This preliminary chapter reviews the most important things you should know before beginning calculus.

- real number system
- Cartesian coordinates in the plane
- equations representing straight lines, circles, and parabolas
- functions and their graphs;
- polynomials and
- trigonometric functions.

P.1. Real Numbers and The Real Line.

- Calculus depends on properties of the real number system.
- Real numbers are numbers that can be expressed as decimals, for instance,

$5 = 5.00000...$ $\qquad \sqrt{2} = 1.4142...$

$-\frac{3}{4} = -0.75000000...$ $\qquad \pi = 3.14159...$

$\frac{1}{3} = 0.3333....$

In each case the three dots (...) indicate that the sequence of decimal digits goes on forever. For the first three numbers above, the patterns of the digits are obvious; we know what all the subsequent digits are. For $\sqrt{2}$ and π there are no obvious patterns!

The Absolute Value

- The absolute value, or magnitude, of a number x denoted $|x|$ read " the absolute value of x"), is defined by the formula

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The vertical lines in the symbol $|x|$ are called absolute value bars.

- Geometrically, $|x|$ represents the distance from x to 0 on the real line.
- More generally, $|x-y|$ represents the distance between the points x and y on the real line.

Example ①: Solve $|2x+5|=3$

Solution: $|2x+5|=3 \Leftrightarrow 2x+5=\pm 3$

Thus, either $2x=-3-5=-8$ or

$2x=3-5=-2$.

The solutions are

$x=-4$ and $x=-1$

Example 2: Solve the equation

$$|x+1| = |x-3|$$

Solution: The equation says that x is equidistant from -1 and 3.

Therefore, x is the point halfway between -1 and 3;

$$x = \frac{(-1+3)}{2} = 1.$$

Alternatively, the given equation says that either $x+1 = x-3$ or $x+1 = -(x-3)$.

The first of these equations has no solutions; the second has the solution $x = 1$.

Example 3: What values of x satisfy the inequality $\left|5 - \frac{2}{x}\right| < 3$?

Solution: We have

$$\left|5 - \frac{2}{x}\right| < 3 \iff -3 < 5 - \frac{2}{x} < 3$$

Subtract 5 from each member

$$-8 < -\frac{2}{x} < -2$$ Divide each member by -2.

Not $a > 0 \Rightarrow \frac{1}{a} > 0$

$$4 > \frac{1}{x} > 1$$ Take reciprocals

$$\frac{1}{4} < x < 1.$$

or

$$-3 < 5 - \frac{2}{x} < 3$$

I $\quad -3 < 5 - \frac{2}{x}$

$\frac{2}{x} < 5 + 3$

$\frac{2}{x} < 8$

$\frac{1}{x} < 4$

$\frac{1}{4} < x$

II $\quad 5 - \frac{2}{x} < 3$

$2 < \frac{2}{x}$

$x < 1$

$$\frac{1}{4} < x < 1$$

Exercises

1- $|x| = 3$

2- $|2t + 5| = 4$

3- $|8 - 3s| = 9$

Solve the equations.

4- $|x+1| > |x-3|$

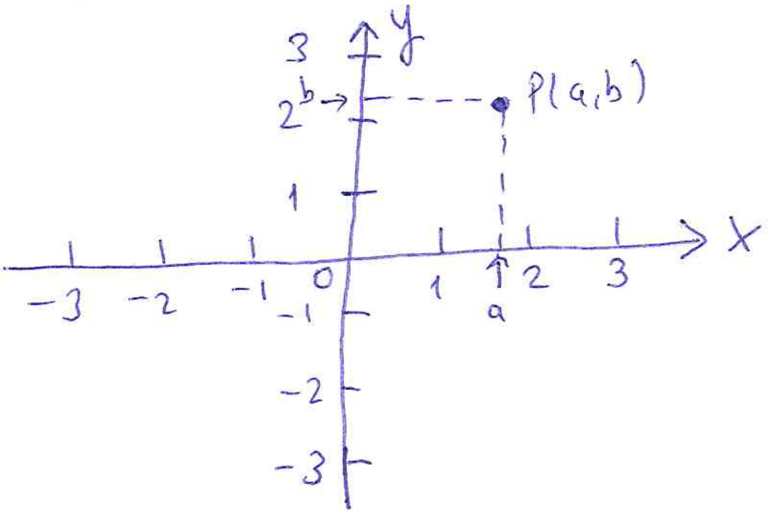
5- $\left|\frac{x}{2} - 1\right| \leq 1$

6- $|3x - 7| < 2$

Write the intervals defined by the given inequality.

Cartesian Coordinates in the Plane

The positions of all points in a plane can be measured with respect to two perpendicular real lines in the plane intersecting at the 0-point of each. These lines are called coordinate axes in the plane.



Origin: The point of intersection of the coordinate axes (the point where x and y are both zero) is called the origin and is often denoted by the letter O.

If P is any point in the plane, we can draw a line through P perpendicular to the x-axis. If a is the value of x where that line intersects the x-axis, we call a the x-coordinate (abscissa) of P.

Similarly, the y-coordinate (ordinate) of P is the value of y where the a line through P perpendicular to the y-axis meets the y-axis.

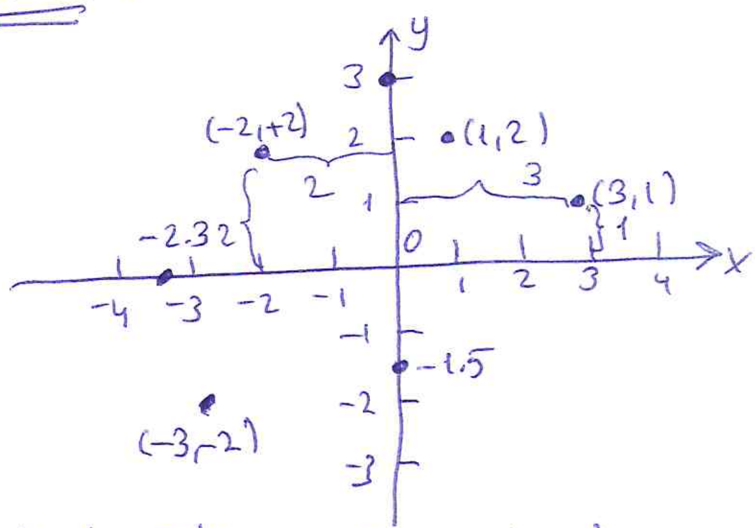
The ordered pair (a,b) is called the coordinate pair, or the Cartesian coordinates, of the point P.

Note that the x-coordinate appears first in a coordinate pair.

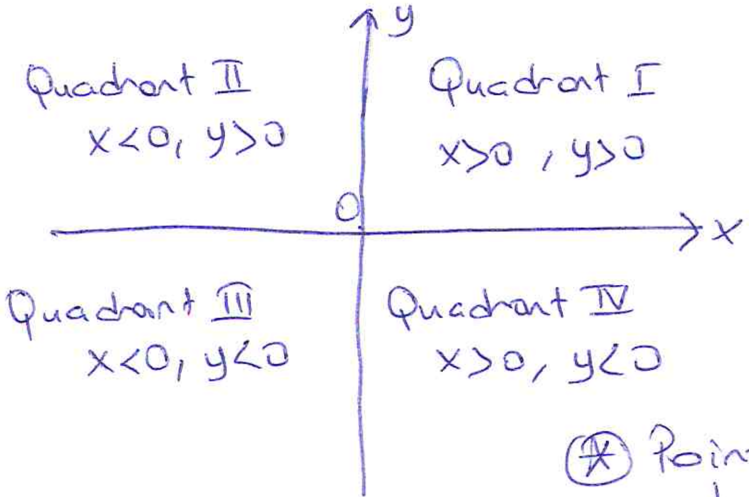
Coordinate pairs are in one-to-one correspondence with points in the plane; each point has a unique coordinate pair, and each coordinate pair determines a unique point.

We call such a set of coordinates axes and the coordinate pairs they determine a <u>Cartesian coordinate system</u> in the plane, after the seventeenth-century philosopher René Descartes, who created analytic (coordinate) geometry. When equipped with such a coordinate system, a plane is called a <u>Cartesian plane</u>.

Note that all points on the x-axis have y-coordinate 0. We usually just write the x-coordinates to label such points. Similarly, points on the y-axis have $x=0$, and we can label such points using their y-coordinates only.



(xy-plane)



The coordinate axes divide the xy-plane into four sections called <u>quadrants</u>.

(*) Points on the coordinate axes belong to no quadrant.

Increments and Distances

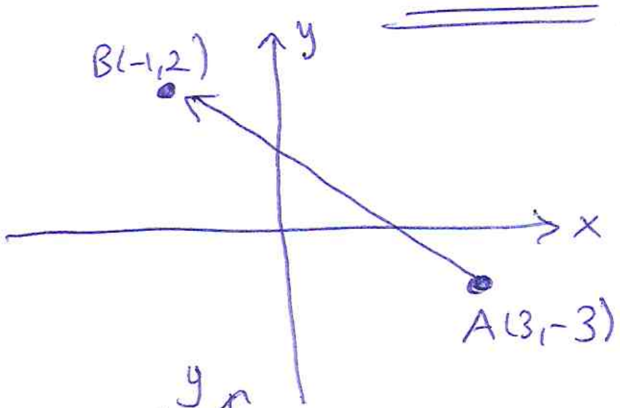
When a particle moves from one point to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

An increment in a variable is the net change in the value of the variable.

If x changes from x_1 to x_2, then the increment in x is

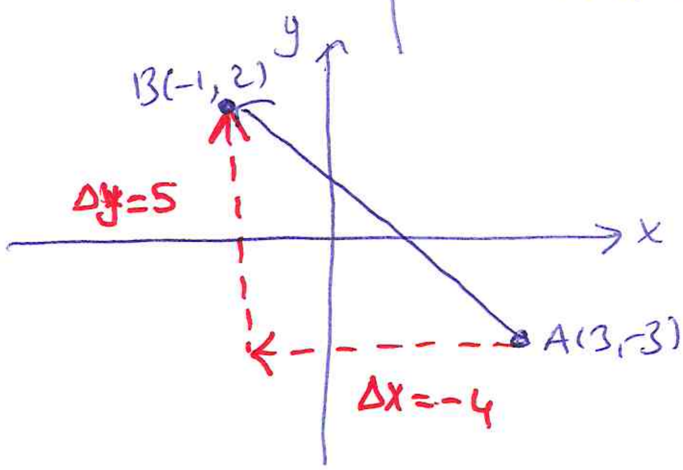
$$\Delta x = x_2 - x_1$$, Δ is the upper case Greek letter delta !

Example:



Find the increments in the coordinates of a particle that moves from $A(3,-3)$ to $B(-1,2)$.

Solution



The increments are:

$$\Delta x = -1 - 3 = -4 \text{ and}$$

$$\Delta y = 2 - (-3) = 5$$

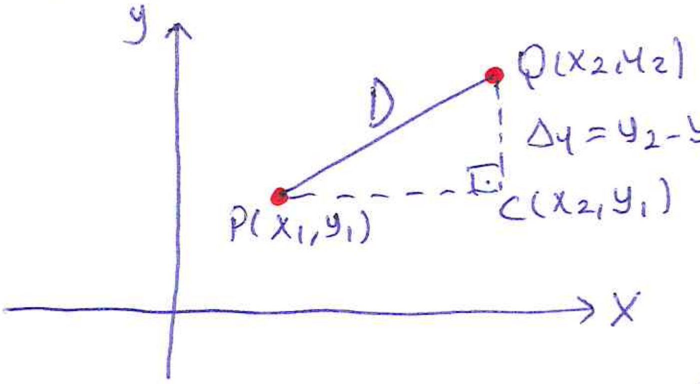
Distace Formula for points in the plane.

The distace D between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

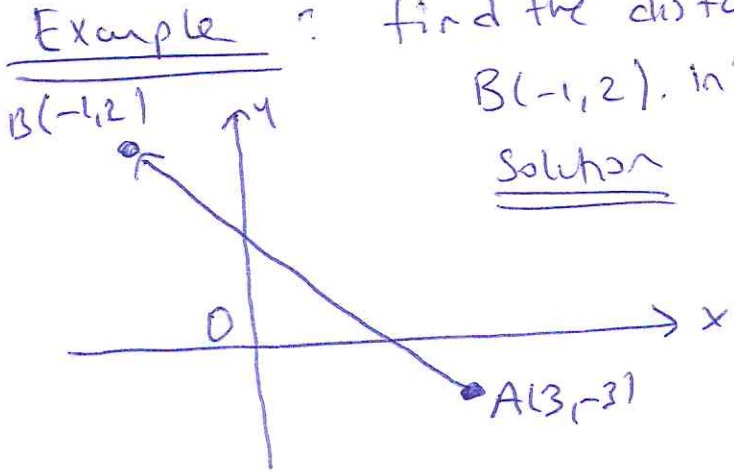
If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plae, the straight line segment PQ is the hypotenuse of a right triagle PCQ, as shown in the figure. The sides PC and CQ of the triagle have lengths

$|\Delta x| = |x_2 - x_1|$ and $|\Delta y| = |y_2 - y_1| \Longrightarrow$



$\Rightarrow$ These are the horizontal distace and vertical distace between P and Q. By the Pythagorean Theorem, the length of PQ is the square root of the sum of the squares of these lengths.

Example: find the distace between $A(3,-3)$ and $B(-1,2)$. in figure below.



Solution

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$D = \sqrt{(-1-3)^2 + (2-(-3))^2}$$

$$D = \sqrt{(-4)^2 + (5)^2}$$

$$D = \sqrt{41} \text{ units}$$

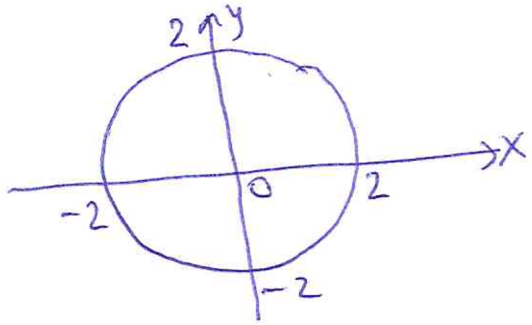
Example : The distance from the origin $O(0,0)$ to a point $P(x,y)$ is

$$D = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}.$$

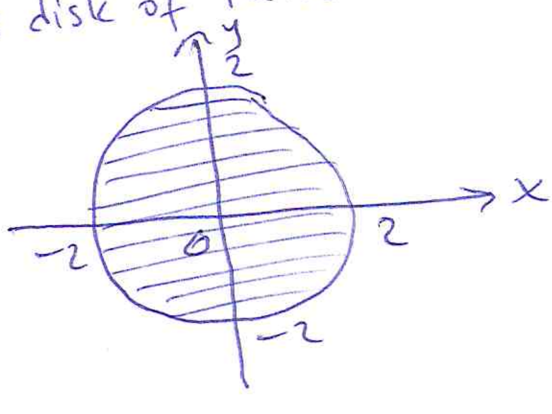
Graphs

The graph of an equation (or inequality) involving the variables x and y is the set of all points $P(x,y)$ whose coordinates satisfy the equation (or inequality).

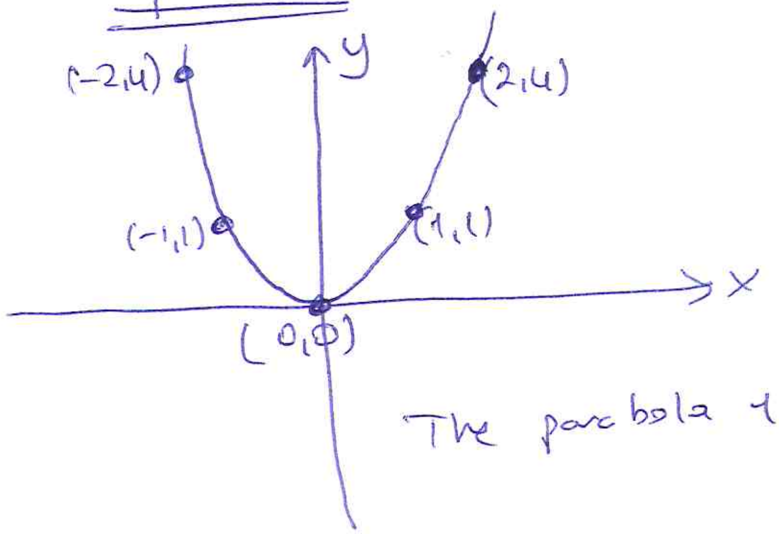
Example : The equation $x^2+y^2=4$ represents all points $P(x,y)$ whose distance from the origin is $\sqrt{x^2+y^2} = \sqrt{4} = 2$. These points lie on the circle of radius 2 centered at the origin. This circle is the graph of the equation $x^2+y^2=4$.



Example : Points (x,y) whose coordinates satisfy the inequality $x^2+y^2 \leq 4$ all have distance ≤ 2 from the origin. The graph of the inequality is therefore the disk of radius 2 centred at the origin.



Example : Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0,0)$, $(1,1)$, $(-1,1)$, $(2,4)$ and $(-2,4)$. These points (and all others satisfying the equation) lie on a smooth curve called a parabola.



The parabola $y = x^2$