Counting and Probability

We can model real-world situations using precise rules, such as equations and functions. But many of our everyday activities are not governed by precise rules but rather involve randomness.

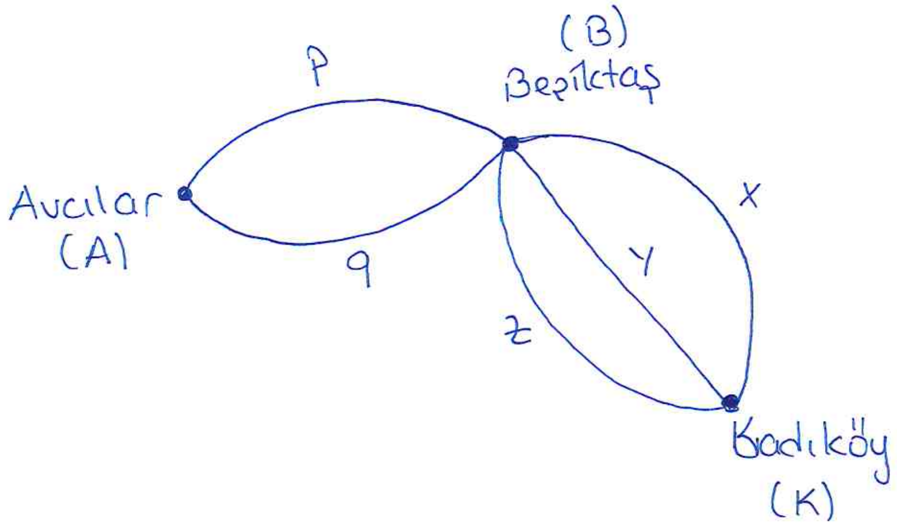
It is remarkable that there are also rules that govern randomness. For instance, if we toss a balanced coin many times, we can be pretty sure that "heads" will show up about half of time. Such patterns in apparently haphazard events allow us to use mathematics to model randomness.

Probability is the mathematical study of chance. The importance of probability in the modern world cannot be overestimated. It is used by business, government medical researchers, political pollsters, and many others.

Counting

* The Fundamental Counting Principle

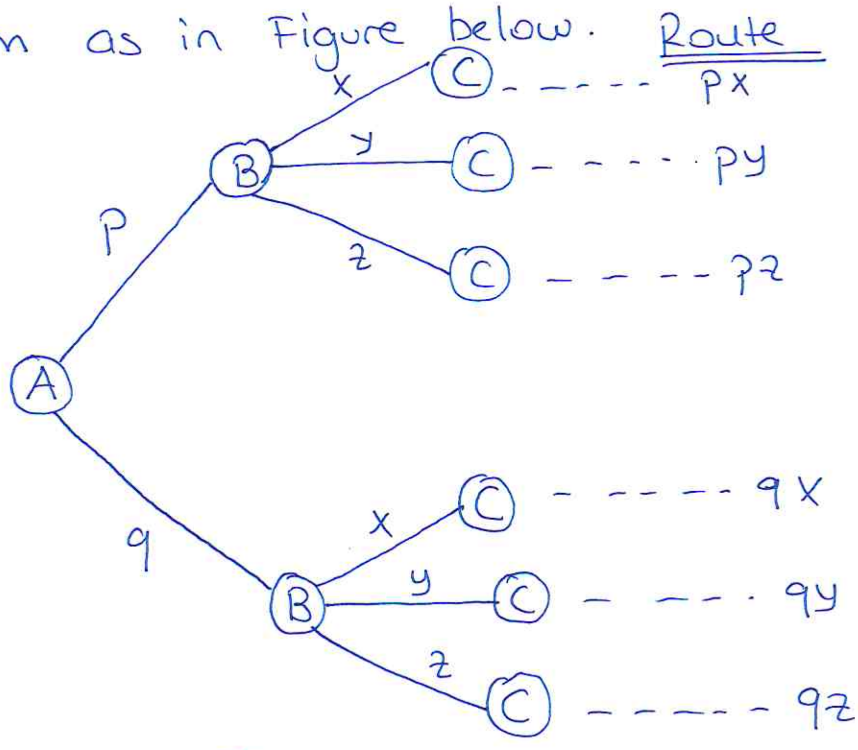
Suppose that three towns — Avcılar, Beşiktaş, Kadıköy — are located in such a way that two roads connect Avcılar to Beşiktaş and three roads connect Beşiktaş to Kadıköy.



How many different routes can one take to travel from Avcılar to Kadıköy via Beşiktaş?

The key to answering this question is to consider the problem in stages:

At the first stage — from Avcılar to Beşiktaş — there are two choices. For each of these choices there are three choices at the second stage — from Beşiktaş to Kadıköy. Thus the number of different routes is $\boxed{2 \times 3 = 6}$. These routes are conveniently enumerated by a tree diagram as in Figure below.



Tree Diagram

The method that we used to solve this problem leads to the following principle.

THE FUNDAMENTAL COUNTING PRINCIPLE

Suppose that two events occur in order. If the first event can occur in m ways and the second can occur in n ways (after the first has occurred), then the two events can occur <u>in order</u> in $m \times n$ ways.

There is an immediate consequence of this principle for any number of events:
If $E_1, E_2, \ldots, E_k$ are events that occur in order and if E_1 can occur in n_1 ways, E_2 in n_2 ways, E_3 in n_3 ways, $\ldots$ E_k in n_k ways, then the events can occur in order in $n_1 \times n_2 \times n_3 \times \cdots \times n_k$ ways.

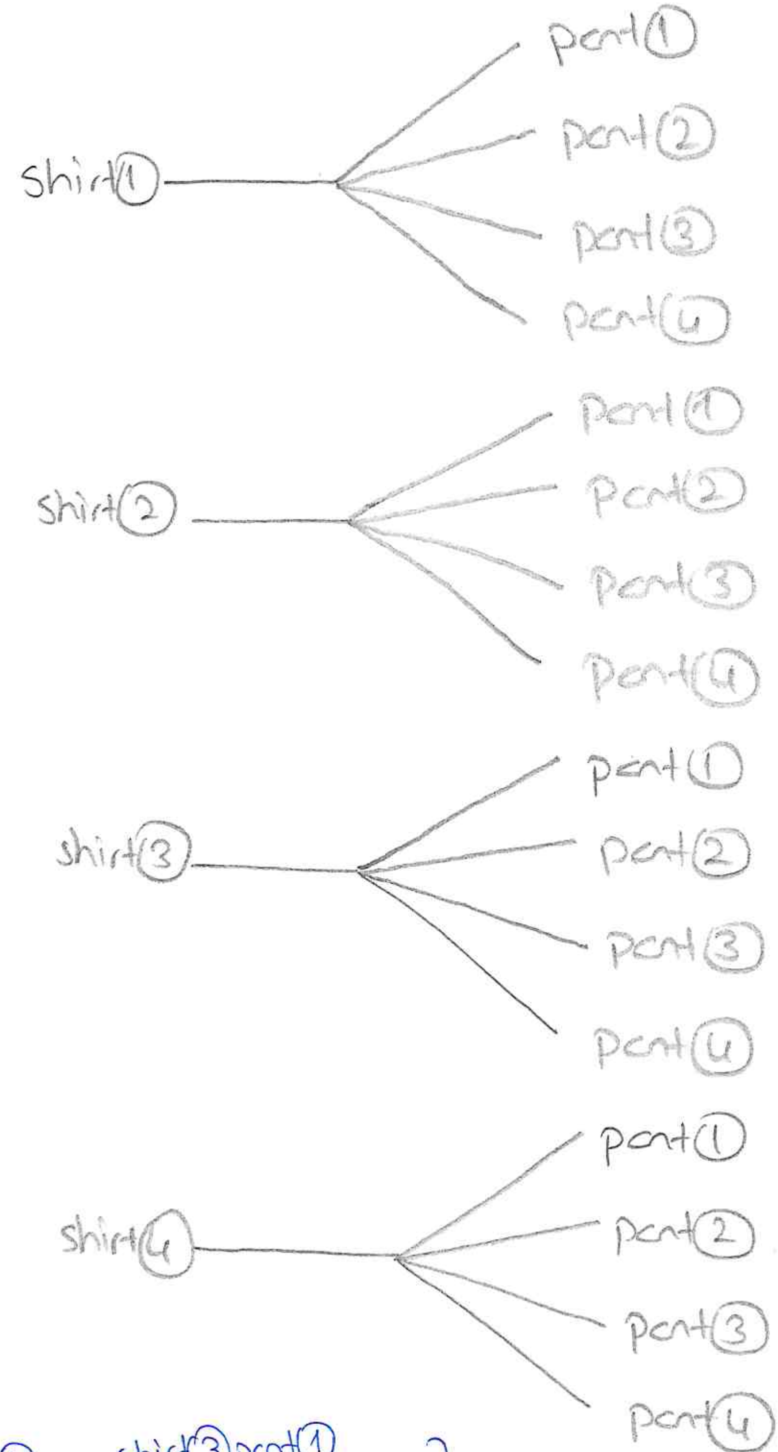
<u>Example ①</u> How many outfits can you make with 4 shirts and 4 pants?

<u>Solution</u> : Solve two ways

Tree Diagram

Fundamental Counting Principle

<u>Tree Diagram</u> :



shirt 1 pant 1
shirt 1 pant 2
shirt 1 pant 3
shirt 1 pant 4

shirt 2 pant 1
shirt 2 pant 2
shirt 2 pant 3
shirt 2 pant 4

shirt 3 pant 1
shirt 3 pant 2
shirt 3 pant 3
shirt 3 pant 4

shirt 4 pant 1
shirt 4 pant 2
shirt 4 pant 3
shirt 4 pant 4

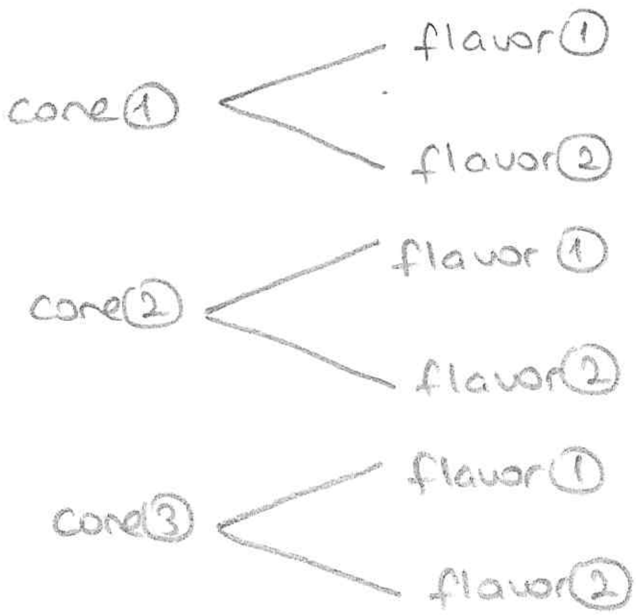
The number of outfits that you can make with 4 shirts and 4 pants is $\boxed{16}$

Fundamental counting principle.

$$\boxed{4 \cdot 4 = 16}$$

Example 2: An ice-cream store offers three types of cones and 2 flavors. How many different single scoop ice-cream cones is it possible to buy at this store?

Solution: Tree Diagram:



cone 1 flavor 1
cone 1 flavor 2
cone 2 flavor 1
cone 2 flavor 2
cone 3 flavor 1
cone 3 flavor 2

it is possible to buy $\boxed{6}$ different single-scoop ice-cream cones

Fundamental Counting Principle:

There are two stages for selecting an ice-cream cone. At the first stage we choose a type of cone, and at the second stage we choose a flavor.

We can think of the different stages as boxes:

Stage 1: Type of Cone

Stage 2: Flavor

The first box can be filled in 3 ways, and the second can be filled in 2 ways:

3	2

Stage 1

Stage 2

By the Fundamental Counting Principle there are $\boxed{3 \times 2 = 6}$ ways of choosing a single-scoop ice-cream cone at this store.

<u>Example</u> (3) : A vendor sells ice cream from a cart on the boardwalk. He offers vanilla, chocolate, strawberry, and pistachio ice cream, served in either a waffle, sugar, or plain cone. How many different single-scoop ice-cream cones can you buy from this vendor?

<u>Solution</u> : By using Fundamental Counting Principle (FCP)

3	4

$\Longleftrightarrow$ $\boxed{3 \times 4 = 12}$

stage 1 :
waffle cone
sugar cone
plain cone

stage 2 :
vanilla
chocolate
strawberry
pistachio

Example 4 : An ice-cream store offers three types of cones and 36 flavors. How many different single-scoop ice-cream cones is it possible to buy at this store?

Solution : By using FCP

3	36

There are $3 \times 36 = 108$ ways of choosing a single-scoop ice-cream cone at this store.

Example 5 : How many ways can a president and vice president be selected from a class of 3 students?

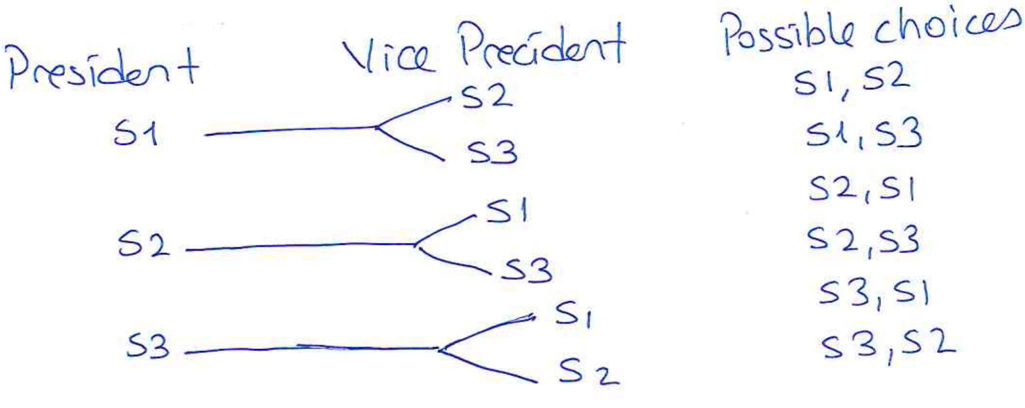
Solution : Initially, there are 3 people to pick as a president. After you pick the president, there are now 2 people that can be picked as VP.

That give $3 \times 2 = 6$ possible choices !

3	2

stage 1 : president

stage 2 : vice president



One person cannot be president and vice president at the same time.

Example 6: How many ways to choose a president and a vice president from a group of 6 members?

Solution:

6	5
president	vice president

$\boxed{6 \times 5 = 30}$ ways of choosing a president and a vice president from a group of 6 members.

Example 7: How many ways to choose a president, vice president, treasurer, and secretary from a group of 24 members?

Solution:

24	23	22	21
president	vice president	treasurer	secretary

$$24 \times 23 \times 22 \times 21 = 255\,024$$

Example 8 : How many three-letter "words" (strings of letters) can be formed by using the 26 letters of the alphabet if repetition of letters

(a) is allowed?

(b) is not allowed?

Solution :

(a)

26	26	26

$\Leftrightarrow 26^3 = 17\,576$

(b)

26	25	24

$\Leftrightarrow 26 \times 25 \times 24 = 15600$

Example 9 : In a certain state, automobile license plates display three letters followed by three digits. How many such plates are possible if repetition of letters (a) is allowed? (b) is not allowed?

Solution : There are six selection stages, one for each letter or digit on the license plate. As in the preceding example, we sketch a box for each stage:

26	26	26	10	10	10

Letters Digits

At the first stage we choose a letter (from 26 possible choices); at the second stage we choose another letter (again from 26 choices); at the third stage we choose another letter (26 choices); at the fourth stage we choose a digit (from 10 possible choices); at the fifth stage we choose a digit (again from 10 choices); and at the sixth stage, we choose another digit (10 choices). By the FCP the number of possible license plates is $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17\,576\,000$.

(b) If repettion of letters is not allowed, then we arrange the choices as follows:

$$\underbrace{\boxed{26}\boxed{25}\boxed{24}}_{\text{Letters}}\underbrace{\boxed{10}\boxed{10}\boxed{10}}_{\text{Digits}}$$

At the first stage we have 26 letters to choose from, but once the first letter has been chosen, there are only 25 letters to choose from at the second stage. Once the first two letters have been chosen, 24 letters are left to choose from for the third stage. The digits are chosen as before. Thus the number of possible license plates in this case is

$$26 \times 25 \times 24 \times 10 \times 10 \times 10 = \underline{\underline{15\,600\,000}}$$