1.2. Draw the approximate workspace for the following robot. Assume the dimensions of the base and other parts of the structure of the robot are as shown.

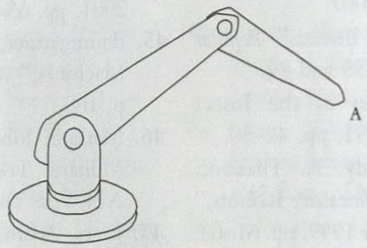

Figure P.1.2

1.3. Draw the approximate workspace for the following robot. Assume the dimensions of the base and other parts of the structure of the robot are as shown.

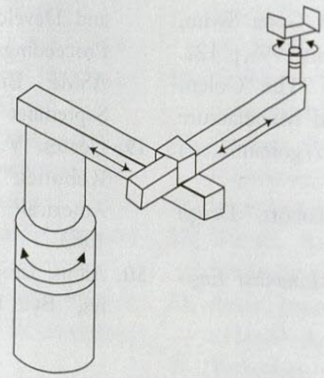

Figure P.1.3

CHAPTER 2

Kinematics of Robots: Position Analysis

2.1 Introduction

In this chapter, we will study forward and inverse kinematics of robots. With forward kinematic equations, we can determine where the robot's end (hand) will be if all joint variables are known. Inverse kinematics enables us to calculate what each joint variable must be in order to locate the hand at a particular point and a particular orientation. Using matrices, we will first establish a method of describing objects, locations, orientations, and movements. Then we will study the forward and inverse kinematics of different robot configurations such as Cartesian, cylindrical, and spherical coordinates. Finally, we will use the Denavit–Hartenberg representation to derive forward and inverse kinematic equations of all possible configurations of robots—regardless of number of joints, order of joints, and presence (or lack) of offsets and twists.

It is important to realize that in practice, manipulator-type robots are delivered with no end effector. In most cases, there may be a gripper attached to the robot; however, depending on the actual application, different end effectors are attached to the robot by the user. Obviously, the end effector's size and length determine where the end of the robot will be. For a short end effector, the end will be at a different location compared to a long end effector. In this chapter, we will assume that the end of the robot is a plate to which the end effector can be attached, as necessary. We will call this the "hand" or the "end plate" of the robot. If necessary, we can always add the length of the end effector to the robot for determining the location and orientation of the end effector. It should be mentioned here that a real robot manipulator, for which the length of the end effector is not defined, will calculate its joint values based on the end plate location and orientation, which may be different from the position and orientation perceived by the user.

2.2 Robots as Mechanisms

Manipulator-type robots are multi-degree-of-freedom (DOF), three-dimensional, open loop, chain mechanisms, and are discussed in this section.

Multi-degree-of-freedom means that robots possess many joints, allowing them to move freely within their envelope. In a 1-DOF system, when the variable is set to a particular value, the mechanism is totally set and all its other variables are known. For example, in the 1-DOF 4-bar mechanism of Figure 2.1, when the crank is set to 120°, the angles of the coupler link and the rocker arm are also known, whereas in a multi-DOF mechanism, all input variables must be individually defined in order to know the remaining parameters. Robots are multi-DOF machines, where each joint variable must be known in order to determine the location of the robot's hand.

Robots are three-dimensional machines if they are to move in space. Although it is possible to have a two-dimensional multi-DOF robot, they are not common (or useful).

Robots are open-loop mechanisms. Unlike mechanisms that are closed-loop (e.g., 4-bar mechanisms), even if all joint variables are set to particular values, there is no guarantee that the hand will be at the given location. This is because deflections in any joint or link will change the location of all subsequent links without feedback. For example, in the 4-bar mechanism of Figure 2.2, when link AB deflects as a result of load F, link BO_2 will also move; therefore, the deflection can be detected. In an open-loop system such as the robot, the deflections will move all succeeding members without any

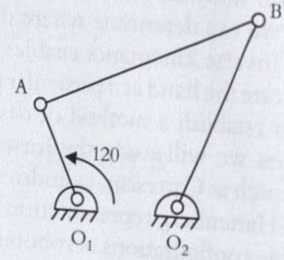

Figure 2.1 A 1-DOF closed-loop 4-bar mechanism.

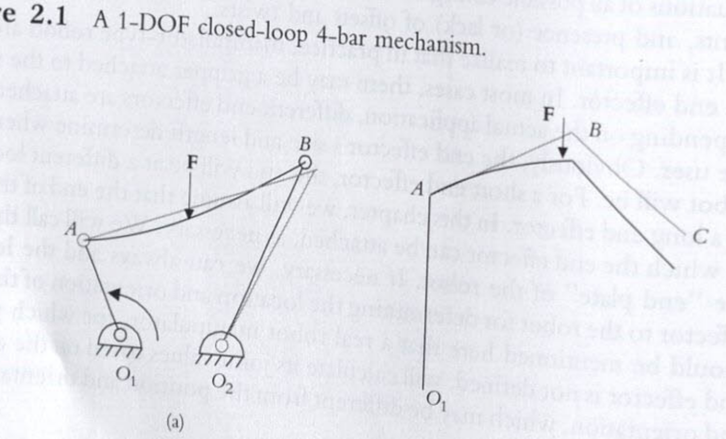

Figure 2.2 Closed-loop (a) versus open-loop (b) mechanisms.

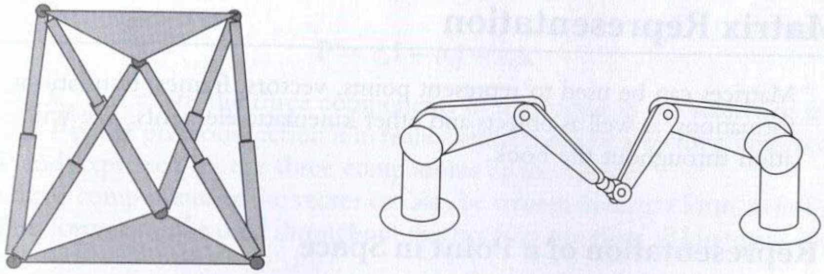

Figure 2.3 Possible parallel manipulator configurations.

feedback. Therefore, in open-loop systems, either all joint and link parameters must continuously be measured, or the end of the system must be monitored; otherwise, the kinematic position of the machine is not completely known. This difference can be expressed by comparing the vector equations describing the relationship between different links of the two mechanisms as follows:

$$\text{For the 4-bar mechanism:}\quad \overline{O_1A} + \overline{AB} = \overline{O_1O_2} + \overline{O_2B} \tag{2.1}$$

$$\text{For the robot:}\quad \overline{O_1A} + \overline{AB} + \overline{BC} = \overline{O_1C} \tag{2.2}$$

As you can see, if there is a deflection in link AB, link O_2B will move accordingly. However, the two sides of Equation (2.1) have changed corresponding to the changes in the links. On the other hand, if link AB of the robot deflects, all subsequent links will move too; however, unless O_1C is measured by other means, the change will not be known. To remedy this problem in open loop robots, either the position of the hand is constantly measured with devices such as a camera, the robot is made into a closed loop system with external means such as the use of secondary arms or laser beams,[1,2,3] or as standard practice, the robot links and joints are made excessively strong to eliminate all deflections. This will render the robot very heavy, massive, and slow, and its specified payload will be very low compared to what it can actually carry.

Alternatives, also called *parallel manipulators*, are based on closed-loop parallel architecture (Figure 2.3). The tradeoff is much-reduced range of motions and workspace.

2.3 Conventions

Throughout this book, we will use the following conventions for describing vectors, frames, transformations, and so on:

Vectors	**i, j, k, x, y, z, n, o, a, p**
Vector components	$n_x, n_y, n_z, a_x, a_y, a_z$
Frames	$F_{xyz}, F_{noa}, xyz, noa, F_{camera}$
Transformations	$T_1, T_2, {}^uT, {}^BP, {}^UT_R$ (transformation of robot relative to the Universe, where Universe is a fixed frame)

2.4 Matrix Representation

Matrices can be used to represent points, vectors, frames, translations, rotations, transformations, as well as objects and other kinematic elements. We will use this representation throughout the book.

2.4.1 Representation of a Point in Space

A point P in space (Figure 2.4) can be represented by its three coordinates relative to a reference frame as:

$$P = a_x\mathbf{i} + b_y\mathbf{j} + c_z\mathbf{k} \tag{2.3}$$

where a_x, b_y, and c_z are the three coordinates of the point represented in the reference frame. Obviously, other coordinate representations can also be used to describe the location of a point in space.

2.4.2 Representation of a Vector in Space

A vector can be represented by three coordinates of its tail and its head. If the vector starts at point A and ends at point B, then it can be represented by $\mathbf{P}_{AB} = (B_x - A_x)\mathbf{i} + (B_y - A_y)\mathbf{j} + (B_z - A_z)\mathbf{k}$. Specifically, if the vector starts at the origin (Figure 2.5),

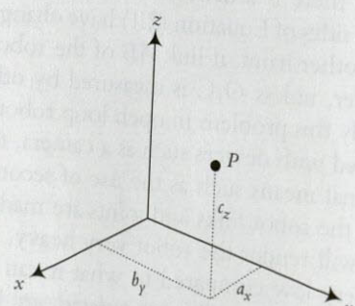

Figure 2.4 Representation of a point in space.

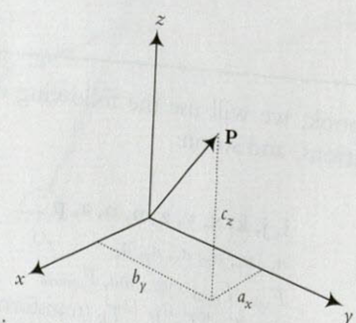

Figure 2.5 Representation of a vector in space.

then:

$$\mathbf{P} = a_x\mathbf{i} + b_y\mathbf{j} + c_z\mathbf{k} \tag{2.4}$$

where a_x, b_y, and c_z are the three components of the vector in the reference frame. In fact, point P in the previous section is in reality represented by a vector connected to it at point P and expressed by the three components of the vector.

The three components of the vector can also be written in matrix form, as in Equation (2.5). This format will be used throughout this book to represent all kinematic elements:

$$\mathbf{P} = \begin{bmatrix} a_x \\ b_y \\ c_z \end{bmatrix} \tag{2.5}$$

This representation can be slightly modified to also include a scale factor w such that if P_x, P_y, and P_z are divided by w, they will yield a_x, b_y, and c_z. Therefore the vector can be written as:

$$\mathbf{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ w \end{bmatrix} \quad \text{where } a_x = \frac{P_x}{w},\ b_y = \frac{P_y}{w},\ \text{etc.} \tag{2.6}$$

w may be any number and, as it changes, it can change the overall size of the vector. This is similar to the zooming function in computer graphics. As the value of w changes, the size of the vector changes accordingly. If w is bigger than 1, all vector components enlarge; if w is smaller than 1, all vector components become smaller.

When w is 1, the size of these components remains unchanged. However, if $w = 0$, then a_x, b_y, and c_z will be infinity. In this case, P_x, P_y, and P_z (as well as a_x, b_y, and c_z) will represent a vector whose length is infinite but nonetheless is in the direction represented by the vector. This means that a *direction vector* can be represented by a scale factor of $w = 0$, where the length is not important, but the direction is represented by the three components of the vector. This will be used throughout the book to represent direction vectors.

In computer graphics applications, the addition of a scale factor allows the user to zoom in or out simply by changing this value. Since the scale factor increases or decreases all vector dimensions accordingly, the size of a vector (or drawing) can be easily changed without the need to redraw it. However, our reason for this inclusion is different, and it will become apparent shortly.

Example 2.1

A vector is described as $\mathbf{P} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Express the vector in matrix form:

(a) With a scale factor of 2.
(b) If it were to describe a direction as a unit vector.

Solution: The vector can be expressed in matrix form with a scale factor of 2 as well as 0 for direction as:

$$\mathbf{P} = \begin{bmatrix} 6 \\ 10 \\ 4 \\ 2 \end{bmatrix} \text{ and } \mathbf{P} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}$$

However, in order to make the vector into a unit vector, we normalize the length to be equal to 1. To do this, each component of the vector is divided by the square root of the sum of the squares of the three components:

$$\lambda = \sqrt{P_x^2 + P_y^2 + P_z^2} = 6.16 \text{ and } P_x = {}^3/_{6.16} = 0.487, \text{ etc. Therefore,}$$

$$\mathbf{P}_{unit} = \begin{bmatrix} 0.487 \\ 0.811 \\ 0.324 \\ 0 \end{bmatrix}$$

Note that $\sqrt{0.487^2 + 0.811^2 + 0.324^2} = 1$. ■

Example 2.2

A vector **p** is 5 units long and is in the direction of a unit vector **q** described below. Express the vector in matrix form.

$$\mathbf{q}_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ q_z \\ 0 \end{bmatrix}$$

Solution: The unit vector's length must be 1. Therefore,

$$\lambda = \sqrt{q_x^2 + q_y^2 + q_z^2} = \sqrt{0.138 + 0.310 + q_z^2} = 1 \quad \rightarrow \quad q_z = 0.743$$

$$\mathbf{q}_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ 0.743 \\ 0 \end{bmatrix} \text{ and } \mathbf{p} = \mathbf{q}_{unit} \times 5 = \begin{bmatrix} 1.855 \\ 2.785 \\ 3.715 \\ 1 \end{bmatrix}$$

■

2.4.3 Representation of a Frame at the Origin of a Fixed Reference Frame

A frame is generally represented by three mutually orthogonal axes (such as x, y, and z). Since we may have more than one frame at any given time, we will use axes x, y, and z to represent the fixed Universe reference frame $F_{x,y,z}$ and a set of axes n, o, and a to represent another (moving) frame $F_{n,o,a}$ relative to the reference frame. This way, there should be no confusion about which frame is referenced.

The letters n, o, and a are derived from the words *normal*, *orientation*, and *approach*. Referring to Figure 2.6, it should be clear that in order to avoid hitting the part while trying to pick it up, the robot would have to approach it along the z-axis of the gripper. In robotic nomenclature, this axis is called *approach-axis* and is referred to as the a-axis. The orientation with which the gripper frame approaches the part is called *orientation-axis*, and it is referred to as the o-axis. Since the x-axis is normal to both, it is referred to as n-axis. Throughout this book, we will refer to a moving frame as $F_{n,o,a}$ with *normal*, *orientation*, and *approach* axes.

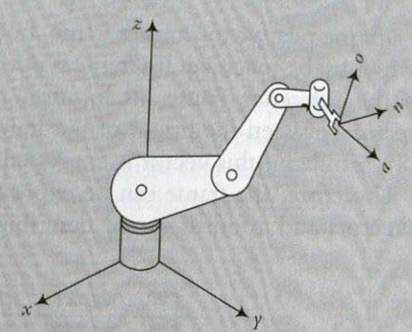

Figure 2.6 The normal-, orientation-, and approach-axis of a moving frame.

Each direction of each axis of a frame $F_{n,o,a}$ located at the origin of a reference frame $F_{x,y,z}$ (Figure 2.7) is represented by its three directional cosines relative to the reference frame as in section 2.4.2. Consequently, the three axes of the frame can be represented by three vectors in matrix form as:

$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} \tag{2.7}$$

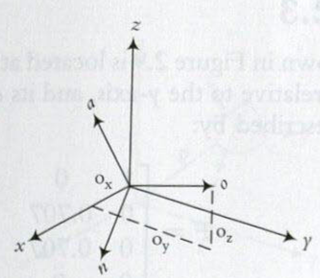

Figure 2.7 Representation of a frame at the origin of the reference frame.

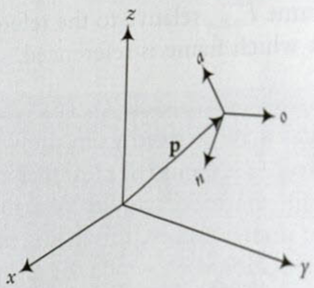

Figure 2.8 Representation of a frame in a frame.

2.4.4 Representation of a Frame Relative to a Fixed Reference Frame

To fully describe a frame relative to another frame, both the location of its origin and the directions of its axes must be specified. If a frame is not at the origin (or, in fact, even if it is at the origin) of the reference frame, its location relative to the reference frame is described by a vector between the origin of the frame and the origin of the reference frame (Figure 2.8). Similarly, this vector is expressed by its components relative to the reference frame. Therefore, the frame can be expressed by three vectors describing its directional unit vectors and a fourth vector describing its location as:

$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

As shown in Equation (2.8), the first three vectors are directional vectors with $w = 0$, representing the directions of the three unit vectors of the frame $F_{n,o,a}$, while the fourth vector with $w = 1$ represents the location of the origin of the frame relative to the reference frame. Unlike the unit vectors, the length of vector p is important. Consequently, we use a scale factor of 1.

A frame may also be represented by a 3×4 matrix without the scale factors, but it is not common. Adding the fourth row of scale factors to the matrix makes it a 4×4 or *homogeneous* matrix.

Example 2.3

The frame F shown in Figure 2.9 is located at 3,5,7 units, with its n-axis parallel to x, its o-axis at 45° relative to the y-axis, and its a-axis at 45° relative to the z-axis. The frame can be described by:

$$F = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■

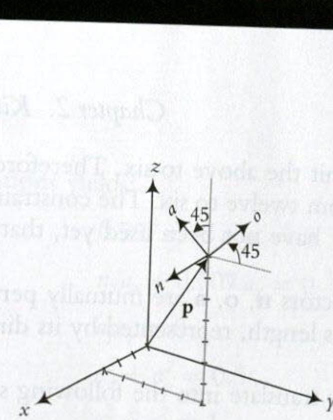

Figure 2.9 An example of representation of a frame.

2.4.5 Representation of a Rigid Body

An object can be represented in space by attaching a frame to it and representing the frame. Since the object is permanently attached to this frame, its position and orientation relative to the frame is always known. As a result, so long as the frame can be described in space, the object's location and orientation relative to the fixed frame will be known (Figure 2.10). As before, a frame can be represented by a matrix, where the origin of the frame and the three vectors representing its orientation relative to the reference frame are expressed. Therefore,

$$F_{object} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

As we discussed in Chapter 1, a point in space has only three degrees of freedom; it can only move along the three reference axes. However, a rigid body in space has six degrees of freedom, meaning that not only can it move along x-, y-, and z-axes, it can also rotate about these three axes. Consequently, all that is needed to completely define an object in space is six pieces of information describing the location of the origin of the object in the reference frame and its orientation about the three axes. However, as can be seen in Equation (2.9), twelve pieces of information are given: nine for orientation, and three for position (this excludes the scale factors on the last row of the matrix because they do not add to this information). Obviously, there must be some constraints present in this

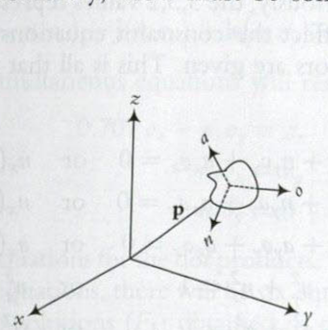

Figure 2.10 Representation of an object in space.

representation to limit the above to six. Therefore, we need 6 constraint equations to reduce the above from twelve to six. The constraints come from the known characteristics of a frame that have not been used yet, that:

- the three unit vectors **n**, **o**, **a** are mutually perpendicular, and
- each unit vector's length, represented by its directional cosines, must be equal to 1.

These constraints translate into the following six constraint equations:

1. $\mathbf{n} \cdot \mathbf{o} = 0$ (the dot-product of **n** and **o** vectors must be zero)
2. $\mathbf{n} \cdot \mathbf{a} = 0$
3. $\mathbf{a} \cdot \mathbf{o} = 0$
4. $|\mathbf{n}| = 1$ (the magnitude of the length of the vector must be 1) (2.10)
5. $|\mathbf{o}| = 1$
6. $|\mathbf{a}| = 1$

As a result, the values representing a frame in a matrix must be such that the above equations remain true. Otherwise, the frame will not be correct. Alternatively, the first three equations in Equation (2.10) can be replaced by a cross product of the three vectors as:

$$\mathbf{n} \times \mathbf{o} = \mathbf{a} \tag{2.11}$$

Since Equation (2.11) includes the correct right-hand-rule relationship too, it is recommended that this equation be used to determine the correct relationship between the three vectors.

Example 2.4

For the following frame, find the values of the missing elements and complete the matrix representation of the frame:

$$F = \begin{bmatrix} ? & 0 & ? & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Obviously, the 5,3,2 values representing the position of the origin of the frame do not affect the constraint equations. Please notice that only 3 values for directional vectors are given. This is all that is needed. Using Equation (2.10), we will get:

$$\begin{aligned}
&n_x o_x + n_y o_y + n_z o_z = 0 && \text{or} \quad n_x(0) + 0.707(o_y) + n_z(o_z) = 0 \\
&n_x a_x + n_y a_y + n_z a_z = 0 && \text{or} \quad n_x(a_x) + 0.707(a_y) + n_z(0) = 0 \\
&a_x o_x + a_y o_y + a_z o_z = 0 && \text{or} \quad a_x(0) + a_y(o_y) + 0(o_z) = 0 \\
&n_x^2 + n_y^2 + n_z^2 = 1 && \text{or} \quad n_x^2 + 0.707^2 + n_z^2 = 1 \\
&o_x^2 + o_y^2 + o_z^2 = 1 && \text{or} \quad 0^2 + o_y^2 + o_z^2 = 1 \\
&a_x^2 + a_y^2 + a_z^2 = 1 && \text{or} \quad a_x^2 + a_y^2 + 0^2 = 1
\end{aligned}$$

Simplifying these equations yields:

$$0.707\, o_y + n_z o_z = 0$$

$$n_x a_x + 0.707\, a_y = 0$$

$$a_y o_y = 0$$

$$n_x^2 + n_z^2 = 0.5$$

$$o_y^2 + o_z^2 = 1$$

$$a_x^2 + a_y^2 = 1$$

Solving these six equations will yield $n_x = \pm 0.707$, $n_z = 0$, $o_y = 0$, $o_z = 1$, $a_x = \pm 0.707$, and $a_y = -0.707$. Notice that both n_x and a_x must have the same sign. The reason for multiple solutions is that with the given parameters, it is possible to have two sets of mutually perpendicular vectors in opposite directions. The final matrix will be:

$$F_1 = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad F_2 = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As you can see, both matrices satisfy all the requirements set by the constraint equations. It is important to realize that the values representing the three direction vectors are not arbitrary but bound by these equations. Therefore, you may not randomly use any desired values in the matrix.

The same problem may be solved using $\mathbf{n} \times \mathbf{o} = \mathbf{a}$, or:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ n_x & n_y & n_z \\ o_x & o_y & o_z \end{vmatrix} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\text{or} \quad \mathbf{i}(n_y o_z - n_z o_y) - \mathbf{j}(n_x o_z - n_z o_x) + \mathbf{k}(n_x o_y - n_y o_x) = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \qquad (2.12)$$

Substituting the values into this equation yields:

$$\mathbf{i}(0.707 o_z - n_z o_y) - \mathbf{j}(n_x o_z) + \mathbf{k}(n_x o_y) = a_x\mathbf{i} + a_y\mathbf{j} + 0\mathbf{k}$$

Solving the three simultaneous equations will result in:

$$0.707\, o_z - n_z o_y = a_x$$

$$-n_x o_z = a_y$$

$$n_x o_y = 0$$

which replace the three equations for the dot products. Together with the three unit-vector length constraint equations, there will be six equations. However, as you will see, only one of the two solutions (F_1) obtained in the first part will satisfy these equations. This is because the dot-product equations are scalar, and therefore, are the same whether the unit vectors are right-handed or left-handed frames, whereas the cross-product equations do indicate the correct right-handed frame configuration. Consequently, it is recommended that the cross-product equation be used. ∎

Example 2.5

Find the missing elements of the following frame representation:

$$F = \begin{bmatrix} ? & 0 & ? & 3 \\ 0.5 & ? & ? & 9 \\ 0 & ? & ? & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$n_x^2 + n_y^2 + n_z^2 = 1 \quad \rightarrow \quad n_x^2 + 0.25 = 1 \quad \rightarrow \quad n_x = 0.866$$

$$\mathbf{n} \cdot \mathbf{o} = 0 \quad \rightarrow \quad (0.866)(0) + (0.5)(o_y) + (0)(o_z) = 0 \quad \rightarrow \quad o_y = 0$$

$$|\mathbf{o}| = 1 \quad \rightarrow \quad o_z = 1$$

$$\mathbf{n} \times \mathbf{o} = \mathbf{a} \quad \rightarrow \quad \mathbf{i}(0.5) - \mathbf{j}(0.866) + \mathbf{k}(0) = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$a_x = 0.5$$

$$a_y = -0.866$$

$$a_z = 0$$

∎

2.5 Homogeneous Transformation Matrices

For a variety of reasons, it is desirable to keep matrices in square form, either 3 × 3 or 4 × 4. First, as we will see later, it is much easier to calculate the inverse of square matrices than rectangular matrices. Second, in order to multiply two matrices, their dimensions must match, such that the number of columns of the first matrix must be the same as the number of rows of the second matrix, as in (m × n) and (n × p), which results in a matrix of (m × p) dimensions. If two matrices, A and B, are square with (m × m) and (m × m) dimensions, we may multiply A by B, or B by A, both resulting in the same (m × m) dimensions. However, if the two matrices are not square, with (m × n) and (n × p) dimensions respectively, A can be multiplied by B, but B may not be multiplied by A, and the result of AB has a dimension different from A and B. Since we will have to multiply many matrices together, in different orders, to find the equations of motion of the robots, we want to have square matrices.

In order to keep representation matrices square, if we represent both orientation and position in the same matrix, we will add the scale factors to the matrix to make it 4 × 4. If we represent the orientation alone, we may either drop the scale factors and use 3 × 3 matrices, or add a fourth column with zeros for position in order to keep the matrix square. Matrices of this form are called homogeneous matrices, and we refer to them as:

$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

2.6 Representation of Transformations

A transformation is defined as making a movement in space. When a frame (a vector, an object, or a moving frame) moves in space relative to a fixed reference frame, we represent this motion in a form similar to a frame representation. This is because transformation is a change in the state of a frame (representing the change in its location and orientation); therefore, it can be represented like a frame. A transformation may be in one of the following forms:

- A pure translation
- A pure rotation about an axis
- A combination of translations and/or rotations

In order to see how these can be represented, we will study each one separately.

2.6.1 Representation of a Pure Translation

If a frame (that may also be representing an object) moves in space without any change in its orientation, the transformation is a pure translation. In this case, the directional unit vectors remain in the same direction, and therefore, do not change. The only thing that changes is the location of the origin of the frame relative to the reference frame, as shown in Figure 2.11. The new location of the frame relative to the fixed reference frame can be found by adding the vector representing the translation to the vector representing the original location of the origin of the frame. In matrix form, the new frame representation may be found by pre-multiplying the frame with a matrix representing the transformation. Since the directional vectors do not change in a pure translation, the transformation T will simply be:

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.14}$$

where d_x, d_y, and d_z are the three components of a pure translation vector $\mathbf{d}$ relative to the x-, y-, and z-axes of the reference frame. The first three columns represent no rotational movement (equivalent of a 1), while the last column represents the translation. The new

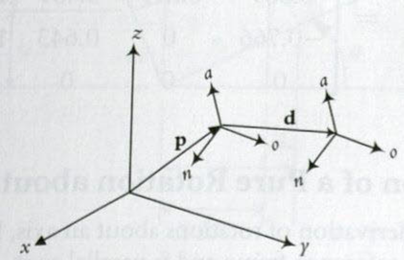

Figure 2.11 Representation of a pure translation in space.

location of the frame will be:

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x + d_x \\ n_y & o_y & a_y & p_y + d_y \\ n_z & o_z & a_z & p_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.15}$$

This equation is also symbolically written as:

$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old} \tag{2.16}$$

First, as you can see, pre-multiplying the frame matrix by the transformation matrix will yield the new location of the frame. Second, notice that the directional vectors remain the same after a pure translation, but the new location of the frame is at **d+p**. Third, notice how homogeneous transformation matrices facilitate the multiplication of matrices, resulting in the same dimensions as before.

Example 2.6

A frame F has been moved 10 units along the y-axis and 5 units along the z-axis of the reference frame. Find the new location of the frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Using Equation (2.15) or (2.16), we get:

$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old} = Trans(0, 10, 5) \times F_{old}$$

and

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 13 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■

2.6.2 Representation of a Pure Rotation about an Axis

To simplify the derivation of rotations about an axis, let's first assume that the frame is at the origin of the reference frame and is parallel to it. We will later expand the results to other rotations as well as combinations of rotations.

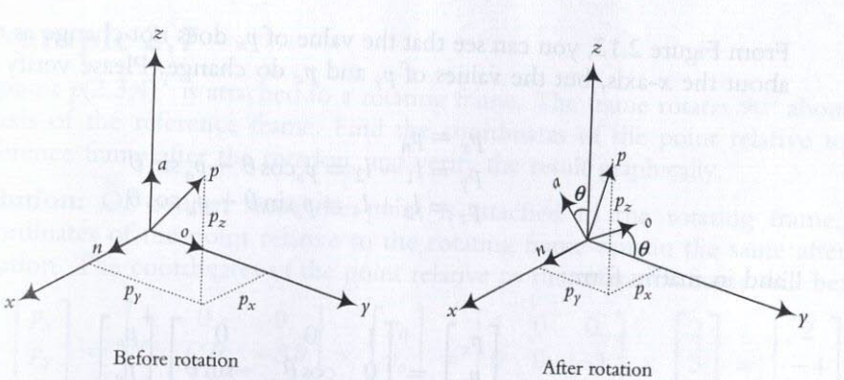

Figure 2.12 Coordinates of a point in a rotating frame before and after rotation.

Let's assume that a frame F_{noa}, located at the origin of the reference frame F_{xyz}, rotates an angle of θ about the x-axis of the reference frame. Let's also assume that attached to the rotating frame F_{noa}, is a point p, with coordinates p_x, p_y, and p_z relative to the reference frame and p_n, p_o, and p_a relative to the moving frame. As the frame rotates about the x-axis, point p attached to the frame will also rotate with it. Before rotation, the coordinates of the point in both frames are the same (remember that the two frames are at the same location and are parallel to each other). After rotation, the p_n, p_o, and p_a coordinates of the point remain the same in the rotating frame F_{noa}, but p_x, p_y, and p_z will be different in the F_{xyz} frame (Figure 2.12). We want to find the new coordinates of the point relative to the fixed reference frame after the moving frame has rotated.

Now let's look at the same coordinates in 2-D as if we were standing on the x-axis. The coordinates of point p are shown before and after rotation in Figure 2.13. The coordinates of point p relative to the reference frame are p_x, p_y, and p_z, while its coordinates relative to the rotating frame (to which the point is attached) remain as p_n, p_o, and p_a.

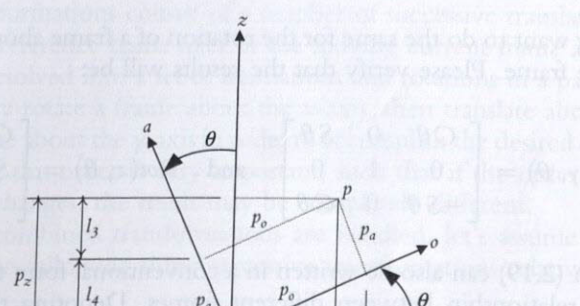

Figure 2.13 Coordinates of a point relative to the reference frame and rotating frame as viewed from the x-axis.

From Figure 2.13, you can see that the value of p_x does not change as the frame rotates about the x-axis, but the values of p_y and p_z do change. Please verify that:

$$\begin{aligned} p_x &= p_n \\ p_y &= l_1 - l_2 = p_o \cos\theta - p_a \sin\theta \\ p_z &= l_3 + l_4 = p_o \sin\theta + p_a \cos\theta \end{aligned} \tag{2.17}$$

and in matrix form:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix} \tag{2.18}$$

This means that the coordinates of the point p (or vector **p**) in the rotated frame must be pre-multiplied by the rotation matrix, as shown, to get the coordinates in the reference frame. This rotation matrix is only for a pure rotation about the x-axis of the reference frame and is denoted as:

$$p_{xyz} = Rot(x, \theta) \times p_{noa} \tag{2.19}$$

Notice that the first column of the rotation matrix in Equation (2.18)—which expresses the location relative to the x-axis—has 1,0,0 values, indicating that the coordinate along the x-axis has not changed.

To simplify writing these matrices, it is customary to designate $C\theta$ to denote $\cos\theta$ and $S\theta$ to denote $\sin\theta$. Therefore, the rotation matrix may be also written as:

$$Rot(x, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \tag{2.20}$$

You may want to do the same for the rotation of a frame about the y- and z-axes of the reference frame. Please verify that the results will be:

$$Rot(y, \theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \quad \text{and} \quad Rot(z, \theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.21}$$

Equation (2.19) can also be written in a conventional form that assists in easily following the relationship between different frames. Denoting the transformation as ${}^{U}T_{R}$ (and reading it as the transformation of frame R relative to frame U (for Universe)), denoting p_{noa} as ${}^{R}p$ (p relative to frame R), and denoting p_{xyz} as ${}^{U}p$ (p relative to frame U), Equation (2.19) simplifies to:

$${}^{U}p = {}^{U}T_{R} \times {}^{R}p \tag{2.22}$$

As you see, canceling the Rs will yield the coordinates of point p relative to U. The same notation will be used throughout this book to relate to multiple transformations.

Example 2.7

A point $p(2,3,4)^T$ is attached to a rotating frame. The frame rotates 90° about the x-axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation, and verify the result graphically.

Solution: Of course, since the point is attached to the rotating frame, the coordinates of the point relative to the rotating frame remain the same after the rotation. The coordinates of the point relative to the reference frame will be:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \times \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

As shown in Figure 2.14, the coordinates of point p relative to the reference frame after rotation are 2, −4, 3, as obtained by the above transformation.

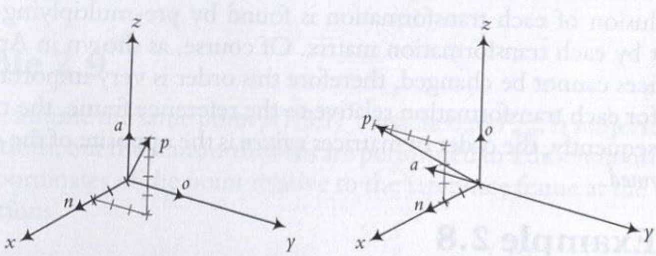

Figure 2.14 Rotation of a frame relative to the x-axis of the reference frame. ■

2.6.3 Representation of Combined Transformations

Combined transformations consist of a number of successive translations and rotations about the fixed reference frame axes or the moving current frame axes. Any transformation can be resolved into a set of translations and rotations in a particular order. For example, we may rotate a frame about the x-axis, then translate about the x-, y-, and z-axes, then rotate about the y-axis in order to accomplish the desired transformation. As we will see later, this order is very important, such that if the order of two successive transformations changes, the result may be completely different.

To see how combined transformations are handled, let's assume that a frame F_{noa} is subjected to the following three successive transformations relative to the reference frame F_{xyz}:

1. Rotation of α degrees about the x-axis,
2. Followed by a translation of $[l_1,l_2,l_3]$ (relative to the x-, y-, and z-axes respectively),
3. Followed by a rotation of β degrees about the y-axis.

Also, let's say that a point p_{noa} is attached to the rotating frame at the origin of the reference frame. As the frame F_{noa} rotates or translates relative to the reference frame, point p within the frame moves as well, and the coordinates of the point relative to the reference frame change. After the first transformation, as we saw in the previous section, the coordinates of point p relative to the reference frame can be calculated by:

$$p_{1,xyz} = Rot(x, \alpha) \times p_{noa} \tag{2.23}$$

where $p_{1,xyz}$ is the coordinates of the point after the first transformation relative to the reference frame. The coordinates of the point relative to the reference frame at the conclusion of the second transformation will be:

$$p_{2,xyz} = Trans(l_1, l_2, l_3) \times p_{1,xyz} = Trans(l_1, l_2, l_3) \times Rot(x, \alpha) \times p_{noa} \tag{2.24}$$

Similarly, after the third transformation, the coordinates of the point relative to the reference frame will be:

$$p_{xyz} = p_{3,xyz} = Rot(y, \beta) \times p_{2,xyz} = Rot(y, \beta) \times Trans(l_1, l_2, l_3) \times Rot(x, \alpha) \times p_{noa}$$

As you can see, the coordinates of the point relative to the reference frame at the conclusion of each transformation is found by pre-multiplying the coordinates of the point by each transformation matrix. Of course, as shown in Appendix A, the order of matrices cannot be changed, therefore this order is very important. You will also notice that for each transformation relative to the reference frame, the matrix is pre-multiplied. Consequently, the order of matrices *written* is the opposite of the order of transformations *performed*.

Example 2.8

A point $p(7,3,1)^T$ is attached to a frame F_{noa} and is subjected to the following transformations. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. Rotation of 90° about the z-axis,
2. Followed by a rotation of 90° about the y-axis,
3. Followed by a translation of [4,−3,7].

Solution: The matrix equation representing the transformation is:

$$p_{xyz} = Trans(4, -3, 7) Rot(y, 90) Rot(z, 90) p_{noa}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

As you can see, the first transformation of 90° about the z-axis rotates the F_{noa} frame as shown in Figure 2.15, followed by the second rotation about the y-axis, followed by the translation relative to the reference frame F_{xyz}. The point p in the frame can then be found relative to the F_{noa} as shown. The final coordinates of the point can be

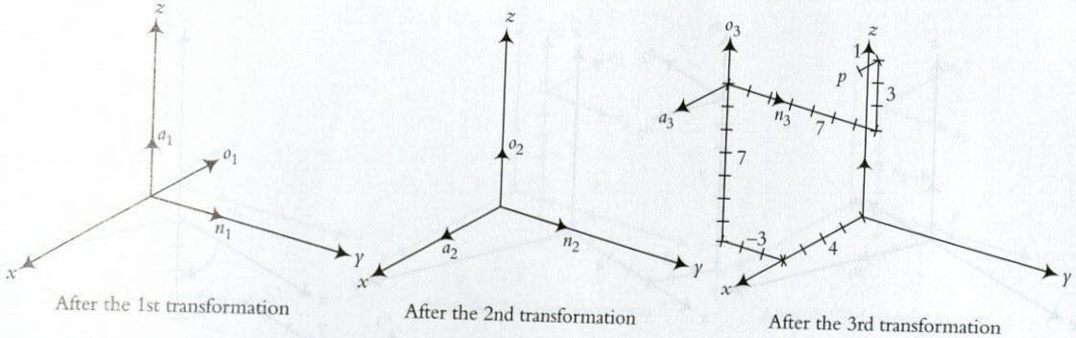

Figure 2.15 Effects of three successive transformations.

traced on the x-, y-, z-axes to be $4+1=5$, $-3+7=4$, and $7+3=10$. Be sure to follow this graphically. ■

Example 2.9

In this case, assume the same point $p(7,3,1)^T$, attached to F_{noa}, is subjected to the same transformations, but the transformations are performed in a different order, as shown. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. A rotation of 90° about the z-axis,
2. Followed by a translation of $[4,-3,7]$,
3. Followed by a rotation of 90° about the y-axis.

Solution: The matrix equation representing the transformation is:

$$p_{xyz} = Rot(y,90)\,Trans(4,-3,7)\,Rot(z,90)\,p_{noa}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

As you can see, although the transformations are exactly the same as in Example 2.8, since the order of transformations is changed, the final coordinates of the point are completely different. This can clearly be demonstrated graphically as in Figure 2.16. In this case, you can see that although the first transformation creates exactly the same change in the frame, the second transformation's result is very different because the translation relative to the reference frame axes will move the rotating frame F_{noa} outwardly. As a result of the third transformation, this frame will rotate about the y-axis, therefore rotating downwardly. The location of point p, attached to the frame is also shown. Please verify that the coordinates of this point

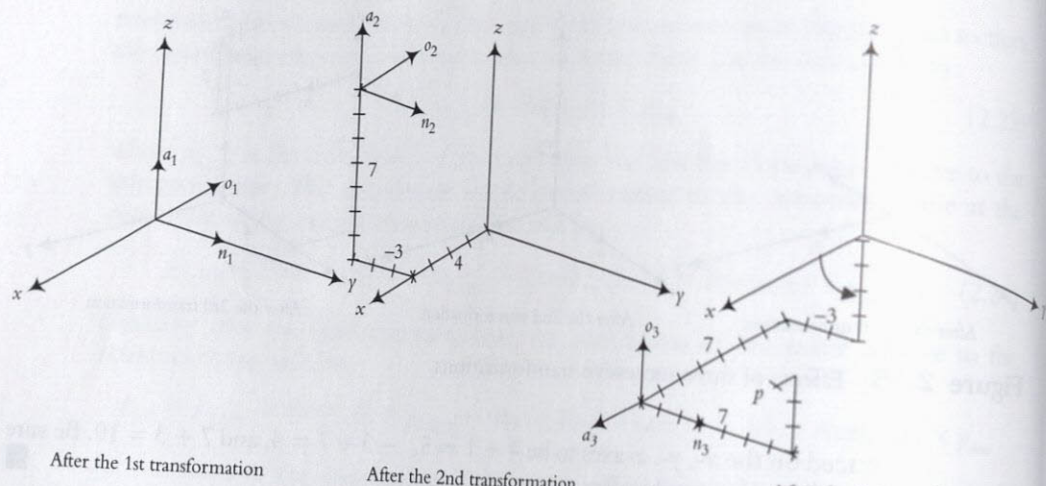

Figure 2.16 Changing the order of transformations will change the final result.

relative to the reference frame are $7+1=8$, $-3+7=4$, and $-4+3=-1$, which is the same as the analytical result. ■

2.6.4 Transformations Relative to the Rotating Frame

All transformations we have discussed so far have been relative to the fixed reference frame. This means that all translations, rotations, and distances (except for the location of a point relative to the moving frame) have been measured relative to the reference frame axes. However, it is possible to make transformations relative to the axes of a moving or current frame. This means that, for example, a rotation of 90° may be made relative to the n-axis of the moving frame (also referred to as the current frame), and not the x-axis of the reference frame. To calculate the changes in the coordinates of a point attached to the current frame relative to the reference frame, the transformation matrix is post-multiplied instead. Note that since the position of a point or an object attached to a moving frame is always measured relative to that moving frame, the position matrix describing the point or object is also always post-multiplied.

Example 2.10

Assume that the same point as in Example 2.9 is now subjected to the same transformations, but all relative to the current moving frame, as listed below. Find the coordinates of the point relative to the reference frame after transformations are completed.

1. A rotation of 90° about the a-axis,
2. Then a translation of [4,−3,7] along n-, o-, a-axes
3. Followed by a rotation of 90° about the o-axis.

Solution: In this case, since the transformations are made relative to the current frame, each transformation matrix is post-multiplied. As a result, the equation

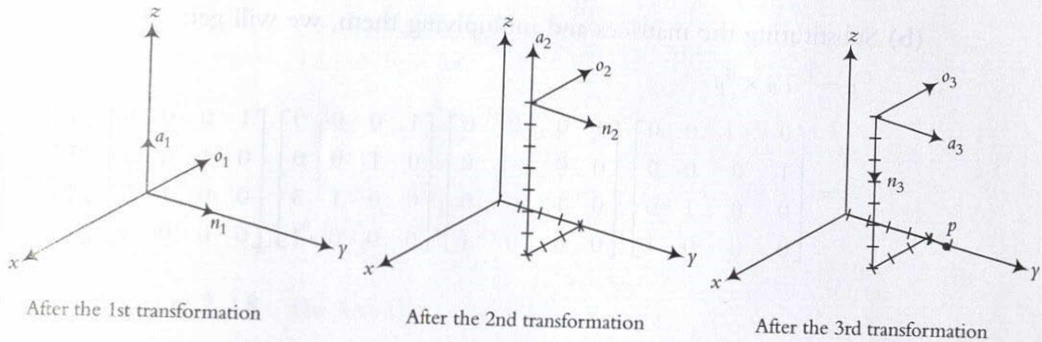

Figure 2.17 Transformations relative to the current frames.

representing the coordinates is:

$$p_{xyz} = Rot(a, 90)Trans(4,-3,7)Rot(o, 90)p_{noa}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

As expected, the result is completely different from the other cases, both because the transformations are made relative to the current frame, and because the order of the matrices is now different. Figure 2.17 shows the results graphically. Notice how the transformations are accomplished relative to the current frames.

Notice how the 7,3,1 coordinates of point p in the current frame will result in 0,5,0 coordinates relative to the reference frame. ■

Example 2.11

A frame B was rotated about the x-axis 90°, then it was translated about the current a-axis 3 inches before it was rotated about the z-axis 90°. Finally, it was translated about current o–axis 5 inches.

(a) Write an equation that describes the motions.

(b) Find the final location of a point $p(1,5,4)^T$ attached to the frame relative to the reference frame.

Solution: In this case, motions alternate relative to the reference frame and current frame.

(a) Pre- or post-multiplying each motion's matrix accordingly, we will get:

$$^{U}T_B = Rot(z, 90)Rot(x, 90)\,Trans(0, 0, 3)\,Trans(0, 5, 0)$$

(b) Substituting the matrices and multiplying them, we will get:

$$^{U}p = {}^{U}T_B \times {}^{B}p$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 10 \\ 1 \end{bmatrix}$$

■

Example 2.12

A frame F was rotated about the y-axis 90°, followed by a rotation about the o-axis of 30°, followed by a translation of 5 units along the n-axis, and finally, a translation of 4 units along the x-axis. Find the total transformation matrix.

Solution: The following set of matrices, written in the proper order to represent transformations relative to the reference frame or the current frame describes the total transformation:

$$T = Trans(4,0,0)Rot(y,90)Rot(o,30)Trans(5,0,0)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.866 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0 & 0.866 & 1.5 \\ 0 & 1 & 0 & 0 \\ -0.866 & 0 & -0.5 & -4.33 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Please verify graphically that this is true.

■

2.7 Inverse of Transformation Matrices

As mentioned earlier, there are many situations where the inverse of a matrix will be needed in robotic analysis. One situation where transformation matrices may be involved can be seen in the following example. Suppose the robot in Figure 2.18 is to be moved toward part P in order to drill a hole in the part. The robot's base position relative to the reference frame U is described by a frame R, the robot's hand is described by frame H, and the end effector (let's say the end of the drill bit that will be used to drill the hole) is described by frame E. The part's position is also described by frame P. The location of the point where the hole will be drilled can be related to the reference frame U through two independent paths: one through the part, one through the robot. Therefore, the

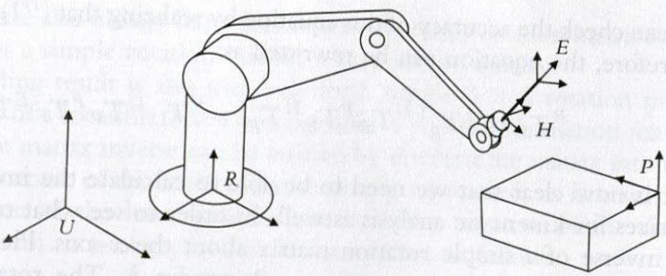

Figure 2.18 The Universe, robot, hand, part, and end effector frames.

following equation can be written:

$$^{U}T_{E} = {}^{U}T_{R}\,{}^{R}T_{H}\,{}^{H}T_{E} = {}^{U}T_{P}\,{}^{P}T_{E} \tag{2.25}$$

The location of point E on the part can be achieved by moving from U to P and from P to E, or it can alternately be achieved by a transformation from U to R, from R to H, and from H to E.

In reality, the transformation of frame R relative to the Universe frame (${}^{U}T_{R}$) is known since the location of the robot's base must be known in any set-up. For example, if a robot is installed in a work cell, the location of the robot's base will be known since it is bolted to a table. Even if the robot is mobile or attached to a conveyor belt, its location at any instant is known because a controller must be following the position of the robot's base at all times. The ${}^{H}T_{E}$, or the transformation of the end effector relative to the robot's hand, is also known since any tool used at the end effector is a known tool and its dimensions and configuration is known. ${}^{U}T_{P}$, or the transformation of the part relative to the universe, is also known since we must know where the part is located if we are to drill a hole in it. This location is known by putting the part in a jig, through the use of a camera and vision system, through the use of a conveyor belt and sensors, or other similar devices. ${}^{P}T_{E}$ is also known since we need to know where the hole is to be drilled on the part. Consequently, the only unknown transformation is ${}^{R}T_{H}$, or the transformation of the robot's hand relative to the robot's base. This means we need to find out what the robot's joint variables—the angle of the revolute joints and the length of the prismatic joints of the robot—must be in order to place the end effector at the hole for drilling. As you can see, it is necessary to calculate this transformation, which will tell us what needs to be accomplished. The transformation will later be used to actually solve for joint angles and link lengths.

To calculate this matrix, unlike in an algebraic equation, we cannot simply divide the right side by the left side of the equation. We need to pre- or post-multiply by inverses of appropriate matrices to eliminate them. As a result, we will have:

$$\left({}^{U}T_{R}\right)^{-1}\left({}^{U}T_{R}\,{}^{R}T_{H}\,{}^{H}T_{E}\right)\left({}^{H}T_{E}\right)^{-1} = \left({}^{U}T_{R}\right)^{-1}\left({}^{U}T_{P}\,{}^{P}T_{E}\right)\left({}^{H}T_{E}\right)^{-1} \tag{2.26}$$

or, since $\left({}^{U}T_{R}\right)^{-1}\left({}^{U}T_{R}\right) = I$ and $\left({}^{H}T_{E}\right)\left({}^{H}T_{E}\right)^{-1} = I$, the left side of Equation (2.26) simplifies to ${}^{R}T_{H}$ and we get:

$$^{R}T_{H} = {}^{U}T_{R}^{-1}\,{}^{U}T_{P}\,{}^{P}T_{E}\,{}^{H}T_{E}^{-1} \tag{2.27}$$