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Complete List of Authors	Yilin Zhang, Canyi Chen and Liping Zhu
Corresponding Authors	Liping Zhu
E-mails	zhu.liping@ruc.edu.cn
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SLICED INDEPENDENCE TEST

Yilin Zhang, Canyi Chen and Liping Zhu

Renmin University of China

Abstract: An ideal independence test is anticipated to possess three properties: zero-independence equivalent, numerically efficient and asymptotically normal. We introduce a slicing procedure to estimate a popular measure of nonlinear dependence, leading the resultant sliced independence test to simultaneously possess all three properties. In addition, the power performance of the sliced independence test improves as the number of observations within each slice increases. The popular rank test corresponds to a special case of the sliced independence test that contains two observations within each slice. The sliced independence test is thus more powerful than the rank test. The size performance of sliced independence test is insensitive to the number of slices, in that the slicing estimation is consistent and asymptotically normal for a wide range of slice numbers. We further adapt the sliced independence test to account for the presence of multivariate control variables. The theoretical properties are confirmed through comprehensive simulations and an application to the astronomical dataset.

Key words and phrases: correlation, measure of association, rank tests.

1. Introduction

Testing for independence between two random variables is a fundamental problem in statistics. Weihs et al. (2016, 2018) stated that, an independence test is anticipated to simultaneously possess the following three properties.

1. *Zero-independence equivalent:* At the population level the dependence metric is equal to zero if and only if the two random variables are independent. This ensures that the independence test is consistent.

2. *Numerically efficient:* The complexity of implementing an independence test is linear or nearly linear in the sample size n, say $O\{n\log(n)\}$. This is almost the minimal computational cost that we have to bear.

3. *Asymptotically normal:* The asymptotic null distribution of an independence test is normal. The asymptotic normality is more desirable for practitioners than the asymptotically distribution-free property.

Many tests are developed in the literature. However, few of them possesses these properties simultaneously. For example, the Pearson correlation (Pearson, 1895), and its variations such as Spearman's rho (Spearman, 1906) and Kendall's tau (Kendall, 1938), are not zero-independence equivalent, although estimating these metrics is numerically efficient. The Hoeffding's index (Hoeffding, 1948) is zero-independence equivalent only if

both random variables are continuous. The independence tests based on Blum et al. (1961)'s correlation and its variations such as Zhou and Zhu (2018) are asymptotically distribution-free. However, implementing these independence tests has the complexity of quadratic order of the sample size n, which is typically regarded as numerically inefficient. In general, the distance correlation (Székely et al., 2007), the projection correlation (Zhu et al., 2017) and the binning approach (Heller et al., 2013) are not numerically efficient, when used to test independence between two random vectors. In addition, their asymptotic null distributions depend on the parent distribution of the two random vectors. A chi-square distribution is suggested to approximate the asymptotic null distribution of the distance correlation test, which is however quite conservative (Székely et al., 2007, page 2783). Using bootstrap or random permutations to approximate asymptotic null distributions is generally regarded as computationally intensive. Several algorithms are proposed to speed up the calculations of distance correlation. In particular, Huang and Huo (2017), Huo and Székely (2016) and Chaudhuri and Hu (2019) improved the computational complexity of calculating distance correlation to the order of $O\{n\log(n)\}$, when both random variables are univariate. Huang and Huo (2017) proposed to approximate the asymptotic null distribution of the distance correlation test through Gamma

distribution, which however lacks rigorous theoretical justification thus far (Gao et al., 2021, page 2012). Dette et al. (2013), Kong et al. (2019) and Chatterjee (2020) introduced independently a dependence metric, which is zero-independence equivalent. It attracts much attention for its simplicity and implementability (Cao and Bickel, 2020; Wiesel, 2021). Dette et al. (2013) and Kong et al. (2019) suggested to estimate this metric with kernel smoother. However, implementing kernel smoothing has the complexity of nearly quadratic order of the sample size, which limits its usefulness in situations where the sample size is extremely large. In addition, its asymptotic null distribution depends upon the kernel function. By contrast, Chatterjee (2020) proposed a rank estimation, which is computationally efficient and asymptotically standard normal. This rank estimation satisfies all desirable properties simultaneously, is thus more appealing than kernel smoothing.

In this article, we introduce a slicing procedure to estimate the dependence metric suggested by Dette et al. (2013), Kong et al. (2019) and Chatterjee (2020). This procedure divides the observations into several slices according to the realizations of one random variable, evaluates the local variation of the other within each slice, and aggregates the variations across all slices to form a slicing estimation. The complexity of implementing the resultant sliced independence test is nearly linear in the sample size,

which is thus numerically efficient. The asymptotic null distribution is standard normal, which does not depend on the parent distributions of the two random variables. The resultant sliced independence test further improves the popular rank test of Chatterjee (2020) from two perspectives. The rank test corresponds to the sliced independence test when there are only two observations within each slice. We show that, the power performance of the sliced independence test improves as the number of observations within each slice increases, even when the total sample size is fixed. As a consequence, our proposed sliced independence test is more powerful than the rank test. In addition, the slicing estimation is consistent and asymptotically normal for a very wide range of the slice number. Therefore, the size performance of the sliced independence test is, surprisingly, very insensitive to the number of slices. The concept of this slicing estimation procedure can be readily generalized to the multivariate case through the K-means clustering (MacQueen, 1967). These theoretical properties are demonstrated through comprehensive simulations and an application to the astronomical dataset. An R package for implementing the sliced independence test will be released on the Comprehensive R Archive Network.

This paper is organized as follows. We propose the slicing procedure and connect it with the rank test in Section 2. We study the asymptotic

properties of the sliced independence test in Section 3, and generalize this slicing procedure to the multivariate case through the K-means clustering in Section 4. We demonstrate the finite-sample performance of sliced independence test through comprehensive simulations and an analysis of the astronomical dataset in Section 5, and conclude this paper in Section 6. All technical proofs are relegated to the on-line Supplementary Material.

2. The Slicing Estimation Procedure

2.1 A Brief Review

Suppose X and Y are two univariate random variables. Define $s(t;X) \stackrel{\text{def}}{=} \text{pr}(Y \geq t \mid X)$. Let T be an independent univariate random variable, with probability mass/density and cumulative distribution functions being $\omega(t)$ and $\mu(t)$, respectively. The support of T is denoted by $\text{supp}(T) \stackrel{\text{def}}{=} \{t : \omega(t) > 0\}$. We assume throughout that $\text{supp}(Y) \subseteq \text{supp}(T)$. It follows immediately that X and Y are independent if and only if $\text{var}\{s(t;X)\} = 0$, for all $t \in \mathbb{R}$. Dette et al. (2013), Kong et al. (2019) and Chatterjee (2020) suggest independently the following metric to quantify the degree of deviation from independence,

$$\mathcal{S}(X,Y) \stackrel{\text{def}}{=} \int \text{var}\{s(t;X)\}\text{d}\mu(t) \Big/ \int \text{var}\{1(Y \geq t)\}\text{d}\mu(t). \quad (2.1)$$

The denominator in (2.1) is to ensure $\mathcal{S}(X,Y)$ to range from 0 to 1. The law of total variance yields immediately that $\mathcal{S}(X,Y)$ is equal to

$$1-\int E\Big[\mathrm{var}\{1(Y\geq t)\mid X\}\Big]\mathrm{d}\mu(t)\Big/\int \mathrm{var}\{1(Y\geq t)\}\mathrm{d}\mu(t). \qquad (2.2)$$

Dette et al. (2013), Kong et al. (2019) and Chatterjee (2020) simply set T to be an independent copy of Y. For now we allow T to be an arbitrary random variable as long as $\mathrm{supp}(Y)\subseteq\mathrm{supp}(T)$. We shall revisit this issue in Study 1 of Section 5. We leave the asymmetry between X and Y in $\mathcal{S}(X,Y)$ deliberately there to study which random variable impacts the other one (Zheng et al., 2012; Cui et al., 2015; Kong et al., 2019).

Kong et al. (2019, Lemma 1) and Chatterjee (2020, Theorem 1) show that this metric possesses several desirable properties at the population level. For example, $\mathcal{S}(X,Y)=0$ if and only if X and Y are independent, and $\mathcal{S}(X,Y)=1$ if and only if Y is a measurable function of X; If (X,Y) is bivariate Gaussian with correlation coefficient ρ, then $\mathcal{S}(X,Y)$ is strictly increasing in $|\rho|$; In addition, $\mathcal{S}(X,Y)$ remains unchanged if we apply strictly monotone transformations to both X and Y.

2.2 The Slicing Procedure

Next we discuss how to estimate $\mathcal{S}(X,Y)$ with a random sample $\{(X_i,Y_i),i=1,\ldots,n\}$. There are two different attempts, kernel smoothing and rank es-

timation, in literature. Dette et al. (2013) and Kong et al. (2019) suggest to estimate $\text{var}\{1(Y \geq t) \mid X\}$ with kernel smoothing, for each given t. The overall complexity of estimating $\mathcal{S}(X, Y)$ through kernel smoothing is in $O(n^2)$ time, which limits its usefulness when n is extremely large. The asymptotic null distribution depend upon the kernel function, which is not desirable either. Chatterjee (2020) propose a rank estimation for $\mathcal{S}(X, Y)$, which has the complexity in $O(n \log n)$ time. In addition, the rank estimation is asymptotically standard normal. In these regards, the rank estimation is thus more appealing than the kernel smoothing.

We introduce a slicing procedure to estimate $\mathcal{S}(X, Y)$, which proceeds as follows. We first order the random sample $\{(X_i, Y_i), i = 1, \ldots, n\}$ according to the values of X_is, which yields $\{(X_{(i)}, Y_{(i)}), i = 1, \ldots, n\}$, where $X_{(1)} \leq \ldots \leq X_{(n)}$ are the ordered values of X_is, and $Y_{(i)}$ is the concomitant of $X_{(i)}$. Next we divide the ordered sample $\{(X_{(i)}, Y_{(i)}), i = 1, \ldots, n\}$ into H slices according to the values of $X_{(i)}$s, such that there are c observations within each slice. We assume for simplicity that $n = Hc$. We rewrite $X_{(h,j)} = X_{(c(h-1)+j)}$ and $Y_{(h,j)} = Y_{(c(h-1)+j)}$ for $j = 1, \ldots, c$ and $h = 1, \ldots, H$. The observations in the h-th slice are $\{(X_{(h,j)}, Y_{(h,j)}), j = 1, \ldots, c\}$. Given t, we estimate $\text{var}\{1(Y \geq t) \mid X\}$ within each slice and

$E\Big[\mathrm{var}\{1(Y \geq t) \mid X\}\Big]$ with

$$H^{-1}\sum_{h=1}^{H}\left[(c-1)^{-1}\sum_{j=1}^{c}\Big\{1(Y_{(h,j)} \geq t) - c^{-1}\sum_{j=1}^{c}1(Y_{(h,j)} \geq t)\Big\}^2\right]$$

$$= \{n(c-1)\}^{-1}\sum_{h=1}^{H}\sum_{j<l}^{c}\big\{1(Y_{(h,j)} \geq t) - 1(Y_{(h,l)} \geq t)\big\}^2.$$

Suppose $\{T_i, i = 1, \ldots, n\}$ is a random sample drawn from $\mu(t)$. Let t run through the values of T_is, which allows us to estimate

$$\int E\Big[\mathrm{var}\{1(Y \geq t) \mid X\}\Big]\mathrm{d}\mu(t) \tag{2.3}$$

in (2.2) with

$$\{n^2(c-1)\}^{-1}\sum_{i=1}^{n}\sum_{h=1}^{H}\sum_{j<l}^{c}\{1(Y_{(h,j)} \geq T_i) - 1(Y_{(h,l)} \geq T_i)\}^2$$

$$= \{n^2(c-1)\}^{-1}\sum_{h=1}^{H}\sum_{j<l}^{c}|r_{(h,j)} - r_{(h,l)}|, \tag{2.4}$$

where $r_{(h,j)}$ stands for the number of T_is such that $Y_{(h,j)} \geq T_i$, for $i = 1, \ldots, n$. In symbols, $r_{(h,j)} = \#\{T_i : Y_{(h,j)} \geq T_i, i = 1, \ldots, n\}$.

Next we turn to the denominator in (2.2). For each given t, we estimate $\mathrm{var}\{1(Y \geq t)\}$ with the standard U-statistic theory (van der Vaart, 1998, chapter 12). To be precise, we estimate the denominator in (2.2) with

$$\{n^2(n-1)\}^{-1}\sum_{i=1}^{n}\sum_{j<k}^{n}\{1(Y_j \geq T_i) - 1(Y_k \geq T_i)\}^2 = \{n^2(n-1)\}^{-1}\sum_{i=1}^{n}R_i(n-R_i),$$

where R_i stands for the number of Y_js such that $Y_j \geq T_i$. Combine the above estimate with (2.4) to form a slicing estimation of $\mathcal{S}(X, Y)$ as follows, and

denote $\widehat{\mathcal{S}}(X,Y)$ as

$$1-(n-1)(c-1)^{-1}\sum_{h=1}^{H}\sum_{j<l}^{c}|r_{(h,j)}-r_{(h,l)}| \Big/ \sum_{i=1}^{n}R_i(n-R_i). \qquad (2.5)$$

The complexity of calculating $\widehat{\mathcal{S}}(X,Y)$ in (2.5) is $O\{n\log(n)\}$.

In the above estimation procedure, we assume implicitly that X is continuous. If X is categorical or discrete taking H distinctive values, say, $X=1,\ldots,H$. We simply divide the random sample $\{(X_i,Y_i), i=1,\ldots,n\}$ into H slices according to the distinctive levels of X. We put the observations for X_i taking the same value in the same slice. The number of observations within each slice is not necessarily the same. We estimate $\mathrm{var}\{1(Y\geq t)\mid X\}$ within each slice, and aggregate over all H slices to form an estimate of (2.3). We omit the details from the present context.

The notion of slicing estimation is originated from Mardia et al. (1979, chapter 12) and Li (1991). We adapt this concept to estimate $\mathcal{S}(X,Y)$. If there are only two observations within each slices, namely, $c=2$, our slicing estimation is in spirit reduced to the popular rank estimation of Chatterjee (2020). A similar observation is also made by Hsing and Carroll (1992). The slicing estimation introduces an annoying tuning parameter H, or equivalently, c. It is natural to ask what kind of role, H, or equivalently, c, plays in estimation or independence testing. This amounts to studying the theoretical properties of our proposed slicing estimation.

3. The Sliced Independence Test

In this section study the asymptotic properties of the slicing estimation by assuming that T is an independent copy of Y. In Study 1 of Section 5, we shall demonstrate that $\mu(t)$ has little impact on either estimation or testing.

We define a family of real-valued functions, $x \mapsto f(t;x)$, for $t \in \mathcal{T}$, to have a uniform total variation of order r over $\mathcal{T}$, if for any finite $B > 0$,

$$\lim_{n\to\infty} n^{-r} \sup_{t\in\mathcal{T},\Pi_n(B)} \sum_{i=1}^{n} |f(t;\widetilde{X}_{(i+1)}) - f(t;\widetilde{X}_{(i)})| = 0, \tag{3.1}$$

where $\Pi_n(B)$ is a collection of all possible n-point partitions of $[-B, B]$ such that $-B \le \widetilde{X}_{(1)} \le \ldots \le \widetilde{X}_{(n)} \le B$. Condition (3.1) is indeed weaker than the uniform bounded variation condition, which requires

$$\lim_{n\to\infty} \sup_{t\in\mathcal{T},\Pi_n(B)} \sum_{i=1}^{n} |f(t;\widetilde{X}_{(i+1)}) - f(t;\widetilde{X}_{(i)})| < \infty.$$

If $f(t;x)$ has bounded first partial derivatives with respect to x, on every finite interval, then condition (3.1) holds for any $r > 0$. We further define $x \mapsto f(t;x)$ to be non-expansive in the metric of $M(x)$ on both sides of B_0, if there exists a non-decreasing real-valued function $M(x)$ and a real number $B_0 > 0$, such that for any two points, say, $\widetilde{X}_1$ and $\widetilde{X}_2$, both in $(-\infty, -B_0]$ or both in $[B_0, \infty)$,

$$|f(t;\widetilde{X}_1) - f(t;\widetilde{X}_2)| \le |M(\widetilde{X}_1) - M(\widetilde{X}_2)|. \tag{3.2}$$

Let $\varepsilon(t;X) \stackrel{\text{def}}{=} 1(Y \geq t) - s(t;X)$, and $V(t_1,t_2;X) \stackrel{\text{def}}{=} \text{cov}\{\varepsilon(X,t_1),\varepsilon(X,t_2) \mid X\}$. We assume the following two conditions on $s(t;x)$ and $V(t_1,t_2;x)$.

(C1) Assume that $x \mapsto s(t;x)$ has a uniform total variation of order $r = 1/2$ and is non-expansive in the metric of $M(x)$ on both sides of a real number $B_0 > 0$, such that $M^2(x)\text{pr}(X > x) \to 0$ as $x \to \infty$.

(C2) Suppose that $x \mapsto V(t_1,t_2;x)$ has a uniform total variation of order $r = 1$.

These conditions concern the variations and tail behaviors of $s(t;x)$ and $V(t_1,t_2;x)$, which are typically regarded as mild and widely used in literature. See, for example, Hsing and Carroll (1992), Zhu and Ng (1995); Zhu et al. (2006), Li and Zhu (2007), Lin et al. (2018) and Kong et al. (2019).

Let Y, T, T_1 and T_2 be independent copies, and " $\xrightarrow{d}$ " stand for "converges in distribution". Define $\theta_1 \stackrel{\text{def}}{=} E\{V(T_1,T_2;X)^2\}$, $\theta_2 \stackrel{\text{def}}{=} E\Big[\text{var}\{1(Y \geq T) \mid T\}\Big]$, $\sigma^2 \stackrel{\text{def}}{=} 2E\Big[\text{cov}^2\{1(Y_1 \geq T), 1(Y_2 \geq T) \mid T\}\Big]/\theta_2^2$ and $\tau^2 \stackrel{\text{def}}{=} \{\zeta_1 + 2\theta_1/(c-1)\}/\theta_2^2$, where $\zeta_1 > 0$ is defined in (S.1.2) of the on-line Supplement Material.

Theorem 1. Assume the number of observations within each slice, c, is fixed.

(i) If X and Y are independent, $\{n(c-1)\}^{1/2}\widehat{\mathcal{S}}(X,Y) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$, as $n \to \infty$. In particular, if Y is a continuous random variable, $\sigma^2 = 4/5$.

(ii) If X and Y are not independent, under Conditions (C1)-(C2), $n^{1/2}\{\widehat{\mathcal{S}}(X,Y) - \mathcal{S}(X,Y)\} \xrightarrow{d} \mathcal{N}(0,\tau^2)$, as $n \to \infty$.

Theorem 1 has several important implications. In particular, the slicing estimation is root-n consistent and asymptotically normal for an arbitrary constant $c \geq 2$. The larger c is, the smaller the asymptotic variance will be. We reject the null hypothesis H_0: X and Y are independent, if $n^{1/2}\widehat{\mathcal{S}}(X,Y)/\sigma \geq z_{1-\alpha}$, at the significance level α, where $z_{1-\alpha}$ is the $(1-\alpha)$-th quantile of standard normal. Let $\Phi(\cdot)$ be the cumulative distribution function of standard normal. The asymptotic power is $1 - \Phi\Big[\{z_{1-\alpha}\,\sigma - n^{1/2}\mathcal{S}(X,Y)\}/\tau\Big]$, which is equal to

$$\Phi\left(\theta_2\mathcal{S}(X,Y)\Big[n/\{\zeta_1 + 2\theta_1/(c-1)\}\Big]^{1/2} - \theta_2 z_{1-\alpha}\Big[4/\{5(c-1)\zeta_1 + 10\theta_1\}\Big]^{1/2}\right). \quad (3.3)$$

It is a strictly monotone increasing function of c. In other words, the larger c is, the more powerful our proposed test will be. The rank test of Chatterjee (2020) corresponds to the sliced independence test with $c = 2$, which indicates that, in general, the sliced independence test is more powerful than the rank test.

The asymptotic power function in (3.3) inspires us to ask whether we

can enhance power performance of the sliced independence test if we allow $c \to \infty$ as $n \to \infty$. Towards this goal, we assume the following conditions.

(C1*) Assume that $x \mapsto s(t;x)$ has a uniform total variation of order $r > 0$ and is non-expansive in the metric of $M(x)$ on both sides of a real number $B_0 > 0$ such that $M^{2+b}(x)\mathrm{pr}(X > x) \to 0$ for $b > 0$, as $x \to \infty$.

(C2*) Let $c = O(n^\alpha)$, where $\alpha = 1/2 - \max\{r, 1/(2+b)\}$.

These conditions are even weaker than (C1) and (C2). Letting c diverge to infinity, we relax the smoothness condition on $x \mapsto s(t;x)$ slightly and completely avoid assuming smoothness conditions on $x \mapsto V(t_1, t_2; x)$. If $x \mapsto s(t;x)$ is L-Lipschitz continuous and X is sub-Gaussian or has a bounded support, then conditions (C1*) and (C2*) hold for any $r > 0$, $b > 0$, $M(x) = Lx$ and $c = o(n^{1/2})$.

Define $\tau_*^2 \stackrel{\text{def}}{=} \zeta_1/\theta_2^2$, which is smaller than $\tau^2 = \{\zeta_1 + 2\theta_1/(c-1)\}/\theta_2^2$.

Theorem 2. Assume the number of observations within each slice, c, diverges.

(i) If X and Y are independent and $c = o(n)$, $(nc)^{1/2}\widehat{\mathcal{S}}(X,Y) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, as $n, c \to \infty$. In particular, $\sigma^2 = 4/5$ if Y is continuous.

(ii) If X and Y are not independent, under Conditions (C1*)-(C2*),

$$n^{1/2}\{\widehat{\mathcal{S}}(X,Y)-\mathcal{S}(X,Y)\} \xrightarrow{d} \mathcal{N}(0,\tau_*^2) \text{ as } n,c\to\infty.$$

The sliced estimation converges at the faster rate of $(nc)^{-1/2}$ than the rank estimate (Chatterjee, 2020). The asymptotic variance decreases as c increases. At the significance level α, the asymptotic power is

$$\Phi\Big[\theta_2\mathcal{S}(X,Y)(n/\zeta_1)^{1/2}-\theta_2 z_{1-\alpha}\{4/(5c\zeta_1)\}^{1/2}\Big],$$

which again increases with c. The power improvement of the sliced independence test over the rank test (Chatterjee, 2020) is substantial when c diverges to infinity.

4. An Extension to Multivariate Control Variables

In this section we generalize the concept of slicing through the K-means clustering procedure (MacQueen, 1967), to account for the presence of multivariate control variables. We use the random vector $\mathbf{x}=(X_1,\ldots,X_p)^{\mathrm{T}}\in\mathbb{R}^p$ to replace the univariate control variable X in $\mathcal{S}(X,Y)$, which leads to

$$\mathcal{S}(\mathbf{x},Y) \overset{\text{def}}{=} \int \mathrm{var}\{s(t;\mathbf{x})\}\mathrm{d}\mu(t)\Big/\int \mathrm{var}\{1(Y\geq t)\}\mathrm{d}\mu(t),$$

where $s(t;\mathbf{x})\overset{\text{def}}{=}\mathrm{pr}(Y\geq t\mid\mathbf{x})$. Similarly, we can verify that $\mathcal{S}(\mathbf{x},Y)$ equals

$$1-\int E\Big[\mathrm{var}\{1(Y\geq t)\mid\mathbf{x}\}\Big]\mathrm{d}\mu(t)\Big/\int \mathrm{var}\{1(Y\geq t)\}\mathrm{d}\mu(t). \qquad (4.1)$$

Both $\mathcal{S}(X, Y)$ and $\mathcal{S}(\mathbf{x}, Y)$ share the zero-independence equivalency property at the population level. However, the slicing procedure used to estimate $\mathcal{S}(X, Y)$ cannot be directly used to estimate $\mathcal{S}(\mathbf{x}, Y)$, unless the sorting algorithm is delicately adapted to account for multivariate observations.

Suppose a random sample $\{(\mathbf{x}_i, Y_i), i = 1, \ldots, n\}$ is available. Instead of using the slicing procedure, in this section we propose to use the K-means clustering approach (MacQueen, 1967) to partitioning the random sample into H clusters, according to the realizations of the control variables, $\{\mathbf{x}_i, i = 1, \ldots, n\}$. We estimate $\text{var}\{1(Y \geq t) \mid \mathbf{x}\}$ within each cluster, and aggregate the resultant estimates to form an estimate of $\mathcal{S}(\mathbf{x}, Y)$.

We implement the K-means clustering approach according to $\{\mathbf{x}_i, i = 1, \ldots, n\}$ only, which proceeds as follows.

1. Randomly choose H points in $\{\mathbf{x}_i, i = 1, \ldots, n\}$ as the initial centers.

2. For each center, identify the points in $\{\mathbf{x}_i, i = 1, \ldots, n\}$ that are "closer" to it than any other center. Update the centers of all clusters.

3. Iterate the above step until convergence.

4. Delete the clusters with a single data point and repeat all above steps.

5. Either (a) absorb the data points in the previously deleted clusters

into the cluster with the nearest center and terminate, or (b) terminate without the data points in the deleted clusters.

The last two steps are implemented to avoid the presence of clusters with a single data point. We implement this K-means clustering approach to partitioning the whole random sample, $\{(\mathbf{x}_i, Y_i), i = 1, \ldots, n\}$, into H clusters, according to the realizations of the control variables, $\{\mathbf{x}_i, i = 1, \ldots, n\}$. The K-means clustering approach can not guarantee that each cluster contains an equal number of observations. Therefore, we assume that the h-th cluster consists of n_h observations, for $h = 1, \ldots, H$. We re-index the random sample as $\{(\mathbf{x}_{(h,j)}, Y_{(h,j)}), j = 1, \ldots, n_h, h = 1, \ldots, H\}$, and estimate

$$\int E\Big[\mathrm{var}\{1(Y \geq t) \mid \mathbf{x}\}\Big]\mathrm{d}\mu(t)$$

in (4.1) with a weighted summation as follows,

$$n^{-1}\sum_{i=1}^{n}\sum_{h=1}^{H}(n_h/n)\left[\sum_{j<l}^{n_h}\{1(Y_{(h,j)} \geq T_i) - 1(Y_{(h,l)} \geq T_i)\}^2/\{n_h(n_h-1)\}\right].$$

Recall that $r_{(h,j)}$ stands for the number of T_is such that $Y_{(h,j)} \geq T_i$, for $i = 1, \ldots, n$. It is straightforward to verify that the above display equals

$$n^{-2}\sum_{h=1}^{H}\sum_{j<l}^{n_h}|r_{(h,j)} - r_{(h,l)}|/(n_h - 1).$$

This motivates us to define that

$$\widehat{\mathcal{S}}(\mathbf{x}, Y) \stackrel{\mathrm{def}}{=} 1 - \sum_{h=1}^{H}\sum_{j<l}^{n_h}|r_{(h,j)} - r_{(h,l)}|/(n_h - 1) \Big/ \sum_{i=1}^{n} R_i(n - R_i)/(n-1),$$

where R_is are defined in Section 2. We further define

$c_n^{-1} \stackrel{\text{def}}{=} \sum_{h=1}^{H} n_h/\{n(n_h-1)\}$, which equals $1/(c-1)$ if $n_h = c$, for all $h = 1, \ldots, H$.

To study the asymptotic behavior of $\widehat{\mathcal{S}}(\mathbf{x}, Y)$ when $\mathbf{x}$ and Y are not independent, we assume the following two conditions.

(C3) There exist two positive constants, C_1 and C_2, such that $\text{pr}(\|\mathbf{x}\| > t) \leq C_1 \exp(-C_2 t^2)$, for all $t \in \mathbb{R}$.

(C4) There exists a positive constant C_3 such that $|s(t; \mathbf{x}_1) - s(t; \mathbf{x}_2)| \leq C_3 \|\mathbf{x}_1 - \mathbf{x}_2\|$, for all $t \in \mathbb{R}$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^p$.

Condition (C3) requires $\mathbf{x}$ be sub-Gaussian, and condition (C4) concerns the smoothness of $x \mapsto s(t; x)$.

Theorem 3. Assume the number of slices H, diverges.

(i) If $\mathbf{x}$ and Y are independent, $(nc_n)^{1/2}\widehat{\mathcal{S}}(\mathbf{x}, Y) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, as $n \to \infty$. In particular, $\sigma^2 = 4/5$ if Y is continuous.

(ii) If $\mathbf{x}$ and Y are not independent and $H = O(n^\delta)$ for some $0 < \delta \leq 1$, under conditions (C3)-(C4), $\widehat{\mathcal{S}}(\mathbf{x}, Y)$ converges in probability to $\mathcal{S}(\mathbf{x}, Y)$, and accordingly, $(nc_n)^{1/2}\widehat{\mathcal{S}}(\mathbf{x}, Y) \to \infty$, as $n \to \infty$.

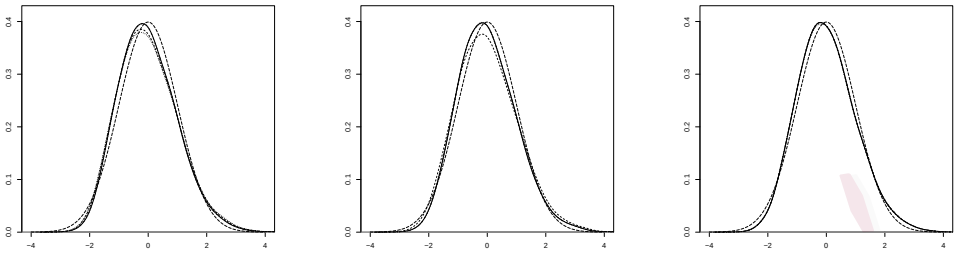
5. Numerical Studies

5.1 Simulations

We first demonstrate the finite-sample performance of the slicing estimation and the sliced independence test through simulations.

Study 1. In the definition of (2.1), there involves a probability measure $\mu(t)$. We evaluate the effect of $\mu(t)$ on the asymptotic null distribution. We draw X_i's and Y_i's independently from uniform, standard normal and $t(1)$ distributions, respectively. We fix $n = 1024$ and $c = 32$. We consider four choices for $\mu(t)$: (i) $T_i = Y_i$; (ii) $T_i \sim \mathcal{N}(0,1)$; (iii) $T_i \sim t(1)$; (iv) T_is is a bootstrap sample of Y_is. We replicate each scenario 10,000 times, and draw the kernel density functions of $Z \stackrel{\text{def}}{=} n^{1/2}\widehat{\mathcal{S}}(X,Y)/\sigma$ in Figure 1. It can be clearly seen that, all kernel densities are pretty close to the reference curve $\mathcal{N}(0,1)$. This is indeed not surprising in that the indicator functions in the slicing estimation (2.4) only vary at Y_is. It also indicates the asymptotic null distribution does not depend upon the parent distribution of (X,Y).

Study 2. Next we evaluate how the number of observations within each slice, c, affects the resulting slicing estimation. We generate $\varepsilon \sim N(0,1)$ and $X \sim \text{uniform}(-1,1)$ independently, and consider the following six dependent structures.



(A) $(X, Y) \overset{\text{iid}}{\sim} \text{unif}(-1, 1)$ (B) $(X, Y) \overset{\text{iid}}{\sim} t(1)$ (C) $(X, Y) \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$

Figure 1: The kernel densities with different choices for $\mu(t)$s, (i) $T_i = Y_i$ (solid), (ii) $T_i \overset{\text{iid}}{\sim} N(0, 1)$ (dashed), (iii) $T_i \overset{\text{i.i.d.}}{\sim} t(1)$ (dotted), (iv) T_is is a bootstrap sample (dotdash). The density function of standard normal is used as a reference curve (longdash).

(A) Log: $Y = C_1 \log(X^2) + \lambda\varepsilon$.

(B) Circular: $Y = Z(1 - X^2)^{1/2} + \lambda C_2\varepsilon$, where Z is independent of X, and $\text{pr}(Z = \pm 1) = 1/2$.

(C) W-shaped: $Y = |X + 0.5|1(X < 0) + |X - 0.5|1(X \geq 0) + \lambda C_3\varepsilon$.

(D) Sinusoid: $Y = \cos(C_4\pi X) + 3\lambda\varepsilon$.

(E) Doppler: $Y = \{X^2(1 - X^2)\}^{1/2} \sin(1.05\pi/X^2) + \lambda C_5\varepsilon$.

(F) HeaviSine: $Y = 4\sin(4\pi X^2) - \text{sign}(X^2 - 0.3) - \text{sign}(0.72 - X^2) + \lambda C_6\varepsilon$.

The structures are also used in similar context. See, for example, Chatterjee (2020), Heller et al. (2013), Kong et al. (2019) and Donoho and Johnstone

(1995). In this study, we fix $(C_1, \ldots, C_6) = (0.05, 0.9, 0.75, 8, 1.5, 24)$, $\lambda = 0.7$, $n = 512$, and vary $c \in \{2, 4, 8, 16\}$. We replicate each scenario 10,000 times. The boxplots of the resultant slicing estimation with different c values are shown in Figure 2. It can be clearly seen that, in terms of the median values of the slicing estimates, $\widehat{\mathcal{S}}(X, Y)$ converges to $\mathcal{S}(X, Y)$ across all scenarios. However, the variances of $\widehat{\mathcal{S}}(X, Y)$ decrease substantially as c increases, which echoes our theoretical results in Theorems 1 and 2.

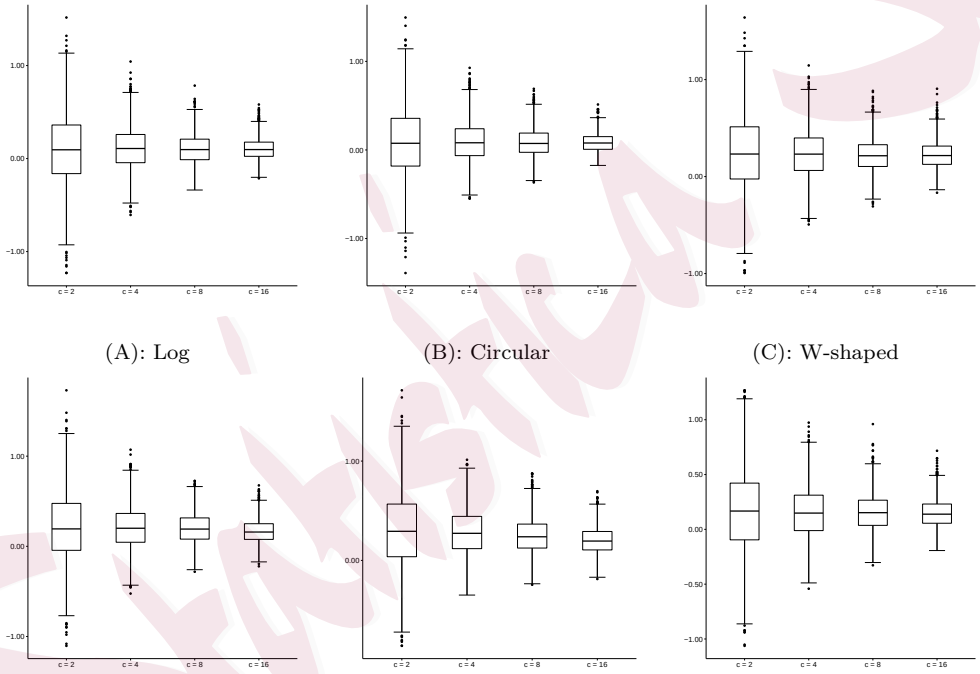


Figure 2: Boxplots of the $\widehat{\mathcal{S}}(X, Y)$, with $c \in \{2, 4, 8, 16\}$ in Study 2.

Study 3. We use the dependence structures in Study 2 to compare the

power performance of our proposed sliced independence test with those of the modified Blum-Kiefer-Rosenblatt correlation test (Zhou and Zhu, 2018), the distance correlation test (Székely et al., 2007), the multivariate test of Heller et al. (2013) and the composite coefficient of determination test of Kong et al. (2019). We remark here that, the composite coefficient of determination is estimated through kernel smoothing, which is computationally intensive. We use 200 random permutations to approximate the asymptotic null distributions for the last three tests. We fix $n = 512$, and vary $c \in \{2, 4, 8, 16\}$ and $\lambda = 0 : 0.1 : 1$. We report the empirical powers at the significance level $\alpha = 0.05$ in Figure 3. Our proposal is apparently superior to its competitors in the oscillatory cases such as the Sinusoid, the HeaviSine and the Doppler. We can also see that, as c increases, the empirical powers of our proposed test are enhanced accordingly. This again confirms theoretical results in Theorems 1 and 2.

Study 4. Next we compare the running time of several popular independence tests in Study 3. We implement the multivariate test of Heller et al. (2013) through the R package `HHG`, and the composite coefficient of determination test with the R code provided by Dr Zhong Wei, one of the authors of Kong et al. (2019). Implementing these two tests is very time-consuming. We terminate them in case their implementations take more

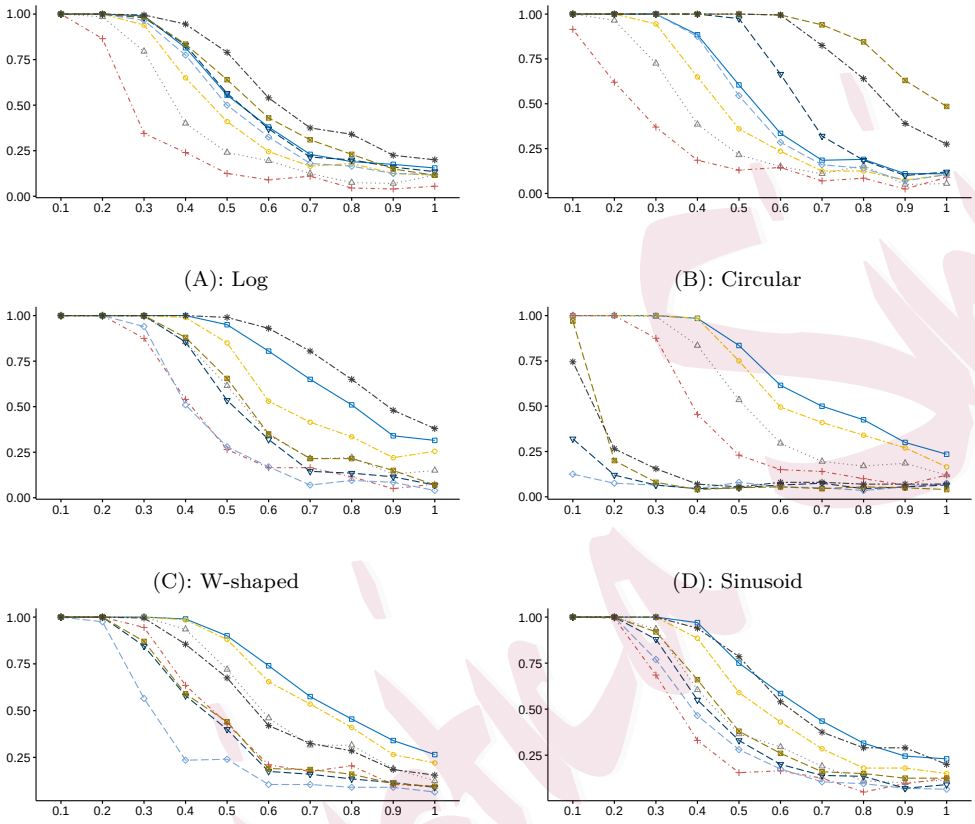


Figure 3: The empirical powers of four independence tests: the sliced independence test with $c = 2$ ($+$), $c = 4$ ($\triangle$), $c = 8$ ($\circ$) and $c = 16$ ($\square$), respectively, the distance correlation test ($\Diamond$), the modified Blum-Kiefer-Rosenblatt correlation test ($\bigtriangledown$), the multivariate test of Heller et al. (2013) ($\boxtimes$) and the composite coefficient of determination test of Kong et al. (2019) ($*$). The horizonal axis stands for λ, and the vertical axis stands for the empirical powers.

than 30 minutes. We also include three versions of the distance correlation test into comparison, which are available at the R packages `energy`, `kpcalg` and `dcov`, respectively. The first is the classic version of the distance correlation test, which is referred to as DC_1 in the caption of Table 1. We refer to the last two versions as DC_2 and DC_3, respectively. For the DC_2 test, the asymptotic null distribution of the distance correlation test is approximated with Gamma approximation in the R package `kpcalg`, where the function `dcov.gamma()` is used. In the DC_3 test, the distance correlation is estimated with the algorithm proposed by Huo and Székely (2016), which is computationally very efficient. To further speed up the DC_3 test, we also use Gamma approximation in the R package `dcov`. In the sliced independence test, we fix the number of slices $c = 16$ and vary the sample size $n \in \{128, 256, 512, 1024, 2048, 4096, 8192\}$. We summarize the averages of the wall-clock time in Table 1 based on 100 replications. It can be clearly seen that, the sliced independence test runs the fastest, followed by the DC_3 test. These two tests have the smallest order of complexity and thus are much more numerically efficient than all other competitors.

Next we conduct a simulation study when the control variables are multivariate. Instead of using the slicing estimation procedure, we use the K-means clustering approach to classify the observations into H clusters.

Table 1: The average wall-clock time (in seconds) over 100 replications for three versions of the distance correlation test (DC_1, DC_2 and DC_3), the modified Blum-Kiefer-Rosenblatt correlation test (MBKR), the multivariate test of Heller et al. (2013) (HHG), the composite coefficient of determination test (CCD) and the sliced independence test (SIT).

n	DC_1	DC_2	DC_3	MBKR	HHG	CCD	SIT
128	0.006	0.032	0.0004	0.004	0.089	1.395	0.00014
256	0.034	0.037	0.0014	0.039	0.248	9.995	0.00017
512	0.109	0.559	0.0034	0.184	0.849	45.934	0.00031
1024	0.812	0.989	0.0174	1.684	3.834	210.100	0.00056
2048	4.540	3.502	0.0708	13.253	14.580	575.938	0.00114
4096	15.823	12.305	0.2249	116.463	$>$ 30mins	$>$ 30mins	0.00215
8192	63.869	51.024	1.3841	899.942	$>$ 30mins	$>$ 30mins	0.00434

Study 5. Let $\mathbf{x} = (X_1, \ldots, X_5)^{\mathrm{T}}$. We generate X_ks independently from the uniform distribution defined on the interval $[-1, 1]$, for $k = 1, \ldots, 5$, and ε from standard normal distribution. Denote $m(\mathbf{x}) \stackrel{\text{def}}{=} (X_1 + \ldots + X_5)/5$. We use the simulated examples used in Study 2, but with X replaced by $m(\mathbf{x})$ throughout. We set $(C_1, \ldots, C_6) = (0.05, 0.05, 0.75, 2, 0.5, 24)$ in this study, and vary $H \in \{8, 16, 32, 64\}$ for our proposal. We include the distance correlation test (Székely et al., 2007) and the multivariate test of Heller et al. (2013) into our comparison. The sample size is fixed at $n = 512$. We vary $\lambda = 0 : 0.1 : 1$, and replicate each experiment 1000 times to compare the power performance of different proposals. The significance level is fixed

at $\alpha = 0.05$, and the simulated results are summarized in Figure 4. It can be clearly seen again that, in this simulation study our proposal is still more powerful than other tests except for the Doppler case.

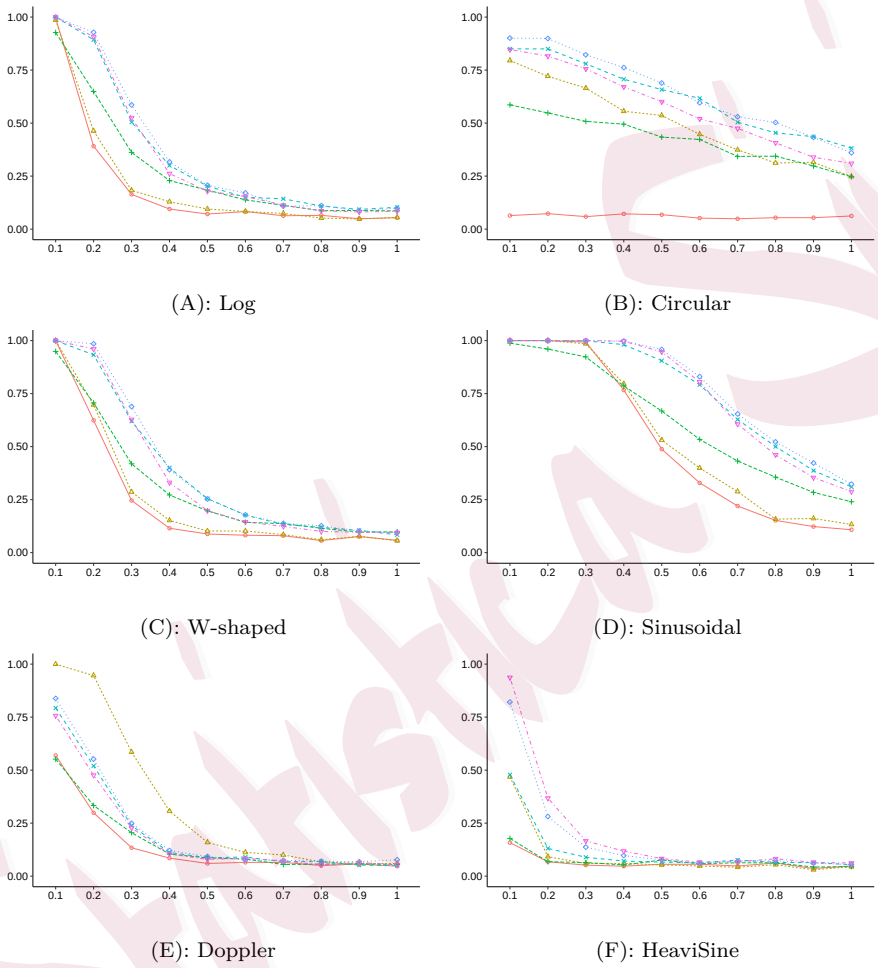


Figure 4: The empirical powers of four independence tests: the sliced independence test with $H = 8$ ($+$), $H = 16$ ($\times$), $H = 32$ ($\Diamond$) and $H = 64$ ($\bigtriangledown$), respectively, the distance correlation test ($\circ$) and the multivariate test of Heller et al. (2013) ($\bigtriangleup$). The horizonal and vertical axes stand respectively for λ and the empirical powers.

5.2 Real Data Analysis

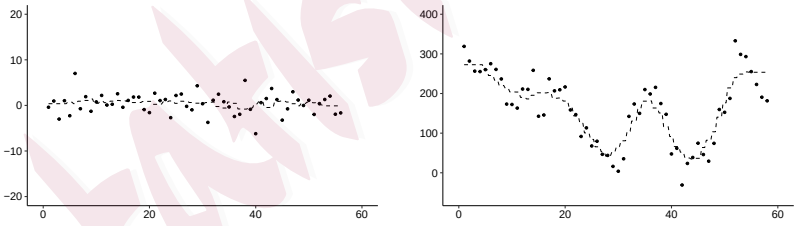
We apply the sliced independence test to an astronomical dataset. The Photometric LSST Astronomical Time-series Classification Challenge (PLAsTiCC) dataset is available on `https://www.kaggle.com/c/PLAsTiCC-2018`. This is a simulated dataset and consists of 15 classes. We consider the r band in class 65 and 88 only. There are 981 objects in class 65 and 370 objects in class 88. For each object, the number of observations, n, ranges from 10 to 60. We remove the objects with less than 30 observations, leaving 313 objects in class 65 and 119 objects in class 88. The target is to examine whether the intensity of brightness (Y) varies over time (X) for each object in these two classes.

We apply the sliced independence test with $c = 2$ and $c = 4$, the distance correlation test (Székely et al., 2007), the modified Blum-Kiefer-Rosenblatt correlation test (Zhou and Zhu, 2018) and the multivariate test of Heller et al. (2013) to this dataset. In Table 2, we report the number of times that we reject the null hypothesis H_0: X and Y are independent, at the significance level $\alpha = 0.05$. The intensity of brightness does not change over time for more than 95% of the objects in class 65. By contrast, all independence tests strongly indicates that, for almost all objects in class 88, the intensity of brightness changes over time.

Table 2: The number of times that the null hypothesis, H_0: X and Y are independent, is rejected at the significance level $\alpha = 0.05$. The distance correlation test, the modified Blum-Kiefer-Rosenblatt correlation test, the multivariate test of Heller et al. (2013) and sliced independence test are denoted by DC, MBKR, HHG and SIT, respectively.

class	DC	MBKR	HHG	SIT $c = 2$	SIT $c = 4$
65	16	11	17	12	10
88	115	118	119	119	119

We present the intensity of brightness of two representative objects, one from each class, in Figure 5, which echoes the results in Table 2. In class 65, most objects exhibit similar pattern that the brightness intensity remains unchanged over time. By contrast, for most objects in class 88, the brightness intensity varies over time.



(A) class 65 (B): class 88

Figure 5: The scatter-plots of the intensity of brightness (on the vertical axis) over time (on the horizontal axis) for one representative object in each class. The dashed line is fitted with k-nearest neighbor regression ($k = 7$).

6. Concluding Remarks

We introduce a slicing procedure to estimate a popular measure of nonlinear dependence. The resultant sliced independence test encompasses the rank test as a special case, has almost the minimal computational complexity, and is asymptotically distribution-free. We show that, as the number of observations within each class increases, the asymptotic variance of the slicing estimation decreases and the power of the independence test improves. In addition, the size performance of the sliced independence test is insensitive to the number of slices. The slicing estimation is consistent for a wide range of slice numbers. We also generalize the concept of slicing through the K-means clustering, to account for multivariate control variables. How to further generalize this concept is challenging if both random variables are multivariate. Investigations along this direction are under way.

Supplementary Materials

The proofs of Theorems 1-3 are relegated to the Supplementary Material.

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Center for Applied Statistics and Institute of Statistics and Big Data, Renmin University of China, Beijing 100872, China.

E-mail: yzhang97@ruc.edu.cn, chency1997@ruc.edu.cn and zhu.liping@ruc.edu.cn