Applied Mathematics
003 – Solving systems of linear equations

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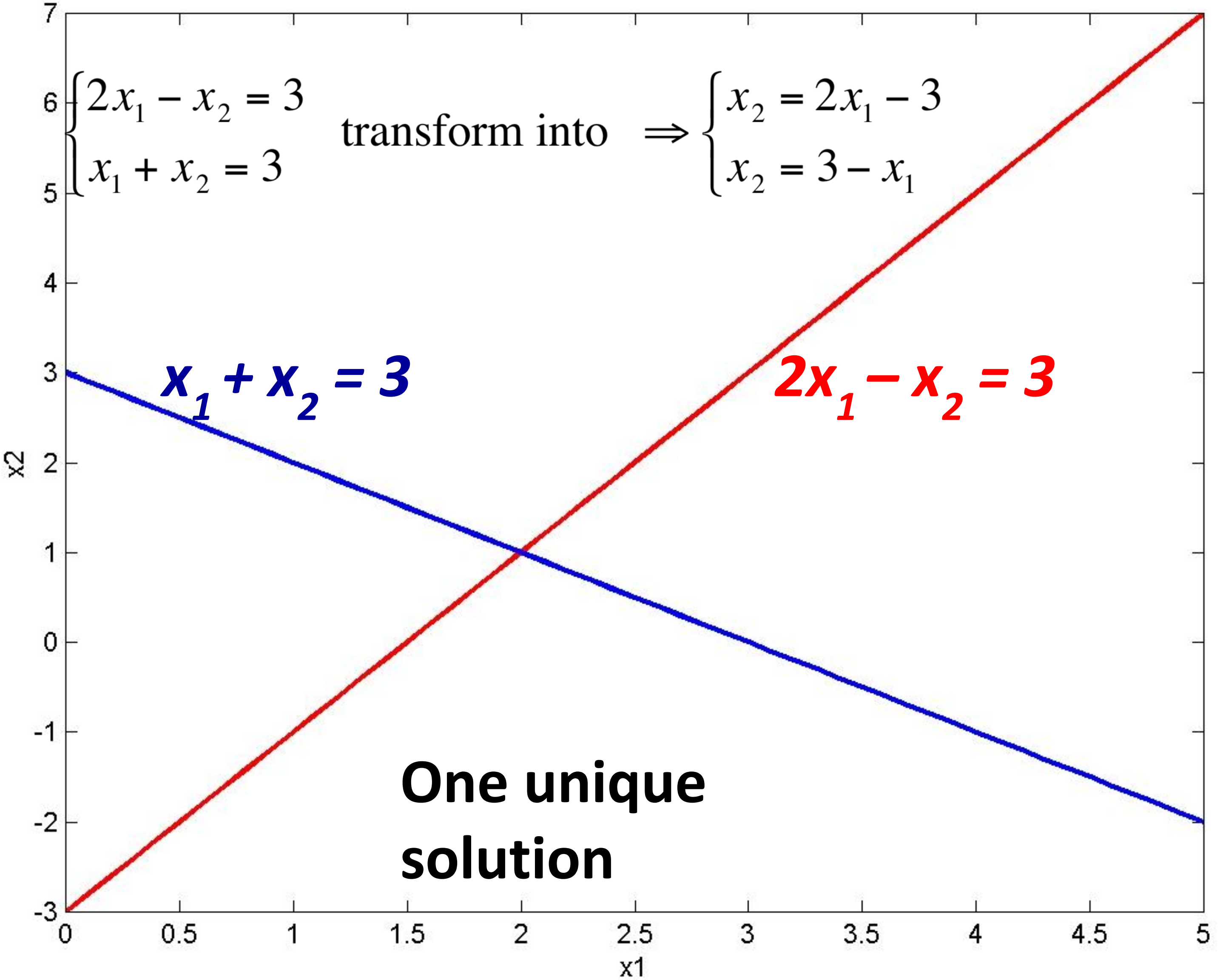
Small Matrices

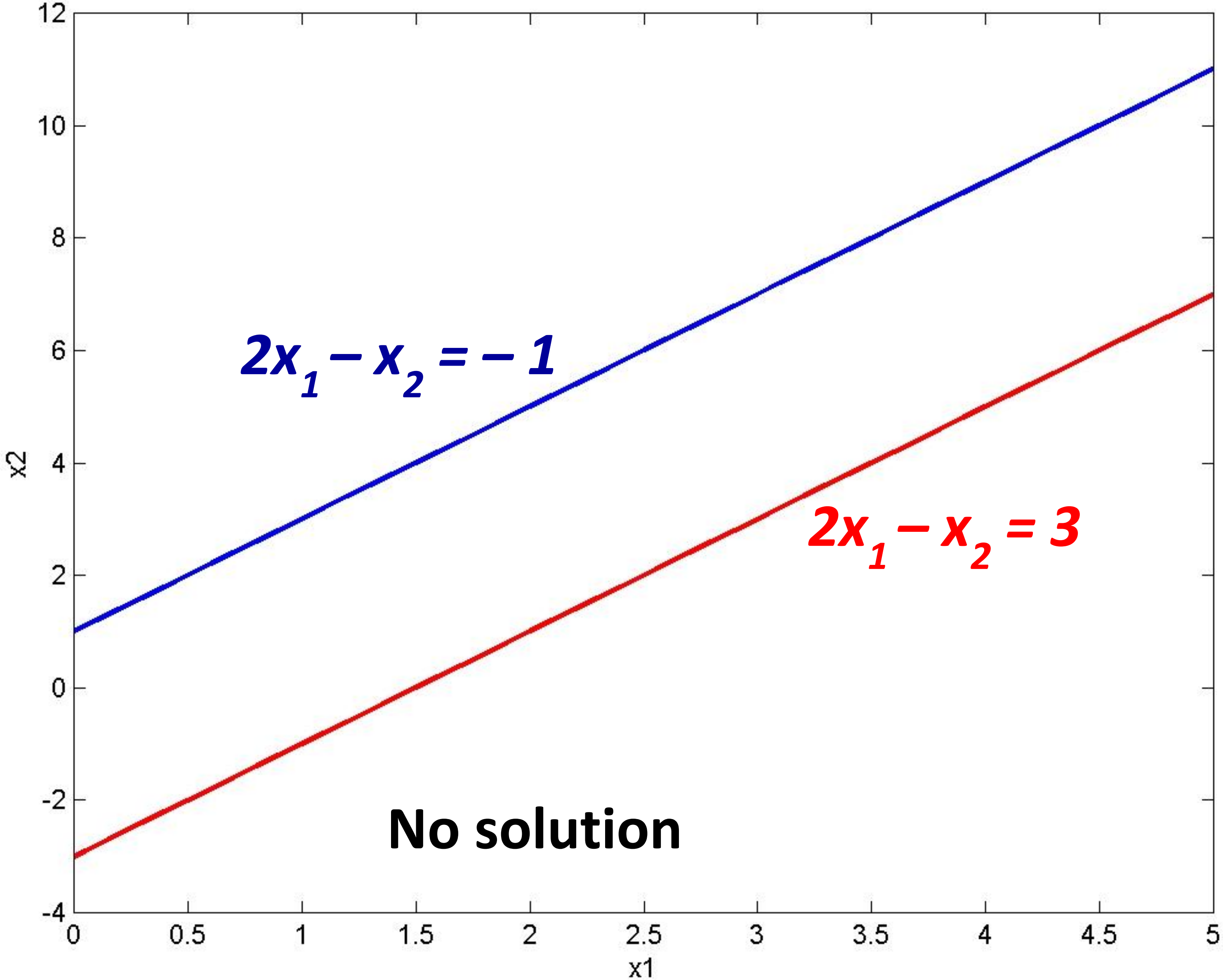
$$\begin{cases} a_{11}x_1 & + a_{12}x_2 & + \cdots & + a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots & + a_{2n}x_n & = b_2 \\ \cdots & & & & \cdots \\ a_{n1}x_1 & + a_{n2}x_2 & + \cdots & + a_{nn}x_n & = b_n \end{cases}$$

- Such systems are called **linear systems of equations**
  - linear = linear in x!, or Ax = b
- Three solution methods
  - Graphical solution (only for small 2x2, or 3x3 systems)
  - Cramer's rule (only practical for small systems)
  - Gaussian Elimination (general method)

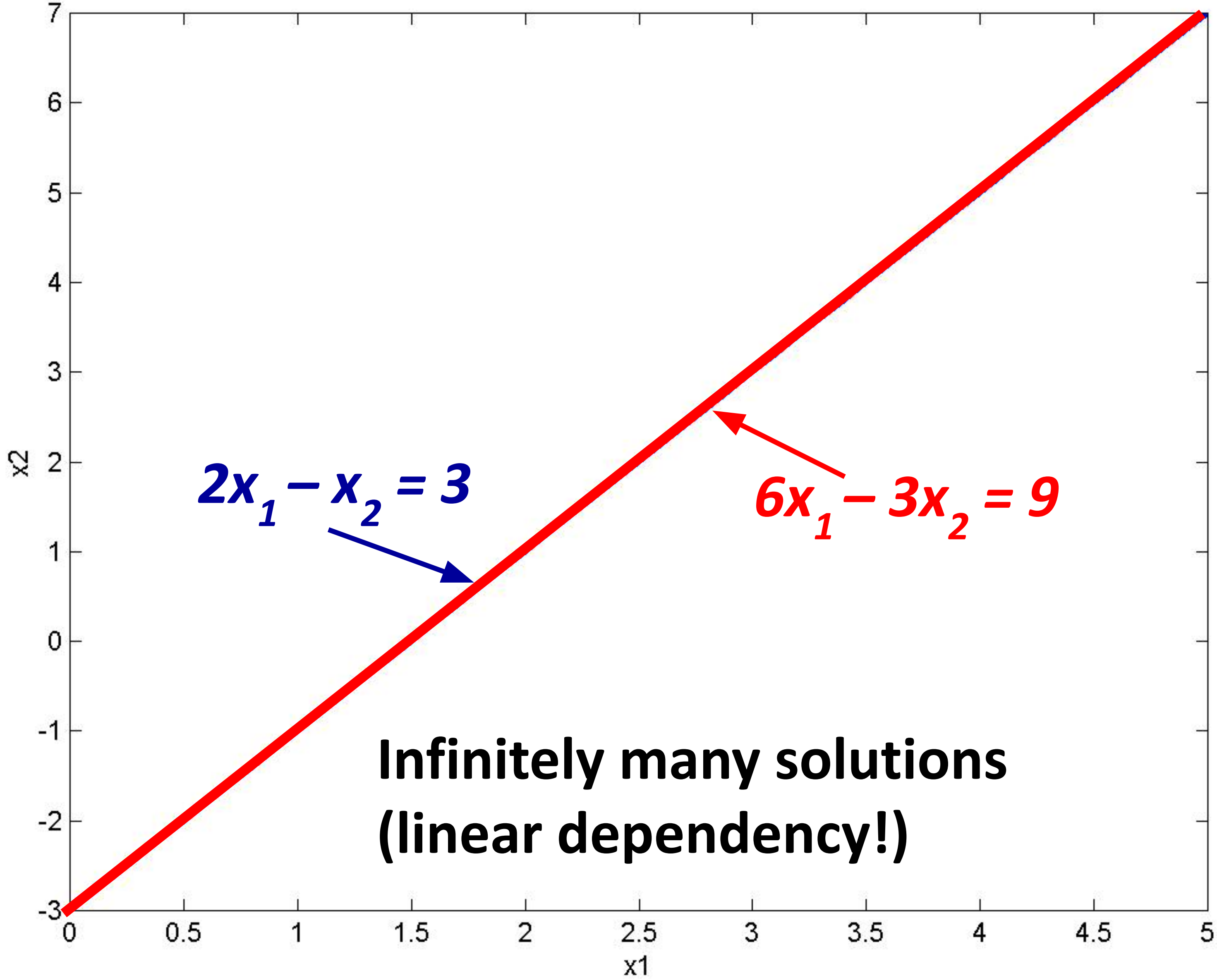
Graphical Method



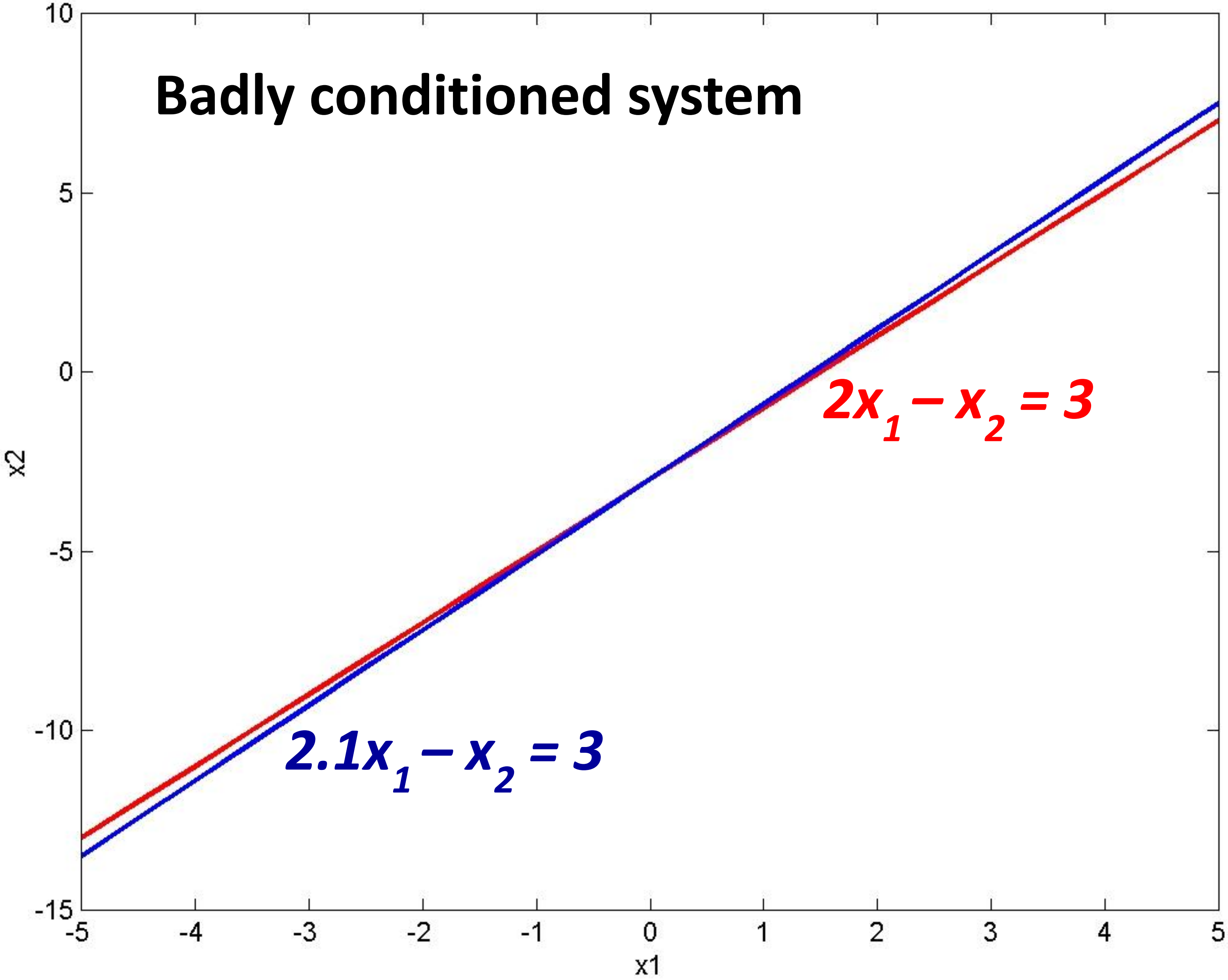
Graphical Method



Graphical Method



Graphical Method



Cramer's Rule

- Use the determinant D of the matrices
- 2 x 2 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3 x 3 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cramer's Rule

- To find x_k for the following system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

- Make a new matrix, where k^{th} column of a's is simply replaced with b's (i.e., $a_{ik} \leftarrow b_i$)
- Then, the solutions x_k are given by: $x_k = \dfrac{D(new\ matrix)}{\mathrm{D}(\mathrm{a}_{\mathrm{ij}})}$

Example

- 3 x 3 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{D_1}{D} = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_2 = \frac{D_2}{D} = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$x_3 = \frac{D_3}{D} = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Ill-Conditioned System

- What happens if the determinant D is very small or zero?
  - **D = det(A) ~ 0**
- Either there will be a **division by zero**
  - this happens, if the system is **linearly dependent**
- Or we will divide by a small number resulting **in numerical instabilities**

Simple Elimination Method

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Eliminate $x_2 \Rightarrow$ $$\begin{cases} a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1 \\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2 \end{cases}$$

Subtract to get

$$a_{22}a_{11}x_1 - a_{12}a_{21}x_1 = a_{22}b_1 - a_{12}b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}} \Rightarrow x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Not very practical for large number (> 4) of equations

Gauss Elimination

CognitiveSystems

- Manipulate equations to eliminate one of the unknowns
- Develop algorithm to do this recursively
- At the end, we will get an **upper triangular matrix**

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & & & \cdots & \cdots \\ & & & & a_{nn} \end{bmatrix}$$

- From this, we can easily find solution by back substitution

Naive Gauss Elimination

- Direct method (no iteration required)
- Consists of the following steps
  - Forward elimination
  - Column-by-column elimination of the below-diagonal elements
  - Reduce to upper triangular matrix
  - Back-substitution

Naive Gauss Elimination

- Using

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

- Multiply the first equation by a_{21} / a_{11} and subtract from second equation

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{n}x_n = b_1$$
$$\cancel{\left(a_{21} - \frac{a_{21}}{a_{11}}a_{11}\right)}x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$
$$\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

Gauss Elimination

- This will give:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$
$$\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

- Repeat this “forward elimination” for **every row** until:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$
$$\ldots$$
$$a'_{n2}x_2 + \ldots + a'_{nn}x_n = b'_n$$

Gauss Elimination

- First equation is **pivot equation**
- a_{11} is **pivot element**
- Now multiply **second** equation by a'_{32}/a'_{22} and subtract from **third** equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2$$

$$\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 + \cdots + \left(a'_{3n} - \frac{a'_{32}}{a'_{22}}a'_{2n}\right)x_n = \left(b'_3 - \frac{a'_{32}}{a'_{22}}b'_2\right)$$

...

Gauss Elimination

- Repeat the elimination of a'_{i2} and get

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\cdots \\
a''_{n3}x_3 + \cdots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

- Continue and get

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\cdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution

- Now we can perform back substitution to get x_k
- By simple division

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

- Substitute this into (n-1)th equation

$$a_{n-1,n-1}^{(n-2)} x_{n-1} + a_{n-1,n}^{(n-2)} x_n = b_{n-1}^{(n-2)}$$

- Solve for x_{n-1}

Back Substitution

CognitiveSystems

- Back substitution: start with x_n
- Repeat the process to solve for x_{n-2} , x_{n-3} , ... x_2, x_1

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

for i = n−1, n−2, …, 1

$$a_{ii}^{(i-1)} \neq 0$$

Elimination of first column

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$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right]$$

$$f_{21} = a_{21} / a_{11}$$
$$f_{31} = a_{31} / a_{11}$$
$$f_{41} = a_{41} / a_{11}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ \boxed{0} & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ \boxed{0} & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ \boxed{0} & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right]$$

$$(2) - f_{21} \times (1)$$
$$(3) - f_{31} \times (1)$$
$$(4) - f_{41} \times (1)$$

Elimination of second column

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right]$$

$$f_{32} = a'_{32} / a'_{22}$$

$$f_{42} = a'_{42} / a'_{22}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & a''_{43} & a''_{44} & b''_4 \end{array}\right]$$

$$(3) - f_{32} \times (2)$$

$$(4) - f_{42} \times (2)$$

Elimination of third column

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & a''_3 \\ 0 & 0 & a''_{43} & a''_{44} & a''_4 \end{array}\right]$$

$$f_{43} = a''_{43} \,/\, a''_{33}$$

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ \boxed{0} & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ \boxed{0} & \boxed{0} & a''_{33} & a''_{34} & b''_3 \\ \boxed{0} & \boxed{0} & \boxed{0} & a'''_{44} & b'''_4 \end{array}\right]$$

Upper triangular matrix

$$(4) - f_{43} \times (3)$$

Back-Substitution

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{array}\right]$$

Upper triangular matrix

$$x_4 = b'''_4 / a'''_{44}$$

$$x_3 = (b''_3 - a''_{34} x_4) / a''_{33}$$

$$x_2 = (b'_2 - a'_{23} x_3 - a'_{24} x_4) / a'_{22}$$

$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3 - a_{14} x_4) / a_{11}$$

$$\boxed{a_{11}, a'_{22}, a''_{33}, a'''_{44} \neq 0}$$

Example

CognitiveSystems

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{array}\right] \qquad \begin{array}{l} f_{21} = -1 \\ f_{31} = 0 \\ f_{41} = 6 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 2 & -10 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} (2) - (1) \times f_{21} \\ (3) - (1) \times f_{31} \\ (4) - (1) \times f_{41} \end{array}$$

Forward Elimination

CognitiveSystems

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 2 & -10 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} f_{32} = 1/2 \\ f_{42} = 1 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & -14 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} (3)-(2)\times f_{32} \\ (4)-(2)\times f_{42} \end{array}$$

Upper Triangular Matrix

CognitiveSystems

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & -14 & -14 & -5 \end{array}\right] \qquad f_{43} = 14$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & -70 & -33 \end{array}\right] \qquad (4) - (3) \times f_{43}$$

Back-Substitution

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & -70 & -33 \end{array}\right]$$

$$x_4 = -33/-70 = 33/70$$

$$x_3 = 4x_4 - 2 = -4/35$$

$$x_2 = -2x_3 = 8/35$$

$$x_1 = 1 - 2x_3 - 3x_4 = -13/70$$

$$\vec{x} = \begin{bmatrix} -13/70 \\ 8/35 \\ -4/35 \\ 33/70 \end{bmatrix}$$

Applications

Application 1

- Finding where planes (one linear equation determines a plane!) intersect

Problem Find the point of intersection of the planes $x - y = 2$, $2x - y - z = 3$, and $x + y + z = 6$ in $\mathbb{R}^3$.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Application 2

- Finding constrained solutions for polynomials

Problem Find a quadratic polynomial $p(x)$ such that $p(1) = 4$, $p(2) = 3$, and $p(3) = 4$.

Suppose that $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, $p(3) = 9a + 3b + c$.

$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 4 \end{cases}$$

Application 3

- Breaking apart complex functions:

Problem Evaluate $\displaystyle\int_0^1 \frac{x(x-3)}{(x-1)^2(x+2)}\,dx.$

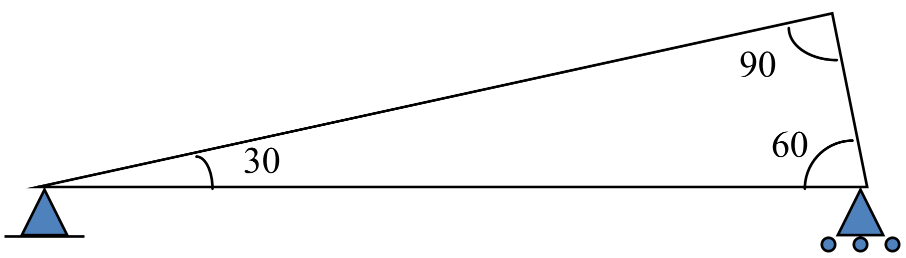
To evaluate the integral, we need to decompose the rational function $R(x) = \frac{x(x-3)}{(x-1)^2(x+2)}$ into the sum of simple fractions:

$$
\begin{aligned}
R(x) &= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2} \\
&= \frac{a(x-1)(x+2) + b(x+2) + c(x-1)^2}{(x-1)^2(x+2)} \\
&= \frac{(a+c)x^2 + (a+b-2c)x + (-2a+2b+c)}{(x-1)^2(x+2)}.
\end{aligned}
$$

$$
\begin{cases}
a + c = 1 \\
a + b - 2c = -3 \\
-2a + 2b + c = 0
\end{cases}
$$

Application 4

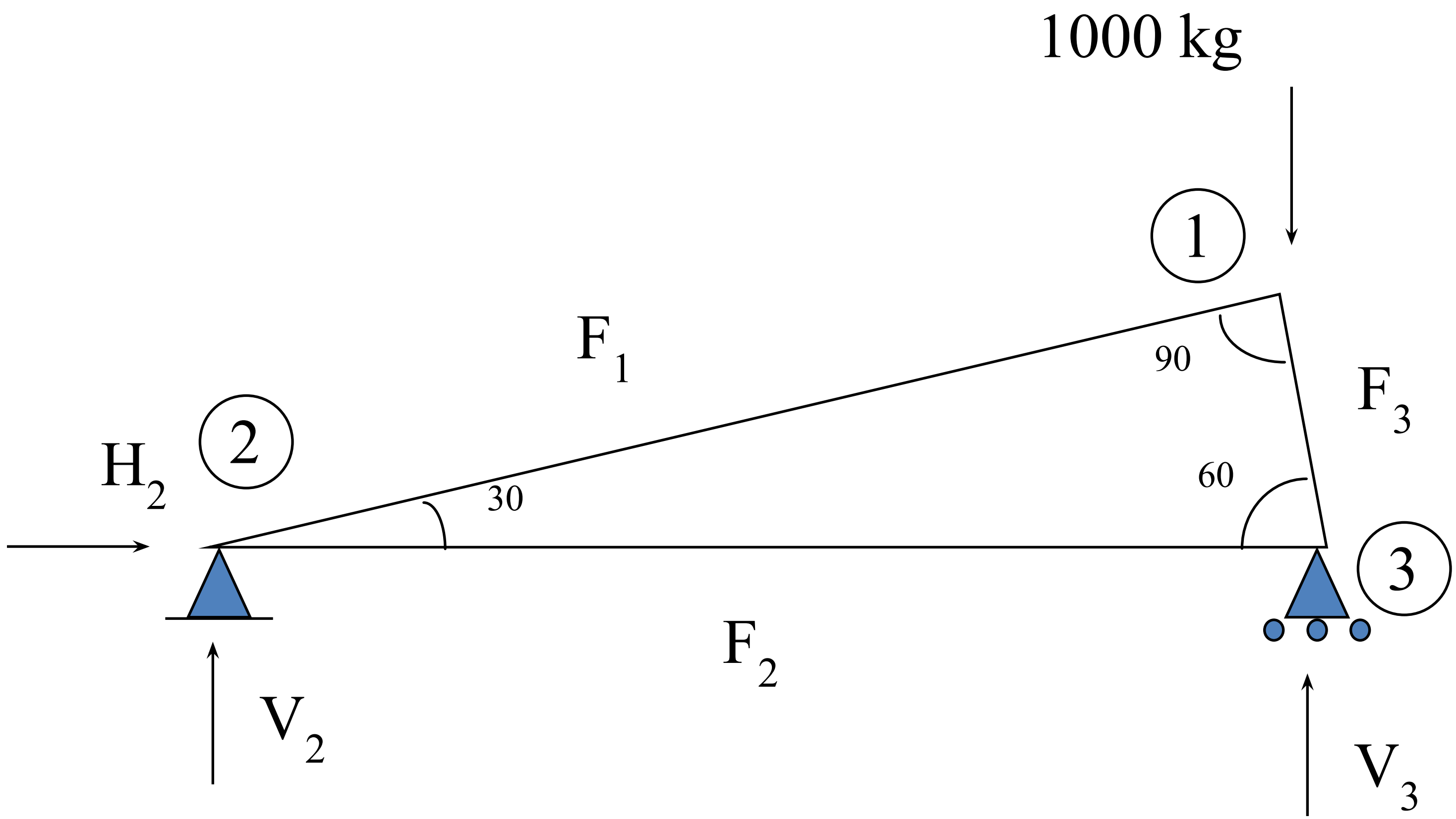
- Consider a problem in structural engineering
- Find the forces and reactions associated with a statically determinant truss



hinge: transmits both vertical and horizontal forces at the surface

roller: transmits vertical forces

Truss – force equilibrium



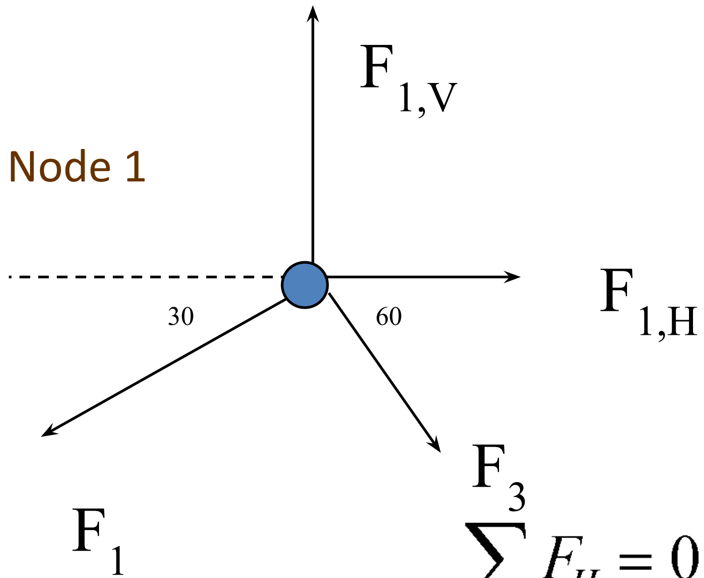
FREE BODY DIAGRAM

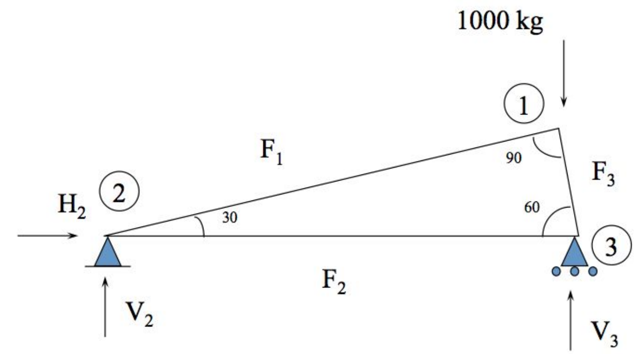
$$\sum F_H = 0$$

$$\sum F_v = 0$$

Truss – node 1

Node 1





$$\sum F_H = 0 = -F_1 \cos 30^\circ + F_3 \cos 60^\circ + F_{1,H}$$

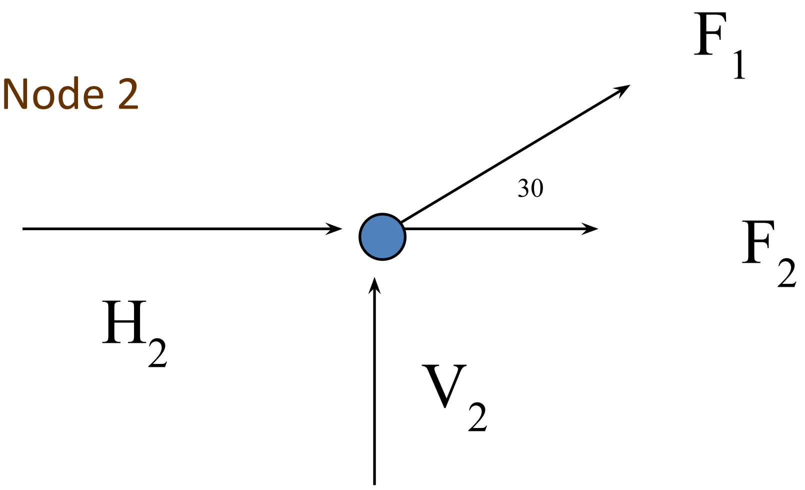
$$\sum F_V = 0 = -F_1 \sin 30^\circ - F_3 \sin 60^\circ + F_{1,V}$$

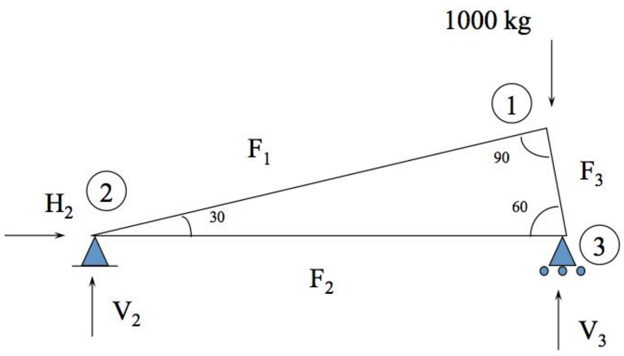
$$-F_1 \cos 30^\circ + F_3 \cos 60^\circ = 0$$

$$-F_1 \sin 30^\circ - F_3 \sin 60^\circ = -1000$$

Truss – node 2

Node 2



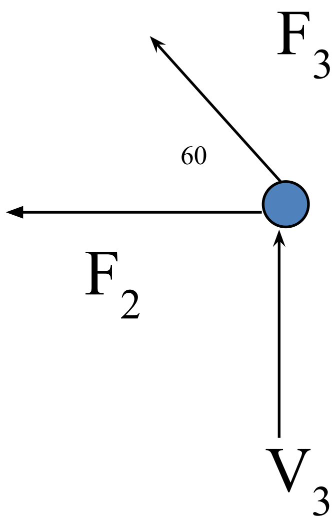


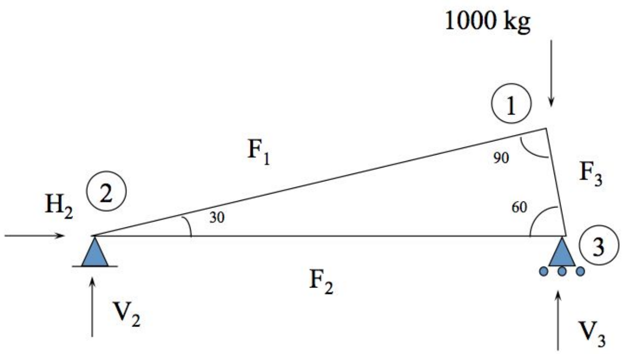
$$\sum F_H = 0 = H_2 + F_2 + F_1 \cos 30^\circ$$

$$\sum F_V = 0 = V_2 + F_1 \sin 30^\circ$$

Truss – node 3

Node 3

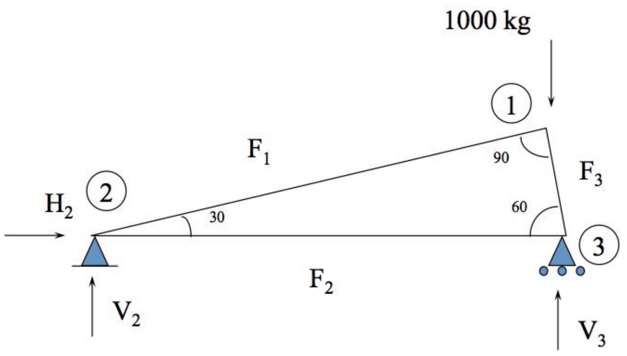




$$\sum F_H = 0 = -F_3 \cos 60^\circ - F_2$$

$$\sum F_V = 0 = F_3 \sin 60^\circ + V_3$$

Truss – all nodes combined



$$-F_1\cos 30^\circ + F_3\cos 60^\circ = 0$$

$$-F_1\sin 30^\circ - F_3\sin 60^\circ = -1000$$

$$H_2 + F_2 + F_1\cos 30^\circ = 0$$

$$V_2 + F_1\sin 30^\circ = 0$$

$$-F_3\cos 60^\circ - F_2 = 0$$

$$F_3\sin 60^\circ + V_3 = 0$$

SIX EQUATIONS

SIX UNKNOWNS

Truss – matrix form

	F_1	F_2	F_3	H_2	V_2	V_3	
1	-cos30	0	cos60	0	0	0	0
2	-sin30	0	-sin60	0	0	0	-1000
3	cos30	1	0	1	0	0	0
4	sin30	0	0	0	1	0	0
5	0	-1	-cos60	0	0	0	0
6	0	0	sin60	0	0	1	0

Truss – matrix form

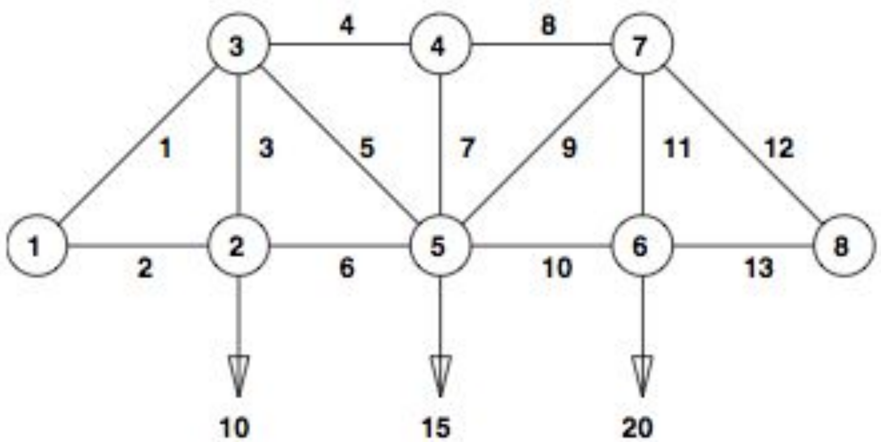
This is the basis for your matrices and the equation [A]{x}={c}

$$\begin{bmatrix} -0.866 & 0 & 0.5 & 0 & 0 & 0 \\ -0.5 & 0 & -0.866 & 0 & 0 & 0 \\ 0.866 & 1 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.866 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ H_2 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

What's the answer? □ see homework assignment!

Statics – force equilibrium

- Shown here is a "truss" having 13 members (the numbered lines) connecting 8 joints (the numbered circles). The indicated loads, in tons, are applied at joints 2, 5, and 6, and we want to determine the **resulting force** on each member of the truss.

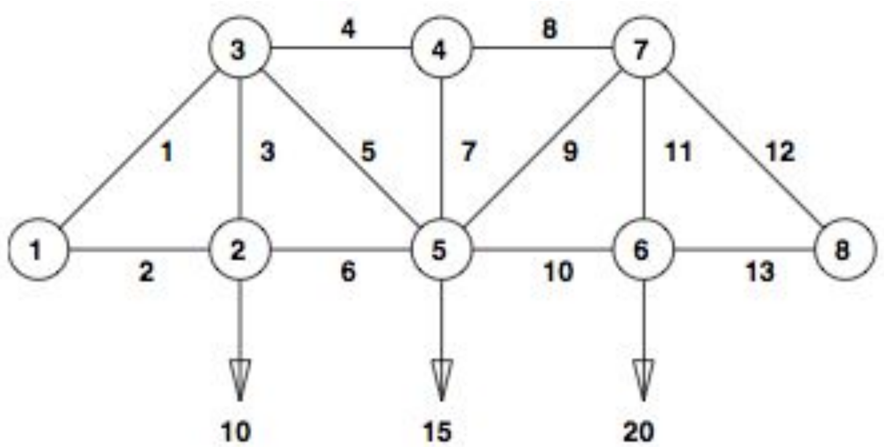


Statics – force equilibrium

- For the truss to be in static equilibrium, there must be **no net force, horizontally or vertically, at any joint**.
- Thus, we can determine the member forces by **equating the horizontal forces** to the left and right at each joint, and similarly **equating the vertical forces** upward and downward at each joint.
- For the eight joints, this would give 16 equations, which is more than the 13 unknown factors to be determined.
  - For the truss to be statically determinate, that is, for there to be a **unique** solution, we assume that joint 1 is rigidly fixed horizontally and vertically and that joint 8 is fixed vertically.

Force equations

- Equating the different forces yields the following system of equations
  - α = 1/sqrt(2) = sin(45deg) = cos(45)



$$
\begin{aligned}
\text{Joint 2:} &\quad f_2 = f_6, \\
&\quad f_3 = 10; \\
\text{Joint 3:} &\quad \alpha f_1 = f_4 + \alpha f_5, \\
&\quad \alpha f_1 + f_3 + \alpha f_5 = 0; \\
\text{Joint 4:} &\quad f_4 = f_8, \\
&\quad f_7 = 0; \\
\text{Joint 5:} &\quad \alpha f_5 + f_6 = \alpha f_9 + f_{10}, \\
&\quad \alpha f_5 + f_7 + \alpha f_9 = 15; \\
\text{Joint 6:} &\quad f_{10} = f_{13}, \\
&\quad f_{11} = 20; \\
\text{Joint 7:} &\quad f_8 + \alpha f_9 = \alpha f_{12}, \\
&\quad \alpha f_9 + f_{11} + \alpha f_{12} = 0; \\
\text{Joint 8:} &\quad f_{13} + \alpha f_{12} = 0.
\end{aligned}
$$

More on Solving Systems of Equations

Dangers with computers!

Solving Linear Systems

$$Ax = b$$

$$7x = 21$$

$$x = \frac{21}{7} = 3$$

$$x = 7^{-1} \times 21 = .142857 \times 21 = 2.99997$$

Floating point arithmetic

$$x = \pm(1 + f) \cdot 2^e$$

$$0 \leq f < 1$$

$$f = (\text{integer} < 2^{52})/2^{52}$$

finite precision

$$-1022 \leq e \leq 1023$$

finite range

$$e = \text{integer}$$

Floating point arithmetic

CognitiveSystems

- `eps` is the distance from 1 to the next larger floating-point number.
  - `eps = pow(2,(-52))=np.finfo(np.float64).eps`
- Important: if you want to test for equality between two numbers, it is usually better to test whether the difference is around or smaller than (a multiple of) `eps`
- Given `a,b` as double:
  - `if (a==b)` is not good!
  - better: `if (abs(a-b)<eps)`

Floating point arithmetic

- Some numbers cannot be expressed exactly!

A frequent instance of roundoff occurs with the simple

```
t = 0.1
```

The mathematical value t stored in `t` is not exactly 0.1 because expressing the decimal fraction 1/10 in binary requires an infinite series. In fact,

$$\frac{1}{10} = \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{0}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{0}{2^{10}} + \frac{0}{2^{11}} + \frac{1}{2^{12}} + \cdots.$$

```
0.3/0.1
```

is not exactly equal to 3 because the actual numerator is a little less than 0.3 and the actual denominator is a little greater than 0.1.

Floating point arithmetic

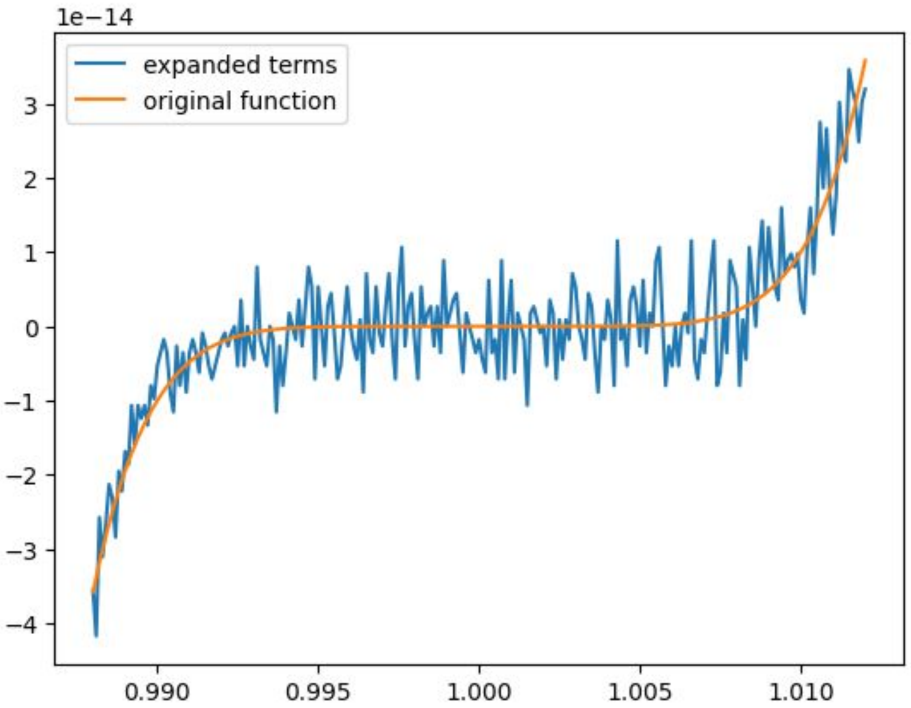
- Some numbers cannot be expressed exactly!

```
x = np.arange(0.988,1.012,0.0001)
y = pow(x,7)-7*pow(x,6)+21*pow(x,5)-35*pow(x,4)+35*pow(x,3)-21*pow(x,2)+7*x-1
```

- This is definitely not a smooth polynomial
- If you use

```
y2 = pow((x-1),7
```

the plot looks smooth



A 3x3 example

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{aligned} 10x_1 - 7x_2 \phantom{{}+6x_3} &= 7 \\ -3x_1 + 2x_2 + 6x_3 &= 4 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

A 3x3 example

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.1 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.1 \\ 2.5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.2 \end{pmatrix}$$

A 3x3 example

$$6.2x_3 = 6.2$$

$$2.5x_2 + (5)(1) = 2.5.$$

$$10x_1 + (-7)(-1) = 7$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

LU - Decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & -0.04 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$LU = PA$$

LU - Decomposition

CognitiveSystems

$$U = M_{n-1}P_{n-1}\cdots M_2P_2M_1P_1A$$

$$L_1L_2\cdots L_{n-1}U = P_{n-1}\cdots P_2P_1A$$

$$L = L_1L_2\cdots L_{n-1}$$
$$P = P_{n-1}\cdots P_2P_1$$

$$LU = PA$$

LU - Decomposition

$$A = \begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.04 & 1 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.04 & 1 \end{pmatrix}$$

Solving on a computer with 3 digit resolution!

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3.901 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

same solution as previous system!

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.001 \\ 2.5 \end{pmatrix}$$

$$(5 + (2.5 \cdot 10^3)(6))x_3 = (2.5 + (2.5 \cdot 10^3)(6.001))$$

$$(5 + 1.5000 \cdot 10^4)x_3 = (2.5 + 1.50025 \cdot 10^4)$$

$$1.5005 \cdot 10^4 x_3 = 1.5004 \cdot 10^4$$

Pivoting!

$$x_3 = \frac{1.5004 \cdot 10^4}{1.5005 \cdot 10^4} = 0.99993$$

$$-0.001x_2 + (6)(0.99993) = 6.001$$

$$x_2 = \frac{1.5 \cdot 10^{-3}}{-1.0 \cdot 10^{-3}} = -1.5$$

$$10x_1 + (-7)(-1.5) = 7$$

$$x_1 = -0.35$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Effect of round-off errors

error

$$e = x - x_*$$

residual

$$r = b - Ax_*$$

Effect of round-off errors

CognitiveSystems

$$\begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}$$

$$\frac{0.780}{0.913} = 0.854 \quad \text{(to three places)}$$

$$\begin{pmatrix} 0.913 & 0.659 \\ 0 & 0.001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254 \\ 0.001 \end{pmatrix}$$

Effect of round-off errors

CognitiveSystems

$$x_2 = \frac{0.001}{0.001} = 1.00 \quad \text{(exactly)},$$

$$\begin{aligned} x_1 &= \frac{0.254 - 0.659x_2}{0.913} \\ &= -0.443 \quad \text{(to three places)}. \end{aligned}$$

$$x_* = \begin{pmatrix} -0.443 \\ 1.000 \end{pmatrix}$$

$$\begin{aligned} r = b - Ax_* &= \begin{pmatrix} 0.217 - ((0.780)(-0.443) + (0.563)(1.00)) \\ 0.254 - ((0.913)(-0.443) + (0.659)(1.00)) \end{pmatrix} \\ &= \begin{pmatrix} -0.000460 \\ -0.000541 \end{pmatrix} \end{aligned}$$

very small residual at this resolution!

Effect of round-off errors

CognitiveSystems

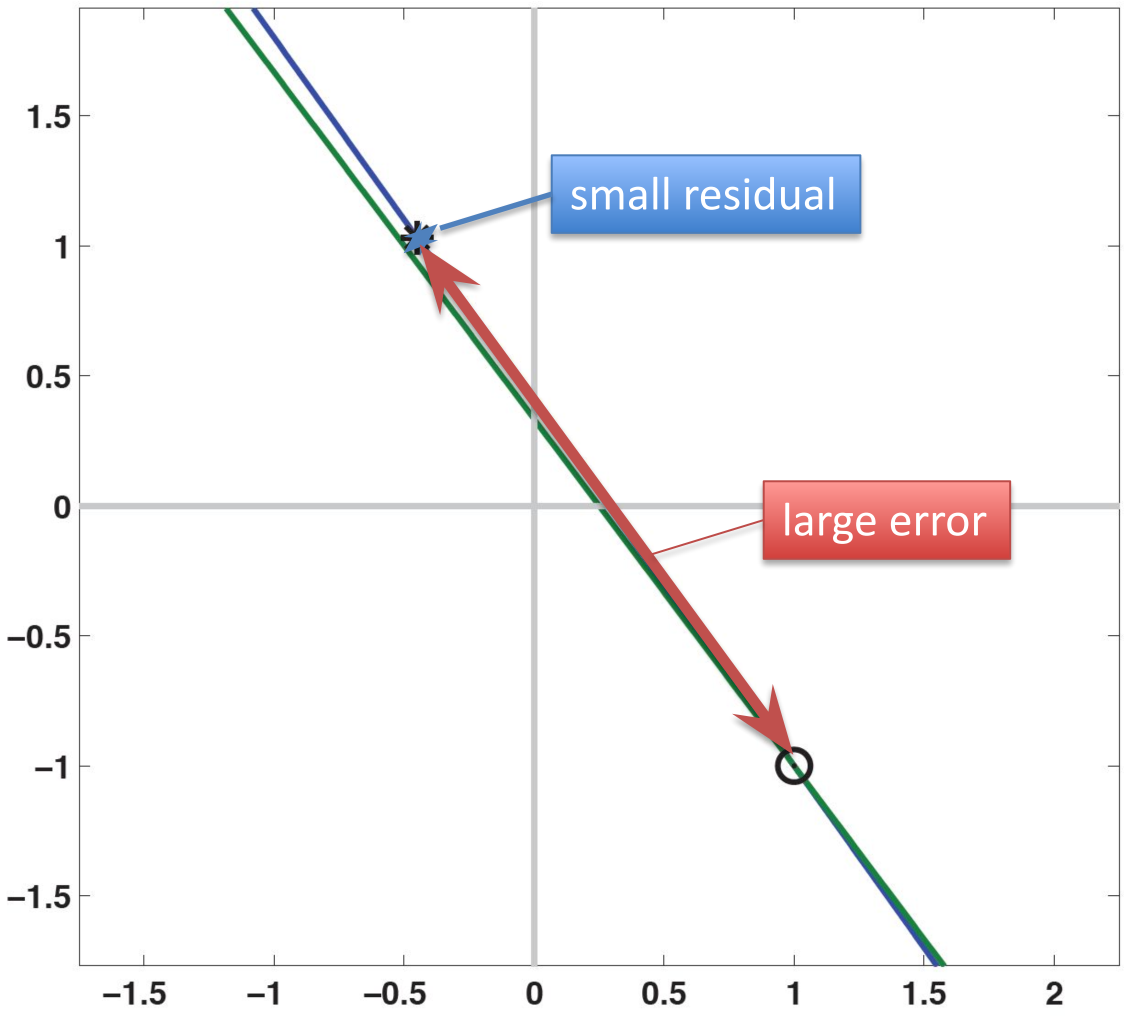
$$\begin{pmatrix} 0.913000 & 0.659000 \\ 0 & 0.000001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254000 \\ -0.000001 \end{pmatrix}$$

$$\left.\begin{aligned} x_2 &= \frac{-0.000001}{0.000001} = -1.00000, \\ x_1 &= \frac{0.254 - 0.659x_2}{0.913} \\ &= 1.00000, \end{aligned}\right\} \quad x_* = \begin{pmatrix} -0.443 \\ 1.000 \end{pmatrix}$$

very different from previous solution at 3-digit resolution!

Effect of round-off errors

- Ill-conditioned system describing two almost parallel lines!



Theoretical result

- Gaussian elimination with partial pivoting is guaranteed to produce small residuals (at machine resolution)
  - **but NOT small error!!**
- Therefore it is essential to check the results of your calculations carefully!

Partial Pivoting

- Problems with Gauss elimination
  - division by zero
  - round off errors
  - ill-conditioned systems
- Use pivoting to avoid this
  - Find the row with largest absolute coefficient below the pivot element
- Switch rows (“partial pivoting”)
  - complete pivoting switches also columns (this is rarely used)

Gauss Elimination with Partial Pivoting

- Forward elimination
- for each equation j, j = 1 to n-1
  - find the maximum element in the current column
  - if the maximum element is not from the current row, then switch the current row with the maximum row (i.e., *pivot*)
  - Now perform the elimination
  - (a) multiply equation j by a_{kj} $/a_{jj}$
  - (b) subtract the result from equation

Partial (Row) Pivoting

$$
\begin{aligned}
x_1 \quad\quad\quad\quad + 2x_3 + 3x_4 &= 1 \\
-x_1 + 2x_2 + 2x_3 - 3x_4 &= -1 \\
x_2 + x_3 + 4x_4 &= 2 \\
6x_1 + 2x_2 + 2x_3 + 4x_4 &= 1
\end{aligned}
$$

$$
[A \,\vdots\, b] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{array}\right]
$$

Forward Elimination

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 1 & 0 & 2 & 3 & 1 \end{array}\right]$$

Interchange rows 1 & 4

$$f_{21} = -1/6$$
$$f_{31} = 0$$
$$f_{41} = 1/6$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & -1/3 & 5/3 & 7/3 & 5/6 \end{array}\right]$$

$$(2) - (1) \times f_{21}$$
$$(3) - (1) \times f_{31}$$
$$(4) - (1) \times f_{41}$$

Forward Elimination

CognitiveSystems

$$\left[\begin{array}{cccc:c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & -1/3 & 5/3 & 7/3 & 5/6 \end{array}\right]$$

No interchange required

$$f_{32} = 3/7$$

$$f_{42} = 1/7$$

$$\left[\begin{array}{cccc:c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 0 & 0 & 5 & 33/14 \\ 0 & 0 & 2 & 2 & 5/7 \end{array}\right]$$

$$(3) - (2) \times f_{32}$$

$$(4) - (2) \times f_{42}$$

Back-Substitution

$$\left[\begin{array}{cccc:c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 0 & 2 & 2 & 5/7 \\ 0 & 0 & 0 & 5 & 33/14 \end{array}\right]$$

Interchange rows 3 & 4

$$f_{43} = 0$$

$$x_4 = (33/14)/5 = 33/70$$

$$x_3 = (5/7 - 2\,x_4)/2 = -4/35$$

$$x_2 = (-5/6 + 7/3\,x_4 - 7/3\,x_3)/(7/3) = 8/35$$

$$x_1 = (1 - 4\,x_4 - 2\,x_3 - 2\,x_2)/6 = -13/70$$

$$\vec{x} = \begin{bmatrix} -13/70 \\ 8/35 \\ -4/35 \\ 33/70 \end{bmatrix}$$