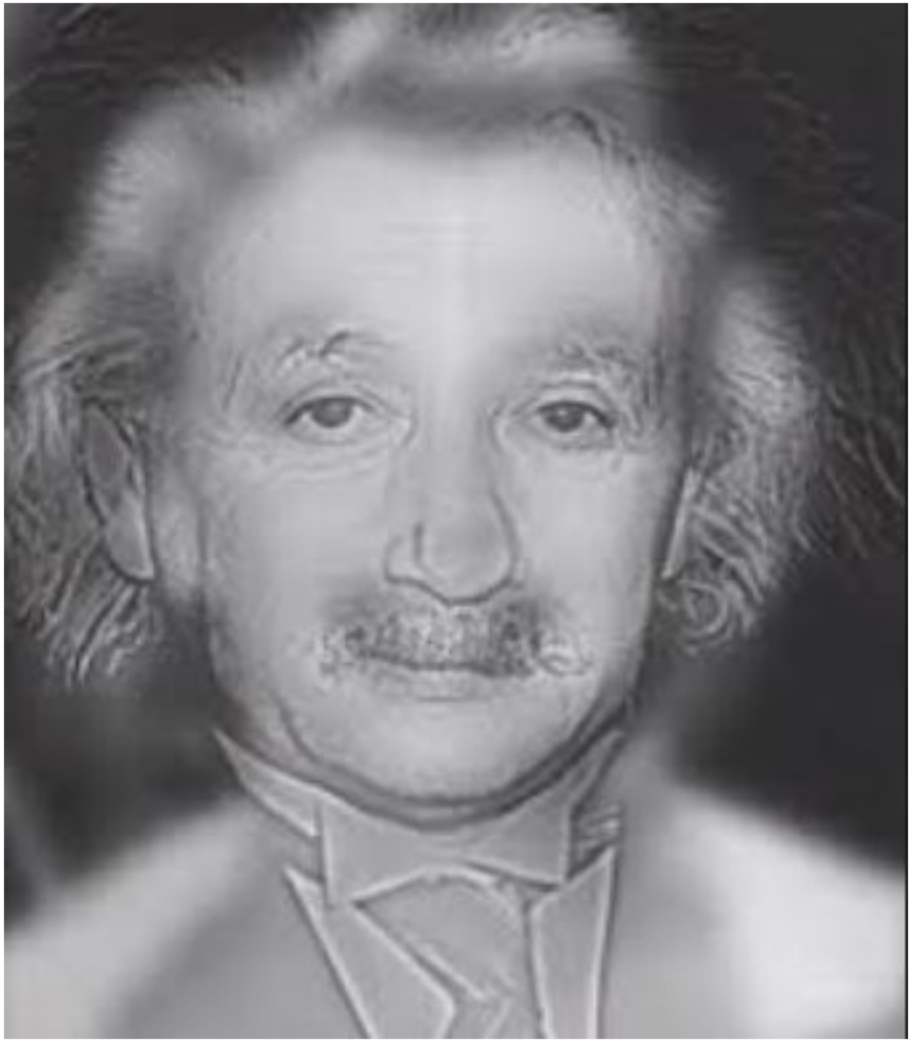


High-passed

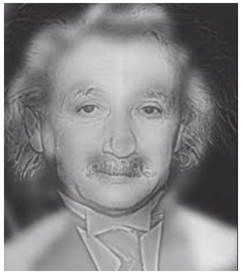


Low passed

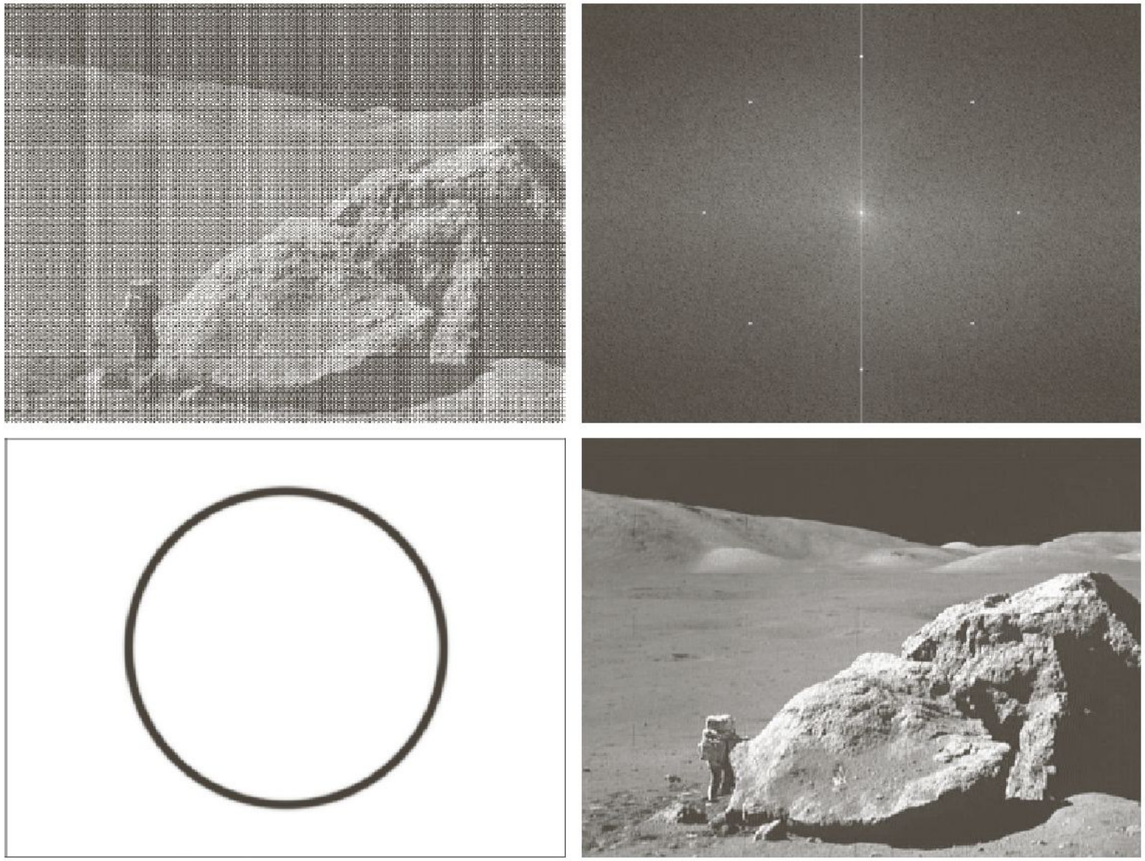
Fun example



Fun example



Example of noise reduction using FT

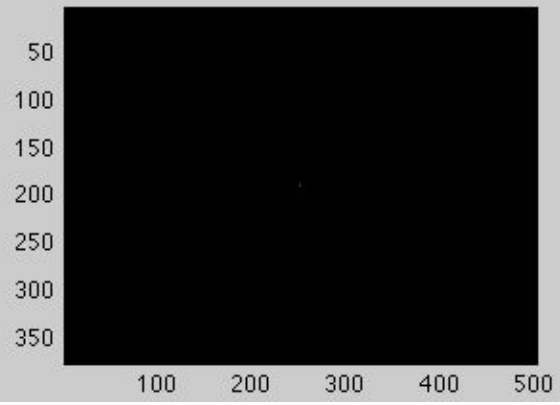


Visualizing DFT

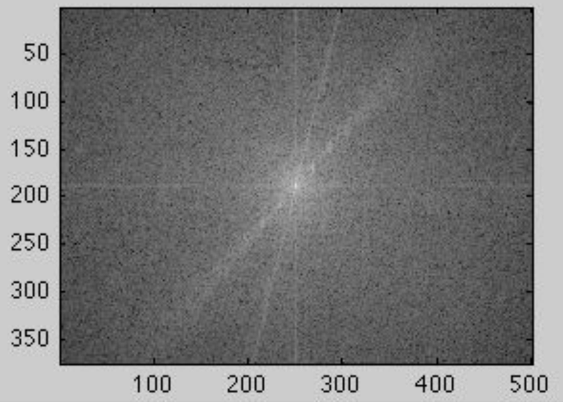
- Typically, we look at the spectrum of F(u,v) = |F(u,v)|, and since the spectrum has a very large range (why?), we apply log to compress
- We also need to shift (0,0) into the center using fftshift, resulting in the following numpy code snippet:

```
fspt = np.fft.fftshift(np.fft.fft2(greyscale_img))
plt.figure()
plt.imshow(np.log(abs(fspt)), cmap='gray');
```

original image



before scaling



after scaling

Why the shift? Visualizing the FFT

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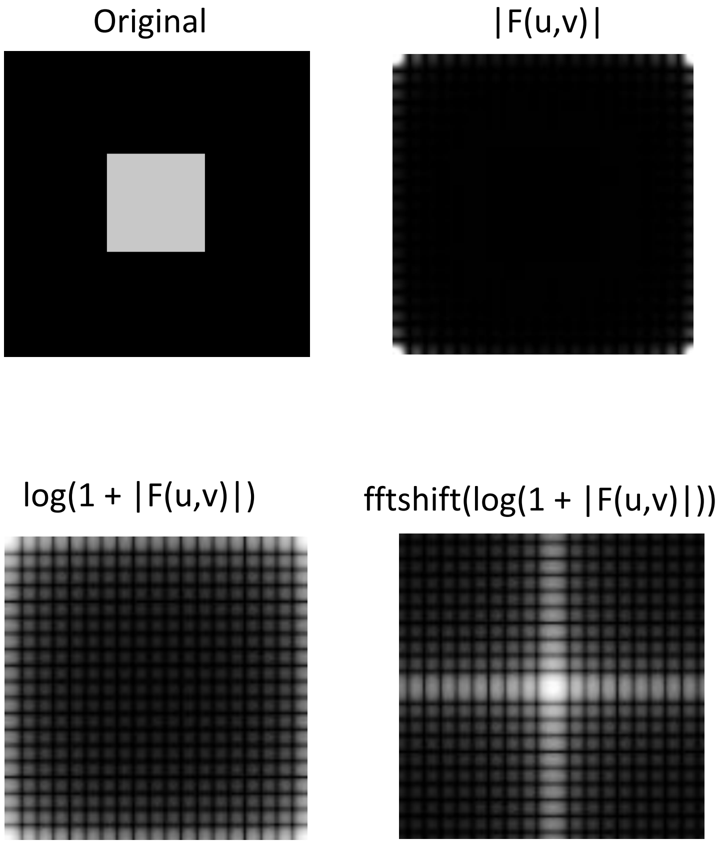
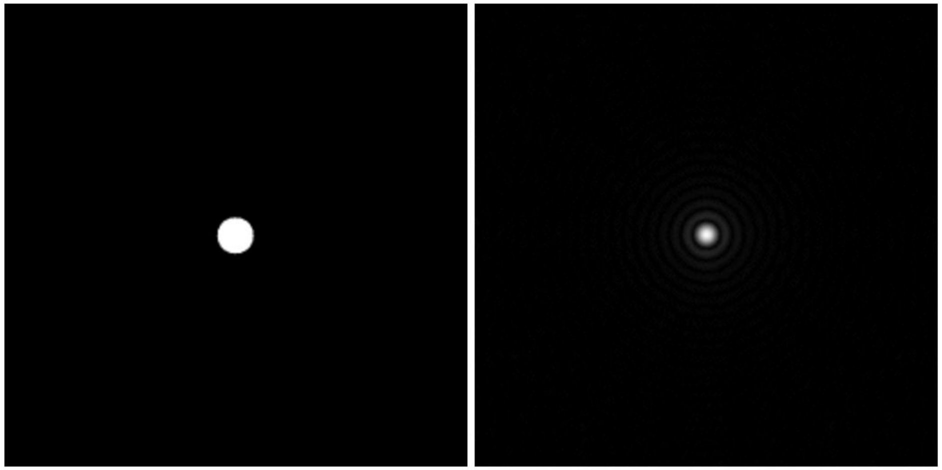
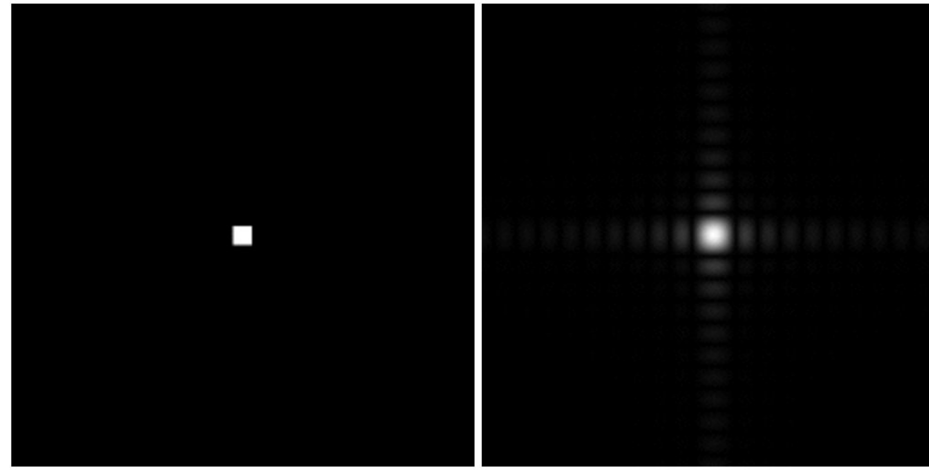


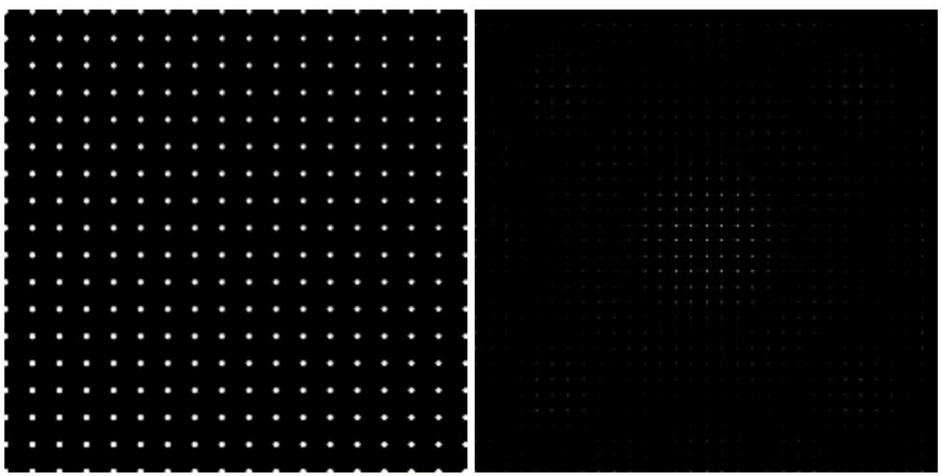
FFTs of some simple patterns



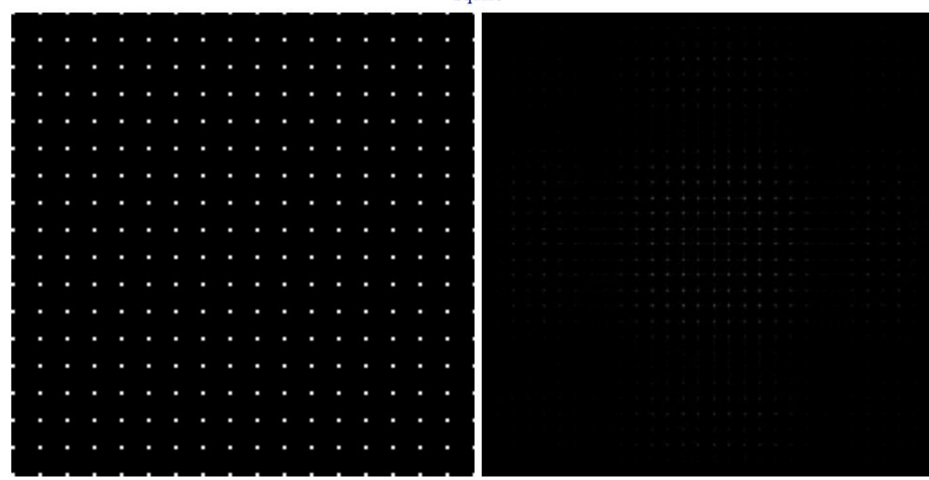
Circle



Square



Periodic circles

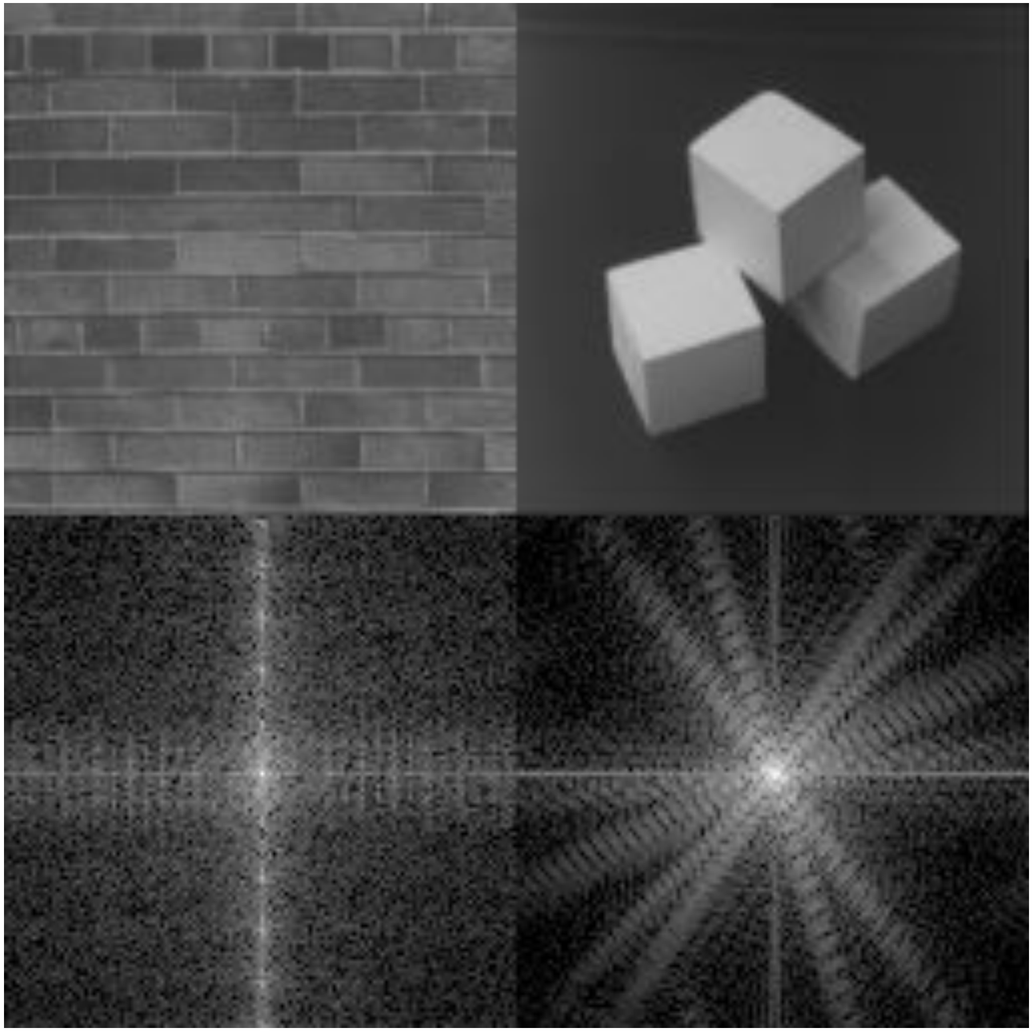


Periodic squares

https://www.cs.auckland.ac.nz/courses/compsci773s1c/lectures/ImageProcessing-html/topic1.htm

Real-world images

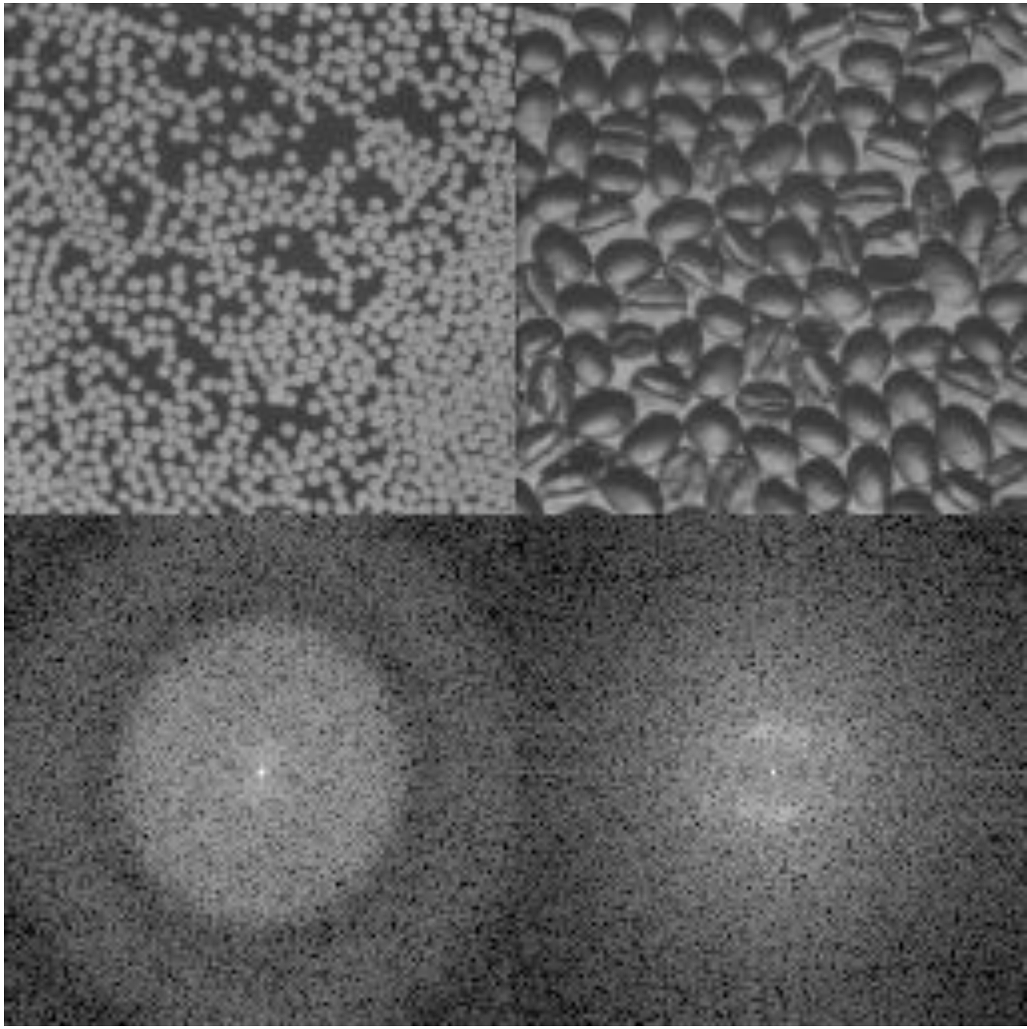
- Edges are clearly visible as patterns in the spectrum
  - In fact, this is used in X-Ray spectroscopy to identify crystal structures of materials



https://www.cs.auckland.ac.nz/courses/compsci773s1c/lectures/ImageProcessing-html/fourier.html

Real-world images

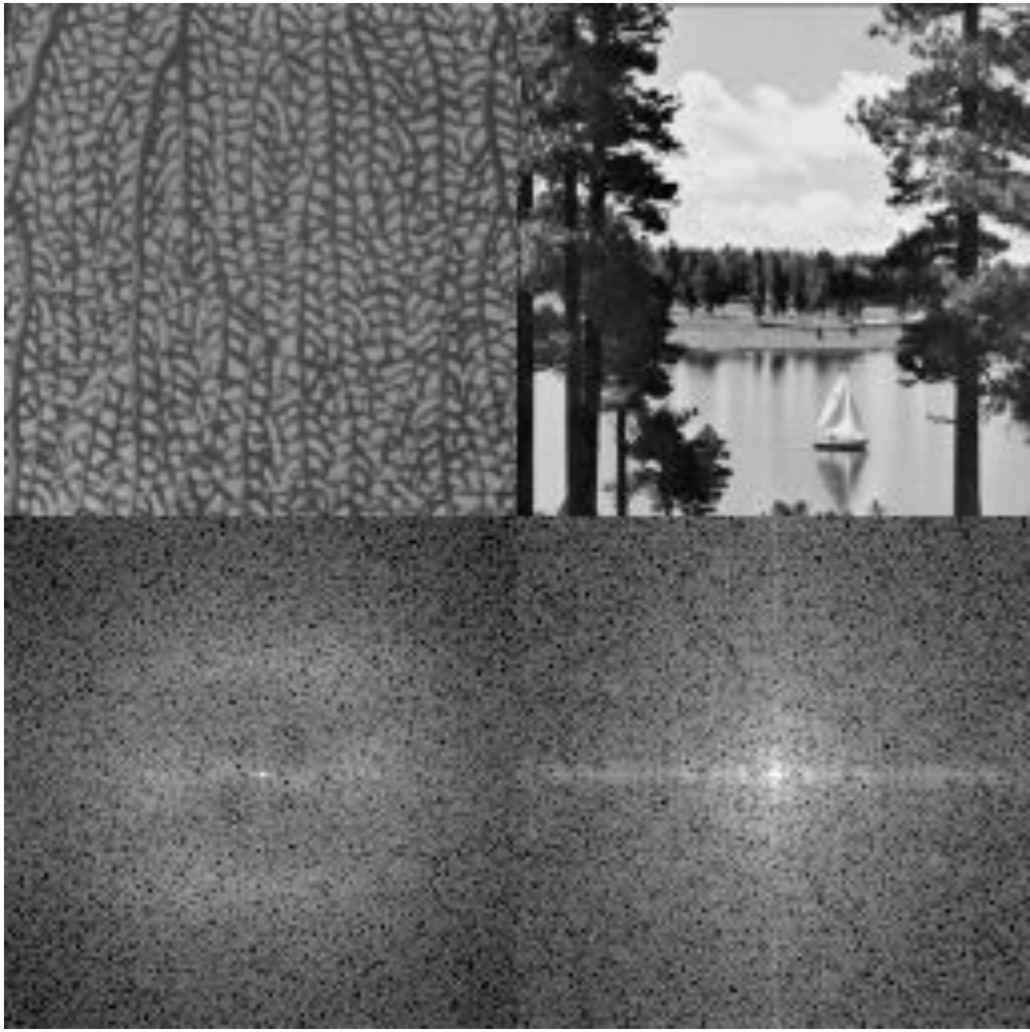
- The halo on the left is due to the FT of one single pellet – the FT spectrum would therefore look same, whether we had one or more pellets!
- The halo on the left is due to the shape of one coffee bean



https://www.cs.auckland.ac.nz/courses/compsci773s1c/lectures/ImageProcessing-html/fourier.html

Real-world images

- The repeated structure in the seafan picture gives rise to the halo in the spectrum
- The horizontal, lower-frequency peaks in the lake picture are due to the tree trunks
- Note also the absence of the cross-artefact in the seafan versus its presence in the lake picture!

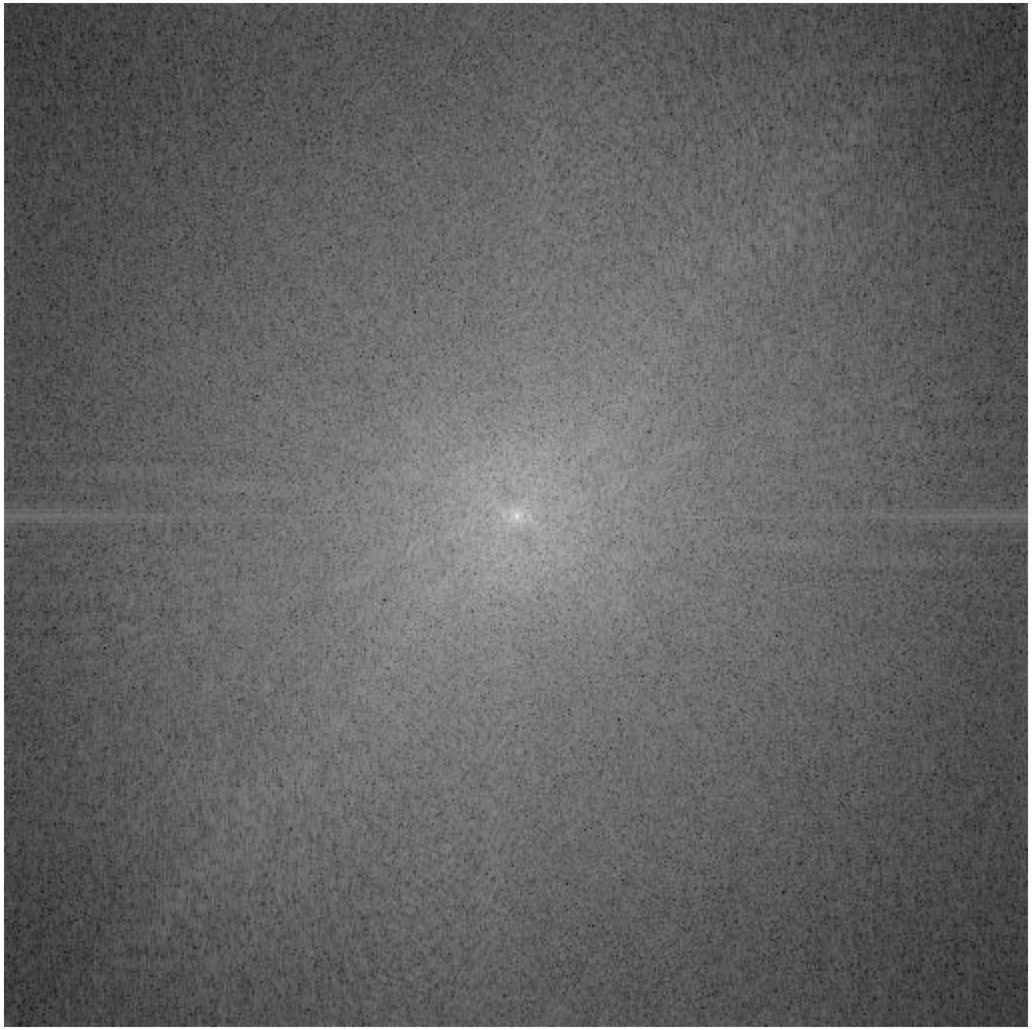


https://www.cs.auckland.ac.nz/courses/compsci773s1c/lectures/ImageProcessing-html/fourier.html

Cheetah – space representation

Slide: Freeman & Durand

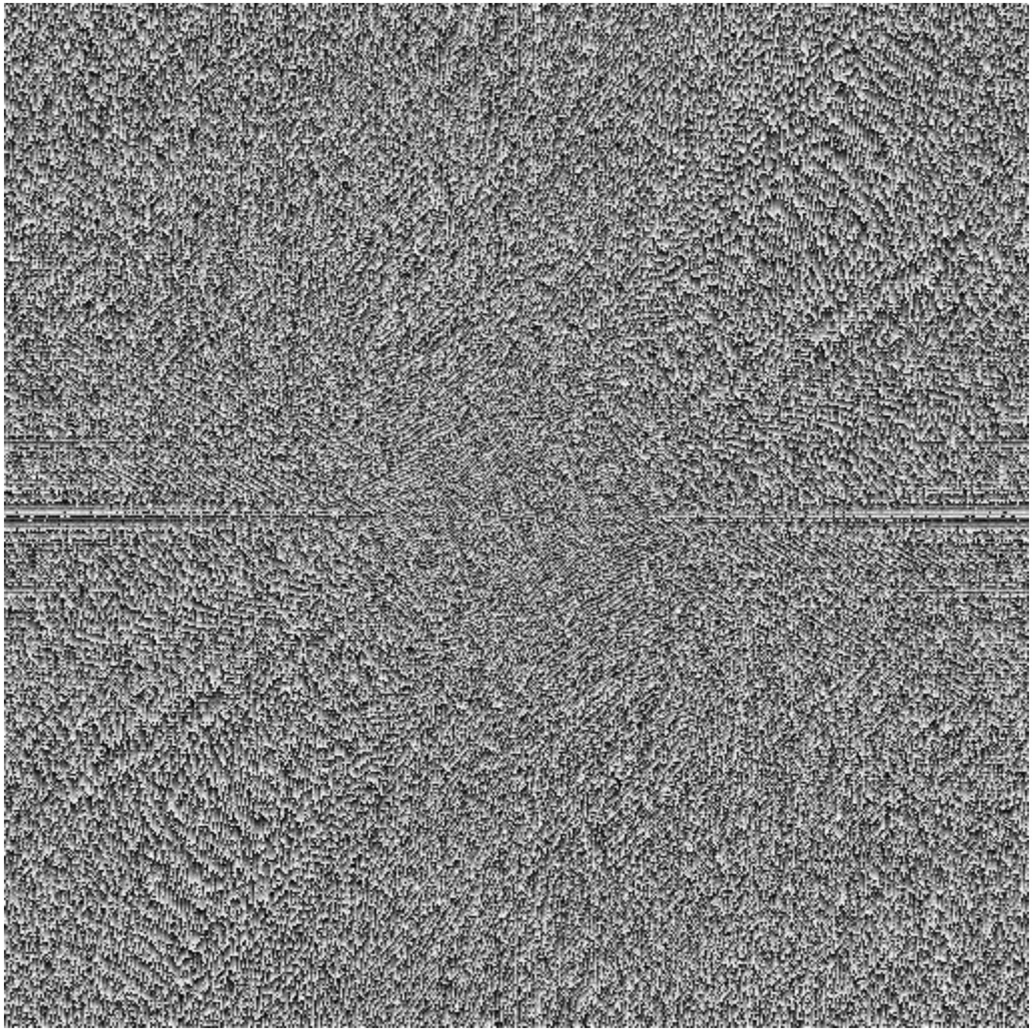
Cheetah – Fourier magnitude



This is the magnitude transform of the cheetah pic

Slide: Freeman & Durand

Cheetah – Fourier phase



This is the phase transform of the cheetah pic

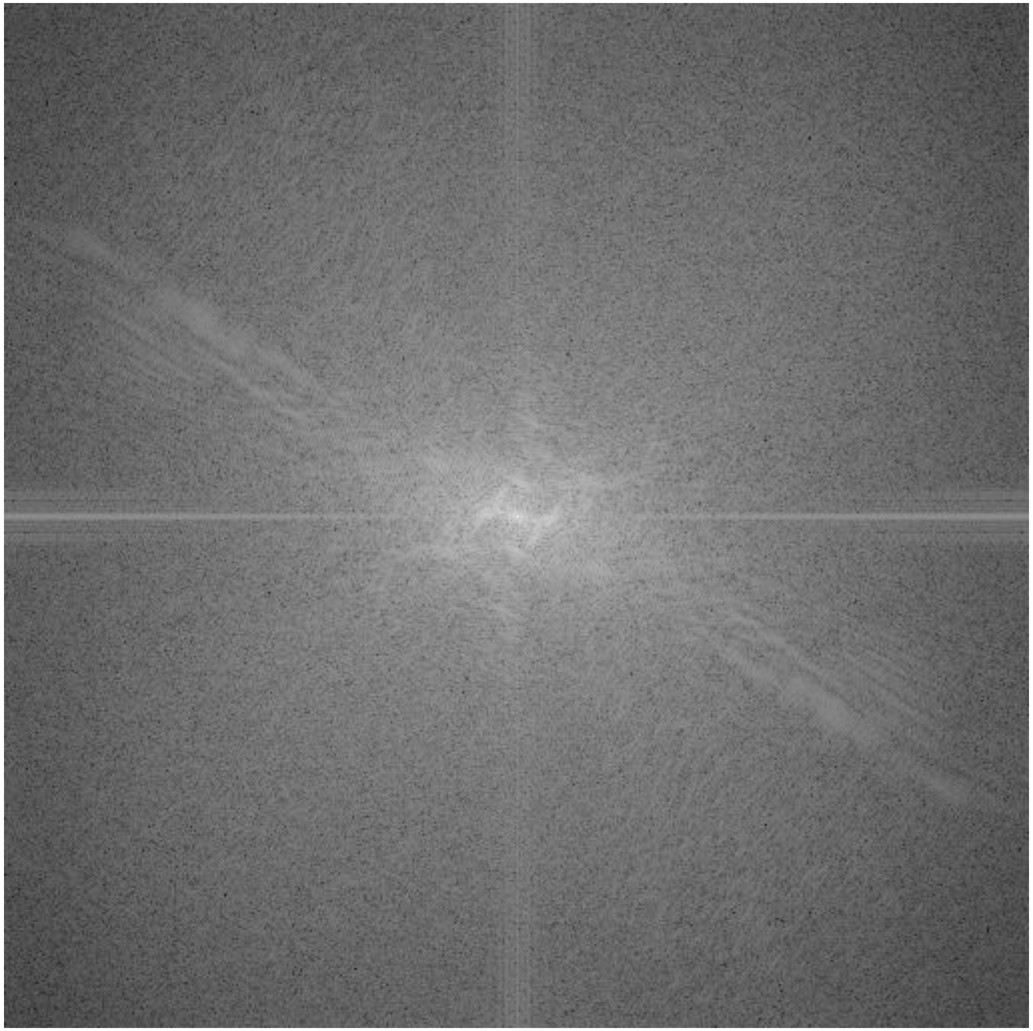
Slide: Freeman & Durand

Zebra – space representation

CognitiveSystems

Slide: Freeman & Durand

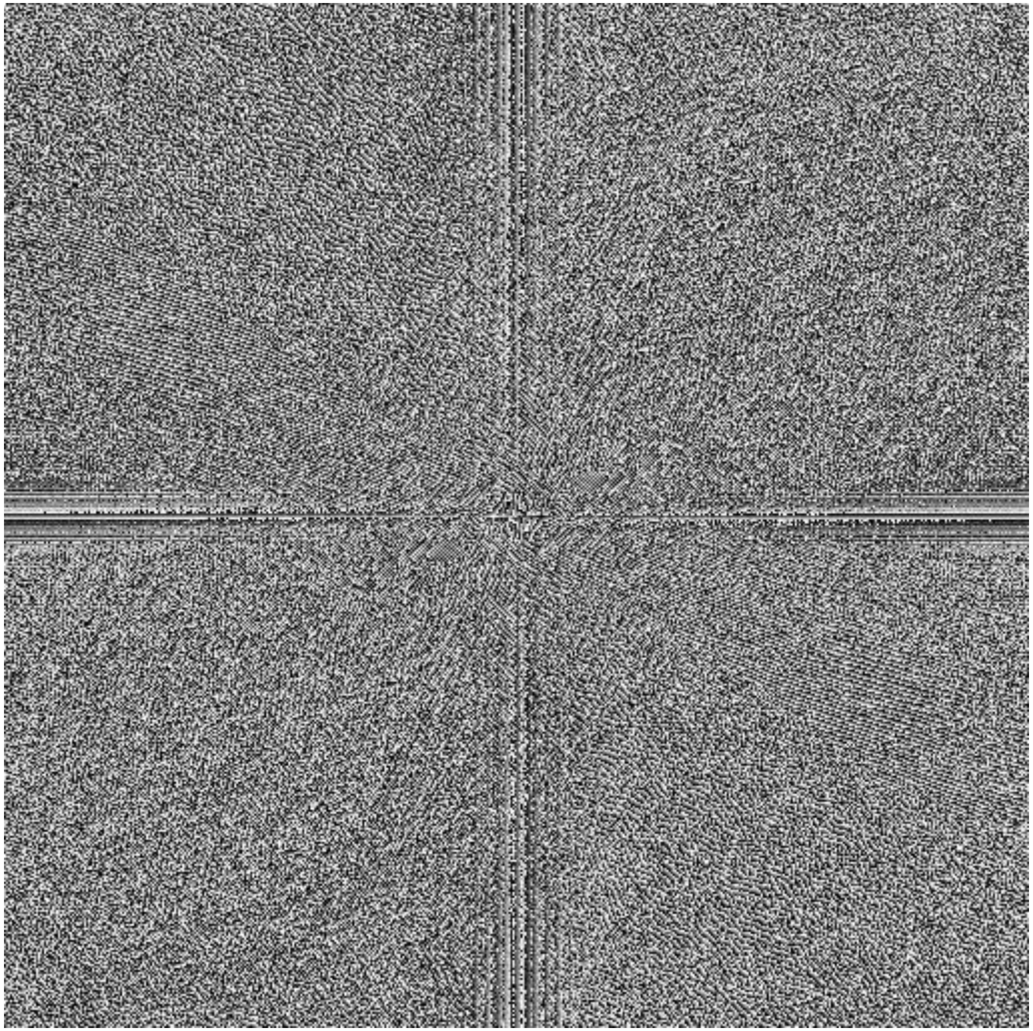
Zebra – Fourier magnitude



This is the magnitude transform of the zebra pic

Slide: Freeman & Durand

Zebra – Fourier phase

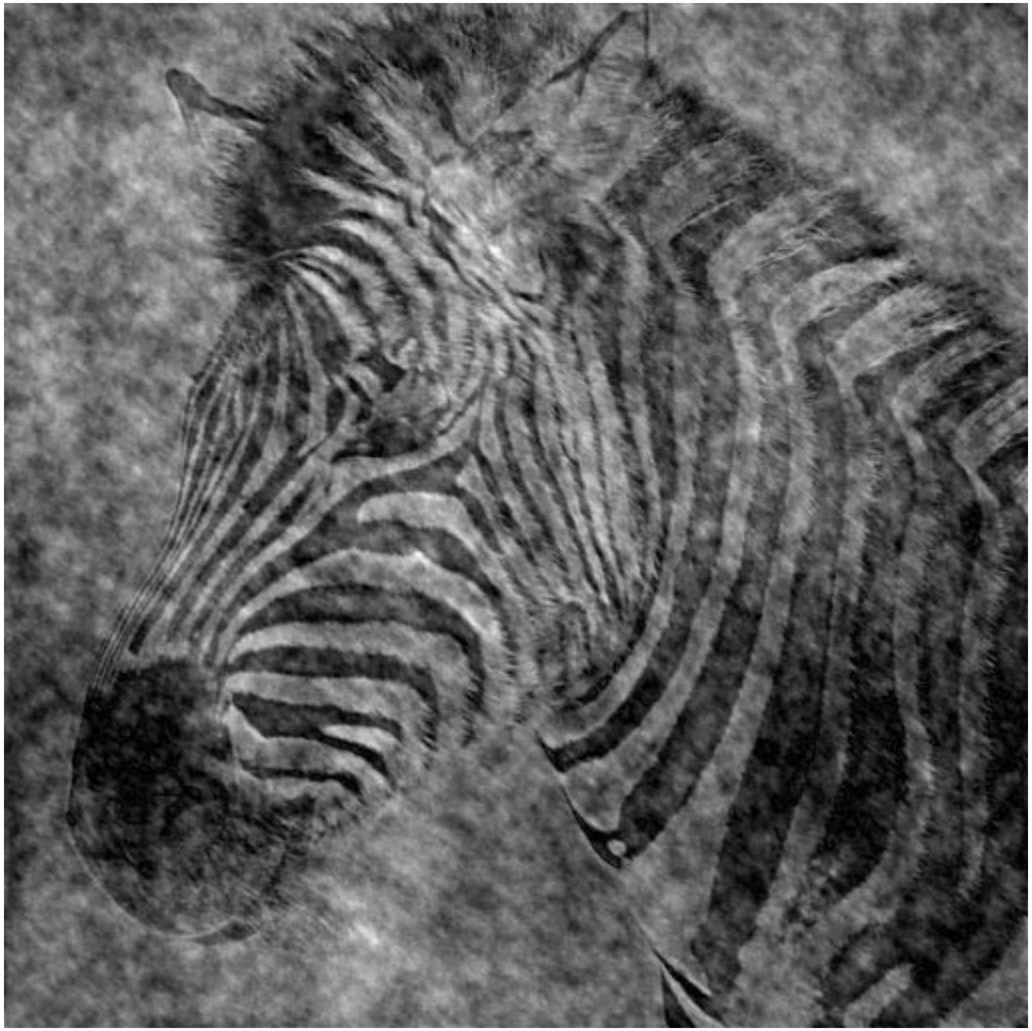


This is the phase transform of the zebra pic

Slide: Freeman & Durand

IFFT(Zebra phase,Cheetah magnitude)

- As we can see, the inverse Fourier transform of the Zebra phase and Cheetah magnitude yields a clearly visible Zebra

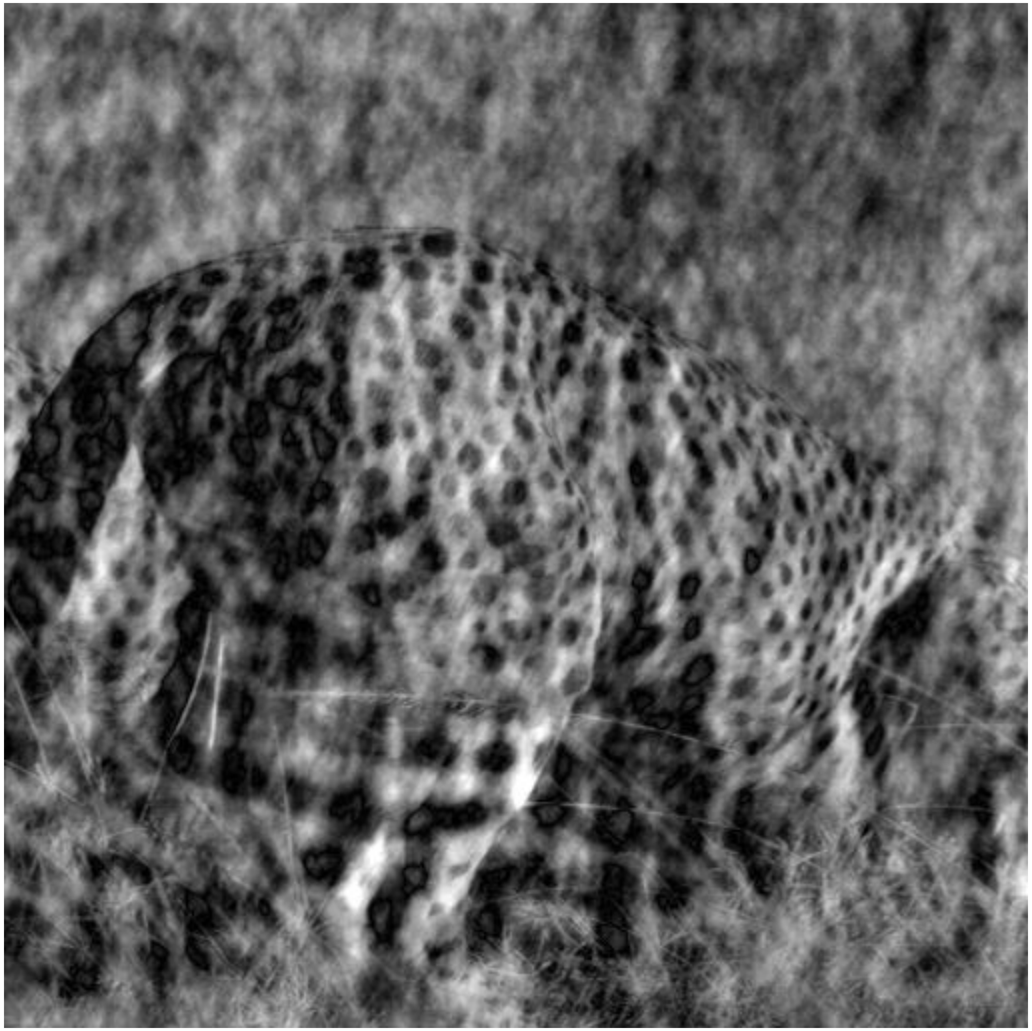


Reconstruction with zebra phase, cheetah magnitude

Slide: Freeman & Durand

IFFT(Cheetah phase, Zebra magnitude)

- As we can see, the inverse Fourier transform of the Cheetah phase and Zebra magnitude yields a clearly visible Cheetah



Reconstruction with cheetah phase, zebra magnitude

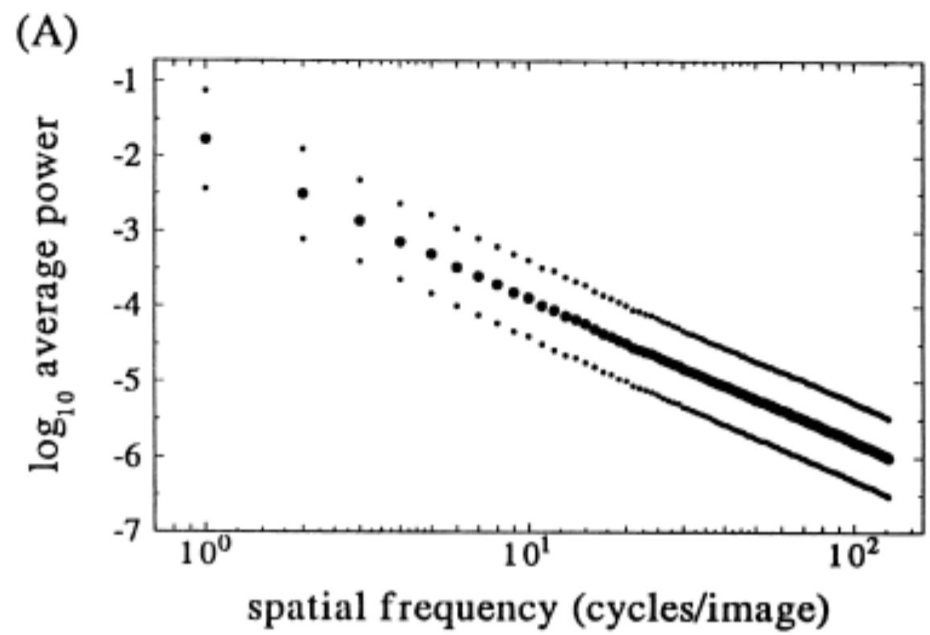
Slide: Freeman & Durand

Phase and Magnitude

- Why is this possible???
  - When you look at natural images, they have very similar power spectra – actually, the same holds for natural sounds, too!
  - this also means that the perceptual system is optimized to exploit these regularities
- What does this mean for FTs of images?
  - Phase matters for **image content**, but magnitude largely does not!

Real-world images

- The visual system needs to exploit structure in images to deal with the enormous amount of data in them
- Barlow already proposed the idea to deal with perception as efficient coding
- Van der Schaaf and Van Hateren extensively investigated the statistics of natural images using the FT
- They found that – on average – the log-log power spectrum shows a straight line with an exponent of -2

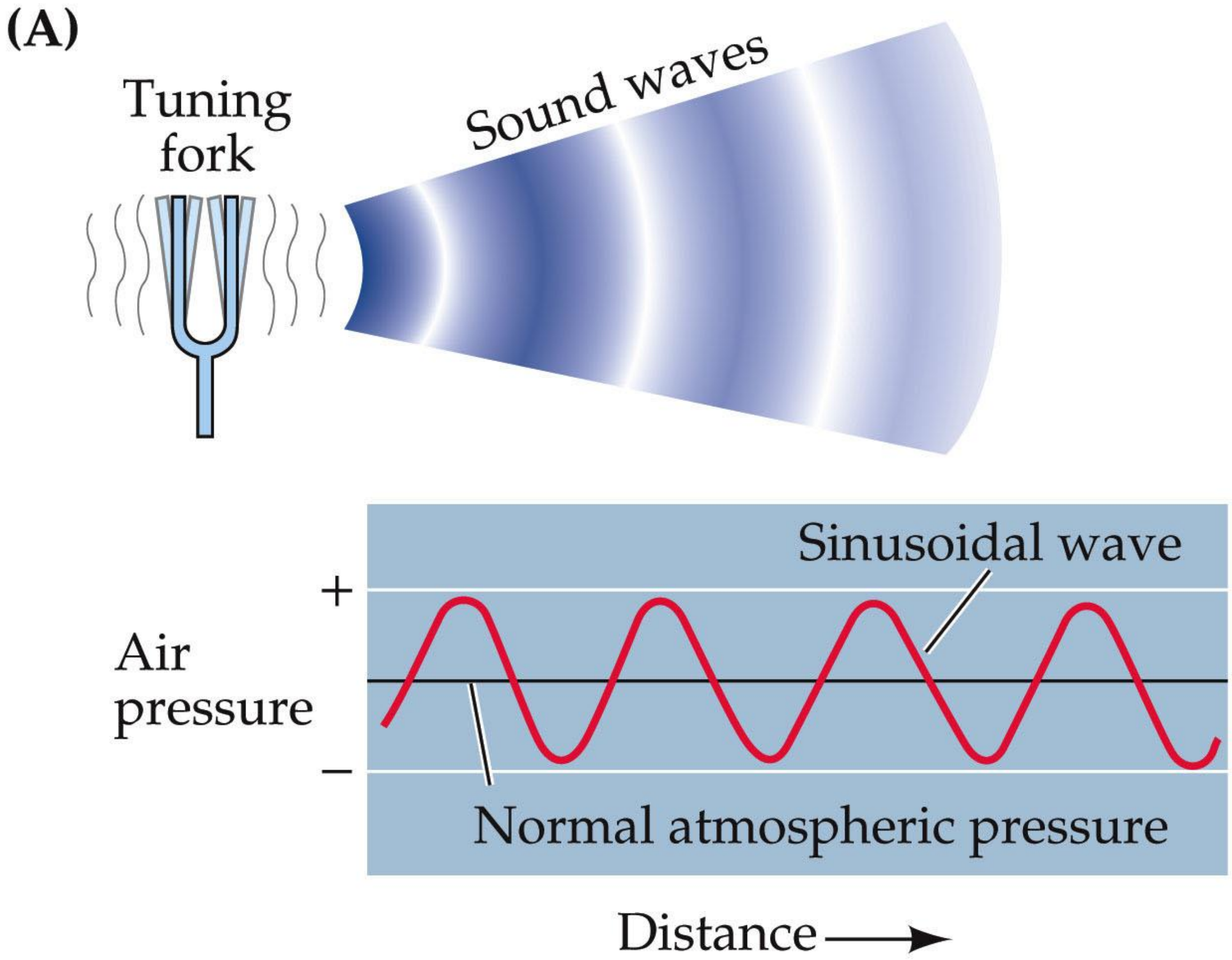


Van Schaaf, Van Hateren, Vision Research, 1996

Examples of image processing

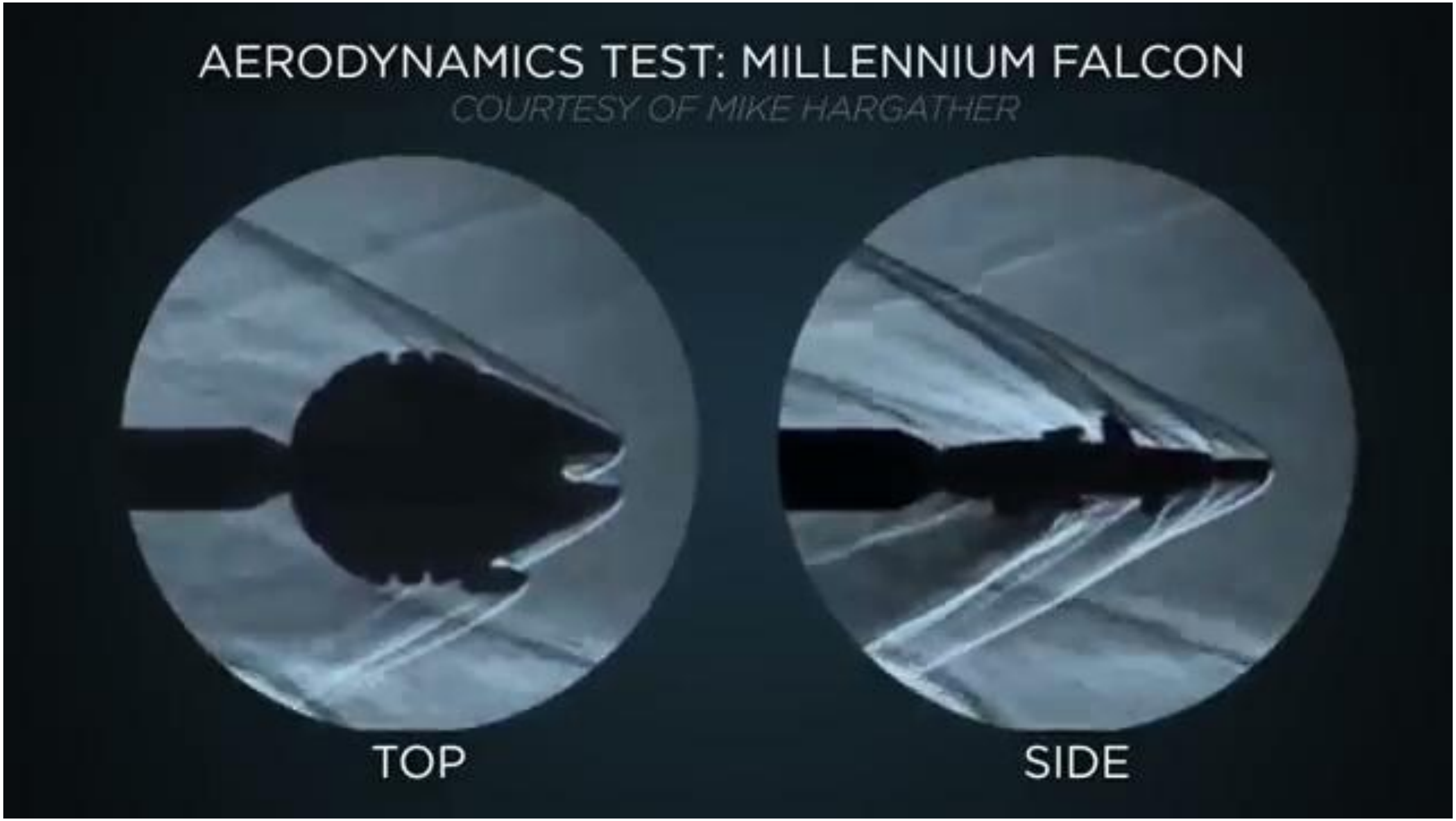
- http://micro.magnet.fsu.edu/primer/java/digitalimaging/processing/fouriertransform/

Sound is a wave



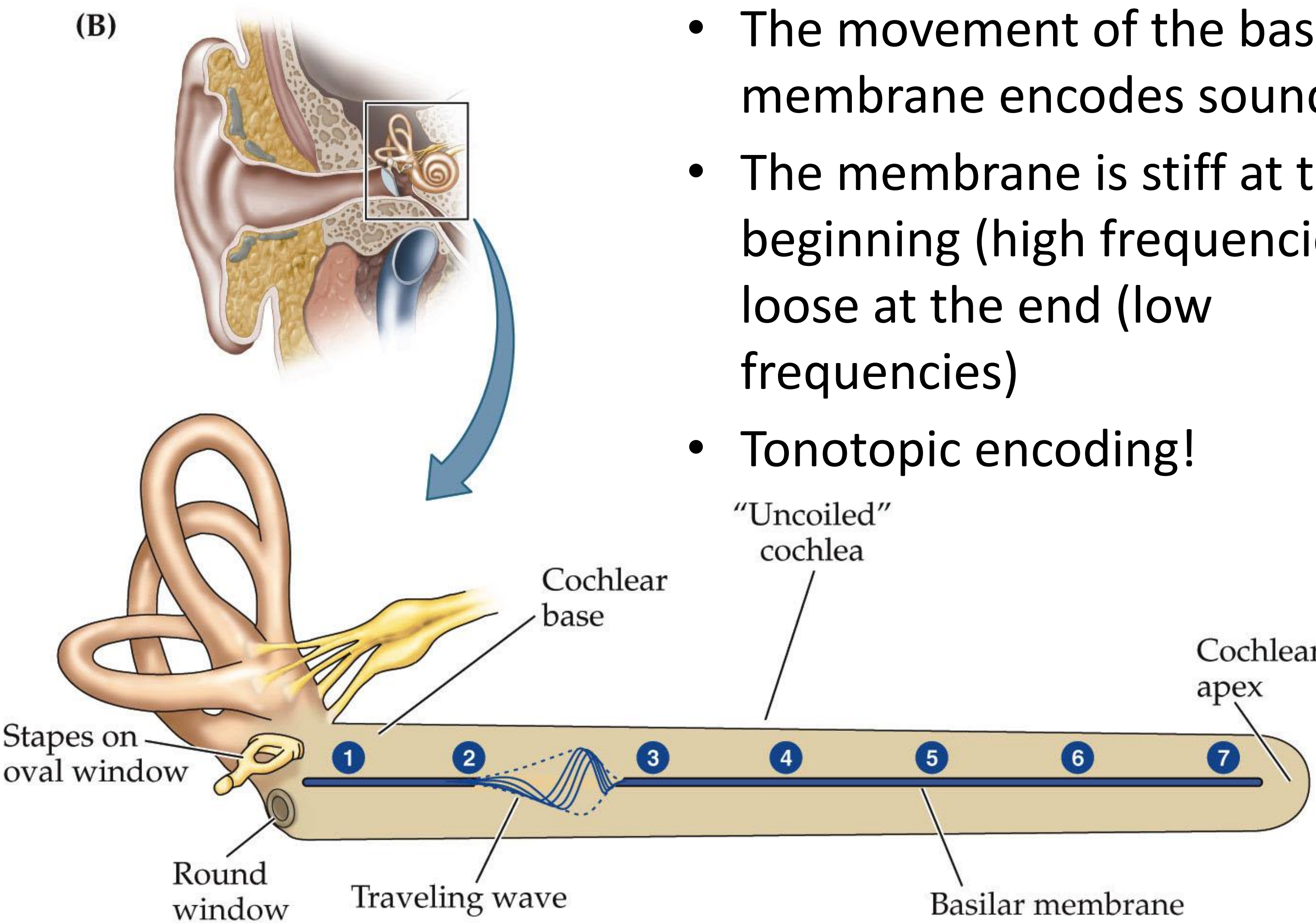
***Principles of Cognitive Neuroscience*, Figure 6.4 (Part 1)**

What does sound look like – Schlieren photography



The ear as a frequency analyzer

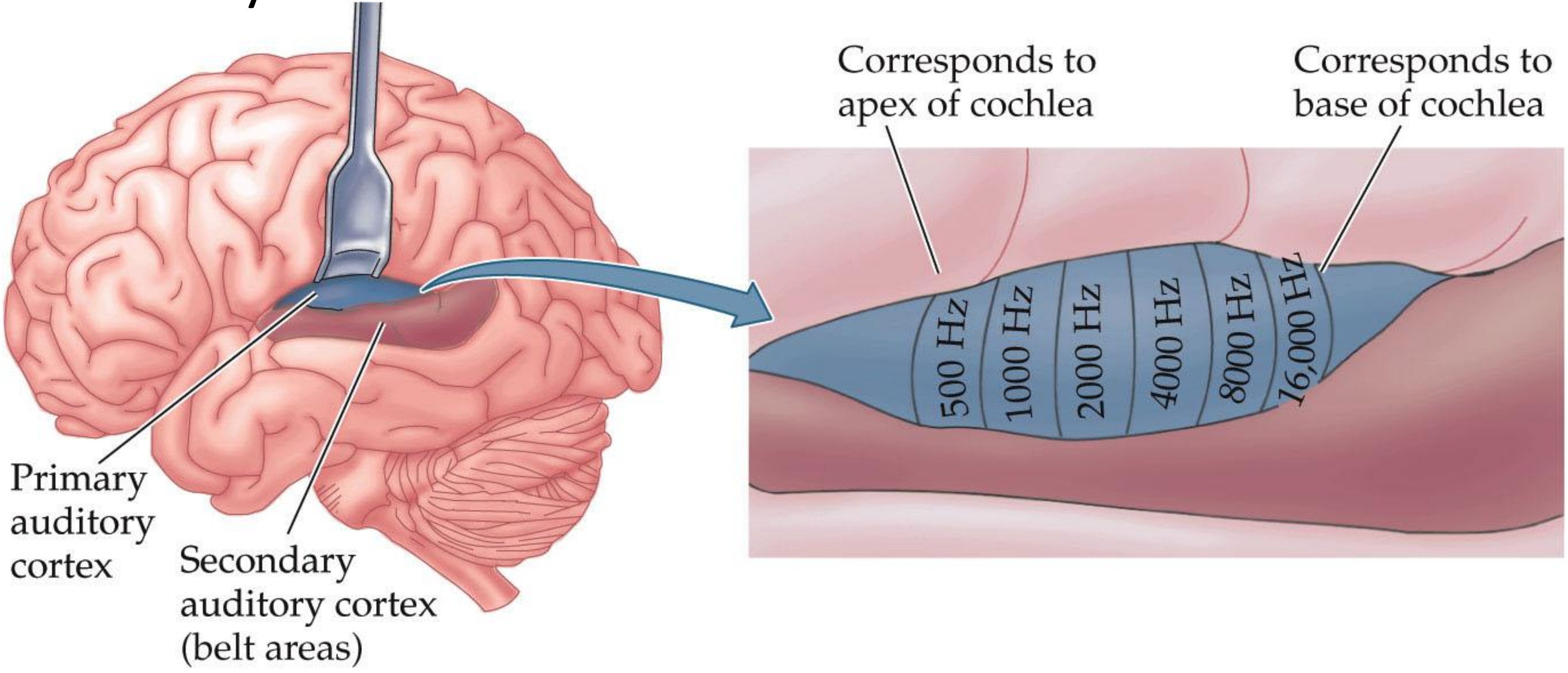
- The movement of the basilar membrane encodes sound
- The membrane is stiff at the beginning (high frequencies) and loose at the end (low frequencies)
- Tonotopic encoding!



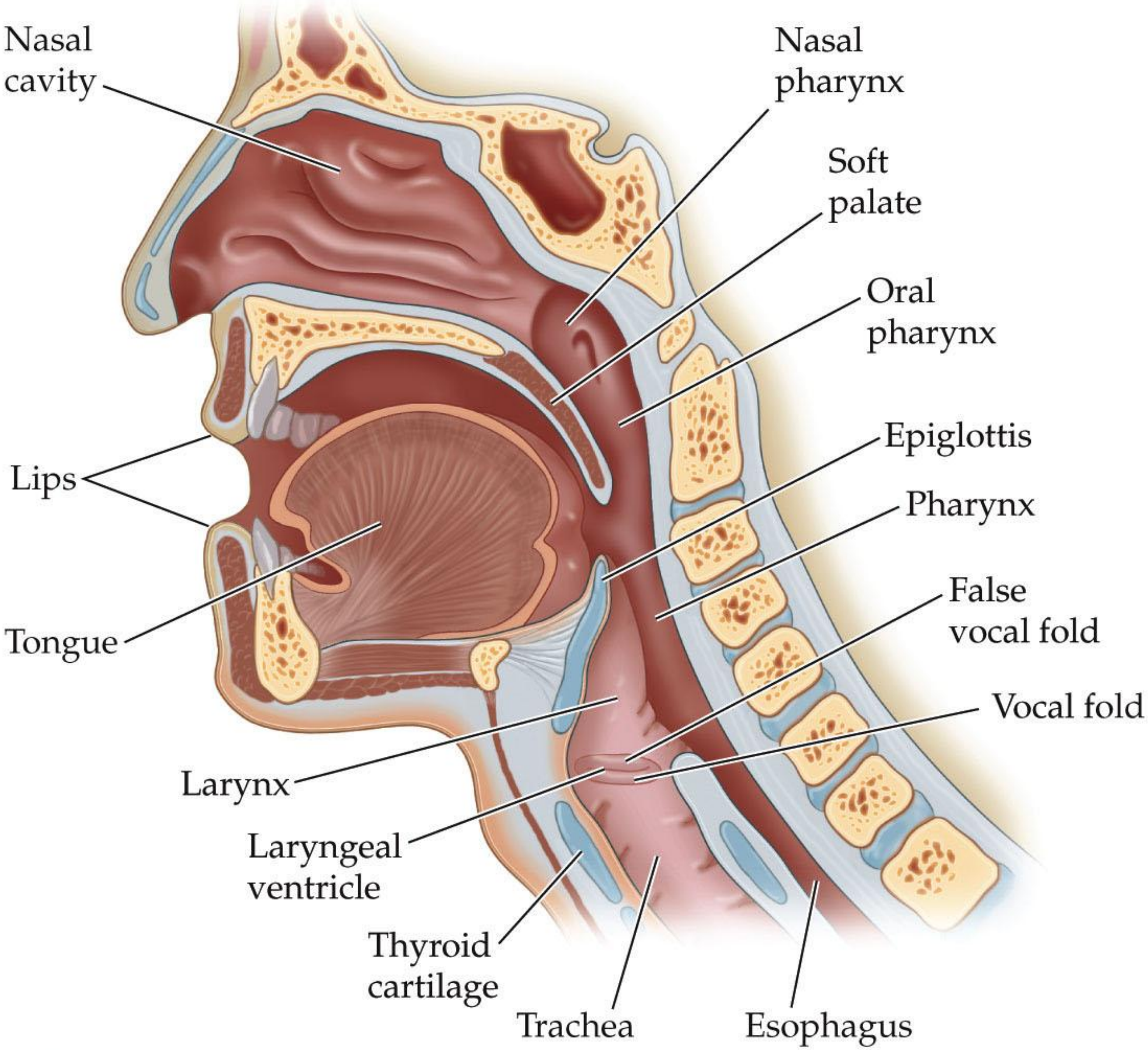
***Principles of Cognitive Neuroscience*, Figure 6.1 (2)**

Tonotopy in the brain

- Tonotopic organization of auditory cortex – highly similar to the retinotopic organization of visual cortex
- The belt areas process higher-order information about sound (auditory entities, such as speech, music, etc.) similarly to the extrastriate areas



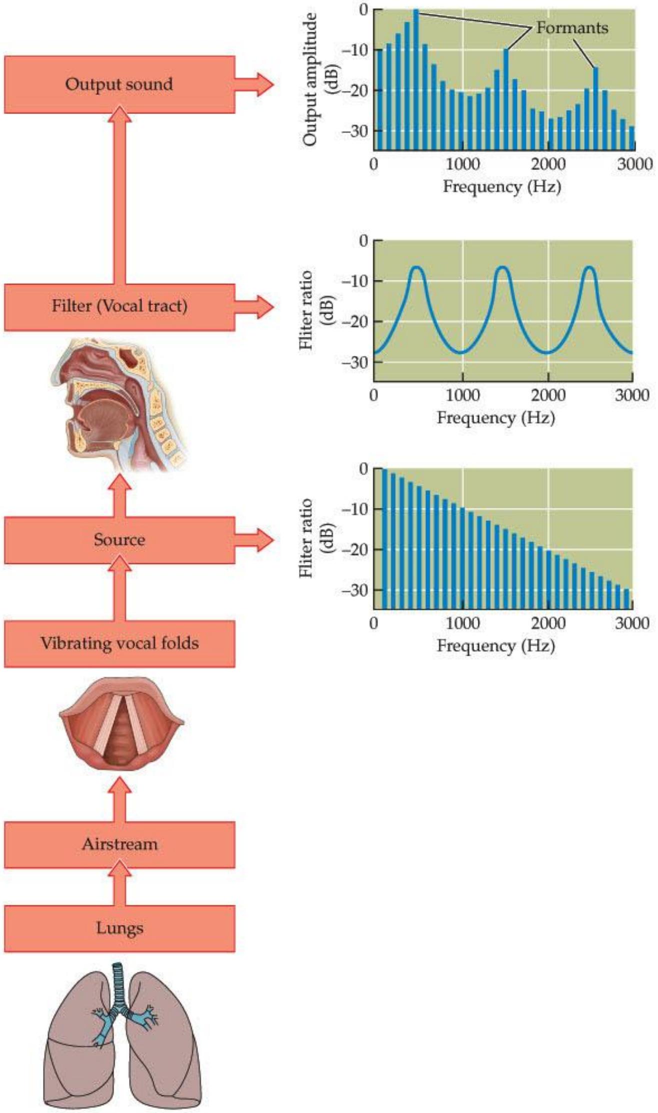
Making speech



Principles of Cognitive Neuroscience, Figure 20.2

Models of how the vocal tract produces vowels

Source-filter model of speech



ɔscience, Figure 20.3

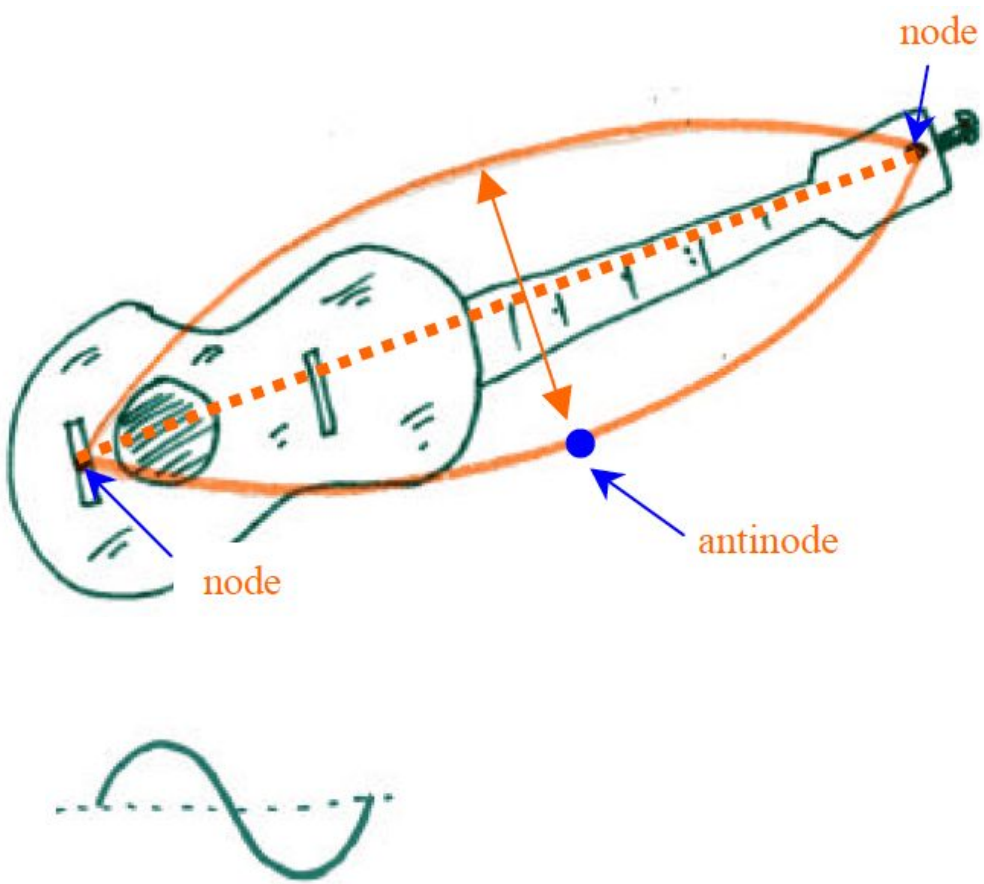
- Miller proposed a source-filter model of speech sound
- Lungs provide air
- Vocal chords produce base vibration for tonal sounds
- The remaining structure act as filters on that vibration

Getting it from waves

- The physical characteristics of sound are very different from our perception
- Try to match the sentence “This is a glad time, indeed” to the soundwaves depicted here!

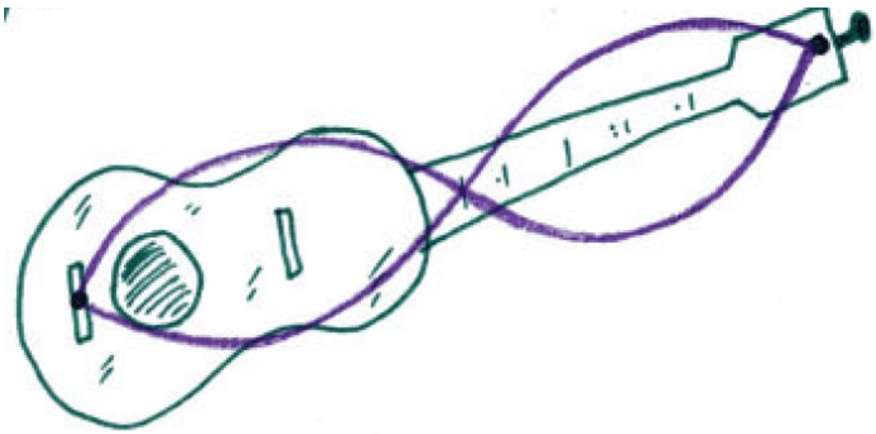
A foray into music

- When you pluck a string on a guitar, the string vibrates like this
- If the length of the guitar-string is L, then the wavelength of this vibration is?
- 2L with frequency f0



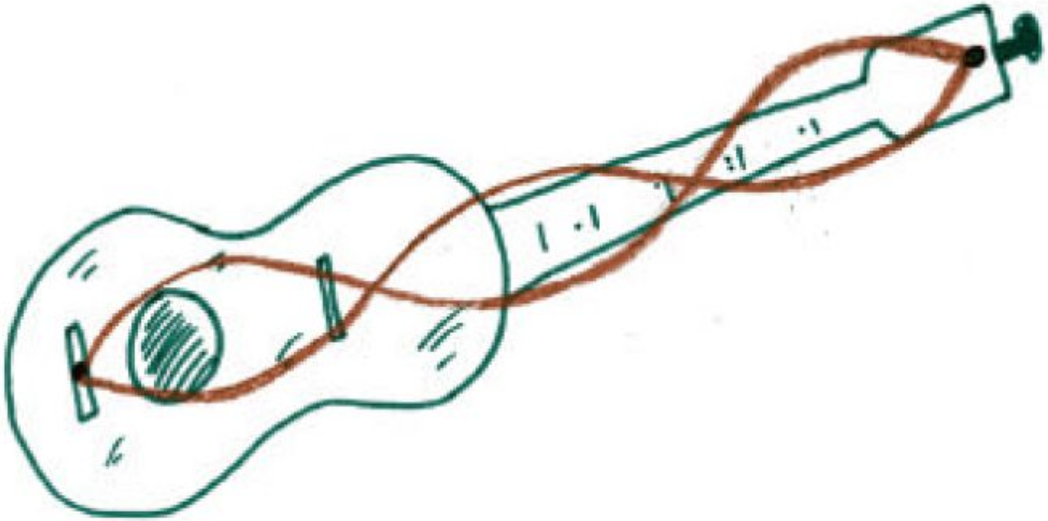
A foray into music

- But, not all parts of the string vibrate at the same frequency
- The next higher frequency that fits onto the fixed string is
- 2 * f0 with wavelength 1/2 * 2*L
- This is called the first overtone (harmonic)



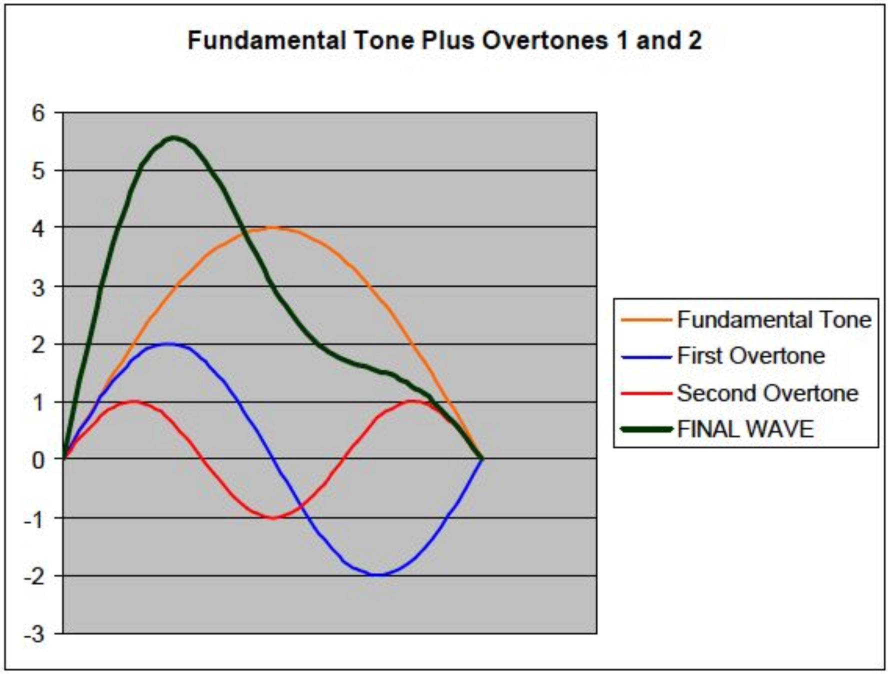
A foray into music

- The next overtone (harmonic) looks like this
- This has 3 * f0 with wavelength 1/3 * 2*L
- This is called the second overtone (harmonic)



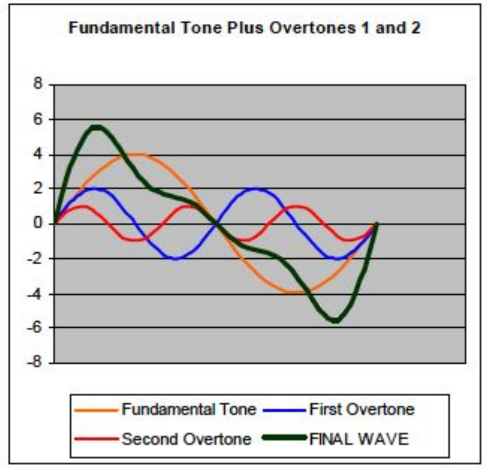
A foray into music

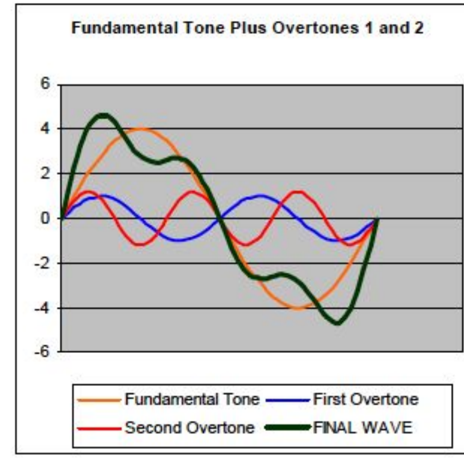
- How do we get music now?
- We add all the waves from all overtones

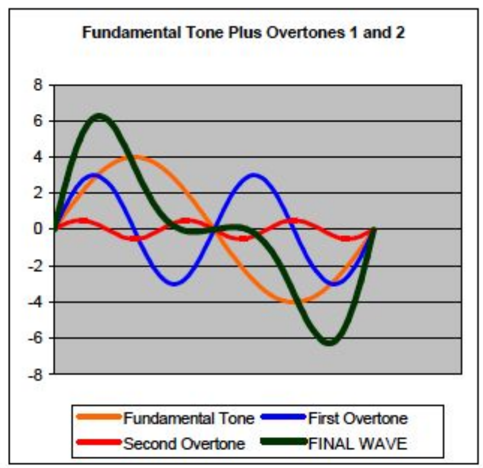


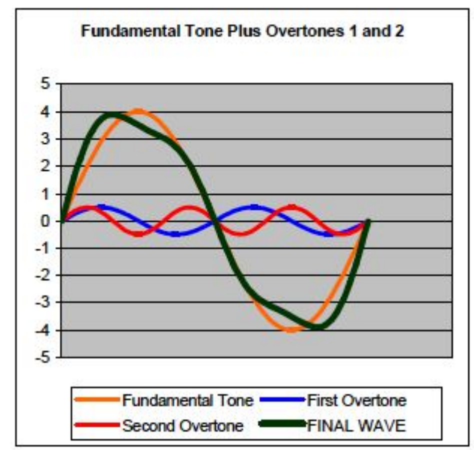
A foray into music

- And why do instruments sound all different?
- Because of the amount of how each overtone contributes to the whole



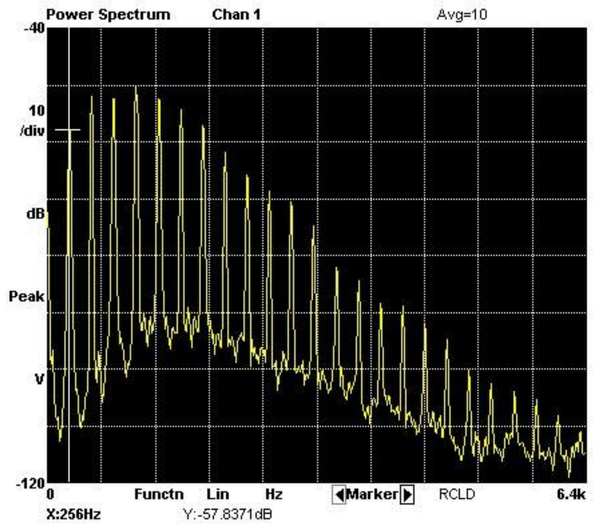




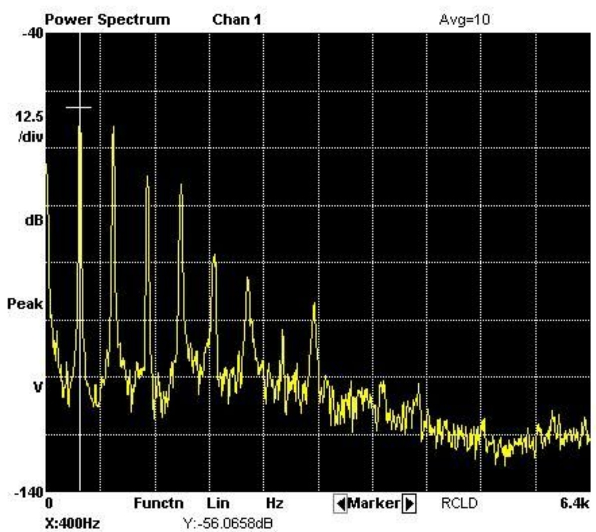


A foray into music

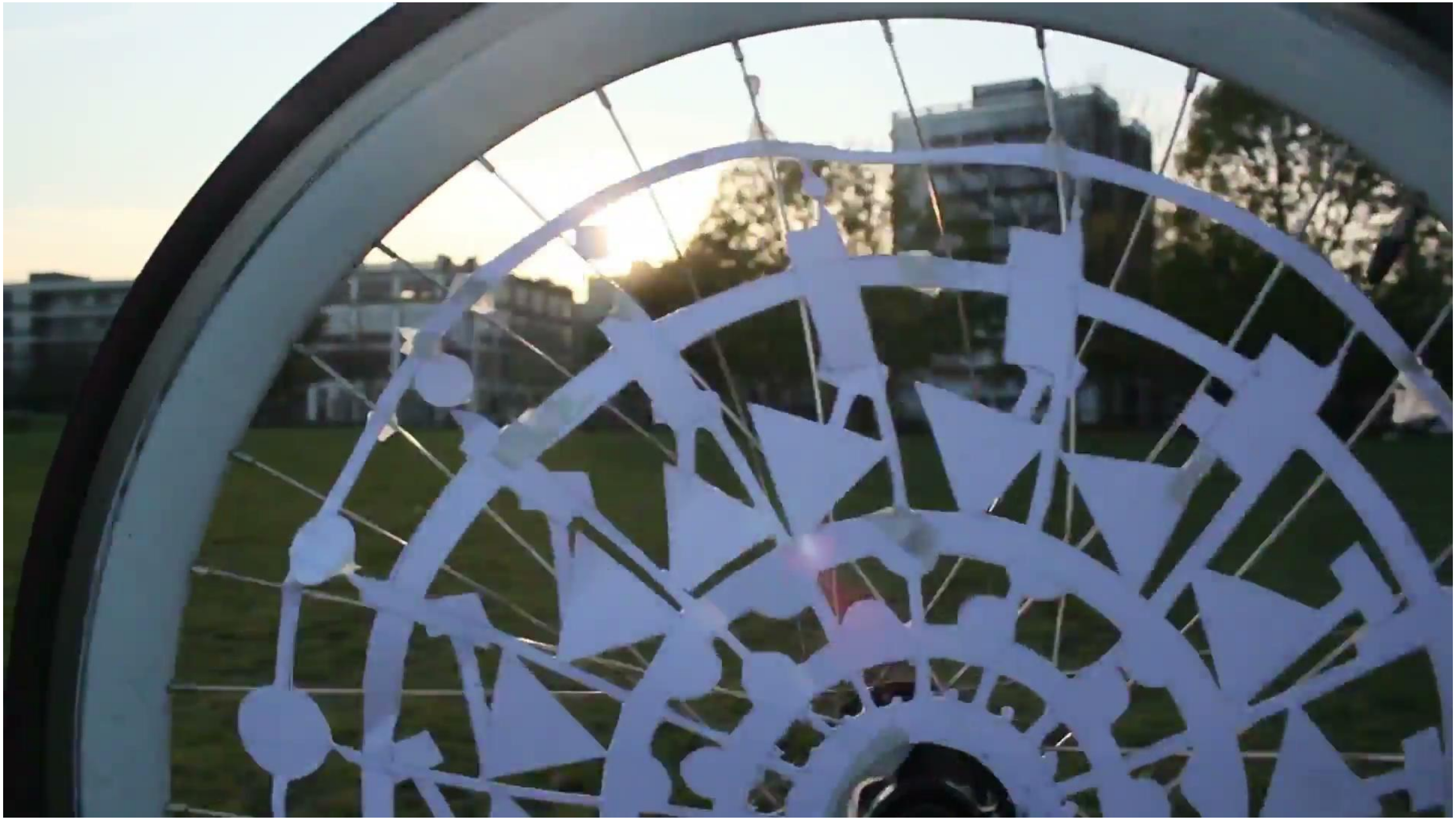
- Power spectrum of a trumpet



- Power spectrum of a clarinet

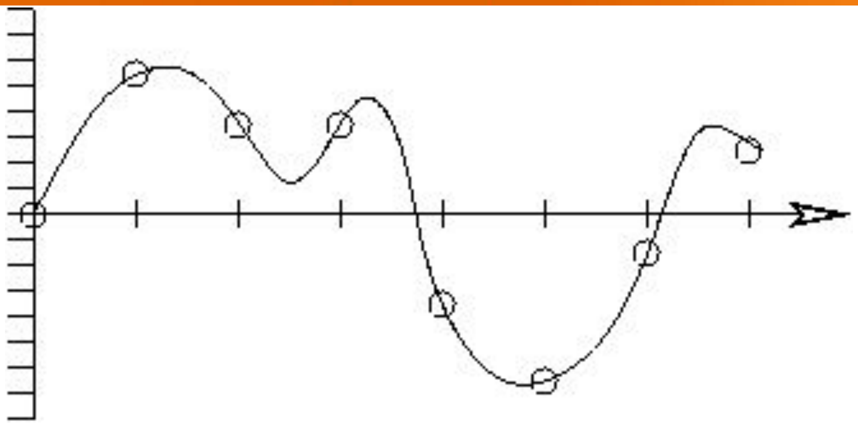


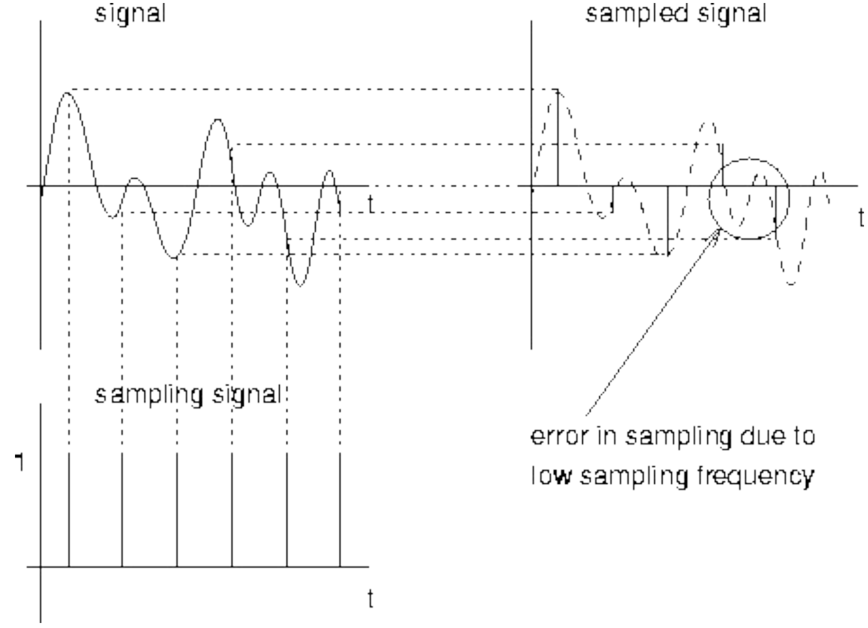
Sampling



Sampling: from analog to digital

- Real signals are continuous, but the computer can only handle discretized data.
  - We therefore have to convert analog to digital and vice versa (**ADC** and **DAC**)
- Sampling then means to measure the analog signal at different moments in time, recording its physical property
  - For EEG-recordings, for example, this is the voltage of the signal
  - This forms an approximation to the original signal



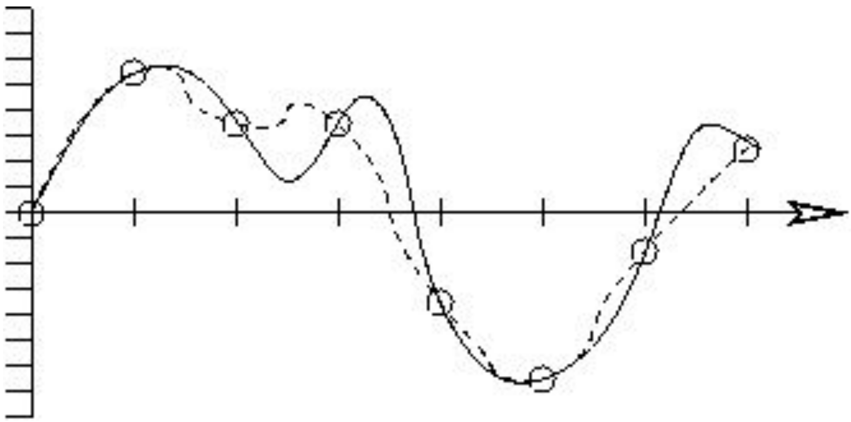


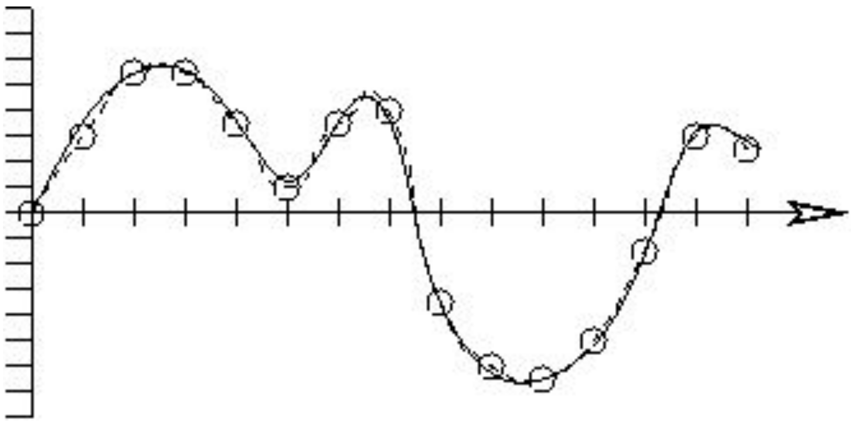
Courtesy of: http://puma.wellesley.edu/~cs110/lectures/M07-analog-and-digital/

Courtesy of: http://www.cs.ucl.ac.uk/staff/jon/mmbook/book/node96.html

Sampling: from digital to analog

- Going the opposite way means to reconstruct the analog signal from the digital signal
  - Interpolation problem!
  - Amounts to drawing a curve through the points
- What curve to choose?
- Multiple curves possible in (a)
  - First part reasonably ok but many errors in the latter part
- In (b), sampling has been doubled
  - Reconstructed curve much better
  - BUT: also needs more “bandwidth”!





Courtesy of: http://puma.wellesley.edu/~cs110/lectures/M07-analog-and-digital/

Nyquist Sampling Theorem

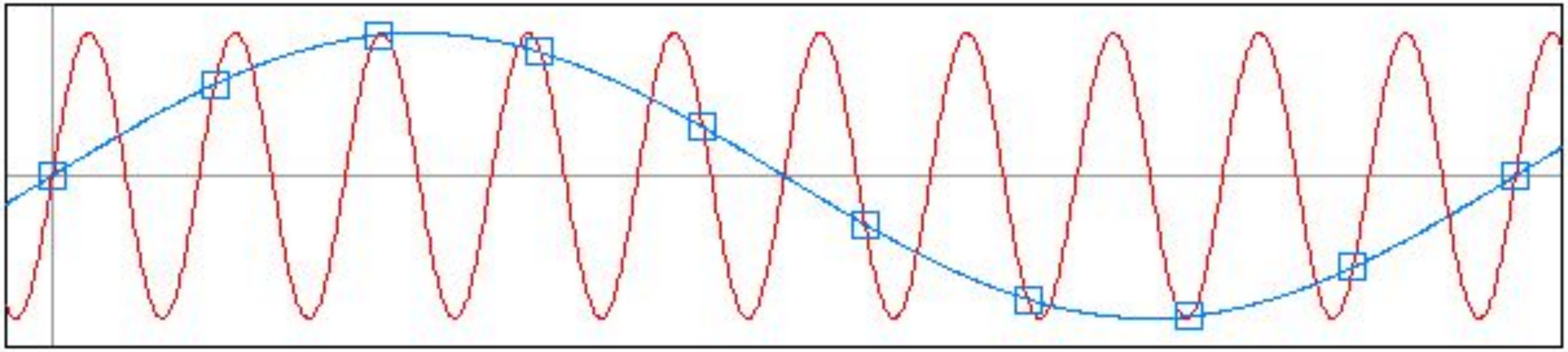
CognitiveSystems

- This theorem answers the question of how often must we sample to faithfully represent the signal using discrete sampling?
  - First articulated by Harry Nyquist and later proved by Claude Shannon
- Solution: **Sample twice as often as the highest frequency that you need to represent!**
- $f_s >= 2 * f_H$ (Nyquist rate)
  - f_s is the sampling frequency and f_H is the highest frequency present in the signal
- For example, highest sound frequency that most people can hear is about 20 KHz (with some sharp ears able to hear up to 22 KHz), we can capture music by sampling at 44 KHz.
  - That's how fast music is sampled for CD-quality music

Courtesy of: http://puma.wellesley.edu/~cs110/lectures/M07-analog-and-digital/

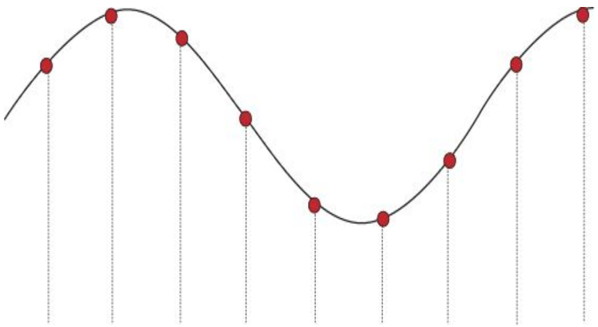
Aliasing

- If the sampling condition is **not** satisfied, then frequencies will overlap
- **Aliasing** is an effect that causes different continuous signals to become indistinguishable (or *aliases* of one another) when sampled.

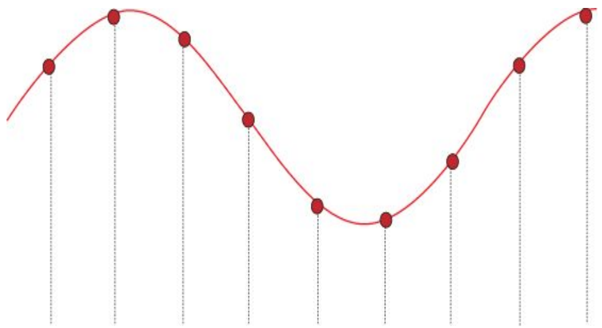


Courtesy of http://en.wikipedia.org/wiki/Aliasing

Critical Sampling versus Aliasing



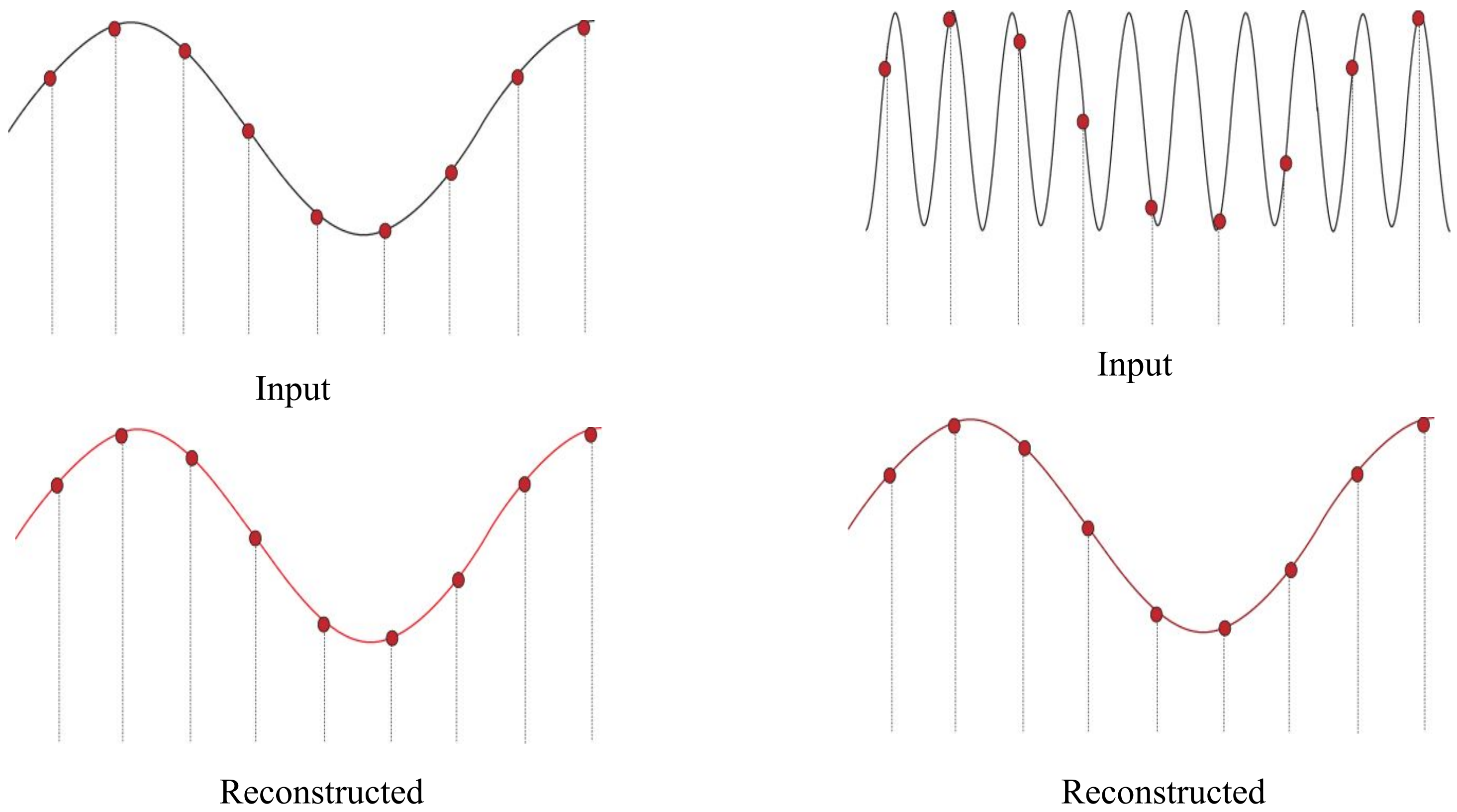
Input



Reconstructed

- **The Nyquist theorem says that you need to sample at at least twice the highest frequency in your signal**

Critical Sampling versus Aliasing



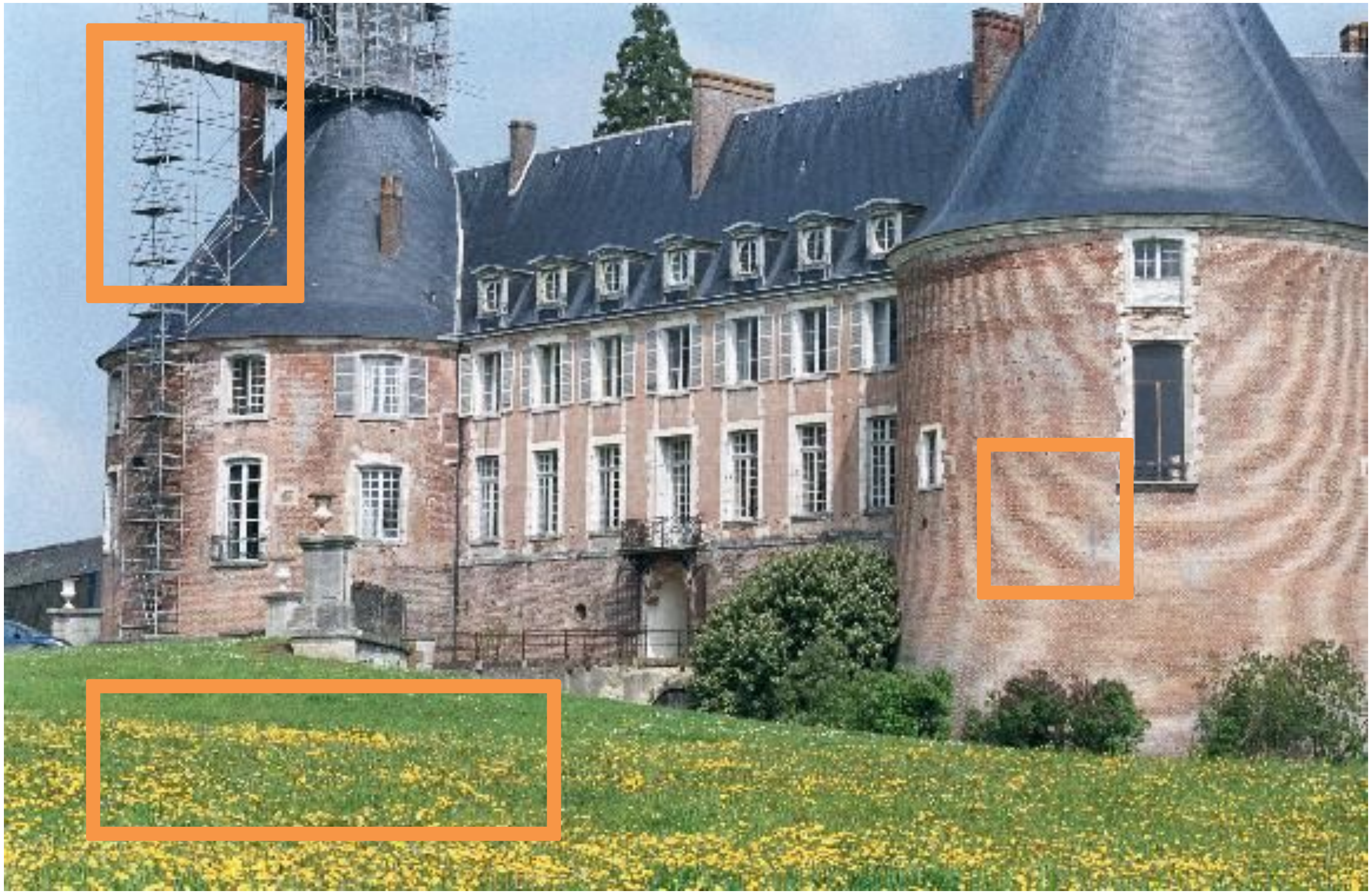
- **The Nyquist theorem says that you need to sample at at least twice the highest frequency in your signal**

Examples of aliasing

- Sunrise – Temporal Aliasing
  - The sun moves east to west in the sky, with 24 hours between sunrises.
  - If one were to take a picture of the sky every 23 hours, the sun would appear to move west to east, with 24 × 23 = 552 hours between sunrises.
- Wagon Wheel effect – Temporal Aliasing
  - The same phenomenon causes spoked wheels to apparently turn at the wrong speed or in the wrong direction when filmed, or illuminated with a flashing light source.
- Moire pattern – Spatial Aliasing
  - Stripes captured on a digital camera would cause aliasing between the stripes and the camera sensor.
    - Distance between the stripes is smaller than what the sensor can capture
    - Solution to this would be to go closer or to use a higher resolution sensor

Courtesy of http://en.wikipedia.org/wiki/Aliasing

Aliasing after downsampling



- Aliasing is present in
  - low-frequency structures on the masonry
  - the coarse flower/grass appearance
  - staircase-like appearance of the scaffolding

https://svi.nl/AntiAliasing

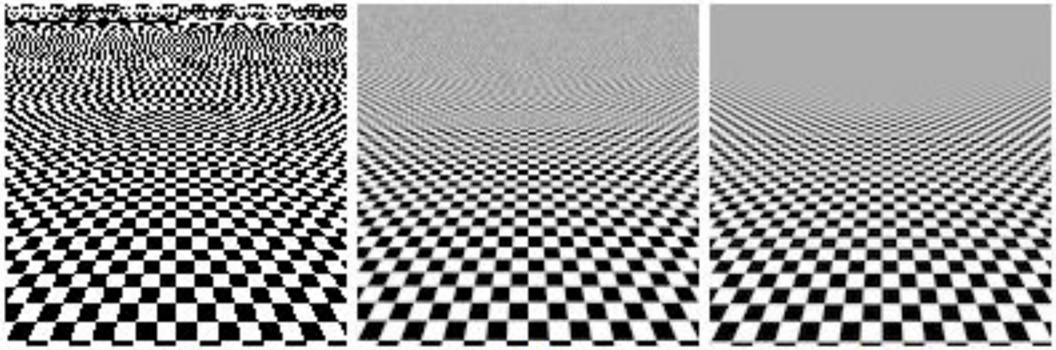
Aliasing

- After applying a band-limiting filter and then downsizing

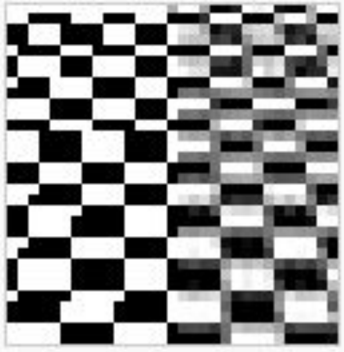
https://svi.nl/AntiAliasing

Aliasing

- To prevent aliasing, two things can be done
  - Increase the sampling rate
  - Introduce an anti-aliasing filter
- Anti-aliasing filter - restricts the bandwidth of the signal to satisfy the sampling condition.
  - This is not satisfiable in reality since a signal will have some energy outside of the bandwidth.
  - The energy can be small enough that the aliasing effects are negligible (not eliminated completely).
- Anti-aliasing filter: low pass filters, band pass filters, non-linear filters
- Always remember to apply an anti-aliasing filter prior to signal down-sampling



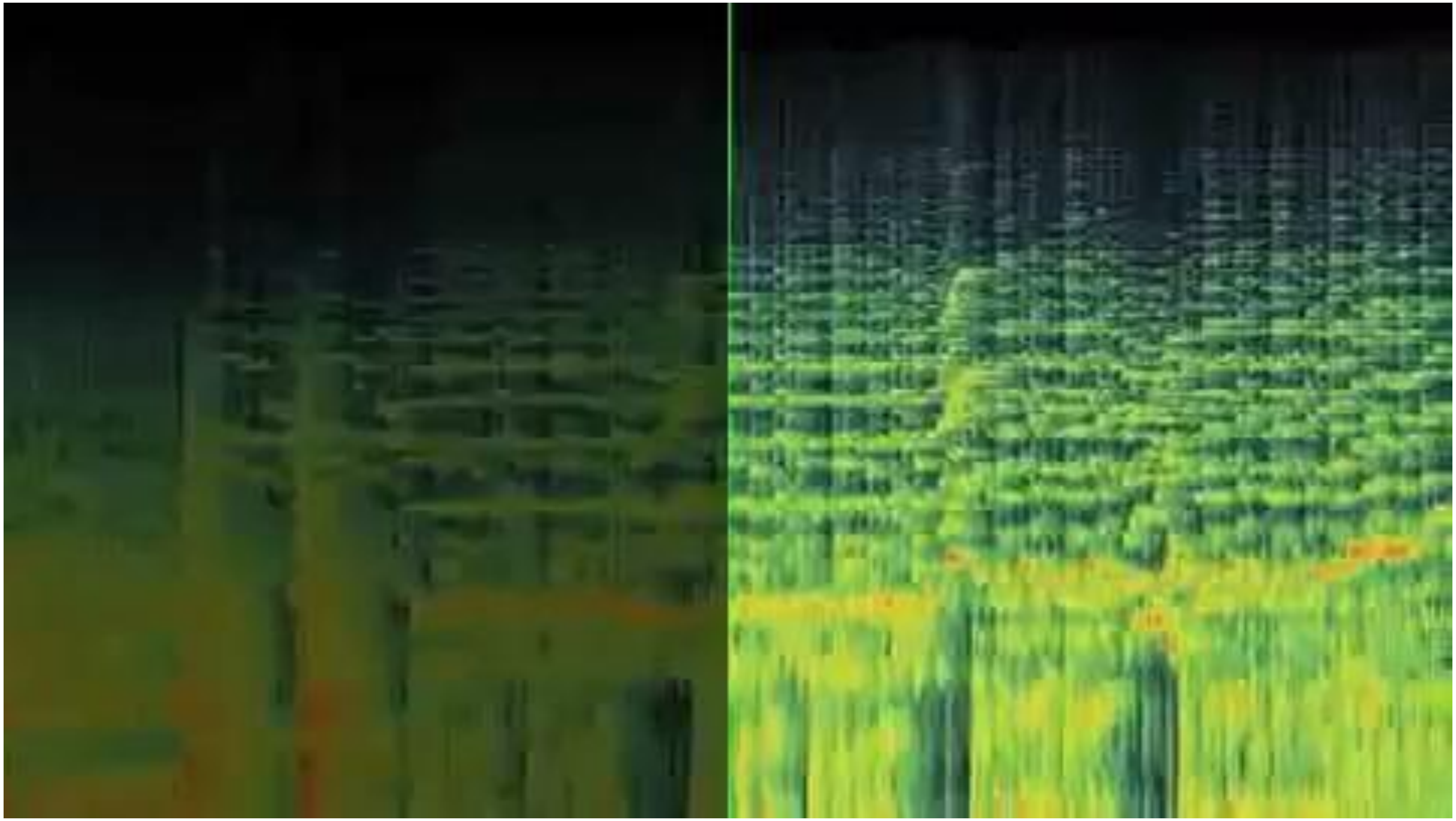
(a) (b) (c)



Spectrograms

CognitiveSystems

- Spectrograms visualize the spectrum over time to show the temporal evolution of frequencies



EEG

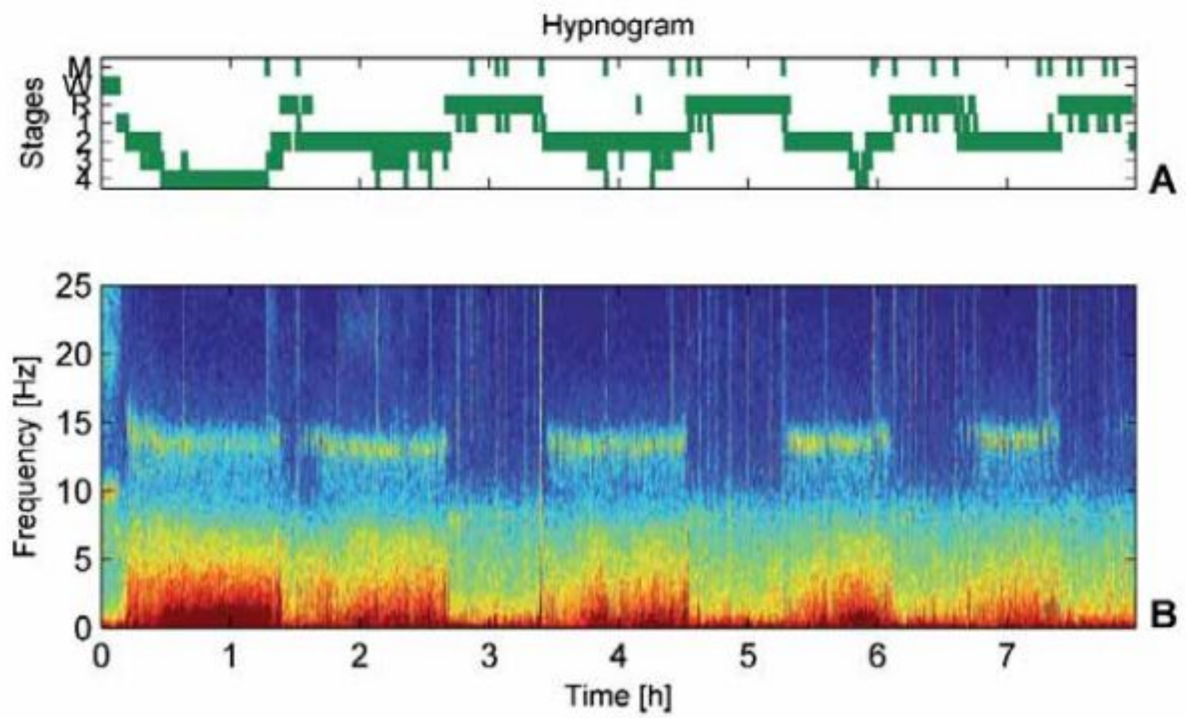


Figure 1: **A** Sleep profile (hypnogram) and **B** color-coded power spectra of consecutive 20-s epochs (average of 5 spectra calculated for 4-s epochs; Hanning window). Data were sampled with 256 Hz. Spectra (derivation C3A2) are color coded on a logarithmic scale (0 dB = 1 μV^2/Hz; -10 dB 20 dB). Sleep stages were visually scored for 20-s epochs (W: waking; M: movement time; R: REM sleep; 1 to 4: non-REM sleep stages 1 to 4).

EEG

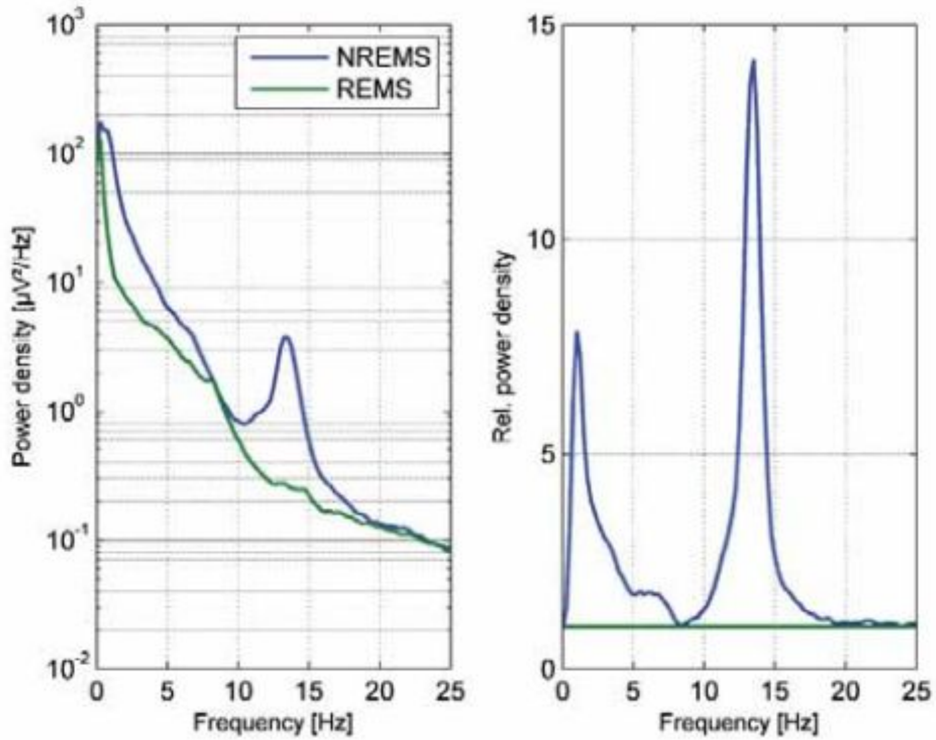
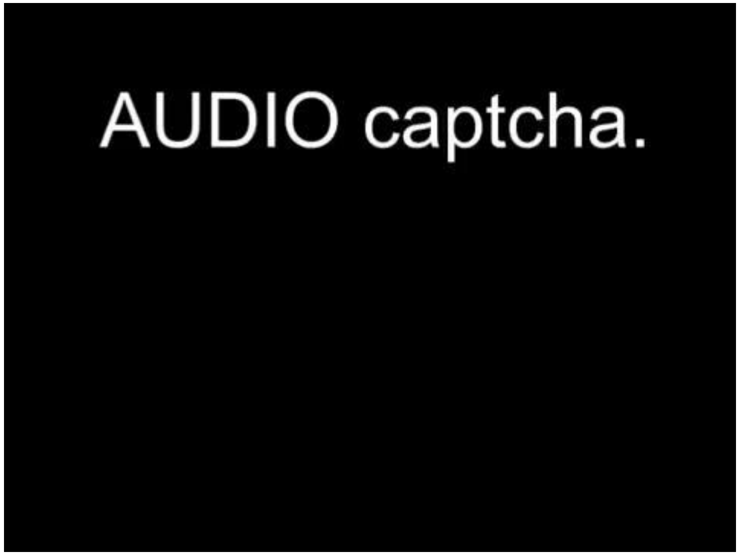
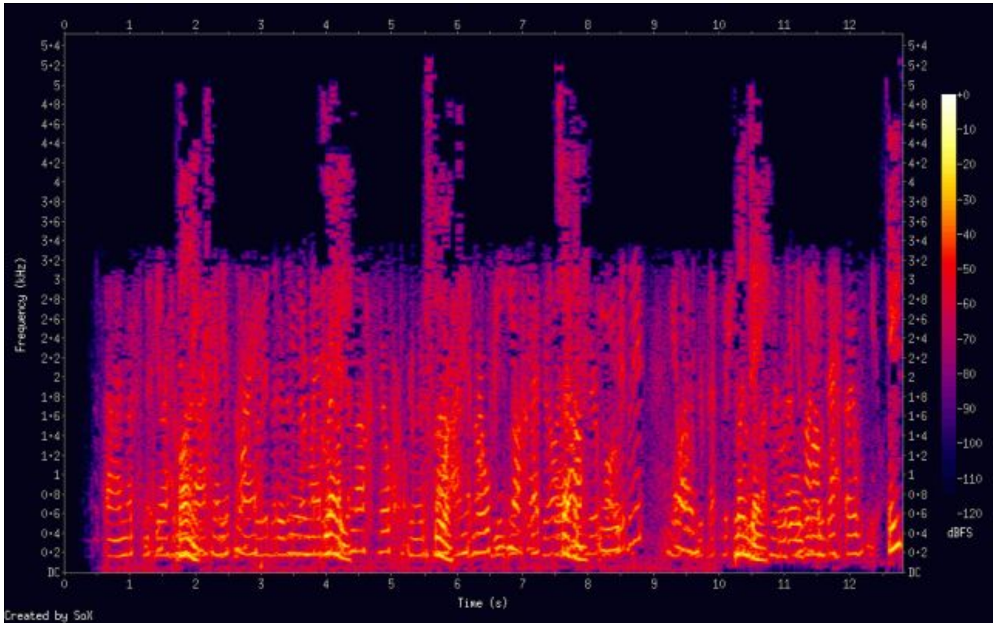


Figure 2: Left: Average spectra of non-REM sleep (stages 2, 3, and 4) and REM sleep. **Right:** Non-REM sleep spectrum plotted relative to the REM sleep spectrum. Only 20-s epochs without artifacts were included in the average spectra. Same recording as in Figure 1.

Spectrograms

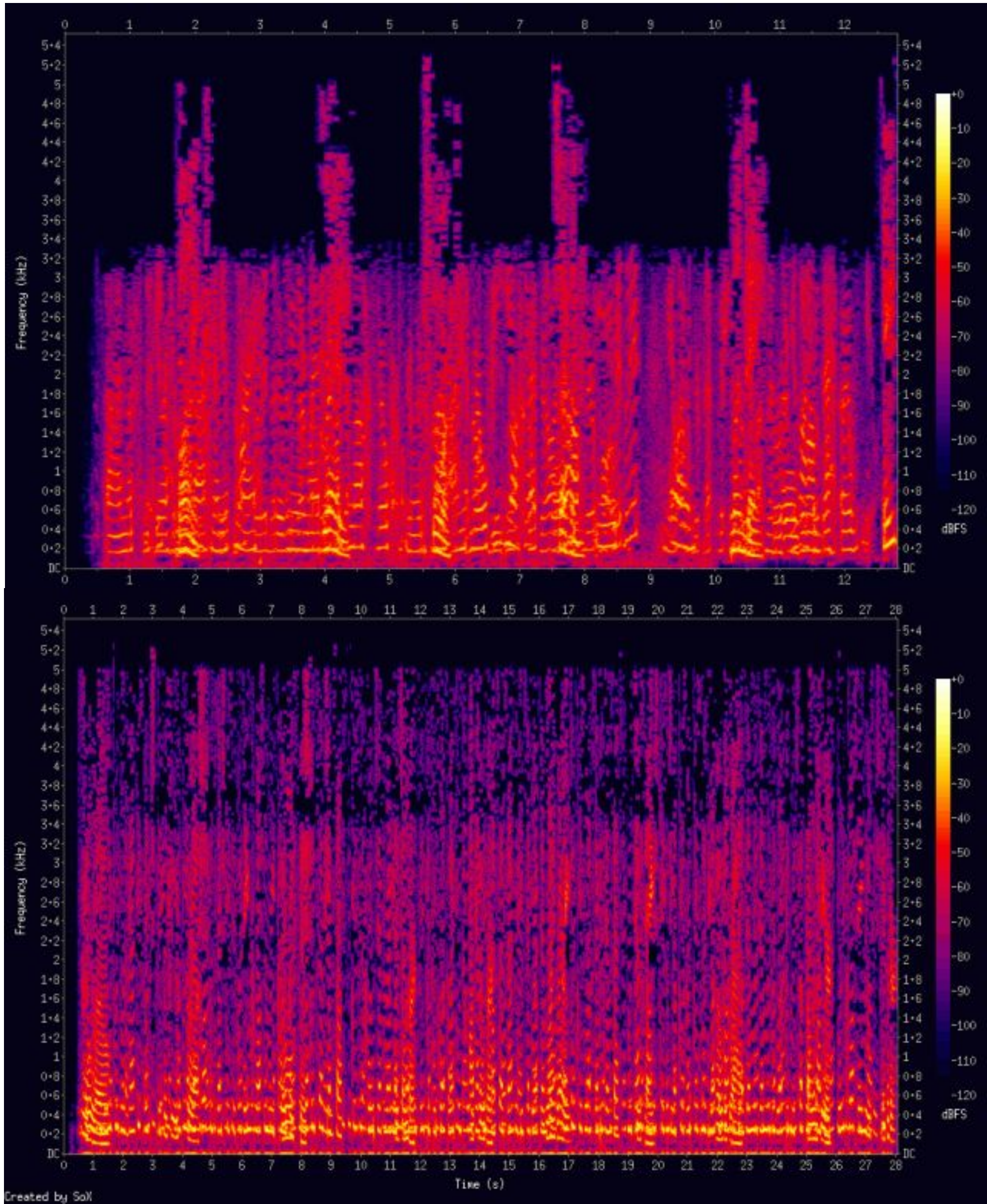
- Google uses speech Captchas for disabled users to check for human versus web-bots inputs
- However, the system was vulnerable to attacks as this spectrogram shows:





Spectrograms

- Shortly before a team from CMU posted their findings, Google announced an upgrade to their system, shown below
- More at http://arstechnica.com/security/2012/05/google-recaptcha-brought-to-its-knees/



Fourier transform - appendix

DFT Properties: (1) Separability

- The 2D DFT can be computed using 1D transforms only:
  - Forward DFT: $$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux+vy}{N})}$$
  - Inverse DFT: $$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux+vy}{N})}$$

DFT Properties: (1) Separability

- Rewrite F(u,v) as follows:

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} e^{-j2\pi(\frac{ux}{N})} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{vy}{N})}$$

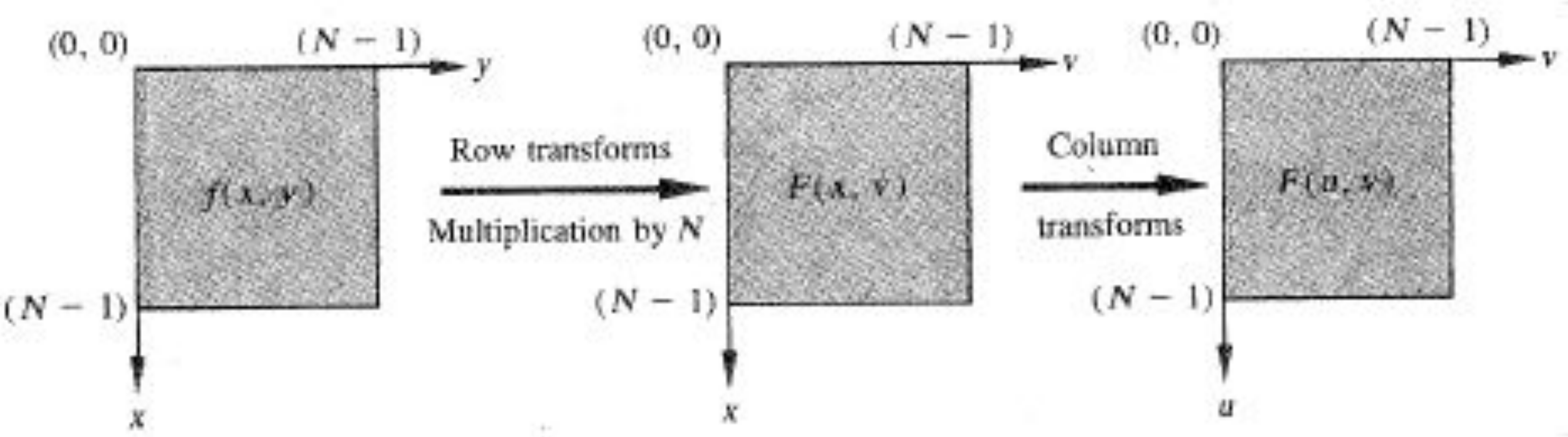
- Let's set:

$$\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{vy}{N})} = F(x,v) = N(\frac{1}{N}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{vy}{N})})$$

- Then:

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} e^{-j2\pi(\frac{ux}{N})} F(x,v)$$

DFT Properties: (1) Separability



DFT Properties: (2) Periodicity and Symmetry

- The DFT and its inverse are periodic with period N

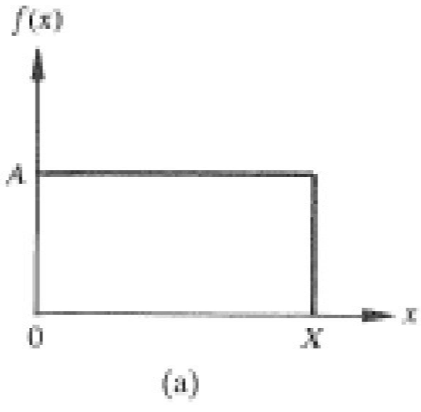
$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$

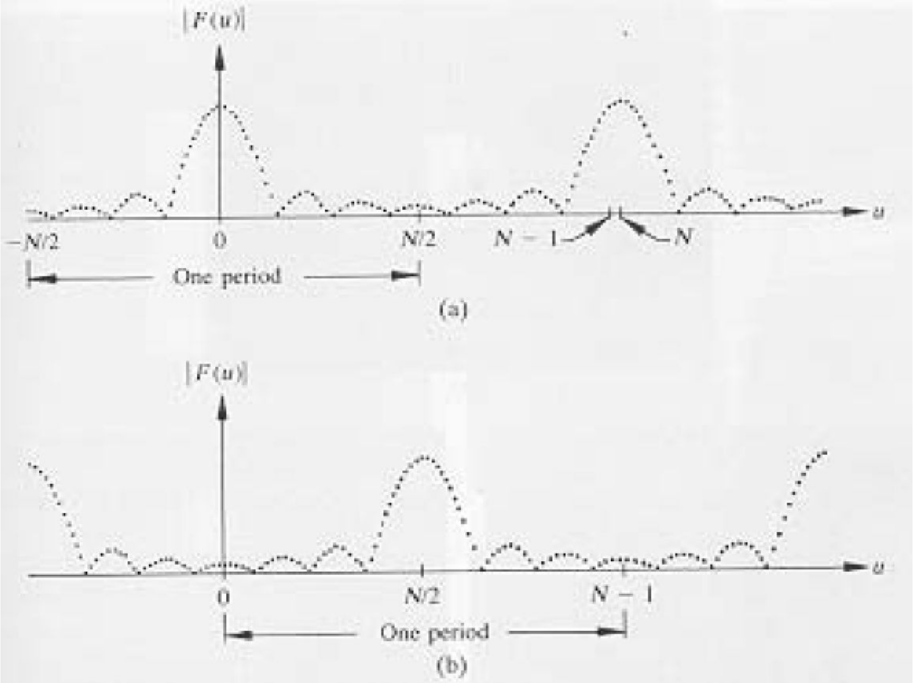
- If f(x,y) is real, then

$$F(u,v) = F^*(-u,-v) \Longrightarrow |F(u,v)| = |F(-u,-v)|$$

DFT Properties: (2) Periodicity and Symmetry

- To display a full period, we need to translate the origin of the transform at **u=N/2** (or at **(N/2,N/2)** in 2D)
- In Matlab this is done with the command fftshift





DFT Properties: (3) Translation

f(x,y) ⟷ F(u,v)

- Translation in the spatial domain

$$f(x - x_0, y - y_0) \iff F(u, v)e^{-j2\pi(\frac{ux_0 + vy_0}{N})} \qquad (1)$$

- Translation in the frequency domain

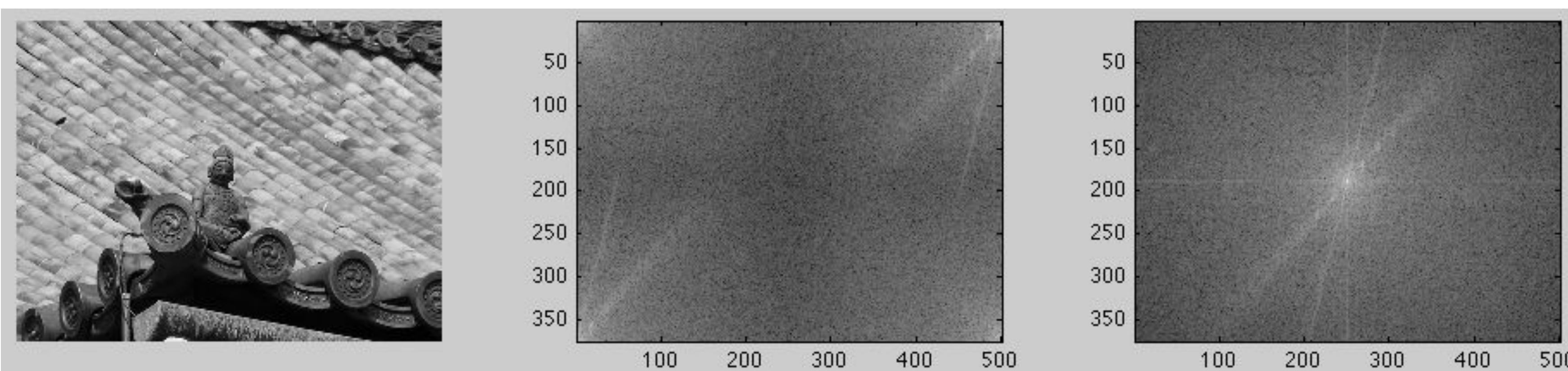
$$f(x, y)e^{j2\pi(\frac{u_0 x + v_0 y}{N})} \iff F(u - u_0, v - v_0) \qquad (2)$$

DFT Properties: (3) Translation

- To move F(u,v) at (N/2, N/2), take $u_0 = v_0 = N/2$

from (2): $$e^{j2\pi\left(\frac{\frac{N}{2}x+\frac{N}{2}y}{N}\right)} = e^{j\pi(x+y)} = (-1)^{x+y}$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-N/2,\, v-N/2)$$

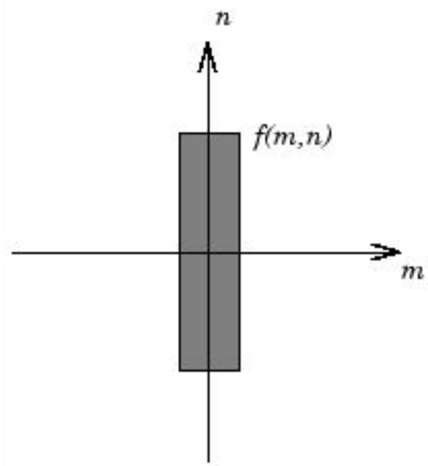


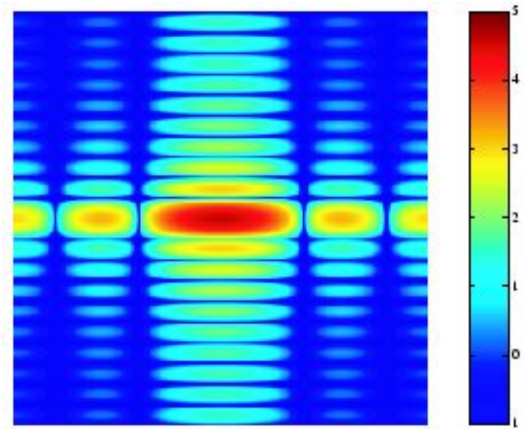
no translation

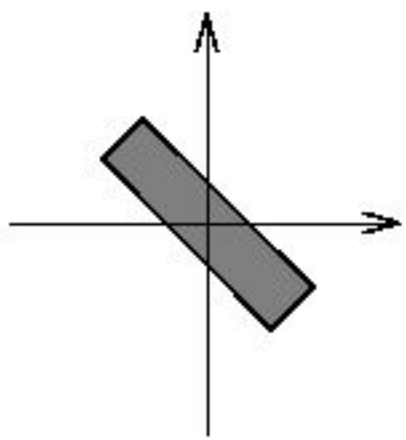
after translation

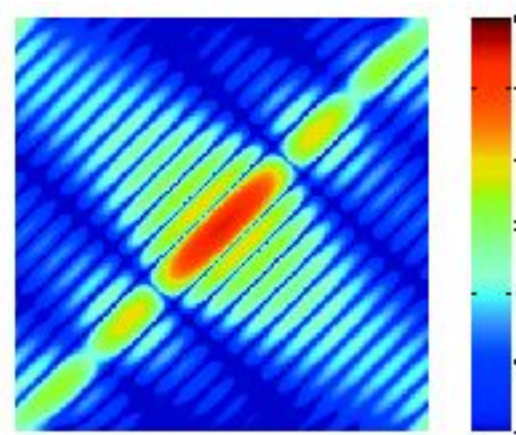
DFT Properties: (4) Rotation

- Rotating f(x,y) by θ rotates F(u,v) by θ









DFT Properties: (5) Distributive

- Fourier transforms are additive

$$F[f(x,y)+g(x,y)]=F[f(x,y)]+F[g(x,y)]$$

- In general, FTs are NOT multiplicative

$$F[f(x,y)g(x,y)]\neq F[f(x,y)]F[g(x,y)]$$

DFT Properties: (6) Scale

- FTs scale

$$af(x, y) \Leftrightarrow aF(u, v)$$

DFT Properties: (7) Average value

Average in the spatial domain:

$$\bar{f}(x,y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

From definition of F(u,v) it follows that F(u,v) at u=0, v=0:

$$F(0,0) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

So we have that:

$$\bar{f}(x,y) = \frac{1}{N} F(0,0)$$

Example: impulse or "delta" function

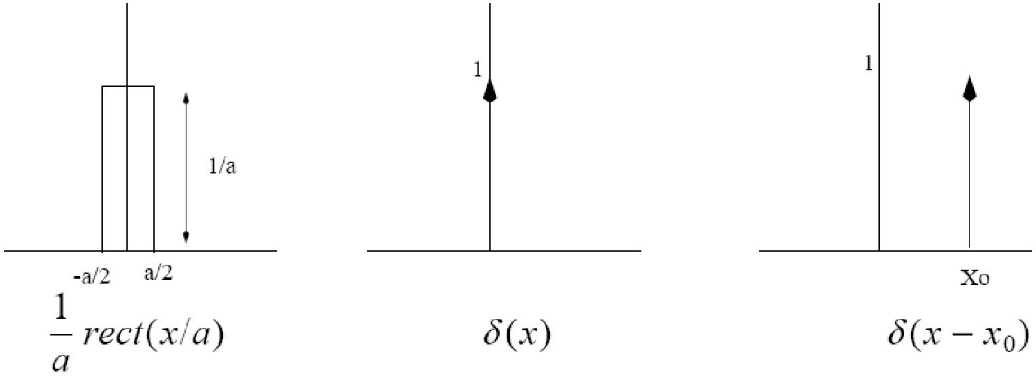
CognitiveSystems

- Definition of delta function:

$$\delta(x) = \lim_{a \to 0} \frac{1}{a} rect(x/a)$$

- Properties:

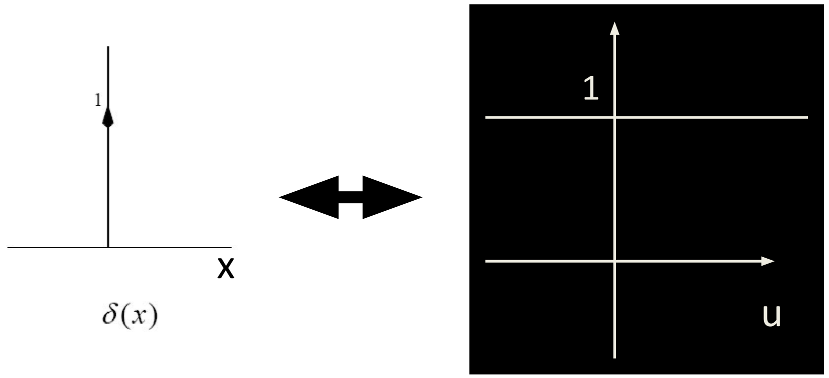
$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad \int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$



Example: impulse or "delta" function

- FT of delta function

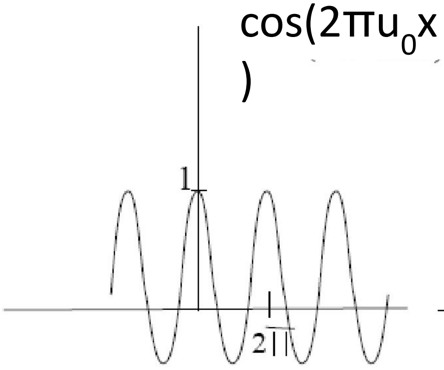
$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi ux} dx = e^0 = 1$$

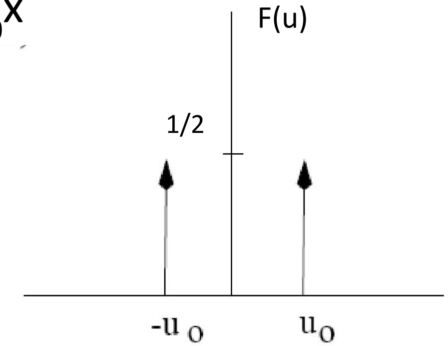


Example: sine and cosine functions

- FT of the cosine function

$$F(\cos(2\pi u_0 x)) = \frac{1}{2}[\delta(u-u_0)+\delta(u+u_0)]$$

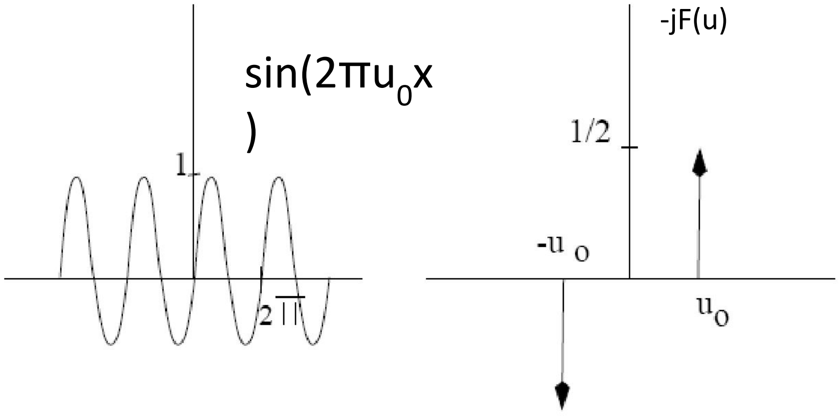




Example: sine and cosine functions

- FT of the sine function

$$F(\sin(2\pi u_0 x)) = \frac{j}{2}\left[\delta(u+u_0) - \delta(u-u_0)\right]$$



Fourier series to Fourier Transform

- Given a function that is zero outside a defined interval [-L/2,L/2], then for any T>=L, we can approximate the function by a Fourier series with coefficients

$$\hat{f}(n/T) = c_n = \int_{-T/2}^{T/2} e^{-2\pi i n x/T} f(x)\, dx$$

and the Fourier series is given by:

$$f(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}(n/T) e^{2\pi i n x/T}.$$

- If we let $\xi_n = n/T$, and we let $\Delta\xi = (n + 1)/T - n/T = 1/T$, then this last sum becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(\xi_n) e^{2\pi i x \xi_n} \Delta\xi. \xrightarrow{T \to \infty} f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi,$$