(09.10.2023)

Straight Lines

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$, respectively, the run and the rise between P_1 and P_2.

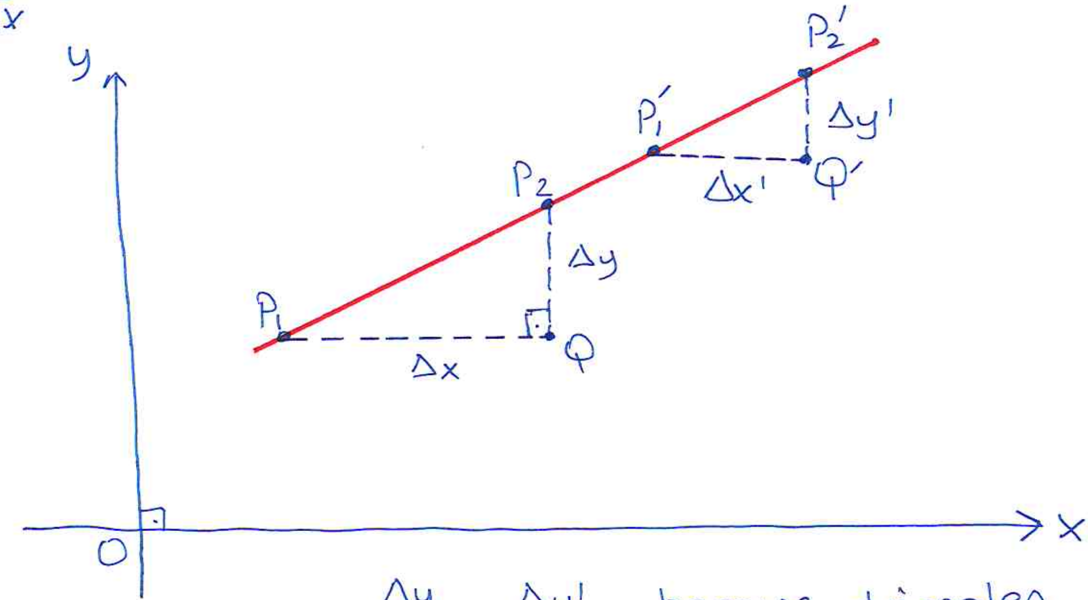
Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the "same value" for every choice of two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line. The constant m

$m = \frac{\Delta y}{\Delta x}$ is called the slope of the nonvertical line.



$\frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x'}$ because triangles

$\overset{\triangle}{P_1QP_2}$ and $\overset{\triangle}{P_1'Q'P_2'}$ are similar !

Example : The slope of the line joining
A (3,−3) and B(−1,2) is

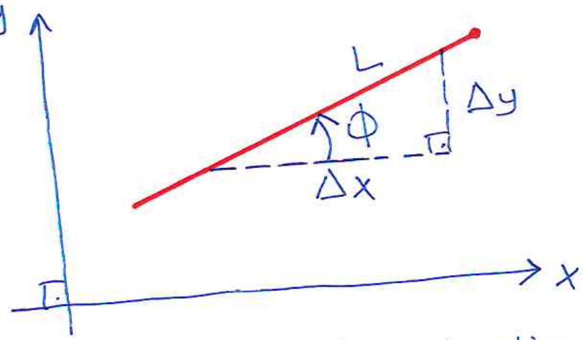
$$m = \frac{\Delta y}{\Delta x} = \frac{2-(-3)}{-1-3} = \frac{5}{-4} = -\frac{5}{4}.$$

* The slope tells us the direction and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right.

* The greater the absolute value of the slope, the steeper the rise or fall.

* Since the run Δx is zero for a vertical line, we cannot form the ratio m; the slope of a vertical line is **undefined** !

* Direction of a line can also be measured by an angle. The *inclination* of a line is the smallest counterclockwise angle from the positive direction of the x-axis to the line.



Line L has inclination ϕ.

* The angle ϕ (the Greek letter ϕ "phi") is the inclination of the line L.

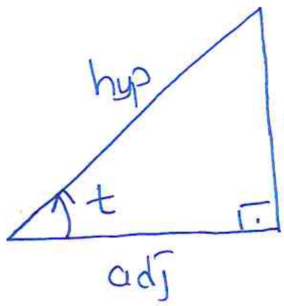
* The inclination ϕ of any line satisfies

$$0^\circ \leq \phi < 180^\circ.$$

* The inclination of a horizontal line is 0° and that of a vertical line is 90°.

* Provided equal scales are used on the coordinate axes, the relationship between the slope m of a nonvertical line and its inclination ϕ is shown in the figure above.

$$m = \frac{\Delta y}{\Delta x} = \tan\phi.$$



$$\cos t = \frac{adj}{hyp} \; ; \; \sin t = \frac{opp}{hyp}$$

$$\cot t = \frac{adj}{opp} \; ; \; \tan t = \frac{opp}{adj}$$

$$\tan t = \frac{\sin t}{\cos t} \; ; \; \cot t = \frac{\cos t}{\sin t}$$

* Parallel lines have the same inclination. If they are not vertical, they must therefore have the same slope. Conversely, lines with equal slopes have the same inclination and so are parallel.

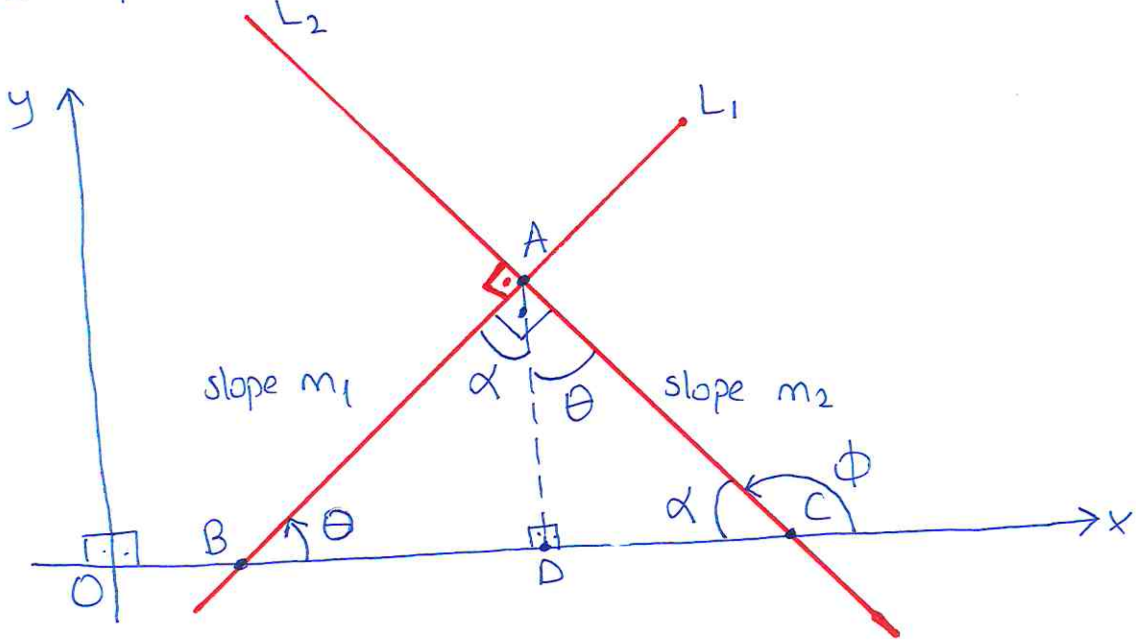
If two nonvertical lines, L_1 and L_2, are perpendicular, their slopes m_1 and m_2 satisfy $\boxed{m_1 m_2 = -1}$, so each slope is the negative reciprocal of the other.

Note: In mathematics, a reciprocal for a number x, denoted by $\frac{1}{x}$ or x^{-1}, is a number which when multiplied by x yields the multiplicative identify 1.

$$m_1 = -\frac{1}{m_2} \quad \text{and} \quad m_2 = -\frac{1}{m_1}.$$

(*) This result also assumes equal scales on two coordinate axes.

Let's prove this result:



$$\tan\theta = m_1 = \frac{AD}{BD} \quad \text{and} \quad m_2 = \tan\phi = \tan\left(\frac{\pi}{2} + \theta\right)$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$m_2 = \tan\left(\frac{\pi}{2}+\theta\right) = \frac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)}$$

$$m_2 = \frac{\sin\frac{\pi}{2}\cos\theta + \sin\theta\,\cancel{\cos\frac{\pi}{2}}^{\,0}}{\underbrace{\cancel{\cos\frac{\pi}{2}}}_{0}\cos\theta - \sin\theta\sin\frac{\pi}{2}}.$$

$$m_2 = -\frac{\cos\theta}{\sin\theta} = -\frac{1}{\tan\theta} = -\frac{1}{m_1} = -\frac{BD}{AD}$$

Since $\overset{\Delta}{ABD}$ is similar to $\overset{\Delta}{CAD}$, we have

$$\frac{AD}{BD} = \frac{DC}{AD}, \text{ and so}$$

$$\frac{BD}{AD} = \frac{AD}{DC}$$

so $\boxed{m_2 = -\frac{AD}{DC}}$

$$m_1 \cdot m_2 = \left(\frac{DC}{AD}\right)\cdot\left(-\frac{AD}{DC}\right) = -1.$$

■.

$\ulcorner$ $\sin(a \pm b) = \sin a\cos b \pm \sin b\cos a$

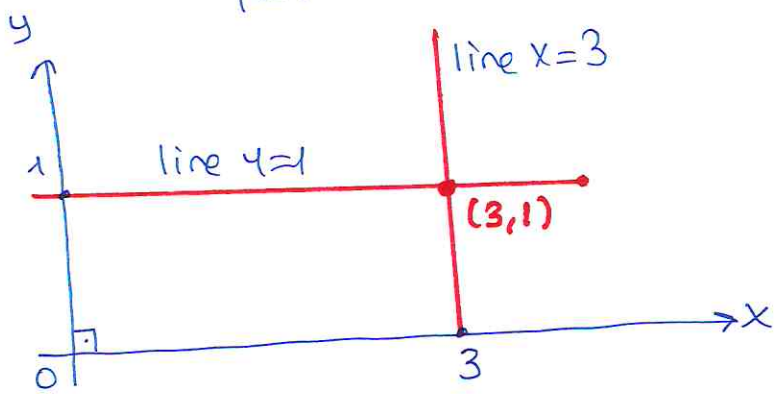
$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$ $\lrcorner$

Equations of Lines

* <u>Straight lines</u> are particularly simple graphs, and their corresponding equations are also simple.

- All points on the vertical line through the point a on the x-axis have their x-coordinates equal to a. Thus $x=a$ is the equation of the line. Similarly, $y=b$ is the equation of the horizontal line meeting the y-axis at b.

<u>Example</u>: The horizontal and vertical lines passing through the point $(3,1)$ have equations $y=1$ and $x=3$, respectively.



The lines $y=1$ and $x=3$.

* To write an equation for a nonvertical straight line L, it is enough to know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x,y)$ is any other point on L, then

$$\frac{y-y_1}{x-x_1} = m,$$

So that

$$y-y_1 = m(x-x_1) \quad \text{or}$$

$$y = m(x-x_1)+y_1$$

The equation $y = m(x - x_1) + y_1$ is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

Example: Find an equation of the line that has slope -2 and passes through the point $(1,4)$.

Solution: $y = m(x - x_1) + y_1$

we substitute $x_1 = 1$, $y_1 = 4$, and $m = -2$ into the point-slope form of the equation and obtain

$$y = -2(x - 1) + 4$$

or

$$y = -2x + 2 + 4$$

$$\boxed{y = -2x + 6}$$

Example: Find an equation of the line through the points $(1, -1)$ and $(3, 5)$.

Solution: The slope of the line is

$$m = \frac{5 - (-1)}{3 - 1} = \frac{6}{2}, \quad \text{if } y_1 = -1,\ y_2 = 5,\ x_1 = 1,\ x_2 = 3$$

$$\boxed{m = 3}$$

$$m = \frac{-1 - 5}{1 - 3}, \quad \text{if } y_1 = 5,\ y_2 = -1,\ x_1 = 3,\ x_2 = 1$$

$$m = \frac{-6}{-2}$$

$$\boxed{m = 3}$$

Either way, $m = 3$!

Let us use the point-slope equation of the line:

$$y = m(x - x_1) + y_1$$

- If we use $(1,-1)$ we get

$$y = 3(x-1) - 1,$$ which simplifies to

$$\boxed{y = 3x - 4}$$

- If we use $(3,5)$ we get

$$y = 3(x-3) + 5$$ which also simplifies to

$$y = 3x - 9 + 5$$

$$\boxed{y = 3x - 4}$$

Either way, $y = 3x - 4$ is ~~an~~ equation of the line.

Example: Does the Point $P(2,1)$ lie on, above, or below the given line $2x + 3y = 6$?

Solution:

$$2x + 3y = 6$$

$$3y = 6 - 2x$$

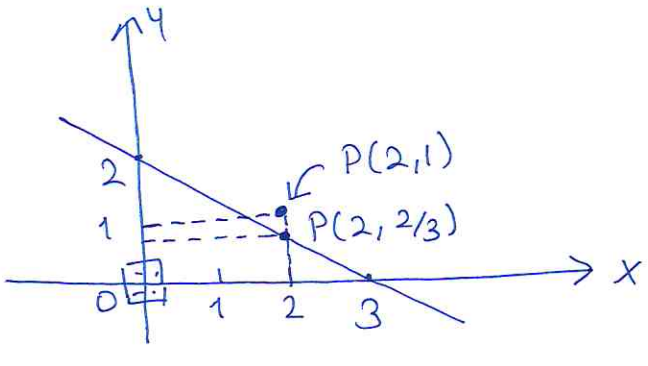
$$y = -\frac{2}{3}x + 2$$

x	y
0	2
3	0
2	$\frac{2}{3}$

$$-2 = \frac{-2x}{3}$$

$$x = 3$$

$$y = -\frac{4}{3} + 2 = \frac{2}{3}$$



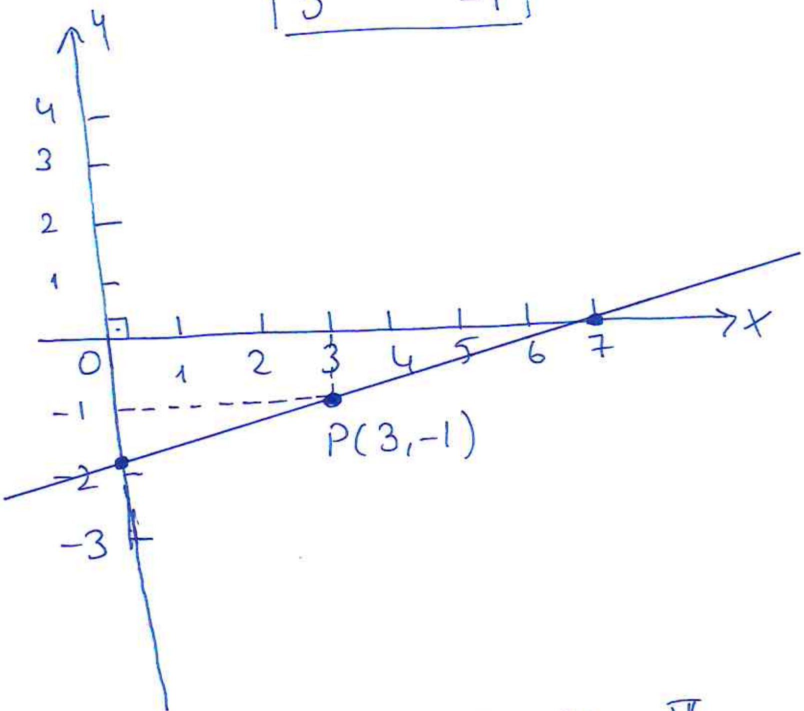
Answer is above the given line.

Example : Does the given point $P(3,-1)$ lie on, above, or below the given line $x - 4y = 7$?

Solution : $x - 4y = 7 \Rightarrow -4y = 7 - x$

$$y = \frac{x}{4} - \frac{7}{4}$$

x	y
0	$-\frac{7}{4}$
7	0
3	-1



Answer is on the line !

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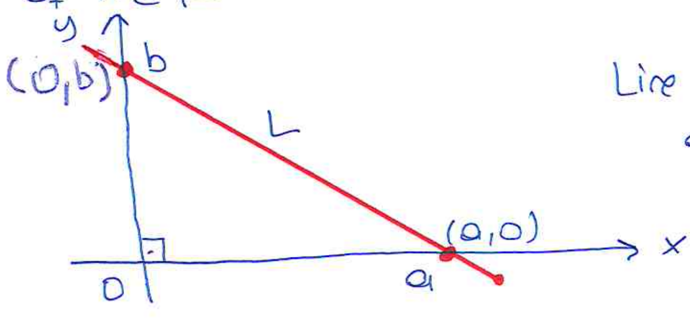
(*) The y-coordinate of the point where a nonvertical line intersects the y-axis is called the y-intercept of the line.

(*) Similarly, the x-intercept of a non-horizontal line is the x-coordinate of the point where it crosses the x-axis.



Line L has x-intercept a and y-intercept b.

* A line with slope m and y-intercept b passes through the point $(0,b)$, so its equation is

$$y = m(x-0)+b \text{ , or more simply,}$$

$$\boxed{y = mx+b}.$$

* A line with slope m and x-intercept a passes through $(a,0)$, and so its equation is

$$y = m(x-a)+0.$$

$$\boxed{y = m(x-a)}.$$

- The equation $y=mx+b$ is called the slope-y-intercept equation of the line with slope m and y-intercept b.
- The equation $y=m(x-a)$ is called the slope-x-intercept equation of the line with slope m and x-intercept a.

Example: Find the slope and the two intercepts of the line with equation $8x+5y=20$.

Solution: Solve the equation for y we get

$$5y = 20-8x$$

$$y = -\frac{8}{5}x+4.$$

Comparing this with the general form $y=mx+b$ of the slope-y-intercept equation, we see that the slope of the line is $m=-8/5$ and the y-intercept is $b=4$.

To find the x-intercept, put $y=0$ and solve for x,

$$0 = -\frac{8x}{5}+4 \Rightarrow \frac{8x}{5}=4 \Rightarrow x=\frac{5}{2}.$$ The x-intercept is $a=5/2$.

Example : Determine the intercepts and sketch the graph of the line

$$\frac{x}{2} - \frac{y}{3} = 1 .$$

Solution :

$$\frac{x}{2} - 1 = \frac{y}{3}$$

$$\frac{y}{3} = \frac{x-2}{2}$$

$$\boxed{y = \frac{3}{2}x - 3}$$

x-intercept ($y=0$)

y-intercept ($x=0$)

- For x-intercept a, $y=0$

$$\cancel{3} = \frac{\cancel{3}x}{2}$$

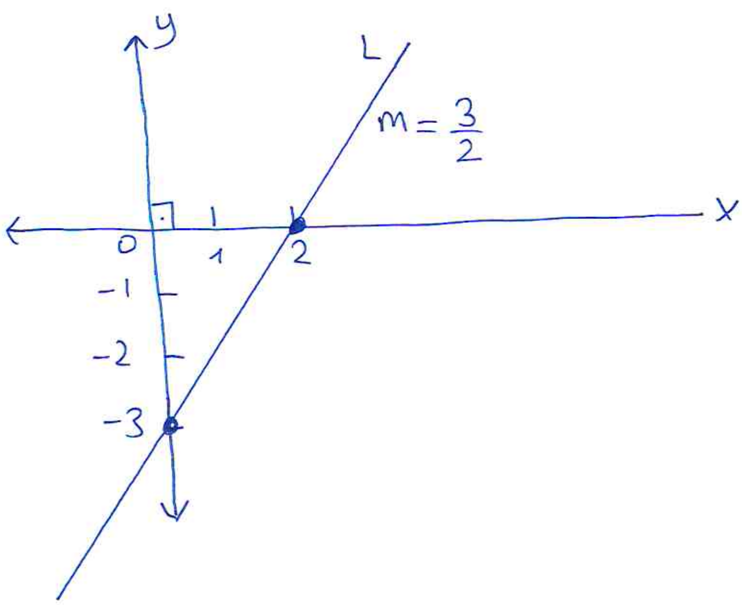
$x=2$ is the x-intercept

- For y-intercept b, $x=0$

$y=-3$ is the y-intercept

-

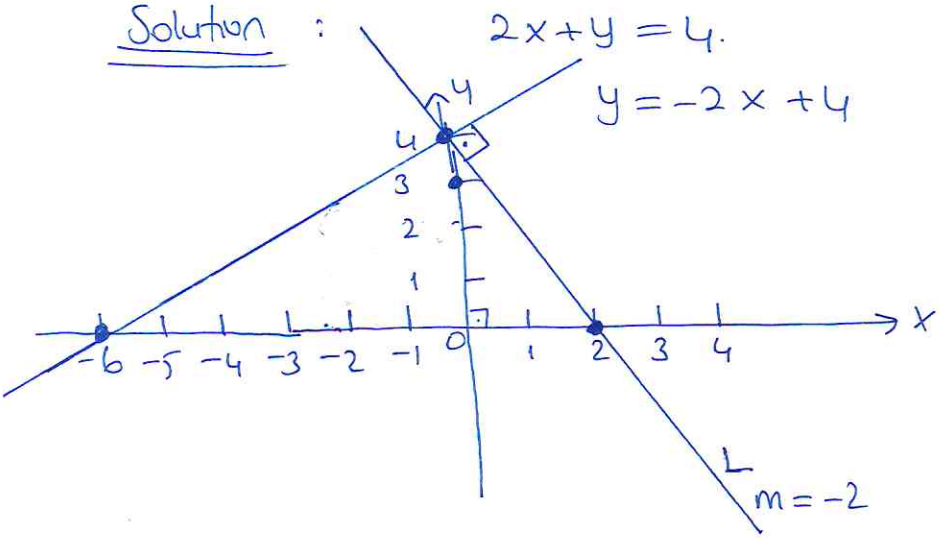
x	y
0	-3
2	0



Example : Find an equation for the line through P that are perpendicular to the given line, $2x+y=4$.

$P(-2,2)$.

Solution :



$2x+y=4$.

$y=-2x+4$

x	y
0	4
2	0

$$m \cdot m_x = -1$$

$$-2 \cdot m_x = -1$$

$$\boxed{m_x = \frac{1}{2}}$$

$y = m(x-x_1)+y_1$; $P(-2,2)$

$$y = \frac{1}{2}(x-(-2))+2$$

$$y = \frac{1}{2}(x+2)+2$$

$$y = \frac{x}{2}+1+2$$

$$\boxed{y = \frac{x}{2}+3} \Rightarrow$$

x	y
0	3
-6	0
-2	2 ✓

(*) The equation $Ax + By = C$ (where A and B are not both zero) is called the "general linear equation" in x and y because its graph always represents a straight line, and any line has an equation in this form.

Example: The relationship between Fahrenheit temperature (F) and Celsius temperature (C) is given by a linear equation of the form $F = mC + b$.

The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus.

$$32 = 0.m + b \quad \text{and} \quad 212 = 100m + b,$$

so $\quad b = 32 \quad$ and $\quad m = \dfrac{(212 - 32)}{100} = \dfrac{18\cancel{0}}{10\cancel{0}} = \dfrac{9}{5}.$

The relationship is given by the linear equation

$$F = \frac{9}{5}C + 32 \quad \text{or} \quad C = \frac{5}{9}(F - 32).$$

Example: The cost printing x copies of a pamphlet is $\$C$, where $C = Ax + B$ for certain constants A and B. If it costs \$5,000 to print 10,000 copies and \$6,000 to print 15,000 copies, how much will it cost to print 100,000 copies?

Solution:

$$C = Ax + B$$

$$
\begin{aligned}
-/\ 5000 &= 10{,}000\,A + B \\
6000 &= 15000\,A + B \\
\hline
1000 &= 5000\,A \implies \boxed{A = 1/5}
\end{aligned}
$$

$$\cancel{5}\cancel{000} = \frac{1}{\cancel{5}} \cdot \overset{2}{\cancel{10}}\cancel{000} + B$$

$$\boxed{B = +3}$$

$$\boxed{C = \frac{1}{5}x + 3}$$

$$C = \frac{1}{\cancel{5}} \cdot \overset{20}{\cancel{100}}000 + 3$$

$$\boxed{C = \$\,23{,}000}$$