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Factorial

* The factorial function (symbol : !) says to multiply all whole numbers from our chosen number down to 1.

<u>Examples</u> :
- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$
- $1! = 1$
- $0! = 1$

We usually say, $6!$, as "6 factorial".

$n \in \mathbb{N} \Rightarrow n! = n(n-1)(n-2) \cdots 3.2.1$

- $n! = n(n-1)!$
- $(n+1)! = (n+1).n.(n-1)!$
- $(n-1)! = (n-1)(n-2)!$

<u>Example 1</u>: $\dfrac{7.9! - 3.8!}{8! + 2.7!} = ?$

<u>Solution</u> :

$$\frac{7.9! - 3.8!}{8! + 2.7!} = \frac{7.9.8.7! - 3.8.7!}{8.7! + 2.7!}$$

$$= \frac{\cancel{7!}(7.9.8 - 3.8)}{\cancel{7!}(8+2)}$$

$$= \frac{504 - 24}{10}$$

$$= 48$$

Example 2 : $\dfrac{(1! + 2! + 3!)! + 8!}{7!} = ?$

Solution : $\dfrac{(1 + 2.1 + 3.2.1)! + 8.7!}{7!}$

$$= \frac{9! + 8.7!}{7!}$$

$$= \frac{9.8.7! + 8.7!}{7!}$$

$$= \frac{(72+8)\cancel{7!}}{\cancel{7!}}$$

$$= 80$$

Example 3 : $\dfrac{(n+1)!}{(n-1)!} = 90$, $n = ?$, $n \in N$

Solution : $\dfrac{(n+1).n.\cancel{(n-1)!}}{\cancel{(n-1)!}} = 90$

$$n.(n+1) = 90.$$

$$n^2 + n - 90 = 0$$

$$10 \quad -9$$

$$(n+10)(n-9) = 0$$

$$\boxed{n = 9}$$

Example 4 : $\dfrac{9! + 10!}{11! + 10!} = ?$

Solution : $\dfrac{9! + 10.9!}{11.10.9! + 10.9!}$

$$= \frac{\cancel{9!}\,(1+10)}{\cancel{9!}\,(110+10)}$$

$$= \frac{11}{120}$$

Example 5 : $\dfrac{12! + 13! + 14!}{14.11!} = ?$

Solution : $\dfrac{12.11! + 13.12.11! + 14.13.12.11!}{14.11!}$

$$= \frac{\cancel{11!}\,(12 + 13.12 + 14.13.12)}{\cancel{11!}.14}$$

$$= \frac{12 + (1 + 14).13.12}{14}$$

$$= \frac{12 + 15.13.12}{14} = \frac{\overset{6}{\cancel{12}}\,(1 + 15.13)}{\cancel{14}_{7}}$$

$$= 168$$

Example 6 : $\dfrac{n! + (n-1)!}{n! - 2(n-2)!} = \dfrac{6}{5}$, $n = ?$, $n \in \mathbb{N}$

Solution : $\dfrac{n.(n-1)(n-2)! + (n-1)(n-2)!}{n.(n-1)(n-2)! - 2(n-2)!} = \dfrac{6}{5}$

$$\frac{\cancel{(n-2)!}\,(n.(n-1) - (n-1))}{\cancel{(n-2)!}\,(n.(n-1) - 2)} = \frac{6}{5}$$

$$\frac{n^2 \cancel{-n} \cancel{-n} + 1}{n^2 - n - 2} \times \frac{6}{5}$$

$$5n^2 + 5 = 6n^2 - 6n - 12$$

$$n^2 - 6n - 7 = 0$$

$$-7 \quad 1$$

$$(n-7)(n+1) = 0$$

$$\boxed{n = 7}$$

Example 7 : $\dfrac{(n+1)! - 2(n-1)!}{n! - (n-1)!} = ?$

Solution : $\dfrac{(n+1)(n).(n-1)! - 2(n-1)!}{n.(n-1)! - (n-1)!}$

$$= \frac{\cancel{(n-1)!}\,((n+1).n - 2)}{\cancel{(n-1)!}\,(n-1)} = \frac{n^2 + n - 2}{n-1}$$

$$= \frac{(n+2)\cancel{(n-1)}}{\cancel{(n-1)}}$$

$$= n+2$$

Permutation

$n \geq r$

$$P(n,r) = \frac{n!}{(n-r)!} \quad \text{or} \quad P(n,r) = n.(n-1)\dots(n-r+1)$$

$P(n,r)$ denote the number of permutations of n distinct objects, taken r at a time.

Example (8) : $3.P(n-1,2) = P(2n,2) - 54$

$n = ?$

Solution : $$3.\frac{(n-1)!}{(n-1-2)!} = \frac{(2n)!}{(2n-2)!} - 54$$

$$3.\frac{(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = \frac{(2n)(2n-1)\cancel{(2n-2)!}}{\cancel{(2n-2)!}} - 54$$

$$3(n-1)(n-2) = (2n)(2n-1) - 54$$

$$3(n^2-2n-n+2) = 4n^2-2n-54$$

$$3n^2-6n-3n+6 = 4n^2-2n-54$$

$$0 = n^2+7n-60$$

$12 \quad -5$

$$0 = (n+12)(n-5)$$

$$\boxed{n=5} \qquad n = \cancel{-12}$$

Example 9 :

- $P(9,3) = ?$
- $P(9,2) = ?$
- $P(8,4) = ?$
- $P(5,1) = ?$

Solution :

- $P(9,3) = \dfrac{9!}{(9-3)!}$

$= \dfrac{9!}{6!}$

$= \dfrac{9.8.7.\cancel{6!}}{\cancel{6!}}$

$= 9.8.7$

- $P(9,2) = \dfrac{9!}{(9-2)!} = \dfrac{9!}{7!} = \dfrac{9.8.\cancel{7!}}{\cancel{7!}} = 9.8$

- $P(8,4) = \dfrac{8!}{(8-4)!} = \dfrac{8!}{4!} = \dfrac{8.7.6.5.\cancel{4!}}{\cancel{4!}}$

$= 8.7.6.5$

- $P(5,1) = \dfrac{5!}{(5-1)!} = \dfrac{5!}{4!}$

$= \dfrac{5.\cancel{4!}}{\cancel{4!}}$

$= 5$

Example 10 : $P(n,2) = 30$, $n = ?$

Solution :

$$\frac{n!}{(n-2)!} = 30$$

$$\frac{n.(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 30$$

$$n^2 - n - 30 = 0$$

$$-6 \quad 5$$

$$(n-6)(n+5) = 0$$

$$\boxed{n=6} \quad \cancel{n=-5}$$

Combination

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!} \quad , \text{where } 0 \le r \le n.$$

Example 11 :
- $C(6,3) = ?$
- $C(9,3) = ?$
- $C(5,2) = ?$

Solution : • $C(6,3) = \dfrac{6!}{(6-3)!\,3!} = \dfrac{6!}{3!\,3!}$

$$= \frac{6.5.4.\cancel{3!}}{3!.\cancel{3!}}$$

$$= \frac{\cancel{6}.5.4}{\cancel{3.2}.1}$$

$$= 20$$

- $C(9,3) = \binom{9}{3} = \dfrac{9!}{(9-3)!\,3!}$

$$= \frac{9!}{6!\ 3!}$$

$$= \frac{{}^{3}\cancel{9} \cdot \overset{4}{\cancel{8}} \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot \cancel{6}_{3}}$$

$$= \underline{\underline{84}}$$

- $C(5,2) = \binom{5}{2}$

$$= \frac{5!}{(5-2)!\,2!}$$

$$= \frac{5!}{3! \cdot 2!}$$

$$= \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = \frac{5 \cdot \cancel{4}^{2}}{\cancel{2}}$$

$$= \underline{\underline{10}}$$

Example 12 : $P(5,2) - C(5,2) = ?$

Solution :

$$\frac{5!}{(5-2)!} - \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!} - \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{3!} \cdot \cancel{2}}$$

$$= 20 - 10$$

$$= \underline{\underline{10}}$$

Example 13 : $\binom{n}{r} = \binom{n}{n-r}$, $0 \leq r \leq n$

prove the above equality.

Solution $\dfrac{n!}{(n-r)!\,r!} = \dfrac{n!}{(n-(n-r))!\,(n-r)!}$

$$\frac{n.(n-1).(n-2)\ldots(n-r+1)\cancel{(n-r)}!}{\cancel{(n-r)!}\,\cancel{r!}} = \frac{n!}{\cancel{r!}\,(n-r)!}$$

$$n.(n-1).(n-2)\ldots(n-r+1) = \frac{n.(n-1).(n-2)\ldots(n-r+1)\cancel{(n-r)}!}{\cancel{(n-r)}!}$$

$$1 = 1$$

Example 14 : $\binom{n+2}{n+1} + \binom{n+2}{n} = 21$, $n = ?$

$$\frac{(n+2)!}{(n+2-(n+1))!\,(n+1)!} + \frac{(n+2)!}{(n+2-n)!\,n!} = 21$$

$$\frac{(n+2)!}{1!\,(n+1)!} + \frac{(n+2)!}{2!\,n!} = 21$$

$$\frac{(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}} + \frac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}.2} = 21$$

$$2n+4+n^2+n+2n+2 = 42$$

$$n^2+5n-36=0$$

$$-4 \quad 9$$

$$(n-4)(n+9)=0 \Rightarrow \boxed{n=4}$$

$\cancel{n=-9}$

Example 15 : $C(n,1) + 2\,C(n,2) = 64$

$n = ?$

Solution : $\dfrac{n!}{(n-1)!\,1!} + 2.\dfrac{n!}{(n-2)!\,2!} = 64.$

$$\frac{n.\cancel{(n-1)!}}{\cancel{(n-1)!}} + \frac{\cancel{2}.\,n.(n-1)\,\cancel{(n-2)!}}{\cancel{(n-2)!}\,\cancel{2}} = 64$$

$$\cancel{n} + n^2 - \cancel{n} = 64$$

$$\sqrt{n^2} = \sqrt{64}$$

$$n = \pm 8 \quad \boxed{n=8} \text{ or } n \neq -8$$

Example 16 : $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

$$\binom{6}{0} = \frac{6!}{(6-0)!\,0!} = \frac{6!}{6!\,1.} = 1$$

$$\binom{6}{1} = \frac{6!}{(6-1)!\,1!} = \frac{6!}{5!\,1!} = 6$$

$$\binom{6}{2} = \frac{6!}{(6-2)!\,2!} = \frac{6!}{4!\,2!} = \frac{5.\cancel{6}^{3}}{\cancel{2}} = 15$$

$$\binom{6}{3} = \frac{6!}{(6-3)!\,3!} = \frac{\cancel{6}.5.4.\cancel{3!}}{\cancel{3!}.\cancel{6}} = 20$$

$$\binom{6}{4} = \frac{6!}{(6-4)!\,4!} = \frac{6!}{2!\,4!} = \frac{\cancel{6}^{3}.5.\cancel{4!}}{\cancel{4!}.\cancel{2}} = 15$$

$$\binom{6}{5} = \frac{6!}{(6-5)!\,5!} = \frac{6.\cancel{5!}}{\cancel{5!}} = 6$$

$$\binom{6}{6} = \frac{6!}{(6-6)!\,6!} = \frac{\cancel{6!}}{0!\,\cancel{6!}} = 1$$

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}=1+6+15+20+15+6+1$$

$$=7+30+20+7$$

$$=14+50$$

$$=64$$

Example (17) : $\binom{9}{x+4}=\binom{9}{2x+2}$

what is the summation of the roots of the equation above?

Solution :

$$\binom{9}{x+4}=\binom{9}{2x+2}$$

(1) $x+4=2x+2$

$$\boxed{x=2}$$

(2) $9-x-4=2x+2$

$$3x=3$$

$$\boxed{x=1}$$

$$2+1=3$$

Example (18) : $6.C(9,r)=P(9,r)$ $r=?$

Solution

$$6.\frac{\cancel{9!}}{\cancel{(9-r)!}\,r!}=\frac{\cancel{9!}}{\cancel{(9-r)!}}$$

$$r!=6.$$

$$\boxed{r=3}$$

Example (19): $P(n,2) = 4\,C(n,1) + C(n,2)$

$n = ?$

Solution:

$$\frac{\cancel{n!}}{(n-2)!} = \frac{4 \cdot \cancel{n!}}{(n-1)!\,1!} + \frac{\cancel{n!}}{(n-2)!\,2!}$$

$$\frac{1}{\cancel{(n-2)!}} = \frac{4}{(n-1)\cancel{(n-2)!}} + \frac{1}{2\cancel{(n-2)!}}$$

$$1 = \frac{4}{n-1} + \frac{1}{2}$$

$$\frac{1}{2} = \frac{4}{n-1}$$

$$n-1 = 8$$

$$n = 9$$

$$\boxed{n = 9}$$