PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Sets (Chapter R, Review, Section R.1, pp. 2–3)

Now Work the 'Are You Prepared?' problems on page 704.

OBJECTIVES
1. Find All the Subsets of a Set (p. 700)
2. Count the Number of Elements in a Set (p. 700)
3. Solve Counting Problems Using the Multiplication Principle (p. 702)

Counting plays a major role in many diverse areas, such as probability, statistics, and computer science; counting techniques are a part of a branch of mathematics called **combinatorics**.

1 Find All the Subsets of a Set

We begin by reviewing the ways in which two sets can be compared.

If two sets A and B have precisely the same elements, we say that A and B are **equal** and write $A = B$.

If each element of a set A is also an element of a set B, we say that A is a **subset** of B and write $A \subseteq B$.

If $A \subseteq B$ and $A \neq B$, we say that A is a **proper subset** of B and write $A \subset B$.

If $A \subseteq B$, every element in set A is also in set B, but B may or may not have additional elements. If $A \subset B$, every element in A is also in B, and B has at least one element not found in A.

Finally, we agree that the empty set, $\varnothing$, is a subset of every set; that is,

$$\varnothing \subseteq A \qquad \text{for any set } A$$

EXAMPLE 1 **Finding All the Subsets of a Set**

Write down all the subsets of the set $\{a, b, c\}$.

Solution To organize the work, write down all the subsets with no elements, then those with one element, then those with two elements, and finally those with three elements. This gives all the subsets. Do you see why?

0 Elements	1 Element	2 Elements	3 Elements
$\varnothing$	$\{a\}, \{b\}, \{c\}$	$\{a, b\}, \{b, c\}, \{a, c\}$	$\{a, b, c\}$

Now Work PROBLEM 9

2 Count the Number of Elements in a Set

In Words
The notation $n(A)$ means "the number of elements in set A."

As you count the number of students in a classroom or the number of pennies in your pocket, what you are really doing is matching, on a one-to-one basis, each object to be counted with the set of counting numbers, $1, 2, 3, \ldots, n$, for some number n. If a set A matched up in this fashion with the set $\{1, 2, \ldots, 25\}$, you would conclude that there are 25 elements in the set A. The notation $n(A) = 25$ is used to indicate that there are 25 elements in the set A.

Because the empty set has no elements, we write

$$n(\varnothing) = 0$$

If the number of elements in a set is a nonnegative integer, the set is **finite**. Otherwise, it is **infinite**. We shall concern ourselves only with finite sets.

Look again at Example 1. A set with 3 elements has $2^3 = 8$ subsets. This result can be generalized.

> If A is a set with n elements, then A has 2^n subsets.

For example, the set $\{a, b, c, d, e\}$ has $2^5 = 32$ subsets.

EXAMPLE 2 **Analyzing Survey Data**

In a survey of 100 college students, 35 were registered in College Algebra, 52 were registered in Computer Science I, and 18 were registered in both courses.

(a) How many students were registered in College Algebra or Computer Science I?
(b) How many were registered in neither course?

Solution (a) First, let A = set of students in College Algebra

B = set of students in Computer Science I

Then the given information tells us that

$$n(A) = 35 \qquad n(B) = 52 \qquad n(A \cap B) = 18$$

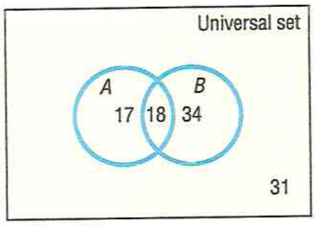


Figure 1

Refer to Figure 1. Since $n(A \cap B) = 18$, the common part of the circles representing set A and set B has 18 elements. In addition, the remaining portion of the circle representing set A will have $35 - 18 = 17$ elements. Similarly, the remaining portion of the circle representing set B has $52 - 18 = 34$ elements. This means that $17 + 18 + 34 = 69$ students were registered in College Algebra or Computer Science I.

(b) Since 100 students were surveyed, it follows that $100 - 69 = 31$ were registered in neither course.

Now Work PROBLEMS 17 AND 27

The solution to Example 2 contains the basis for a general counting formula. If we count the elements in each of two sets A and B, we necessarily count twice any elements that are in both A and B—that is, those elements in $A \cap B$. To count correctly the elements that are in A or B—that is, to find $n(A \cup B)$—we need to subtract those in $A \cap B$ from $n(A) + n(B)$.

THEOREM

Counting Formula

If A and B are finite sets,

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)} \qquad (1)$$

Refer to Example 2. Using formula (1), we have

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 35 + 52 - 18 \\ &= 69 \end{aligned}$$

There are 69 students registered in College Algebra or Computer Science I.

A special case of the counting formula (1) occurs if A and B have no elements in common. In this case, $A \cap B = \varnothing$, so $n(A \cap B) = 0$.

THEOREM

Addition Principle of Counting

If two sets A and B have no elements in common, that is,

$$\boxed{\text{if } A \cap B = \varnothing, \text{ then } n(A \cup B) = n(A) + n(B)} \qquad (2)$$

Formula (2) can be generalized.

THEOREM

General Addition Principle of Counting

If, for n sets $A_1, A_2, \ldots, A_n$, no two have elements in common, then

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = n(A_1) + n(A_2) + \cdots + n(A_n) \quad (3)$$

EXAMPLE 3 **Counting**

Table 1 lists the level of education for all United States residents 25 years of age or older in 2014.

Table 1

Level of Education	Number of U.S. Residents at Least 25 Years Old
Not a high school graduate	24,458,000
High school graduate	62,240,000
Some college, but no degree	34,919,000
Associate's degree	20,790,000
Bachelor's degree	42,256,000
Advanced degree	24,623,000

Source: U.S. Census Bureau

(a) How many U.S. residents 25 years of age or older had an associate's degree or a bachelor's degree?

(b) How many U.S. residents 25 years of age or older had an associate's degree, a bachelor's degree, or an advanced degree?

Solution Let A represent the set of associate's degree holders, B represent the set of bachelor's degree holders, and C represent the set of advanced degree holders. No two of the sets A, B, and C have elements in common (although the holder of an advanced degree certainly also holds a bachelor's degree, the individual would be part of the set for which the highest degree has been conferred). Then

$$n(A) = 20{,}790{,}000 \quad n(B) = 42{,}256{,}000 \quad n(C) = 24{,}623{,}000$$

(a) Using formula (2),

$$n(A \cup B) = n(A) + n(B) = 20{,}790{,}000 + 42{,}256{,}000 = 63{,}046{,}000$$

There were 63,046,000 U.S. residents 25 years of age or older who had an associate's degree or a bachelor's degree.

(b) Using formula (3),

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) \\ &= 20{,}790{,}000 + 42{,}256{,}000 + 24{,}623{,}000 \\ &= 87{,}669{,}000 \end{aligned}$$

There were 87,669,000 U.S. residents 25 years of age or older who had an associate's degree, a bachelor's degree, or an advanced degree.

Now Work PROBLEM 31

3 Solve Counting Problems Using the Multiplication Principle

EXAMPLE 4 **Counting the Number of Possible Meals**

The fixed-price dinner at Mabenka Restaurant provides the following choices:

Appetizer: soup or salad
Entrée: baked chicken, broiled beef patty, beef liver, or roast beef au jus
Dessert: ice cream or cheese cake

How many different meals can be ordered?

Solution Ordering such a meal requires three separate decisions:

Choose an Appetizer	**Choose an Entrée**	**Choose a Dessert**
2 choices	4 choices	2 choices

Look at the **tree diagram** in Figure 2. Note that for each choice of appetizer, there are 4 choices of entrées. And for each of these $2 \cdot 4 = 8$ choices, there are 2 choices for dessert. A total of

$$2 \cdot 4 \cdot 2 = 16$$

different meals can be ordered.

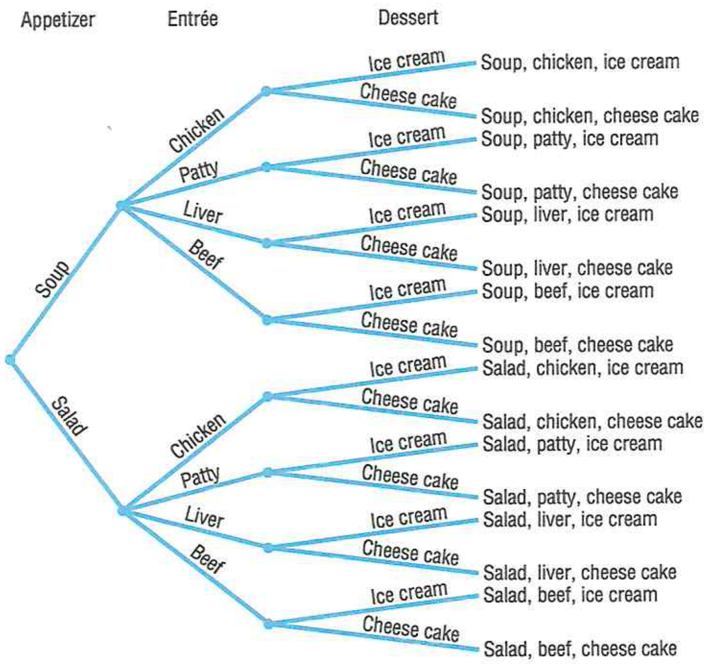


Figure 2

Example 4 demonstrates a general principle of counting.

THEOREM

Multiplication Principle of Counting

If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, the task of making these selections can be done in

$$\boxed{p \cdot q \cdot r \cdot \ldots}$$

different ways.

EXAMPLE 5 **Forming Codes**

How many two-symbol code words can be formed if the first symbol is an uppercase letter and the second symbol is a digit?

Solution It sometimes helps to begin by listing some of the possibilities. The code consists of an uppercase letter followed by a digit, so some possibilities are A1, A2, B3, X0, and so on. The task consists of making two selections: The first selection requires choosing an uppercase letter (26 choices), and the second task requires choosing a digit (10 choices). By the Multiplication Principle, there are

$$26 \cdot 10 = 260$$

different code words of the type described.

Now Work PROBLEM 23

10.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The ________ of A and B consists of all elements in either A or B or both. (pp. 2–3)
2. The ________ of A with B consists of all elements in both A and B. (pp. 2–3)
3. ***True or False*** The intersection of two sets is always a subset of their union. (pp. 2–3)
4. ***True or False*** If A is a set, the complement of A is the set of all the elements in the universal set that are not in A. (pp. 2–3)

Concepts and Vocabulary

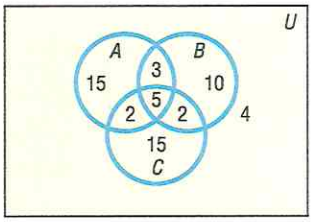
5. If each element of a set A is also an element of a set B, we say that A is a ________ of B and write A ______ B.
6. If the number of elements in a set is a nonnegative integer, we say that the set is ________.
7. The Counting Formula states that if A and B are finite sets, then $n(A \cup B) =$ ________________________.
8. ***True or False*** If a task consists of a sequence of three choices in which there are p selections for the first choice, q selections for the second choice, and r selections for the third choice, then the task of making these selections can be done in $p \cdot q \cdot r$ different ways.

Skill Building

9. Write down all the subsets of $\{a, b, c, d\}$.
10. Write down all the subsets of $\{a, b, c, d, e\}$.
11. If $n(A) = 15, n(B) = 20$, and $n(A \cap B) = 10$, find $n(A \cup B)$.
12. If $n(A) = 30, n(B) = 40$, and $n(A \cup B) = 45$, find $n(A \cap B)$.
13. If $n(A \cup B) = 50, n(A \cap B) = 10$, and $n(B) = 20$, find $n(A)$.
14. If $n(A \cup B) = 60, n(A \cap B) = 40$, and $n(A) = n(B)$, find $n(A)$.

In Problems 15–22, use the information given in the figure.

15. How many are in set A?
16. How many are in set B?
17. How many are in A or B?
18. How many are in A and B?
19. How many are in A but not C?
20. How many are not in A?
21. How many are in A and B and C?
22. How many are in A or B or C?



Applications and Extensions

23. **Shirts and Ties** A man has 5 shirts and 3 ties. How many different shirt-and-tie arrangements can he wear?
24. **Blouses and Skirts** A woman has 5 blouses and 8 skirts. How many different outfits can she wear?
25. **Four-digit Numbers** How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0? Repeated digits are allowed.
26. **Five-digit Numbers** How many five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0 or 1? Repeated digits are allowed.
27. **Analyzing Survey Data** In a consumer survey of 500 people, 200 indicated that they would be buying a major appliance within the next month, 150 indicated that they would buy a car, and 25 said that they would purchase both a major appliance and a car. How many will purchase neither? How many will purchase only a car?
28. **Analyzing Survey Data** In a student survey, 200 indicated that they would attend Summer Session I, and 150 indicated Summer Session II. If 75 students plan to attend both summer sessions, and 275 indicated that they would attend neither session, how many students participated in the survey?
29. **Analyzing Survey Data** In a survey of 100 investors in the stock market,
    - 50 owned shares in IBM
    - 40 owned shares in AT&T
    - 45 owned shares in GE
    - 20 owned shares in both IBM and GE
    - 15 owned shares in both AT&T and GE
    - 20 owned shares in both IBM and AT&T
    - 5 owned shares in all three

    (a) How many of the investors surveyed did not have shares in any of the three companies?
    (b) How many owned just IBM shares?
    (c) How many owned just GE shares?
    (d) How many owned neither IBM nor GE?
    (e) How many owned either IBM or AT&T but no GE?
30. **Classifying Blood Types** Human blood is classified as either Rh+ or Rh−. Blood is also classified by type: A, if it contains an A antigen but not a B antigen; B, if it contains a B antigen but not an A antigen; AB, if it contains both A and B antigens; and O, if it contains neither antigen. Draw a Venn diagram illustrating the various blood types. Based on this classification, how many different kinds of blood are there?

31. **Demographics** The following data represent the marital status of males 18 years old and older in the U.S. in 2014.

Marital Status	Number (in millions)
Married	65.7
Widowed	3.1
Divorced	10.7
Never married	36.3

Source: Current Population Survey

(a) Determine the number of males 18 years old and older who are widowed or divorced.
(b) Determine the number of males 18 years old and older who are married, widowed, or divorced.

32. **Demographics** The following data represent the marital status of U.S. females 18 years old and older in 2014.

Marital Status	Number (in millions)
Married	66.7
Widowed	11.2
Divorced	14.6
Never married	31.0

Source: Current Population Survey

(a) Determine the number of females 18 years old and older who are widowed or divorced.
(b) Determine the number of females 18 years old and older who are married, widowed, or divorced.

33. **Stock Portfolios** As a financial planner, you are asked to select one stock each from the following groups: 8 Dow Jones stocks, 15 NASDAQ stocks, and 4 global stocks. How many different portfolios are possible?

Explaining Concepts: Discussion and Writing

34. Make up a problem different from any found in the text that requires the addition principle of counting to solve. Give it to a friend to solve and critique.

35. Investigate the notion of counting as it relates to infinite sets. Write an essay on your findings.

Retain Your Knowledge

Problems 36–39 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

36. Graph $(x-2)^2 + (y+1)^2 = 9$.

37. Given that the point (3, 8) is on the graph of $y = f(x)$, what is the corresponding point on the graph of $y = -2f(x+3) + 5$?

38. Find all the real zeros of the function

$$f(x) = (x-2)(x^2 - 3x - 10)$$

39. Solve: $\log_3 x + \log_3 2 = -2$

'Are You Prepared?' Answers

1. union 2. intersection 3. True 4. True

10.2 Permutations and Combinations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Factorial (Section 9.1, p. 657)

Now Work the 'Are You Prepared?' problems on page 711.

OBJECTIVES

1 Solve Counting Problems Using Permutations Involving n Distinct Objects

DEFINITION A **permutation** is an ordered arrangement of r objects chosen from n objects.