**REQUIRED ELEMENTS**

In this section, we discuss the required elements for the report.

*Force Model & Center of integration* (10 pts)

Choose the frame and center body of integration for the spacecraft (e.g. the Sun or the Earth, EMO2000 or EME2000). There is no right or wrong answer, but explain clearly. Based on your decision, report the full dynamical model of the spacecraft and proceed with your software implementation of the different force models.

*Radiometric measurement / partials* (10 pts)

Derive and provide the $\widetilde{H}$ measurement mapping matrix of the problem, i.e. the partials of the range and range rate measurements with respect to the spacecraft state (r and v) AND parameters you selected in your augmented state vector. You can do this completely by hand or use the Matlab symbolic toolbox. If you selected the Sun as your center body of integration, do not forget to adjust your radiometric measurement models to account for that fact.

I derived the Force Model and $\widetilde{H}$ matrix that we should put on the report.

# For the report

## Frame: EMO2000

## Augmented State Vector $\vec{X}$

$$\vec{X} = \begin{bmatrix} \vec{r} \\ \vec{v} \\ K_{SRP} \\ \phi_4 \\ \lambda_4 \\ \dot{\rho}_{bias} \\ \Phi_{6x6} \end{bmatrix}$$

$\phi_4$ : latitude of station #4 (Antarctica)

$\lambda_4$ : Logitude of station #4 (Antarctica)

## Equations of Motion and Force Model (10 pts)

$$\dot{\vec{r}} = \vec{v} \quad , \quad \dot{\vec{v}} = \vec{a}_{sun} + \vec{a}_{Earth} + \vec{a}_{SRP}$$

- $$\boxed{\vec{a}_{sun} = -\mu_s \frac{\vec{r}}{\|r\|^3}}$$

$$\vec{r}_{SC/E} = \vec{r}_{SC/Sun} - \vec{r}_{E/sun}$$

- $$\boxed{\vec{a}_{Earth} = -\mu_E \left( \frac{\vec{r}_{SC/E}}{\|\vec{r}_{SC/E}\|^3} - \frac{\vec{r}_{E/sun}}{\|r_{E/sun}\|^3} \right)}$$

- $$\boxed{\vec{a}_{SRP} = (K_{SRP})(P)(1+\rho_r)\left(\frac{A}{m}\right)\left(\frac{1}{1000}\right)\left(\frac{\vec{r}_{SC}}{\|r_{SC}\|}\right)}$$

Modeling $\vec{a}_{SRP}$;

For 1AU:

$$\vec{a}_{SRP} = P_{SRP} \frac{(1+p_r)A}{m} \hat{s}$$

$\hat{S}$ : sun to S/C unit vector

$$\hat{S} = \frac{\vec{r}_{sc}}{\|r_{sc}\|}$$

$$P_{SRP} = 4.54 \times 10^{-6} \frac{N}{m^2}$$

In our case distance is not constant and 1AU:

we define: $P = P_{SRP} \left(\frac{AU}{r}\right)^2$ (inverse square law)

$$\vec{a}_{SRP} = (K_{SRP})(a_{SRP,mag})$$

$$a_{SRP,mag} = (P)(1+P_r)\left(\frac{A_{sc}}{m_{sc}}\right)\left(\frac{1}{1000}\right)$$

$$\boxed{\bullet\ \vec{a}_{SRP} = (K_{SRP})\underbrace{(P)(1+P_r)\left(\frac{A_{sc}}{m_{sc}}\right)\left(\frac{1}{1000}\right)}_{a_{SRP,mag}}\underbrace{\left(\frac{\vec{r}_{sc}}{\|r_{sc}\|}\right)}_{\hat{S}}}$$

also can be expressed shortly

$$\vec{a}_{SRP} = K_{SRP}\, a_{SRP,mag}\, \hat{S}$$

# Define $\tilde{H}$ measurement mapping matrix (10 pts)

$$\vec{\rho} = \vec{r}_{SC,ECEF} - \vec{r}_{Station,ECEF} \quad , \quad \vec{\dot{\rho}} = \vec{V}_{SC} - \vec{V}_{Station}$$

$$\hat{\rho} = \frac{\vec{\rho}}{\|\rho\|} \quad , \quad \underbrace{\dot{\rho}}_{\text{scalar}_{1\times1}} = \underbrace{\hat{\rho}^T}_{1\times3} \underbrace{\left(\vec{V}_{SC,ECEF} - \vec{V}_{Station,ECEF}\right)}_{3\times1}$$

$$\tilde{H} = \frac{\partial \vec{G}}{\partial \vec{X}} = \begin{bmatrix} \frac{\partial \rho}{\partial \vec{r}} & \frac{\partial \rho}{\partial \vec{v}} & \frac{\partial \rho}{\partial \mu_{SRP}} & \frac{\partial \rho}{\partial \phi_4} & \frac{\partial \rho}{\partial \lambda_4} & \frac{\partial \rho}{\partial \dot{\rho}_{bias}} \\ \frac{\partial \dot{\rho}}{\partial \vec{r}} & \frac{\partial \dot{\rho}}{\partial \vec{v}} & \frac{\partial \dot{\rho}}{\partial \mu_{SRP}} & \frac{\partial \dot{\rho}}{\partial \phi_4} & \frac{\partial \dot{\rho}}{\partial \lambda_4} & \frac{\partial \dot{\rho}}{\partial \dot{\rho}_{bias}} \end{bmatrix}$$

we know

for Range Partials w.r.t S/C position and velocity;

$$\boxed{\frac{\partial \rho}{\partial \vec{r}} = \hat{\rho}^T \quad , \quad \frac{\partial \rho}{\partial \vec{v}} = \vec{0}_{1\times3}}$$

for range rate Partials w.r.t S/C position and velocity;

$$\boxed{\frac{\partial \dot{\rho}}{\partial \vec{r}} = \frac{(\vec{v} - \vec{v}_s)^T}{\|\vec{r} - \vec{r}_s\|} \left[ I_{3\times3} - \frac{(\vec{r} - \vec{r}_s) \cdot (\vec{r} - \vec{r}_s)^T}{\|\vec{r} - \vec{r}_s\|^2} \right]}$$

$$\boxed{\frac{\partial \dot{\rho}}{\partial \vec{v}} = \hat{\rho}^T}$$

Take the derivatives of other Partials Let's go block by block:

$$\boxed{\frac{\partial \rho}{\partial K_{SRP}} = 0}$$ (range measurements does not explicitly depend on the SRP scale factor)

$$\frac{\partial \rho}{\partial \phi_4} \rightarrow \frac{\partial \rho}{\partial \phi_4} = \underbrace{\frac{\partial \rho}{\partial \vec{\rho}}}_{\hat{\rho}^T} \frac{\partial \vec{\rho}}{\partial \phi_4}$$ Chain Rule

$$\frac{\partial \vec{\rho}}{\partial \phi_4} = \frac{\partial(\vec{r}_{sc} - \vec{r}_{station})}{\partial \phi_4} = \underbrace{\frac{\partial \vec{r}_{sc}}{\partial \phi_4}}_{0} - \frac{\partial \vec{r}_{station}}{\partial \phi_4}$$

position of the S/C does not depend where the station is on Earth

$$\frac{\partial \vec{\rho}}{\partial \phi_4} = -\frac{\partial \vec{r}_{station}}{\partial \phi_4}$$

Station #4, ECEF: $\vec{r}_{station} = R_E \begin{bmatrix} \cos\phi_4 \cos\lambda_4 \\ \cos\phi_4 \sin\lambda_4 \\ \sin\phi_4 \end{bmatrix}$

$$\frac{\partial \rho}{\partial \phi_4} = -\hat{\rho}^T \frac{\partial \vec{r}_{station}}{\partial \phi_4} = -\hat{\rho}^T \frac{\partial \begin{bmatrix} R_E\cos\phi_4 \cos\lambda_4 \\ R_E\cos\phi_4 \sin\lambda_4 \\ R_E\sin\phi_4 \end{bmatrix}}{\partial \phi_4}$$

We get; $$\boxed{\frac{\partial \rho}{\partial \phi_4} = -\hat{\rho}^T R_E \begin{bmatrix} -\sin\phi_4 \cos\lambda_4 \\ -\sin\phi_4 \sin\lambda_4 \\ \cos\lambda_4 \end{bmatrix}}$$

$$\frac{\partial \rho}{\partial \lambda_4} \rightarrow \frac{\partial \rho}{\partial \lambda_4} = \underbrace{\frac{\partial \rho}{\partial \vec{\rho}}}_{\hat{\rho}^T} \frac{\partial \vec{\rho}}{\partial \lambda_4}$$

$$\frac{\partial \vec{\rho}}{\partial \lambda_4} = \frac{\partial (\vec{r}_{sc} - \vec{r}_{station})}{\partial \lambda_4} = - \frac{\partial \vec{r}_{station}}{\partial \lambda_4}$$

$$\frac{\partial \rho}{\partial \lambda_4} = -\hat{\rho}^T \frac{\partial \vec{r}_{station}}{\partial \lambda_4} = -\hat{\rho}^T \frac{\partial \begin{bmatrix} R_E \cos\phi_4 \cos\lambda_4 \\ R_E \cos\phi_4 \sin\lambda_4 \\ R_E \sin\phi_4 \end{bmatrix}}{\partial \lambda_4}$$

$$\boxed{\frac{\partial \rho}{\partial \lambda_4} = -\hat{\rho}^T R_E \begin{bmatrix} -\cos\phi_4 \sin\lambda_4 \\ \cos\phi_4 \cos\lambda_4 \\ 0 \end{bmatrix}}$$

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$$\boxed{\frac{\partial \rho}{\partial \dot{\rho}_{bias}} = 0}$$

$$\rho = \rho(\vec{r}_{sc}, \vec{r}_{station})$$

no dependency between $\rho$ and $\dot{\rho}_{bias}$.

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$$\boxed{\frac{\partial \dot{\rho}}{\partial k_{SRP}} = 0}$$

$$\dot{\rho} = \hat{\rho}^T (\vec{V}_{sc} - \vec{V}_{station})$$

$$\dot{\rho} = \dot{\rho}(r, v, \phi_4, \lambda_4)$$

no dependency between $\dot{\rho}$ and $k_{SRP}$

$$\frac{\partial \dot{\rho}}{\partial \phi_4} \longrightarrow \dot{\rho} = \hat{\rho}^T (\vec{V}_{sc} - \vec{V}_{station})$$

Let $\hat{\rho} = A$
$\vec{V}_{sc} - \vec{V}_{station} = B$ $\Big\}$ $\dot{\rho} = A^T B$

Remember the chain rule; $\frac{d}{dx}(u^T v) = \left(\frac{du}{dx}\right)^T v + u^T \frac{dv}{dx}$

Applying this to our case;

$$\frac{\partial \dot{\rho}}{\partial \phi_4} = \frac{\partial (A^T B)}{\partial \phi_4} = \left(\frac{\partial A}{\partial \phi_4}\right)^T B + A^T \frac{\partial B}{\partial \phi_4}$$

write again;

$$\frac{\partial \dot{\rho}}{\partial \phi_4} = \left(\frac{\partial \hat{\rho}}{\partial \phi_4}\right)^T (\vec{V}_{sc} - \vec{V}_{station}) + \hat{\rho}^T \frac{\partial (\vec{V}_{sc} - \vec{V}_{station})}{\partial \phi_4}$$

$$\frac{\partial (\vec{V}_{sc} - \vec{V}_{station})}{\partial \phi_4} = \underbrace{\frac{\partial \vec{V}_{sc}}{\partial \phi_4}}_{0} - \frac{\partial \vec{V}_{station}}{\partial \phi_4}$$

So; $$\frac{\partial (\vec{V}_{sc} - \vec{V}_{station})}{\partial \phi_4} = -\frac{\partial \vec{V}_{station}}{\partial \phi_4}$$

$$\frac{\partial \dot{\rho}}{\partial \phi_4} = \left(\frac{\partial \hat{\rho}}{\partial \phi_4}\right)^T (\vec{V}_{sc} - \vec{V}_{station}) - \hat{\rho}^T \frac{\partial \vec{V}_{station}}{\partial \phi_4}$$

$$\left(\frac{\partial \hat{\rho}}{\partial \phi_4}\right) = -\frac{1}{\rho}\left(I - \hat{\rho}\hat{\rho}^T\right)\frac{\partial \vec{r}_{station}}{\partial \phi_4}$$

$$= -\frac{1}{\rho}\left(I - \hat{\rho}\hat{\rho}^T\right) R_E \begin{bmatrix} -\sin\phi_4 \cos\lambda_4 \\ -\sin\phi_4 \sin\lambda_4 \\ \cos\phi_4 \end{bmatrix}$$

take the transpose:

$$\left(\frac{\partial \hat{\rho}}{\partial \phi_4}\right)^T = -\frac{1}{\rho}\left(I - \hat{\rho}\hat{\rho}^T\right)\left(R_E \left[-\sin\phi_4\cos\lambda_4 \quad -\sin\phi_4\sin\lambda_4 \quad \cos\phi_4\right]\right)$$

$$\frac{\partial \vec{V}_{station}}{\partial \phi_4} = \frac{\partial\left(\vec{\omega}_E \times \vec{r}_{station}\right)}{\partial \phi_4} = \omega_E R_E \begin{bmatrix} \sin\phi_4 \sin\lambda_4 \\ -\sin\phi_4 \cos\lambda_4 \\ 0 \end{bmatrix}$$

Put all of them together;

$$\frac{\partial \dot{\rho}}{\partial \phi_4} = \left(\frac{\partial \hat{\rho}}{\partial \phi_4}\right)^T\left(\vec{V}_{SC} - \vec{V}_{station}\right) - \hat{\rho}^T \frac{\partial \vec{V}_{station}}{\partial \phi_4}$$

$$\boxed{\frac{\partial \dot{\rho}}{\partial \phi_4} = -\frac{1}{\rho}\left(I - \hat{\rho}\hat{\rho}^T\right) R_E \begin{bmatrix} -\sin\phi_4 \cos\lambda_4 \\ -\sin\phi_4 \sin\lambda_4 \\ \cos\phi_4 \end{bmatrix}^T \left(\vec{V}_{SC} - \vec{V}_{station}\right) - \hat{\rho}^T \omega_E R_E \begin{bmatrix} \sin\phi_4 \sin\lambda_4 \\ -\sin\phi_4 \cos\lambda_4 \\ 0 \end{bmatrix}}$$

$$\frac{\partial \dot{\rho}}{\partial \phi_4} = -\frac{1}{\rho}\left(R_E \left[-\sin\phi_4\cos\lambda_4 \quad -\sin\phi_4\sin\lambda_4 \quad \cos\phi_4\right]\right)\left(I - \hat{\rho}\hat{\rho}^T\right)\left(\mathbf{v}_{SC} - \mathbf{v}_{st}\right) - \hat{\rho}^T \omega_E R_E \begin{bmatrix} \sin\phi_4 \sin\lambda_4 \\ -\sin\phi_4 \cos\lambda_4 \\ 0 \end{bmatrix}$$

It will be similar for $\dfrac{\partial \dot{\rho}}{\partial \lambda_4}$;

$$\frac{\partial \dot{\rho}}{\partial \lambda_4} = \left(\frac{\partial \hat{\rho}}{\partial \lambda_4}\right)^T (\vec{V}_{SC} - \vec{V}_{station}) - \hat{\rho}^T \frac{\partial \vec{V}_{station}}{\partial \lambda_4}$$

$$\boxed{\frac{\partial \dot{\rho}}{\partial \lambda_4} = -\frac{1}{\rho}\left(I - \hat{\rho}\hat{\rho}^T\right) R_E \begin{bmatrix} -\cos\phi_4 \sin\lambda_4 \\ \cos\phi_4 \cos\lambda_4 \\ 0 \end{bmatrix}^T (\vec{V}_{SC} - \vec{V}_{station}) - \hat{\rho}^T \omega_E R_E \begin{bmatrix} -\cos\phi_4 \cos\lambda_4 \\ -\cos\phi_4 \sin\lambda_4 \\ 0 \end{bmatrix}}$$

$$\frac{\partial \dot{\rho}}{\partial \lambda_4} = -\frac{1}{\rho}\Big(R_E \left[-\cos\phi_4 \sin\lambda_4 \quad \cos\phi_4 \cos\lambda_4 \quad 0\right]\Big)\left(I - \hat{\rho}\hat{\rho}^T\right)(\mathbf{v}_{SC} - \mathbf{v}_{st}) - \hat{\rho}^T \omega_E R_E \begin{bmatrix} -\cos\phi_4 \cos\lambda_4 \\ -\cos\phi_4 \sin\lambda_4 \\ 0 \end{bmatrix}$$

$$\frac{\partial \dot{\rho}}{\partial \dot{\rho}_{bias}} \Rightarrow \qquad \underbrace{\dot{\rho}}_{\text{real}} = \underbrace{\dot{\rho}_{computed}}_{\substack{\text{model} \\ \text{(computed)}}} + \dot{\rho}_{bias}$$

$$\dot{\rho} = \hat{\rho}^T(\vec{V}_{SC} - \vec{V}_{station}) + \dot{\rho}_{bias}$$

Attention ! For first 6 days there is $\dot{\rho}_{bias}$

$$\boxed{\frac{\partial \dot{\rho}}{\partial \dot{\rho}_{bias}} = \frac{\partial\left(\hat{\rho}^T(\vec{V}_{SC} - \vec{V}_{station}) + \dot{\rho}_{bias}\right)}{\partial \dot{\rho}_{bias}} = 1}$$

After 6 days there is no $\dot{\rho}_{bias}$

$$\dot{\rho} = \dot{\rho}_{computed} + \cancelto{0}{\dot{\rho}_{bias}}$$

$$\dot{\rho} = \hat{\rho}^T(\vec{V}_{SC} - \vec{V}_{station})$$

$$\boxed{\frac{\partial \dot{\rho}}{\partial \dot{\rho}_{bias}} = \frac{\partial\left(\hat{\rho}^T(\vec{V}_{SC} - \vec{V}_{station})\right)}{\partial \dot{\rho}_{bias}} = 0}$$

Now we have all elements for H matrix.

$\rho$ partials:

$$\frac{\partial \rho}{\partial \mathbf{r}} = \hat{\rho}^T$$

$$\frac{\partial \rho}{\partial \mathbf{v}} = 0_{1\times 3}$$

$$\frac{\partial \rho}{\partial k_{SRP}} = 0$$

$$\frac{\partial \rho}{\partial \phi_4} = -\hat{\rho}^T \frac{\partial \mathbf{r}_{st}}{\partial \phi_4}$$

$$\frac{\partial \rho}{\partial \lambda_4} = -\hat{\rho}^T \frac{\partial \mathbf{r}_{st}}{\partial \lambda_4}$$

$$\frac{\partial \rho}{\partial \dot{\rho}_{bias}} = 0$$

$\dot{\rho}$ partials

$$\frac{\partial \dot{\rho}}{\partial \mathbf{r}} = \frac{1}{\rho}\left(\mathbf{v}_{SC} - \mathbf{v}_{st}\right)^T \left(I - \hat{\rho}\hat{\rho}^T\right)$$

$$\frac{\partial \dot{\rho}}{\partial \mathbf{v}} = \hat{\rho}^T$$

$$\frac{\partial \dot{\rho}}{\partial k_{SRP}} = 0$$

$$\frac{\partial \dot{\rho}}{\partial \phi_4} = -\frac{1}{\rho}\left(\partial \mathbf{r}_{st}/\partial \phi_4\right)^T \left(I - \hat{\rho}\hat{\rho}^T\right)\left(\mathbf{v}_{SC} - \mathbf{v}_{st}\right) - \hat{\rho}^T \frac{\partial \mathbf{v}_{st}}{\partial \phi_4}$$

$$\frac{\partial \dot{\rho}}{\partial \lambda_4} = -\frac{1}{\rho}\left(\partial \mathbf{r}_{st}/\partial \lambda_4\right)^T \left(I - \hat{\rho}\hat{\rho}^T\right)\left(\mathbf{v}_{SC} - \mathbf{v}_{st}\right) - \hat{\rho}^T \frac{\partial \mathbf{v}_{st}}{\partial \lambda_4}$$

$$\frac{\partial \dot{\rho}}{\partial \dot{\rho}_{bias}} = \begin{cases} 1, & t \leq 6 \text{ gün} \\ 0, & t > 6 \text{ gün} \end{cases}$$

$$\tilde{H} = \frac{\partial \vec{G}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial \rho}{\partial \vec{r}} & \frac{\partial \rho}{\partial \vec{v}} & \frac{\partial \rho}{\partial k_{SRP}} & \frac{\partial \rho}{\partial \phi_4} & \frac{\partial \rho}{\partial \lambda_4} & \frac{\partial \rho}{\partial \dot{\rho}_{bias}} \\ \frac{\partial \dot{\rho}}{\partial \vec{r}} & \frac{\partial \dot{\rho}}{\partial \vec{v}} & \frac{\partial \dot{\rho}}{\partial k_{SRP}} & \frac{\partial \dot{\rho}}{\partial \phi_4} & \frac{\partial \dot{\rho}}{\partial \lambda_4} & \frac{\partial \dot{\rho}}{\partial \dot{\rho}_{bias}} \end{bmatrix}$$