The Problem

Let $X_1, \dots, X_n$ be independent and identically distributed (iid) for some distribution with parameter θ. An estimator $\hat{\theta}$ of parameter θ is said to be consistent if

$$\lim_{n\to\infty} \mathbb{P}(|\hat{\theta} - \theta| \geq \varepsilon) = 0 \quad \text{or equivalently} \quad \lim_{n\to\infty} \mathbb{P}(|\hat{\theta} - \theta| \leq \varepsilon) = 1,$$

for all $\varepsilon > 0$. Intuitively what this means is that as the sample size increases, the probability that the estimator is the different from the true parameter tends towards 0. Or equivalently the probability that the estimator is the same as the true parameter tends to 1.

Let $Y_1, \dots, Y_n$ be iid and $Y_i \sim \text{Expon}\,(1/\lambda)$. Consider the estimators of λ: $\tilde{\lambda}_1 = 1/n \sum_{i=1}^n Y_i$ and $\tilde{\lambda}_2 = n \min\{Y_1, \dots, Y_n\}$.

The Analysis

It can be shown (although we won't) that both $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are unbiased. That is, $\mathbb{E}(\tilde{\lambda}_1) = \lambda = \mathbb{E}(\tilde{\lambda}_2)$.

To see that $\tilde{\lambda}_1$ is consistent, we can first calculate $\text{Var}(\tilde{\lambda}_1) = \lambda^2/n$, and then use Chebyshev's Inequality to obtain

$$\mathbb{P}(|\tilde{\lambda}_1 - \mathbb{E}(\tilde{\lambda}_1)| \geq \varepsilon) \leq \text{Var}(\tilde{\lambda}_1)/\varepsilon^2 \implies \mathbb{P}(|\tilde{\lambda}_1 - \lambda| \geq \varepsilon) \leq \lambda^2/n\varepsilon^2.$$

Thus by taking the limit as $n \to \infty$ of both sides we obtain

$$\lim_{n\to\infty} \mathbb{P}(|\tilde{\lambda}_1 - \lambda| \geq \varepsilon) \leq 0 \implies \lim_{n\to\infty} \mathbb{P}(|\tilde{\lambda}_1 - \lambda| \geq \varepsilon) = 0,$$

since probability is non-negative.

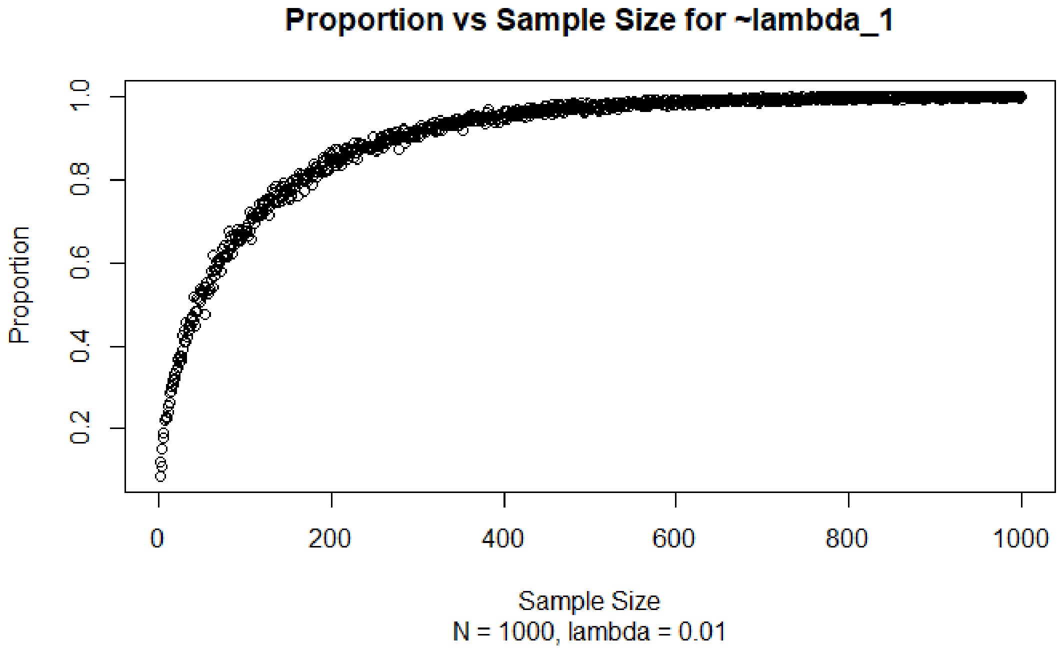
To see that $\tilde{\lambda}_2$ is inconsistent, notice that

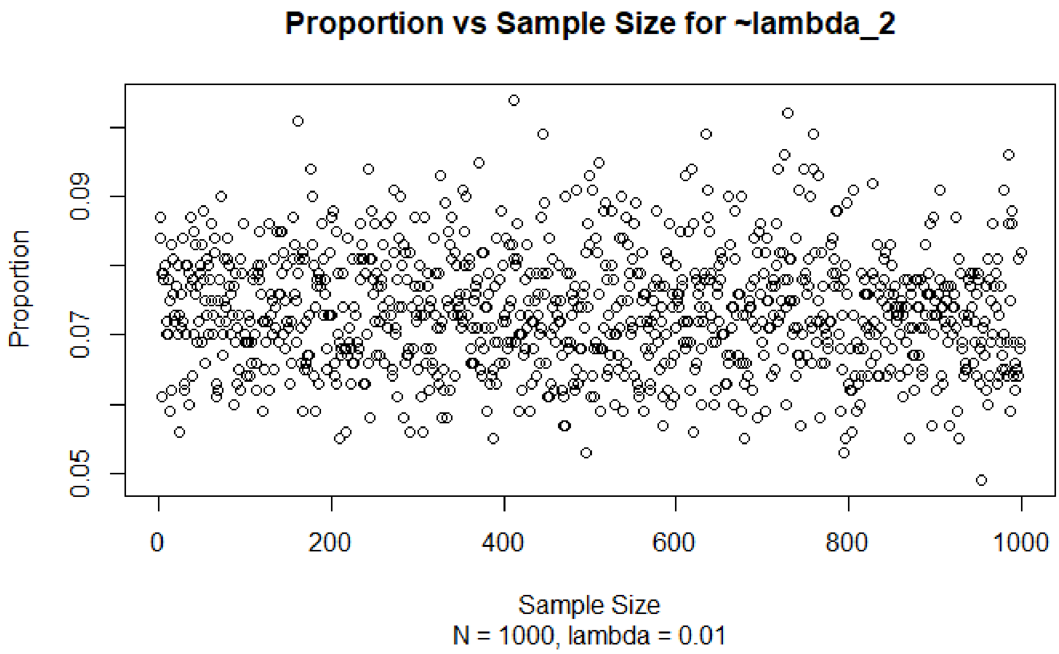
$$\lim_{n\to\infty} \mathbb{P}(|\tilde{\lambda}_2 - \lambda| \geq \varepsilon) = 0 \iff \lim_{n\to\infty} \mathbb{P}(|\tilde{\lambda}_2 - \lambda|^2 \geq \varepsilon^2) = 0.$$

Thus we need only find some $\varepsilon > 0$ for which $\lim_{n\to\infty} \mathbb{P}(|\tilde{\lambda}_2 - \lambda|^2 \geq \varepsilon^2) \neq 0$. It can be shown that $\text{Var}(\tilde{\lambda}_2) = \lambda^2$ We define a new random variable $Z_n = |\tilde{\lambda}_2 - \lambda|^2 = (\tilde{\lambda}_2 - \lambda)^2$, and we see $\mathbb{E}(Z_n) = \text{Var}(\tilde{\lambda}_2) = \lambda^2 \underset{n\to\infty}{\not\longrightarrow} 0$. Since Z_n is positive, so is its expectation. Thus $\mathbb{E}(Z_n)$ is bounded below by some $\delta^2 > 0$. Now notice that the expectation of Z_n is bounded above by the maximum of the image of Z_n. Thus there must exist at least one element in the sample space of Z_n (namely the element that maximizes Z_n) such that $Z_n \geq \delta^2$. Therefore $\mathbb{P}(Z_n \geq \delta^2) = \mathbb{P}(|\tilde{\lambda}_2 - \lambda|^2 \geq \delta^2) \neq 0$. In this limit this remains true. Thus $\lim_{n\to\infty} \mathbb{P}(|\tilde{\lambda}_2 - \lambda|^2 \geq \varepsilon^2) = 0$ is not true for all $\varepsilon > 0$ (take $\varepsilon = \delta$ as a counter example). Therefore $\tilde{\lambda}_2$ is inconsistent.

The Simulation

While the analysis may be correct, it is not particularly convincing unless one is familiar with the probability theory. This is where the simulation is helpful. By generating 1000 samples of sizes from $n = 1$ to $n = 1000$, we can calculate the proportion of samples for which the estimator was "within ε" of the true parameter. By then plotting this proportion against sample size we should see convergence for $\tilde{\lambda}_1$ but no convergence for $\tilde{\lambda}_2$. Note that I chose a small value of λ for my true parameter so that it converged more quickly and my laptop could finish the simulation. Indeed as n gets larger, the proportion of $\tilde{\lambda}_1$ that is "within ε" of λ is close to 1. But the proportion of $\tilde{\lambda}_2$ that is "within ε" of λ does not converge to anything.


Proportion vs Sample Size for ~lambda_1
Proportion
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000
Sample Size
N = 1000, lambda = 0.01


Proportion vs Sample Size for ~lambda_2
Proportion
0.05
0.07
0.09
0
200
400
600
800
1000
Sample Size
N = 1000, lambda = 0.01