Energy driven pattern formation in a non-local Cahn-Hilliard energy

by

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Abstract

Résumé en anglais

We study the asymptotic behavior of a non-local Cahn-Hilliard energy, often referred to as the Ohta-Kawasaki energy in the context of di-block copolymer melts. In that model, two phases appear, and they interact via a non-local Coulombic type energy. We focus on the regime where one of the phases has very small volume fraction, thus creating "droplets" of that phase in a sea of the other phase. We compute the Γ-limit of the leading order energy and yield averaged information for almost minimizers, namely that the density of the minority phase forms droplets which are almost spherical, with the same radii, and are uniformly distributed throughout the domain. We then derive a next order Γ-limit energy which determines the geometric arrangement of the droplets. Without thus appealing at all to the Euler-Lagrange equation, we establish here for all configurations which have "almost minimal energy," the asymptotic roundness and radius of the droplets, and the fact that they asymptotically shrink to points whose arrangement should minimize this energy, in some averaged sense; this leads to expecting to see triangular lattices of droplets. In addition, we prove that the density of droplets of a priori non-minimizing stationary points of the energy is also aysmptotically uniform even in dimensions $n \geq 2$.

We also study a non-local isoperimetric problem in $\mathbb{R}^2$. We show that the connected critical points are determined by perimeter alone, under mild assumptions on the boundary, in the small energy/mass regime. These results differ from the recent results of Julin and Muratov-Knupfer in that they concern general critical points rather than global minimizers to the energy, making it a non-local extension of the well known fact by Alexandrov that the only compact, connected, constant curvature curve in the plane is the circle. Our method demonstrates that not only does the perimeter dominate the non-locality when minimizing this energy, but also that the change in perimeter slaves to the change of the non-local term in this scaling regime.

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Introduction

This thesis focuses on problems from the Calculus of Variations, focusing on the energetic description of pattern formation in systems with *competing short and long range interactions*, in particular the Ohta-Kawasaki energy (see [87]) which is a canonical mathematical model in the studies of energy-driven pattern formation. Closely related to the Ohta-Kawasaki energy, this thesis addresses a particular geometric variational problem which concerns the critical points of a non-local isoperimetric functional [56], which is a non-local perturbation to the classical isoperimetric problem.

After an overview of the two main types of problems considered in this thesis and a summary of our main results, we will present a more detailed, yet still heuristic, description of our results with future research directions in Section I.1 and I.2. Section I.1 will describe our main results related to energy driven pattern formation, which are the subject of Chapters 1, 2 and 4. Section I.2 will describe our main results concerning the critical points of a non-local isoperimetric functional, which is the subject of Chapter 3. As each chapter has its own introduction, this introduction to the thesis serves to provide a general overview and a brief, informal description of the results contained in each chapter.

Pattern formation driven by energy minimization

In this thesis, we use methods in the Calculus of Variations to address patterns observed for a non-local Cahn-Hilliard energy with Coulombic repulsion, also known as the Ohta-Kawasaki model in connection with di-block copolymer systems (see Sections I.1.1 and I.1.2 for definitions of these energies). In that model, two phases appear, which interact via a non-local Coulombic type energy. Physically, this can be viewed as an example of energy-driven pattern formation driven by competing short-range and long-range forces. Mathematically, it amounts to a competition between perimeter and a Coulombic repulsive term. The interplay between these competing terms turns out to be quite delicate, resulting in highly ordered patterns consisting of (approximately equally sized) droplets (see Figure 1a.). Despite the abundance of physical systems where this model appears and the apparent simplicity of the model, the mathematical theory remains in its infancy.

Our goal was to provide rigorous mathematical evidence for this observed pattern formation in the Ohta-Kawsaki model. We focused on the regime where one of the phases has very small volume fraction, thus creating small "droplets" of that phase in a "sea" of the other phase (cf. Figure 1a.). There is strong numerical evidence of this pattern formation [30, 79], some mathematical analysis which hints at the periodic structures [2, 108], and a study of exact minimizers in 2D [80]; however a full variational characterization of the energy describing these patterns in this scaling regime is lacking. The natural way to express such a result is via the very robust scheme of Γ-convergence (see [14] for instance). This approach however is rendered difficult due to the emergence of more than two well-separated spatial scales appearing in the Ohta-Kawasaki model. To this end, we have the following three results:

- *Leading order energy. Droplet density*: In Chapter 1 (see [54] for main paper) we prove that, after a suitable re-scaling, the leading order term in the Γ-limit expansion of the Ohta-Kawasaki energy functional is a quadratic functional of the limit droplet density (see (I3)). A consequence of our results is that minimizers (or almost minimizers) of the energy form droplets of the minority phase which are almost all asymptotically round, have the same radius, and are uniformly distributed in the domain. This result is outlined in section I.1.4 of this introduction, and presented in Chapter 1. This describes joint work with Cyrill Muratov and Sylvia Serfaty.

- *Next order correction. Renormalized energy*: In Chapter 2 (see [55] for main paper), by a non-trivial adaptation of the methods in [100] for the magnetic Ginzburg-Landau model, we derive a second order Γ-limit expansion (see (I7)) for a "sharp-interface version" of the energy (see section I.1.2), which is a Coulombic renormalized energy for an infinite number of point charges in the plane, first derived in [100]. As a consequence, we establish for all configurations which have "almost minimal energy" at a suitable scale, the asymptotic roundness and radius of the droplets. We also show that the droplets asymptotically shrink to points whose geometric arrangement should minimize this Coulombic energy in some averaged sense; this leads one to expect to see triangular lattices of droplets (see Figure 1b.), which is widely believed to be the ground state for this model. This result is outlined in Section I.1.5 and is the subject of Chapter 2. This describes joint work with Cyrill Muratov and Sylvia Serfaty.

- *Non-minimizing stationary points*: In Chapter 4, we address the asymptotic behavior of non-minimizing stationary points in dimensions $n \geq 2$ left open by the study of the Γ-convergence of the energy established in Chapters 1 and 2, which provides information only for almost minimizing sequences when $n = 2$. In particular, we prove that (asymptotically) stationary points satisfy a force balance condition which implies that the minority phase distributes itself uniformly in the background majority phase. Our proof uses and generalizes the framework of Sandier-Serfaty [99,102], used in the context of stationary points of the Ginzburg-Landau model, to higher dimensions. When $n = 2$ we also are able to conclude that the droplets in the sharp interface energy become asymptotically round when the number of droplets is constrained to be finite and have bounded isoperimetric deficit. As part of our result, we establish the $C^{3,\alpha}$ regularity of the reduced boundary of stationary points of the sharp-interface energy, showing stationary points satisfy the Euler-Lagrange equation strongly on the reduced boundary. The part of this chapter concerning regularity of the reduced boundary is joint work with Alexander Volkmann.

The above analysis has many close connections to the magnetic Ginzburg-Landau model, where similar results exist due to Sandier and Serfaty [97–100]. A key difference is that the droplets, unlike the vortices in Ginzburg-Landau, are not quantized. This creates major technical difficulties which we must overcome.

A non-local isoperimetric problem

In Chapter 3, we study a non-local perturbation to the classical isoperimetric problem in the plane [56] (see (I9)). Physically, the problem can be seen as an example of energy driven minimization induced by a competition between surface tension and the self-interaction energy of a uniform charge distribution. Recently there has been much interest in the existence and characterization of minimizers of this problem [65,81,82]. It has been shown that the unique global minimizer, is once again the ball, for sufficiently small masses. Furthermore, minimizers fail to exist for large enough masses [81,82]. Left open in this analysis is the characterization of *non-minimizing* critical points. While there are results characterizing critical points of the perimeter (ie. CMC surfaces) [3,66], and some results characterizing those of the non-local self interaction energy (ie. equipotential surfaces) [43,91], there is so far no rigorous analysis concerning the characterization of critical points for the interpolation of the two. Such a problem can be seen as an example of an over-determined boundary value problem, which have been studied extensively, and which originated with the pioneering work of Serrin [105]. We partially resolve this question in the plane, showing the only connected critical point is the ball under mild assumptions on the boundary in the small mass/energy regime. This result is outlined in Section I.2 and is the subject of Chapter 3.

We now present a more detailed description of our results concerning energy driven pattern formation (Section I.1) and the non-local isoperimetric problem (Section I.2).

I.1 Pattern formation driven by energy minimization

Energy-driven pattern formation is a ubiquitous phenomenon seen in many settings. Examples include microstructures created by coherent phase transitions, folding patterns seen in thin elastic sheets, dislocation patterns seen in crystal plasticity and vortices appearing in type-II superconductors. In these and other areas, we strive to understand why structures seen in nature are preferred by the relevant variational principles.

One method to obtain such information is by studying necessary conditions for minimizers of the energy. This is generally quite difficult, however, as it requires solving a non-linear partial differential equation. Moreover it provides no information about 'almost minimizers' whose energy is close, but not equal, to the minimal energy. A more robust technique which provides information about such configurations is the method of Γ-convergence [14], which requires developing methods for obtaining upper and lower bounds on the energy. Upper bounds are often easy, using the patterns seen in nature. Lower bounds are more difficult, and the search for an ansatz-free lower bound is often the heart of the matter.

We used the framework of Γ-convergence to study the Ohta-Kawasaki energy and its sharp interface analogue (Sections I.1.1 and I.1.2). This resembles the viewpoint adopted in studies of plasticity [49, 119], micromagnetics [69], type-II superconductors [98–100] and thin films [39]. Our results concern both the original diffuse interface version of the Ohta-Kawasaki energy (Section I.1.1) and its sharp interface version (Section I.1.2). We present the existing results concerning these energies (Section I.1.3), then my three main results (Sections I.1.4, I.1.5 and I.1.6) and finally future directions (Section I.1.7).

I.1.1 The diffuse-interface Ohta-Kawasaki energy

The energy functional has the following form:

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x, y)(u(y) - \bar{u})\, dx\, dy, \tag{I1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $V(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg-Landau potential $V(u) = \frac{1}{4}(1 - u^2)^2$, $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, i.e.,

G_0 solves

$$-\Delta G_0(x,y) = \delta(x-y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x,y)\,dx = 0, \tag{I.1.1}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|}\int_\Omega u\,dx = \bar{u}. \tag{I.1.2}$$

We study (I1) specifically when $\Omega = \mathbb{T}^n := [0,1)^n$ is the n-dimensional flat torus, and we are interested in the regime where $\epsilon \ll 1$ and $\bar{u} \approx -1$. The parameter $\epsilon > 0$ in (I1) determines both the scale of the short-range interaction and the interfacial energy between the regions with different values of u when ϵ is sufciently small. When ϵ is sufficiently small, one begins to observe highly ordered patterns consisting of domain structures (see Fig. 1a)), which are of particular physical interest. These patterns consist of regions in which u is close to one of the minima of the potential V, separated by narrow domain walls.

I.1.2 The sharp-interface Ohta-Kawasaki energy

The following “sharp interface version” of (I1) was introduced by Muratov [80] when $\Omega = \mathbb{T}^n$:

$$E^\epsilon[u] = \epsilon \int_{\mathbb{T}^n} |\nabla u| dx + \iint_{\mathbb{T}^n \times \mathbb{T}^n} (u(x) - \bar{u}) G(x-y)(u(y) - \bar{u}) dx\, dy, \qquad \text{(I2)}$$

where now $u \in \mathcal{A}$ defined as

$$\mathcal{A} := \{u \in BV(\mathbb{T}^n; \{-1, +1\})\}, \qquad \text{(I.1.3)}$$

G is now the *screened* Green's potential

$$-\Delta G + \kappa^2 G = \delta(x-y) \text{ on } \mathbb{T}^n,$$

with $\kappa = 1/\sqrt{V''(1)}$. The charge neutrality condition (cf. equation (4.1.3)) is no longer imposed, i.e. $\int_{\mathbb{T}} u \neq \bar{u}$. This is related to the fact that the charge of the minority phase is expected to partially redistribute itself into the majority phase to ensure screening of the induced non-local field (see Chapter 1 for a more detailed discussion). In [80] it is shown (with V appropriately scaled) that

$$\lim_{\epsilon \to 0} \frac{\min \mathcal{E}^\epsilon}{\min E^\epsilon} = 1.$$

We study the behavior of (I1) and (I2) in the *small volume fraction regime*, ie.

$$\bar{u} = -1 + o_\epsilon(1) \text{ as } \epsilon \to 0.$$

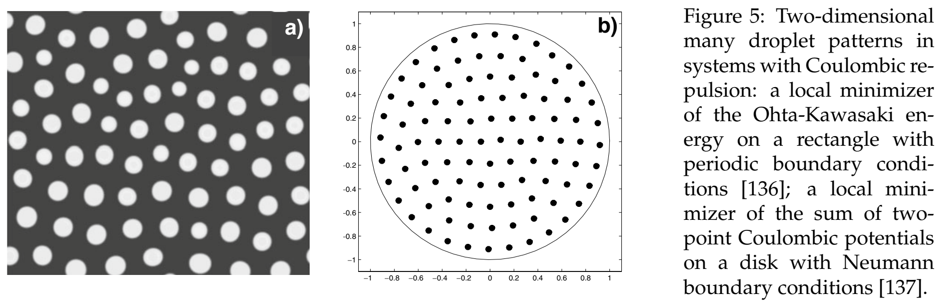

Figure 1: Local minimizers of the Ohta-Kawasaki energy and renormalized energy (taken from [79, 93]).

The following results were known prior to this work.

I.1.3 Existing results

- The results of [2] and [108] establish equidistribution of energy of minimizers in any dimension for (I2) and (I1) respectively. More precisely, they show that by blowing up coordinates at a suitable scale, and averaging the energy over larger and larger balls, one obtains the same averaged energy regardless of the blowup center. This hints at expecting a periodic structure for exact minimizers, albeit in an averaged sense.

- In [28, 29], after a suitable rescaling, it is shown that when the number of droplets is constrained to be finite, the functionals (I1) and (I2) Γ-converge to a Coulombic interaction energy for a finite number of point charges, which determines the locations and magnitudes of the limiting droplets.

- In [80], the asymptotics of (I1) of exact minimizers was studied in the limit $\epsilon \to 0$. Here rather strong information was obtained, showing that the droplets all converge uniformly to balls with the same radius which are uniformly distributed throughout the domain.

I.1.4 Main result I: Leading order energy. Droplet density

In Chapter 1, our main result (cf. **Theorem 1**) states that when $n = 2$ we have:

$$\epsilon^{-4/3}|\ln\epsilon|^{2/3}E^{\epsilon} \xrightarrow{\Gamma} E^{0}(\mu) := \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^{2}}\right)\int d\mu + \iint_{\mathbb{T}^{2}\times\mathbb{T}^{2}} \mu(y)G(x-y)\mu(x)dx\,dy, \tag{I3}$$

where the topology of convergence is weak convergence in the sense of measures of the "renormalized droplet density":

$$\mu^{\epsilon} := \epsilon^{-4/3}|\ln\epsilon|^{-2/3}(1+u^{\epsilon}) \overset{*}{\rightharpoonup} \mu, \tag{I4}$$

and $\bar{\delta}$ is a constant corresponding to the rescaled volume fraction of minority phase. This statement requires two parts, the first of which is an ansatz free lower bound (cf. **Part i) of Theorem 1**) of the left hand side of (I3) by the right side, and the 'recovery sequence': given a prescribed μ, construct a sequence u^{ϵ} such that the right side of (I3) is also an upper bound (cf. **Part ii) of Theorem 1**). An immediate consequence of comparing upper and lower bounds is the following first order energetic expansion for 'almost minimizers'

u^ϵ (ie. $\epsilon^{-4/3}|\ln\epsilon|^{2/3}(E^\epsilon - \min E^\epsilon) = o_\epsilon(1)$):

$$\epsilon^{-4/3}|\ln\epsilon|^{2/3}E^\epsilon[u^\epsilon] = \min E^0 + o_\epsilon(1). \tag{I5}$$

Moreover we establish the following consequences for almost minimizers in **Theorem 2** and **Corollary 1.2.2**:

- The renormalized droplet density given by the left side of (I4) converges to a constant density. This means that the droplets are evenly spread out in the domain on average. When $\bar{\delta}$ is sufficiently small (below some critical $\bar{\delta}_c$), the density is zero and otherwise is a positive constant.

- There are roughly $|\ln\epsilon|$ droplets when $\bar{\delta} > \bar{\delta}_c$ and on average they are spherical with radius $3^{1/3}$.

The idea of the proof of the lower bound is to decompose u into its connected components by writing $u = -1 + 2\sum_i \chi_{\Omega_i}$ where $\bigcup_i \Omega_i = \{u > 0\}$ (using results in [5]) and to optimize the droplet shape and size, in order to obtain optimal lower bounds on the left hand side of (I3). A key point in the analysis is the logarithmic scaling of the Green's potential G when $n = 2$, which allows the perimeter to dominate the self-interaction energy to leading order. A corresponding upper bound is then obtained by a direct construction. The connection to the original diffuse interface energy (I1) is made through matching upper and lower bounds after applying a non-linear filter to u^ϵ (cf. **Theorem 3**). This is joint work with Cyrill Muratov and Sylvia Serfaty.

I.1.5 Main result II: Next order correction. Renormalized energy

In Chapter 2 we subtract off the leading order contribution to the energy, and study the Γ convergence of

$$F^{\epsilon}[u] := \epsilon^{-4/3}|\ln \epsilon|^{1/3}\ell^{-2}E^{\epsilon}[u] - |\ln \epsilon| \min E^0 + \frac{1}{4 \cdot 3^{1/3}}(\bar{\delta} - \bar{\delta}_c)(\ln|\ln \epsilon| + \ln 9). \tag{I6}$$

Our main result (cf. **Theorem 4**) states that

$$F^{\epsilon} \xrightarrow{\Gamma} F^0[P] := 3^{4/3} \int W(\Lambda) dP(\Lambda) + \frac{3^{2/3}\bar{\mu}}{8}, \tag{I7}$$

when $n = 2$. Here W is the renormalized energy developed in [100], which is a Coulombic type interaction energy for an infinite number of point charges in the plane, which we have denoted Λ. The measure P is, formally speaking, a probability measure on 'blow up' centers of the domain $\mathbb{T}^2$. The second constant term on the right side of (I7) comes from the contribution of energy 'inside' each droplet and is related to the constant density $\bar{\mu}$ which is the minimizer of $E^0(\mu)$, defined by (I3).

The first term on the right side of (I7) can be interpreted as follows. One zooms in at the expected droplet distance (which is $|\ln \epsilon|^{-1/2}$) at a random point. For 'most' blow up centers, the droplets converge and shrink to points in the plane to a configuration of point charges Λ. Then one computes this energy for Λ via W, and averages it out over all blow up centers by integrating against

P. This uses the abstract scheme of Sandier-Serfaty [100] which was used to obtain a similar Γ-convergence result for the magnetic Ginzburg-Landau model. We, however, have the added difficulty that we no longer have control on the size of the droplets as is the case with Ginzburg-Landau where the vortices are quantized. This creates major technical differences which we must overcome.

Statement (I7) once again requires two parts for the full statement: an ansatz free lower bound (cf. **Part i) of Theorem 4**), and a corresponding 'recovery sequence' upper bound (cf. **Part ii) of Theorem 4**). By comparing upper and lower bounds we have the following immediate corollary which is the next order expansion of (I5) for 'almost minimizers' (ie. $F[u^\epsilon] = \min F^0[P] + o_\epsilon(1)$) (cf. Theorem 5):

$$\epsilon^{-4/3}|\ln\epsilon|^{1/3}\ell^{-2}E^\epsilon[u] - |\ln\epsilon|\min E^0 + \frac{1}{4\cdot 3^{1/3}}(\bar{\delta}-\bar{\delta}_c)(\ln|\ln\epsilon| + \ln 9) \quad \text{(I.1.4)}$$
$$= 3^{4/3}\min W + \frac{3^{2/3}\bar{\mu}}{8} + o_\epsilon(1). \qquad \text{(I8)}$$

By comparing upper and lower bounds, we obtain the following corollaries for almost minimizers, summarized in **Theorem 7**:

- Zooming in at random and blowing up at the scale $|\ln\epsilon|^{-1/2}$, P-almost surely the droplets asymptotically converge to points which minimize W. The conjectured minimizer of W is the triangular lattice and it is shown in [100] that the minimizer amongst lattice configurations of W is the triangular one (see Figure 1b) for a numerical simulation of a minimizer of a closely related energy). Thus one expects to see for most blow up centers, a triangular lattice structure.

- For most blow up centers (P-almost surely), the appropriately rescaled droplet shape and volume converge uniformly to spheres of radius $3^{1/3}$. This is joint work with Cyrill Muratov and Sylvia Serfaty.

I.1.6 Main result III: Non-minimizing stationary points

In Chapter 4 we study the asymptotics of non-minimizing stationary points of the energies (I1) and (I2) in dimensions $n \geq 2$. If one perturbs the location of a single droplet in (I1) or (I2), one would expect a first order change in energy unless the droplets are geometrically evenly distributed in some sense. Our main result shows, in appropriate sense, that this is indeed the case (note that we have already established this fact for minimizing critical points via the previous two main results in Section I.1.4 and I.1.5).

A stationary point of (I1) for $u \in \mathcal{A}$ or (I2) for $u \in \mathcal{A}_{\bar{u}}$ satisfies

$$\frac{d}{dt}\Big|_{t=0} E^{\epsilon}(u \circ \phi_t) = 0, \qquad \frac{d}{dt}\Big|_{t=0} \mathcal{E}^{\epsilon}(u \circ \phi_t) = 0,$$

respectively, where $\phi_t(x) = x + tX$ and X is a C^1 vector field such that $\phi_t \circ u \in \mathcal{A}$ in the former case and $\phi_t \circ u \in \mathcal{A}_{\bar{u}}$ in the latter for $t \in (-\varepsilon, \varepsilon)$ and some $\varepsilon > 0$. Ignoring the contributions from the interfacial terms appearing in (I1) and (I2), we obtain the following, expressed by **Theorem 14** for (I2) and **Theorem 17** for (I1):

$$\mathrm{Per}(\{u^\varepsilon = 1\}) \to 0 \text{ as } \varepsilon \to 0 \Rightarrow \mu^\epsilon \nabla v^\epsilon \to \mu \nabla v = 0 \text{ as } \varepsilon \to 0, \tag{I.1.5}$$

in an appropriately weak sense, where μ^ϵ is given by (I4), $v^\epsilon(x) = \int_{\mathbb{T}^n} G(x - y)\mu^\epsilon(y)dy$ and $\mathrm{Per}(\{u^\varepsilon = 1\})$ denotes the H^{n-1} measure of the boundary of $\{u^\varepsilon = 1\}$. Since we only know that $\mu^\epsilon \overset{*}{\rightharpoonup} \mu$ weakly in $(C^0)^*$ and $\nabla v^\epsilon \rightharpoonup \nabla v$ distributionally, we cannot simply pass to the limit as $\epsilon \to 0$ in the above. It turns out we can indeed conclude (I.1.5), due to the phenomenon of "vorticity concentration cancellation", similar to that which occurs when passing to the weak limit in the 2D Euler equations [10, 33, 36, 37, 120]. These techniques have similarly been adapted in the context of the magnetic Ginzburg-Landau model [99, 102]. Here we have additional difficulties however, since we have non-trivial contributions appearing from varying the interfacial terms apperaing in (I1) and (I2). We overcome this difficulty by showing that these terms are concentrated on very small 1-capacity sets (see Chapter 4 or [41] for definitions), and thus do not contribute to the final limit in a significant way, using and generalizing the framework of Sandier-Serfaty [99, 102]. Our main results (cf. **Theorems 14** and **17**) work in any dimension $n \geq 2$. We have the following characterizations of the condition (I.1.5), which are listed in the full statement of **Theorem 14**:

1. If $\mu \in L^p$ for large enough p, then $\mu \equiv \mathbf{1}dx$, the n-dimensional uniform Lebesgue measure on $\mathbb{T}^n$.

2. If $\mu = \sum_{i=1}^{d} b_i \delta_{a_i}$ then setting $v(x) = -\frac{1}{\alpha_{n-1}}\Phi(x - a_i) + H_i(x)$ where α_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$, Φ is the solution to $-\Delta\Phi = \omega_{n-1}\delta(x)$ in $\mathbb{R}^n$ and H_i is smooth in a neighborhood of a_i, we have

$$\nabla H_i(a_i) = 0, \tag{I.1.6}$$

which is known as the "vanishing gradient property" in the context of Ginzburg-Landa [11].

Both of the above conditions imply that the droplets satisfy a "force balance", ie. the total force on each droplet from the others is zero. We also present results in Chapter 4 which show that when the number of droplets is constrained to finite, and has a bounded isoperimetric deficit ratio, that they are asymptotically round as $\varepsilon \to 0$ (cf. **Theorem 16**).

I.1.7 Future work

Our first two results in Chapters 1 and 2 focus on three dimensional materials which are spatially homogeneous in one direction, thus essentially reducing the problem to two spatial dimensions. This fails to address the very important and physically relevant case of three spatial dimensions. The tools which currently exist to address the two dimensional problem break down in three dimensions, necessitating completely new techniques. It is expected that one should observe precisely the same pattern formation in the fully three dimensional model in this scaling regime, however there has been almost no rigorous analysis. Our results in Chapter 4 are a modest step in this direction, however we would like to understand a more complete picture of the qualtitative behavior of minimizers of (I1) and (I2) in dimensions $n > 2$.

Two exceptions prior to our have already been mentioned: [2], where they prove equidistribution of energy in any dimension and [28, 29], whose results extend our work in the case of a *finite number of droplets* to any spatial dimension. One of the main technical difficulties in going from two to three dimensions is extending Jerrard's ball construction [64] to higher dimensions, which is one of the main tools needed in the analysis of Chapter 2. This would then extend the notion of the Coulombic renormalized energy for an infinite number of point charges introduced by Sandier-Serfaty [100] to any dimension, which would open analysis to a wide variety of problems where only two dimensional results are known. Such an example is the behavior of Coulombic gases recently studied by Sandier-Serfaty [97] whose study is still restricted to two dimensions, relying on the analysis of [100] which is also two dimensional.

The Swift-Hohenberg equation, despite being a fourth order energy, exhibits pattern formation very closely resembling that observed in the Ohta-Kawsaki model (see [60] for instance). It is currently a work in progress with Christian Seis to use the methods of Γ-convergence to provide rigorous evidence for this pattern formation.

There is a very close similarity with the droplets appearing in the Ohta-Kawasaki model and dislocations which appear in the study of crystal plasticity [49,119]. Recent work has begun to consider the situation where the number of dislocations is unbounded [119]. A natural extension of my thesis work would be to derive a Coulombic renormalized energy which describes the geometric arrangement of the dislocations appearing in these materials.

I.2 A non-local isoperimetric problem

The work of Serrin [105] was the first to consider an over-determined boundary value problem. In particular he asked the following question: If one has a linear, elliptic PDE on a bounded domain with zero Dirichlet, and constant Neumann boundary conditions, what must the domain be? He answered this question, showing the only possibility was the ball. This has led to a an extensive collection of problems with variations, and generalizations of this. More recently, non-local versions of this problem were considered [43,91]. We studied a particular over-determined boundary value problem, which is to characterize the critical points of a non-local isoperimetric functional which arises in determining the optimal droplet shape in the full 3D Ohta-Kawasaki energy. This functional is defined by

$$I[\Omega] = \mathrm{Per}(\Omega) + \iint_{\Omega\times\Omega} K(x-y)dx\, dy, \tag{I9}$$

where K is a kernel in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\mathrm{Per}(\Omega)$ denotes the perimeter of the set Ω and $I[\Omega]$ is defined for $\Omega \in \mathcal{A}_m$ where

$$\mathcal{A}_m := \{\Omega \subset \mathbb{R}^n : \mathrm{Per}(\Omega) < +\infty \text{ and } |\Omega| = m\}.$$

Minimizing the above functional with respect to a fixed volume constraint physically corresponds to a competition between surface tension and self-interaction energy of a uniform charge distribution. It is expected that, when the mass is small, the surface tension should dominate, thus leading one to expect the unique global minimizer should be 'close' to a ball in some sense. This is the spirit of the work of Figalli-Maggi [42], who study a non-isotropic perimeter functional with a gravitational potential perturbation. They show that minimizers are uniformly close to minimizers of the perimeter functional, also known as Wulff shapes. In [65, 81, 82], it is shown that the unique global minimizer of (I9), when the mass is sufficiently small, is once again the ball. Moreover in [81, 82], they prove that minimizers fail to exist for large enough masses in dimension $n \geq 2$.

This analysis fails to address the characterization of non-minimizing critical points of (I9). Here a critical point of I is defined as a set Ω such that variations of the domain in the direction of the normal do not cause a first order change in the energy. Such a problem can be interpreted as an over-determined boundary value problem in the spirit of Serrin [105], since critical points are easily seen to satisfy

$$\kappa(y) + \phi_\Omega(y) = \text{ constant on } \partial\Omega,$$

where κ is the mean curvature of $\partial\Omega$ and $\phi_\Omega(y) = \int_\Omega K(x-y)dy$ is the potential generated by the uniform charge in Ω.

Contrary to the work of [105] however, our problem is fundamentally non-local in nature, making methods such as the moving plane method difficult to implement. We therefore proceed by finding a 'direction' for which admissible variations in the domain cause a first order change in the energy. The convenient direction to make the variations of the domain is the evolution through area-preserving curve-shortening flow, first introduced by Gage [47].

Since we are no longer focusing on minimizing critical points, we must naturally impose bounds on the energy itself as well as the mass. More precisely we define the following parameters

$$\bar{\eta} := \begin{cases} m^{1/2}L^2(1+|\log L|) & \text{for} \quad K(x,y) = -\frac{1}{2\pi}\log|x-y|, \\ m^{1/2}L^{2-\alpha} & \text{for} \quad K(x,y) = \frac{1}{|x-y|^\alpha} \quad \alpha \in (0,1), \end{cases}$$

where $L = |\partial\Omega|$. A simple scaling analysis of (I9) reveals

$$\left\langle \frac{d(I-L)}{dL}, \zeta \right\rangle = \frac{\int_{\partial\Omega} \phi_\Omega(y)\zeta(y)dS(y)}{\int_{\partial\Omega} \kappa(y)\zeta(y)dS(y)} \sim \bar{\eta}, \tag{I.2.1}$$

where I is defined by (I9), dS is surface measure on $\partial\Omega$ and with some abuse of notation $\left\langle \frac{d(I-L)}{dL}, \zeta \right\rangle$ denotes the variation of Ω induced by the normal velocity $\zeta : \partial\Omega \to \mathbb{R}$. Thus $\bar{\eta}$ represents the rate of change of the non-local term in the energy with respect to the length of the boundary. Our result can thus be stated formally as saying that when the the change of the non-local term is small compared to a change in length, the critical points can be classified entirely in terms of those of the length term in (I9), and thus are constant curvature curves.

The main result of Chapter 3 (cf. **Theorem 17**) specifically states that that there exists a $\bar{\eta}_{cr} > 0$ such that

$$\bar{\eta} < \bar{\eta}_{cr} \Rightarrow \text{ The only connected critical point is the ball .} \tag{I10}$$

We also show in any dmiension $n \geq 2$ that stationary points of (I9) have a reduced boundary which is $C^{3,\alpha}$ for some $\alpha > 0$ (cf. **Theorem 15, Chapter 4**), although this result is contained in Chapter 4. Theorem 15 is joint work with Alexander Volkmann.

Future work

A first natural conjecture is that the assumption of connectedness of the critical points is unnecessary. While we prove that a critical point cannot have two components which can be separated by a hyperplane (see remark following Theorem 17 in Chapter 3), this is not an ideal assumption. The next natural extension of this problem would be to prove the result in higher dimensions where the result should continue to hold.

Somewhat related to this question of rigidity is the question of stability for equipotential surfaces. More specifically, if ϕ_Ω is close to a constant on $\partial\Omega$ (so that Ω is 'almost' an equipotential surface), is Ω close to a ball in some sense? It is known that when ϕ_Ω = constant on $\partial\Omega$, then Ω is the ball [43], but this result does not address stability. Such stability results are known for the original Serrin problem [105], where instead of the over-determined part of the problem being convolution with the Newtonian potential, a constant Neumann boundary condition is imposed. A stability result in this spirit is known for the original Serrin problem [16], but our problem, being non-local in nature, necessitates new techniques for a similar statement.

Chapter 1

The Γ limit of the Ohta-Kawasaki energy. I. Droplet Density

1.1 Introduction

In the studies of energy-driven pattern formation, one often encounters variational problems with competing terms operating on different spatial scales [61,62,73,83,106,111,114]. Despite the fundamental importance of these problems to a multitude of physical systems, their detailed mathematical studies are fairly recent (see e.g. [21, 23–26, 44, 68, 99]). To a great extent this fact is related to the emerging multiscale structure of the energy minimizing patterns and the associated difficulty of their description [23, 25, 38, 70, 77]. In particular, the popular approach of Γ-convergence [14] is rendered difficult due to the emergence of more than two well-separated spatial scales in suitable asymptotic limits (see e.g. [23–26, 38, 70, 77, 100]).

These issues can be readily seen in the case of the Ohta-Kawasaki model, a canonical mathematical model in the studies of energy-driven pattern forming systems. This model, originally proposed in [87] to describe different morphologies observed in diblock copolymer melts (see e.g. [9]) is defined by the energy functional (up to a choice of scales)

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x, y)(u(y) - \bar{u})\, dx\, dy, \tag{1.1.1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $W(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg-Landau potential $W(u) = \frac{1}{4}(1 - u^2)^2$, $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, i.e., G_0 solves

$$-\Delta G_0(x, y) = \delta(x - y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x, y)\, dx = 0, \tag{1.1.2}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|}\int_\Omega u\,dx = \bar{u}. \tag{1.1.3}$$

Let us point out that in addition to a number of polymer systems [34, 86, 110], this model is also applicable to many other physical systems due to the fundamental nature of the Coulombic non-local term in (4.1.1) [19, 40, 58, 73, 79, 84]. Because of this Coulomb interaction, we also like to think of u as a density of "charge".

The Ohta-Kawasaki functional admits the following "sharp-interface" version:

$$E[u] = \frac{\varepsilon}{2}\int_\Omega |\nabla u|\,dx + \frac{1}{2}\int_\Omega\int_\Omega (u(x)-\bar{u})G(x,y)(u(y)-\bar{u})\,dx\,dy, \tag{1.1.4}$$

where $G(x,y)$ is the *screened* Green's function of the Laplacian, i.e., it solves the Neumann problem for the equation

$$-\Delta G + \kappa^2 G = \delta(x-y), \tag{1.1.5}$$

where $\kappa := 1/\sqrt{W''(1)} > 0$. Note also that in contrast to the diffuse interface energy in (4.1.1), for the sharp interface energy in (1.1.4) the charge neutrality constraint in (4.1.3) is no longer imposed. This is due to the fact that in a minimizer of the diffuse interface energy the charge of the minority phase is expected to partially redistribute into the majority phase to ensure screening of the induced non-local field (see a more detailed discussion in the following

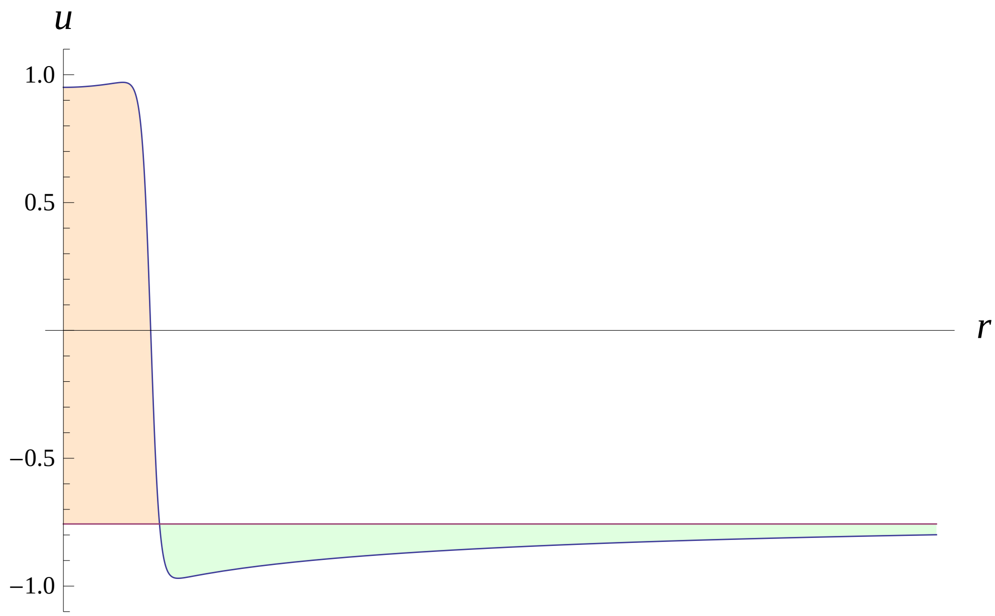

Figure 1.1: Two-dimensional multi-droplet patterns in systems with Coulombic repulsion: a local minimizer of the Ohta-Kawasaki energy on a rectangle with periodic boundary conditions; a local minimizer of the sum of two-point Coulombic potentials on a disk with Neumann boundary conditions. Taken from [79, 93].

section).

The two terms in the energy (1.1.4) are competing: the second term favors u to be constant equal to its average $\bar{u}$, but since u is valued in $\{+1, -1\}$ this means in effect that it is advantageous for u to oscillate rapidly between the two phases $u = +1$ and $u = -1$; the first term penalizes the perimeter of the interface between the two phases, and thus opposes too much spreading and oscillation. The competition between these two selects a lengthscale, which is a function of ε. In the diffuse interface version (4.1.1), the sharp transitions between $\{u = +1\}$ and $\{u = -1\}$ are replaced by smooth transitions at the scale $\varepsilon > 0$ as soon as $\varepsilon \ll 1$.

In one space dimension and in the particular case $\bar{u} = 0$ (symmetric phases) the behavior of the energy has been completely understood can be understood the work of Müller [77]: the minimizer u is periodic and alternates between $u = +1$ and $u = -1$ at scale $\varepsilon^{1/3}$ (for other one-dimensional results, see also [94, 118]). In higher dimensions the patterns of minimizers are much more complex and are not well understood. The behavior depends on the volume fraction between the phases, i.e. on the constant $\bar{u}$ chosen, and also on the dimension. When $\bar{u} < 0$, we call $u = -1$ the majority phase and $u = +1$ the minority phase, and conversely when $\bar{u} > 0$. In two dimensions, numerical simulations lead to expecting round "droplets" of the minority phase surrounded by a "sea" of the majority phase (see Fig. 1.1) at sufficiently small asymmetries between the majority and the minority phases (i.e., for $\bar{u}$ sufficiently far away from zero) [78, 79, 87, 93]. The situation is less clear for $\bar{u}$ close to zero, although it is commonly believed that in this case the minimizers are one-dimensional stripe patterns [27, 78, 79, 87].

In all cases, minimizers are intuitively expected to be periodic. However, at the moment this seems to be very difficult to prove. The only general result in that direction to date is that of Alberti, Choksi and Otto [2], which proves that the energy of minimizers of the sharp interface energy from (1.1.4) with no screening (with $\kappa = 0$ and the neutrality condition from (4.1.3)) is uniformly distributed in the limit where the size of the domain Ω goes to infinity (see also [21, 108]). Their results, however, do not provide any further information about the structure of the energy-minimizing patterns.

Note in passing that the question of proving any periodicity of minimizers for multi-dimensional energies is unsolved even for systems of point particles forming simple crystals (see e.g. [70, 100]), with a notable exception of certain two-dimensional particle systems with short-range interactions which somehow reduce to packing problems [90, 112, 116]. Naturally, the situation can be expected to be more complicated for pattern forming systems in which the constitutive elements are "soft" objects, such as, e.g., droplets of the minority phase in the matrix of the majority phase in the Ohta-Kawasaki model.

Here we are going to focus on the situation where one phase is in strong majority with respect to the other, which is imposed by taking $\bar{u}$ very close to -1 as $\varepsilon \to 0$. Thus we can expect a distribution of small droplets of $u = +1$ surrounded by a sea of $u = -1$. In this regime, Choksi and Peletier analyzed the asymptotic properties of a suitably rescaled version of the sharp interface energy (1.1.4) with no screening in [28], as well as (4.1.1) in [29]. They work in the setting of a fixed domain Ω, and in a regime where the number of droplets remains finite as $\varepsilon \to 0$. They showed that the energy minimizing patterns concentrate to a finite number of point masses, whose magnitudes and locations are determined via a Γ-expansion of the energy [15]. We note that Γ-convergence of (4.1.1) to the functional (1.1.4) with no screening for fixed volume fractions was established by Ren and Wei in [94], who also analyzed local minimizers of the sharp interface energy in the strong asymmetry regime in two space dimensions [93].

All these works are in the finite domain Ω setting, while we are generally interested in the *large volume* (macroscopic) limit, i.e., the regime when the number of droplets tends to infinity. A rather comprehensive study of the behavior of the minimizers for the Ohta-Kawasaki energy in macroscopically large domains was recently performed in [80], still in the regime $\bar{u}$ close to 1. There the two-dimensional Ohta-Kawasaki energy was considered in the case when Ω is a unit square with periodic boundary conditions. The interesting regime corresponds to the parameters $\varepsilon \ll 1$ and $1 + \bar{u} = O(\varepsilon^{2/3} |\ln \varepsilon|^{1/3}) \ll 1$. It is shown in [80] that under these assumptions on the parameters, (1.1.4) gives the correct asymptotic limit of the minimal energy in (4.1.1). Moreover, it is shown that when $\bar{\delta} := \varepsilon^{-2/3} |\ln \varepsilon|^{-1/3} (1 + \bar{u})$ becomes greater than a certain critical constant $\bar{\delta}_c$, the minimizers of E in (1.1.4) consist of $O(|\ln \varepsilon|)$ simply connected, nearly round droplets of radius $\simeq 3^{1/3} \varepsilon^{1/3} |\ln \varepsilon|^{-1/3}$, and uniformly distributed throughout the domain [80]. Thus, the following hierarchy of lengthscales is established in the considered regime:

$$\varepsilon \ll \varepsilon^{1/3} |\ln \varepsilon|^{-1/3} \ll |\ln \varepsilon|^{-1/2} \ll 1, \tag{1.1.6}$$

where the scales above correspond to the width of the interface, the radius of the droplets, the average distance between the droplets, and the screening length, respectively. The multiscale nature of the energy minimizing pattern is readily apparent from (1.1.6).

The analysis of [80] makes heavy use of the minimality condition for (1.1.4) and, in particular, the Euler-Lagrange equation associated with the energy. One is thus naturally led to asking whether the qualitative properties of the minimizers established in [80] (roundness of the droplets, identical radii, uniform distribution) carry over to, e.g., almost minimizers of E, for which no Euler-Lagrange equation is available. More broadly, it is natural to ask how robust the properties of the energy minimizing patterns are with respect to various perturbations of the energy, for example, how the picture presented above is affected when the charge density $\bar{u}$ is spatially modulated. A natural way to approach these questions is via Γ-convergence. However, for a multiscale problem such as the one we are considering the proper setting for studying Γ-limits of the functionals in (4.1.1) or (1.1.4) is presently lacking. The purpose of this paper is to formulate such a setting and extract the leading order term in the Γ-expansion of the energy in (4.1.1). In our forthcoming paper [55], we obtain the next order term in the Γ-expansion, using the method of "lower bounds for 2-scale energies" via Γ-convergence introduced in [100].

For simplicity, as in [80] we consider the energy defined on a torus (a flat square with periodic boundary conditions). The main question for setting up the Γ-limit in the present context is to choose a suitable metric for Γ-convergence. In this paper we show that the right metric to consider is the weak convergence of measures for a suitably rescaled sequence of characteristic functions associated with droplets (see the next section for precise definitions and statements of theorems). Then, up to a rescaling, both the energy $\mathcal{E}$ from (4.1.1) and E from (1.1.4) Γ-converge to a quadratic functional in terms of the limit measure, with the quadratic term generated by the screened Coulomb kernel from (2.1.6) and the linear term depending explicitly on $\bar{\delta}$ and κ.

We note that the obtained limit variational problem is strictly convex and its unique minimizer is a measure with constant density across the domain Ω. In particular, this implies equidistribution of mass and energy for the minimizers of the diffuse interface energy $\mathcal{E}$ in (4.1.1) in the considered regime, which is a new result. In our companion paper [55], we further address the mutual arrangement of the droplets in the energy minimizing patterns, using the formalism developed recently for Ginzburg-Landau vortices [100]. In fact, the problem under consideration, and its mathematical treatment (here as well as in [55]), share several important features with the two-dimensional magnetic Ginzburg-Landau model from the theory of superconductivity [99]. There the role of droplets is played by the Ginzburg-Landau vortices, which in the appropriate limits also become uniformly distributed throughout the domain [98]. We note, however, that the approach developed in [98, 99] cannot be carried over directly to the problem under consideration, since the vortices are more rigid than their droplet counterparts: the topological degrees of the vortices are quantized and can only take integer values, while their counterparts here, which are the droplet volumes, are not. Thus we also have to consider the possibility of many very small droplets. Developing a control on the droplet volumes from above and below is one of the key ingredient of the proofs presented below, and relies on the control of their perimeter via the energy. Let us also mention yet another closely related system from the studies of ferromagnetism, where the role of vortices is played by the slender needle-like domains of opposite magnetization at the onset magnetization reversal [67].

Our paper is organized as follows. In Sec. 1.2, we introduce the considered scaling regime and state our main results; in Sec. 1.3.2 we prove the Γ-convergence result in the sharp interface setting; and in Sec. 1.4 we treat the Γ-limit for the case of the diffuse interface energy.

Some notations. We use the usual notation $(u^\varepsilon) \in \mathcal{A}$ of Γ-convergence to denote sequences of functions $u^\varepsilon \in \mathcal{A}$ as $\varepsilon = \varepsilon_n \to 0$, where $\mathcal{A}$ is an admissible class. We also use the notation $\mu \in \mathcal{M}(\Omega)$ to denote a positive finite Radon measure $d\mu$ on the domain Ω. With a slight abuse of notation, we will often speak of μ as the "density" on Ω. The symbols $H^1(\Omega)$, $BV(\Omega)$, $C(\Omega)$ and $H^{-1}(\Omega)$ denote the usual Sobolev space, space of functions of bounded variation, space of continuous functions, and the dual of $H^1(\Omega)$, respectively.

1.2 Statement of results

In this paper we are mainly concerned with the two-dimensional version of the sharp interface energy in (1.1.4) in the regime near the onset of non-trivial energy minimizing patterns. Throughout the rest of the paper the domain Ω is assumed to be a flat two-dimensional torus of length ℓ, i.e., when $\Omega = \mathbb{T}^2_\ell = [0, \ell)^2$, with periodic boundary conditions. Given the parameters $\kappa > 0$, $\bar{\delta} > 0$ and $\ell > 0$, for every $\varepsilon > 0$ we define

$$\bar{u}^\varepsilon := -1 + \varepsilon^{2/3} |\ln \varepsilon|^{1/3} \bar{\delta}. \tag{1.2.1}$$

Then the sharp interface version of the Ohta-Kawasaki energy (cf. (1.1.4)) can be written as

$$E^\varepsilon[u] = \frac{\varepsilon}{2}\int_{\mathbb{T}_\ell^2} |\nabla u|\, dx + \frac{1}{2}\int_{\mathbb{T}_\ell^2} (u - \bar{u}^\varepsilon)(-\Delta + \kappa^2)^{-1}(u - \bar{u}^\varepsilon)\, dx, \qquad (1.2.2)$$

for all $u \in \mathcal{A}$, where

$$\mathcal{A} := BV(\mathbb{T}_\ell^2; \{-1, 1\}). \qquad (1.2.3)$$

We wish to understand the asymptotic properties of the energy E^ε in (4.1.4) as $\varepsilon \to 0$ when all other parameters are fixed. We later relate our conclusions based on the study of this energy to its diffuse interface version, which under the same scaling assumptions takes the form

$$\mathcal{E}^\varepsilon[u] = \int_{\mathbb{T}_\ell^2} \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u) \ + \frac{1}{2}(u - \bar{u}^\varepsilon)(-\Delta)^{-1}(u - \bar{u}^\varepsilon) \right) dx, \qquad (1.2.4)$$

for all $u \in \mathcal{A}^\varepsilon$, where

$$\mathcal{A}^\varepsilon := \left\{ u \in H^1(\mathbb{T}_\ell^2) : \frac{1}{\ell^2}\int_{\mathbb{T}_\ell^2} u\, dx = \bar{u}^\varepsilon \right\}. \qquad (1.2.5)$$

We note that on the level of the energy minimizers the relation between the two functionals was established in [80].

The energy E^ε may alternatively be written in terms of the level sets of u. Indeed, the set $\Omega^+ := \{u = +1\}$ is a set of finite perimeter (for precise definitions, see e.g. [6]), and when $|\partial\Omega^+|$ is sufficiently small, the set Ω^+ may be uniquely decomposed into at most countable union of connected components Ω_i^+, where the boundaries $\partial\Omega_i^+$ of each connected component are Jordan curves which are essentially disjoint (after an extension to the whole of $\mathbb{R}^2$ and up to negligible sets, see e.g. [5, Corollary 1 and Theorem 8]). In the context of Γ-convergence the sets Ω_i^+ may be viewed as a suitable generalization of the droplets introduced earlier in the studies of energy minimizing patterns [80]. Note, however, that the sets Ω_i^+ lack the regularity properties of the energy minimizers in [80] and may in general be fairly ill-behaved (in particular, they do not have to be simply connected). Nevertheless, they are fundamental for the description of the low energy states associated with E^ε and, in particular, will be shown to be close, in some average sense, to disks of prescribed radii for almost minimizers of energy.

In terms of the droplets, from the above discussion we have

$$u = -1 + 2\sum_i \chi_{\Omega_i^+} \qquad \text{a.e. in } \mathbb{T}_\ell^2, \tag{1.2.6}$$

where $\chi_{\Omega_i^+}$ are the characteristic functions of Ω_i^+, and, furthermore

$$\begin{aligned} E^\varepsilon[u] = {} & \frac{\ell^2(1+\bar{u}^\varepsilon)^2}{2\kappa^2} \\ & + \sum_i \left\{ \varepsilon|\partial\Omega_i^+| - 2\kappa^{-2}(1+\bar{u}^\varepsilon)|\Omega_i^+| \right\} + 2\sum_{i,j} \int_{\Omega_i^+}\int_{\Omega_j^+} G(x-y)\,dx\,dy, \end{aligned} \tag{1.2.7}$$

where G solves

$$-\Delta G(x) + \kappa^2 G(x) = \delta(x) \qquad \text{in} \quad \mathbb{T}_\ell^2, \tag{1.2.8}$$

and we took into account the translational symmetry of the problem in $\mathbb{T}_\ell^2$. Moreover, since the optimal configurations for Ω_i^+ are expected to consist of droplets of size of order $\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$ (see (1.1.6) and the discussion around), it is convenient to introduce the rescaled area and perimeter of each droplet:

$$A_i := \varepsilon^{-2/3}|\ln\varepsilon|^{2/3}|\Omega_i^+|, \qquad P_i := \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}|\partial\Omega_i^+|. \tag{1.2.9}$$

Similarly, let us introduce the suitably rescaled measure $d\mu$ associated with the droplets:

$$d\mu(x) := \varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}\sum_i \chi_{\Omega_i^+}(x)dx. \tag{1.2.10}$$

Then the energy $E^\varepsilon[u]$ may be rewritten as

$$E^\varepsilon[u] = \varepsilon^{4/3}|\ln\varepsilon|^{2/3}\left(\frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \bar{E}^\varepsilon[u]\right), \tag{1.2.11}$$

where

$$\bar{E}^\varepsilon[u] := \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i - \frac{2\bar{\delta}}{\kappa^2}A_i\right) + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y). \tag{1.2.12}$$

Observe that for the minimizers we have $\bar{E}^\varepsilon = O(1)$, $A_i = O(1)$ and $P_i = O(1)$, (and even more precisely $A_i \sim \pi 3^{2/3}$ and $P_i = 2\pi 3^{1/3}$) the number of droplets is $N = O(|\ln\varepsilon|)$ and μ closely approximates the sum of Dirac masses with weights of order $|\ln\varepsilon|^{-1}$ [80]. Observe also that if $(u^\varepsilon) \in \mathcal{A}$ is such that $\limsup_{\varepsilon\to 0} \bar{E}^\varepsilon[u^\varepsilon] < +\infty$, we have

$$\limsup_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|}\sum_i P_i^\varepsilon < +\infty, \qquad \limsup_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon < +\infty, \tag{1.2.13}$$

where $\{A_i^\varepsilon\}$, $\{P_i^\varepsilon\}$ and μ^ε are associated with u^ε. Indeed, using the fact that

$$\frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon = \int_{\mathbb{T}_\ell^2} d\mu^\varepsilon, \tag{1.2.14}$$

and, therefore, by positive definiteness of G we have

$$\frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \bar{E}^\varepsilon[u] \geq 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y) \geq 2\kappa^{-2}\ell^{-2}|\ln\varepsilon|^{-2}\left(\sum_i A_i^\varepsilon\right)^2. \tag{1.2.15}$$

We now state our Γ-convergence result, which is obtained for configurations (u^ε) that obey the optimal energy scaling, i.e. when $\bar{E}^\varepsilon[u^\varepsilon]$ remains bounded as $\varepsilon \to 0$. The result is obtained with the help of the framework established in [98], where an analogous result for the Ginzburg-Landau functional was obtained. What we show is that the limit functional E^0 depends only on the limit density μ of the droplets (more precisely, on a general positive measure $d\mu$). In passing to the limit the second term (2.2.4) remains unchanged, while the first term is converted into a term proportional to the integral of the measure. The proportionality constant is non-trivially determined by the optimal droplet profile that will be discussed later on. We give the statement of the result in terms of the original screened sharp interface energy E^ε, which is defined in terms of $u \in \mathcal{A}$. In the proof, we work instead with the equivalent energy $\bar{E}^\varepsilon$, which is defined through $\{A_i^\varepsilon\}$, $\{P_i^\varepsilon\}$ and μ^ε (cf. (2.2.3) and (2.2.4)).

Theorem 1. **(Γ-convergence of E^ε)** *Fix $\bar{\delta} \in \mathbb{R}$, $\kappa > 0$ and $\ell > 0$, and let E^ε be defined by (4.1.4) with $\bar{u}_\varepsilon$ given by* (4.2.4). *Then we have that*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{2/3}E^\varepsilon \xrightarrow{\Gamma} E^0[\mu] := \frac{\bar{\delta}^2\ell^2}{2\kappa^2}+\left(3^{2/3}-\frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu+2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y),$$

where $\mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$. More precisely, as $\varepsilon \to 0$ we have

i) (Lower Bound) Let $(u^\varepsilon) \in \mathcal{A}$ be such that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-4/3}|\ln\varepsilon|^{2/3}E^\varepsilon[u^\varepsilon] < +\infty, \tag{1.2.16}$$

let $d\mu^\varepsilon(x) := \frac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u^\varepsilon(x))dx$, and let v^ε satisfy

$$-\Delta v^\varepsilon + \kappa^2 v^\varepsilon = \mu^\varepsilon \qquad in \quad \mathbb{T}_\ell^2. \tag{1.2.17}$$

Then, up to extraction of subsequences, we have

$$\mu^\varepsilon \rightharpoonup \mu \ in\ (C(\mathbb{T}_\ell^2))^*, \ \ v^\varepsilon \rightharpoonup v \ in\ H^1(\mathbb{T}_\ell^2),$$

where $\mu \in \mathcal{M}(\mathbb{T}^2_\ell) \cap H^{-1}(\mathbb{T}^2_\ell)$ *and* $v \in H^1(\mathbb{T}^2_\ell)$ *satisfy*

$$-\Delta v + \kappa^2 v = \mu \qquad in \quad \mathbb{T}^2_\ell. \tag{1.2.18}$$

Moreover, we have

$$\liminf_{\varepsilon \to 0} \varepsilon^{-4/3} |\ln \varepsilon|^{2/3} E^\varepsilon[u^\varepsilon] \geq E^0[\mu].$$

ii) (Upper Bound) Conversely, given $\mu \in H^{-1}(\mathbb{T}^2_\ell) \cap \mathcal{M}(\mathbb{T}^2_\ell)$ *and* $v \in H^1(\mathbb{T}^2_\ell)$ *solving* (1.2.18)*, there exist* $(u^\varepsilon) \in \mathcal{A}$ *such that for the corresponding* μ^ε*,* v^ε *as in (2.2.7) we have*

$$\mu^\varepsilon \rightharpoonup \mu \ in \ (C(\mathbb{T}^2_\ell))^*, \quad v^\varepsilon \rightharpoonup v \ in \ H^1(\mathbb{T}^2_\ell)$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon^{-4/3} |\ln \varepsilon|^{2/3} E^\varepsilon[u^\varepsilon] \leq E^0[\mu].$$

We note that the limit energy E^0 obtained in Theorem 1 may be viewed as the *homogenized* (or mean field) version of the non-local part of the energy in the definition of E^ε associated with the limit charge density μ of the droplets, plus a term associated with the self-energy of the droplets. The functional E^0 is strictly convex, so there exists a unique minimizer $\bar{\mu} \in H^{-1}(\mathbb{T}^2_\ell) \cap (C(\mathbb{T}^2_\ell))^*$ of E^0, which is easily seen to be either $\bar{\mu} = 0$ for $\bar{\delta} \leq \frac{1}{2} 3^{2/3} \kappa^2$ or $\bar{\mu} = \frac{1}{2}(\bar{\delta} - \frac{1}{2} 3^{2/3} \kappa^2)$ otherwise. The latter can also be seen immediately from the Remark 1.2.1 below, which gives a local characterization of the limit energy E^0.

Remark 1.2.1. *The limit energy E^0 in Theorem 1 becomes local when written in terms of the limit potential v from* (1.2.18)*:*

$$E^0[\mu] = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} + \left(3^{2/3}\kappa^2 - 2\bar{\delta}\right) \int v\, dx + 2 \int_{\mathbb{T}_\ell^2} \left(|\nabla v|^2 + \kappa^2 v^2\right) dx. \qquad (1.2.19)$$

Also, by the usual properties of Γ-convergence [14], the optimal density $\bar{\mu}$ above is exhibited by the minimizers of E^ε in the limit $\varepsilon \to 0$, in agreement with [80, Theorem 2.2]:

Corollary 1.2.2. *For given $\bar{\delta} \in \mathbb{R}$, $\kappa > 0$ and $\ell > 0$, let $\bar{u}^\varepsilon$ be given by* (4.2.4) *and let $(u^\varepsilon) \in \mathcal{A}$ be minimizers of E^ε. Then*

$$\mu^\varepsilon \rightharpoonup \begin{cases} 0 \\ \frac{1}{2}(\bar{\delta} - \bar{\delta}_c) \end{cases} \quad in\ (C(\mathbb{T}_\ell^2))^*, \qquad \varepsilon^{-4/3} |\ln \varepsilon|^{2/3} \ell^{-2} \min E^\varepsilon \to \begin{cases} \frac{\bar{\delta}^2}{2\kappa^2} \\ \frac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c), \end{cases} \qquad (1.2.20)$$

when $\bar{\delta} \le \bar{\delta}_c$ or $\bar{\delta} > \bar{\delta}_c$, respectively, with $\bar{\delta}_c := \frac{1}{2} 3^{2/3} \kappa^2$.

In particular, since the minimal energy scales with the area of $\mathbb{T}_\ell^2$, it is an *extensive quantity.*

We next give a definition of almost minimizers with prescribed limit density, for which a number of further results may be obtained. These can be viewed, e.g., as almost minimizers of E^ε in the presence of an external potential. We note that in view of strict convexity of E^0, minimizing $E^0[\mu] - \int_{\mathbb{T}_\ell^2} \varphi(x) d\mu(x)$ for a given $\varphi \in H^1(\mathbb{T}_\ell^2)$ one obtains a one-to-one correspondence between the minimizing density μ and the potential φ. It then makes sense to talk about almost minimizers of the energy E^ε with prescribed limit density μ by viewing them as almost minimizers of $E^\varepsilon - \int_{\mathbb{T}_\ell^2} \varphi d\mu^\varepsilon$. Also, observe that almost minimizers with the particular prescribed density $\bar{\mu}$ from Corollary 1.2.2 are simply almost minimizers of E^ε. Below we give a precise definition.

Definition 1.2.3. *For given $\bar{\delta} \in \mathbb{R}$, $\kappa > 0$, $\ell > 0$ and a given $\mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$, we will call every recovery sequence $(u^\varepsilon) \in \mathcal{A}$ in Theorem 1(ii) almost minimizers of E^ε with prescribed limit density μ.*

For almost minimizers with prescribed limit density, we show that in the limit $\varepsilon \to 0$ most of the droplets, with the exception of possibly many tiny droplets, converge to disks of radius $r = \varepsilon^{1/3}|\ln \varepsilon|^{-1/3}3^{1/3}$. More precisely, we have the following result.

Theorem 2. *Let $(u^\varepsilon) \in \mathcal{A}$ be a sequence of almost minimizers of E^ε with prescribed limit density μ. For every $\gamma \in (0,1)$ define the set $I_\gamma^\varepsilon := \{i \in \mathbb{N} : 3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}\}$. Then*

$$\limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right) = 0, \tag{1.2.21}$$

$$\limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \left(\sum_{i \in I_\gamma^\varepsilon} \left| A_i^\varepsilon - 3^{2/3}\pi \right|^2 \right) = 0, \tag{1.2.22}$$

$$\limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \left(\sum_{i \notin I_\gamma^\varepsilon} A_i^\varepsilon \right) = 0, \tag{1.2.23}$$

where $\{A_i^\varepsilon\}$ and $\{P_i^\varepsilon\}$ are given by (2.2.2) with $u = u^\varepsilon$.

As an immediate consequence of Theorem 2 and (1.2.14), we have the following result.

Corollary 1.2.4. *Under the assumptions of Theorem 2, when* $\int_{\mathbb{T}_\ell^2} d\mu > 0$ *we have*

$$\lim_{\varepsilon \to 0} \frac{3^{2/3}\pi |I_\gamma^\varepsilon|}{|\ln \varepsilon|} = \int_{\mathbb{T}_\ell^2} d\mu, \qquad \lim_{\varepsilon \to 0} \frac{P_i^\varepsilon}{\sqrt{4\pi A_i^\varepsilon}} = 1 \quad \forall i \in I_\gamma^\varepsilon, \tag{1.2.24}$$

where $|I_\gamma^\varepsilon|$ *denotes the cardinality of* I_γ^ε*.*

This result generalizes the one in [80], where it was found that in the case of the minimizers *all* the droplets are uniformly close to disks of the optimal radius $r = \varepsilon^{1/3}|\ln \varepsilon|^{-1/3}3^{1/3}$. What we showed here is that this result holds for almost all droplets in the case of almost minimizers, in the sense that in the limit almost all the mass concentrates in the droplets of optimal area and vanishing isoperimetric deficit. We note that the density μ is also the limit of the number density of the droplets, up to a normalization constant, once the droplets of vanishing area have been discarded.

The result that almost all droplets in almost minimizers with prescribed limit density have asymptotically the *same* size, even if the limit density is not constant in $\mathbb{T}_\ell^2$ appears to be quite surprising. In fact, this observation is consistent with the expectation that quantum mechanical charged particle systems form Wigner crystals at low particle densities [59,73,117]. Let us point out that the Ohta-Kawasaki energy $\mathcal{E}^\varepsilon$ bares a number of similarities with the classical Thomas-Fermi-Dirac-Weizsäcker model arising in the context of density functional theory of quantum systems (see e.g. [70, 71]).

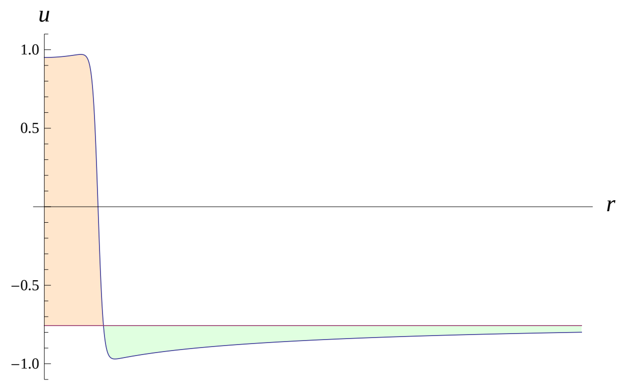

Figure 1.2: A qualitative form of the u-profile for a single droplet from the Euler-Lagrange equation associated with $\mathcal{E}$. The horizontal line shows the level corresponding to $\bar{u}$. Charge is transferred from the region where $u < \bar{u}$ (depletion shown in green) to the region where $u > \bar{u}$ (excess shown in orange).

We now turn to relating the results obtained so far for the screened sharp interface energy E^ε to the original diffuse interface energy $\mathcal{E}^\varepsilon$. On the level of the minimal energy, the asymptotic equivalence of the energies, namely, that for every $\delta > 0$

$$(1-\delta)\min E^\varepsilon \leq \min \mathcal{E}^\varepsilon \leq (1-\delta)\min E^\varepsilon \qquad (1.2.25)$$

for $\varepsilon \ll 1$, in the considered regime was established in [80, Theorem 2.3]. The main idea of the proof in [80] is for a given function $u^\varepsilon \in \mathcal{A}^\varepsilon$ to establish an approximate lower bound for $\mathcal{E}^\varepsilon[u^\varepsilon]$ in terms of $(1-\delta)E^\varepsilon[\tilde{u}^\varepsilon]$ for some $\tilde{u}^\varepsilon \in \mathcal{A}$, with $\delta > 0$ which can be chosen arbitrarily small for $\varepsilon \ll 1$. The matching approximate upper bound is then obtained by a suitable lifting of the minimizer $u^\varepsilon \in \mathcal{A}$ of E^ε into $\mathcal{A}^\varepsilon$.

Here we show that the procedure outlined above may also be applied to almost minimizers of E^0 in a suitably modified version of Definition 1.2.3 involving $\mathcal{E}^\varepsilon$. We note right away, however, that it is not possible to simply replace E^ε with $\mathcal{E}^\varepsilon$ in Definition 1.2.3. The reason for this is the presence of the mass constraint in the definition of the admissible class $\mathcal{A}^\varepsilon$ for $\mathcal{E}^\varepsilon$. This implies, for example, that any sequence of almost minimizers $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ of $\mathcal{E}^\varepsilon$ must satisfy $\ell^{-2} \int_{\mathbb{T}_\ell^2} d\mu^\varepsilon = \frac{1}{2}\bar{\delta}$, while, according to Corollary 1.2.2, for sequences of almost minimizers $(u^\varepsilon) \in \mathcal{A}$ of E^ε we have $\ell^{-2} \int_{\mathbb{T}_\ell^2} d\mu^\varepsilon \to \bar{\mu} \neq \frac{1}{2}\bar{\delta}$. This phenomenon is intimately related to the effect of screening of the Coulombic field from the droplets by the compensating charges that move into their vicinity [79]. For a single radially symmetric droplet the solution of the Euler-Lagrange equation associated with $\mathcal{E}^\varepsilon$ will have the form shown in Fig. 1.2.

In order to be able to extract the limit behavior of the energy, we need to take into consideration the redistribution of charge discussed above and define almost minimizers with prescribed limit density that belong to $\mathcal{A}^\varepsilon$ and for which the screening charges are removed from the consideration of convergence to the limit density. Hence, we introduce the following definition.

Given a candidate function $u^\varepsilon \in \mathcal{A}^\varepsilon$, we define a new function

$$u_0^\varepsilon(x) := \begin{cases} +1, & u^\varepsilon(x) > 0, \\ -1, & u^\varepsilon(x) \le 0, \end{cases} \tag{1.2.26}$$

The jump set of u_0^ε coincides with the zero level set of u^ε. This introduces a nonlinear filtering operation that eliminates the effect of the small deviations of

u^ε from ± 1 in almost minimizers on the limit density (compare also with [67]). The density is now defined as

$$d\mu_0^\varepsilon := \tfrac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u_0^\varepsilon(x))dx. \tag{1.2.27}$$

We can follow the ideas of [80] to establish an analog of Theorem 1 for the diffuse interface energy. To avoid many technical assumptions, we formulate the result for a specific choice of W. A general result may be easily reconstructed. Also, we make a technical assumption to avoid dealing with the case $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} > 1$, when spiky configurations in which $|u^\varepsilon|$ significantly exceeds 1 in regions of vanishing size may appear. We note that this condition is satisfied by the minimizers of $\mathcal{E}^\varepsilon$ [80, Proposition 4.1].

Theorem 3. **(Γ-convergence of $\mathcal{E}^\varepsilon$)** *Fix $\bar\delta \in \mathbb{R}$ and $\ell > 0$, and let $\mathcal{E}^\varepsilon$ be defined by (4.7.1) with $W(u) = \frac{9}{32}(1-u^2)^2$ and $\bar u^\varepsilon$ given by* (4.2.4)*. Then we have that*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{2/3}\mathcal{E}^\varepsilon \xrightarrow{\Gamma} E^0[\mu] := \frac{\bar\delta^2\ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar\delta}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y),$$

where $\mu \in \mathcal{M}(\mathbb{T}_\ell^2)\cap H^{-1}(\mathbb{T}_\ell^2)$ and $\kappa = \frac{2}{3}$. More precisely, we have

i) (Lower Bound) Let $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ be such that $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq 1$ *and*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-4/3}|\ln\varepsilon|^{2/3}\mathcal{E}^\varepsilon[u^\varepsilon] < +\infty, \tag{1.2.28}$$

and let $\mu_0^\varepsilon(x)$ be defined by (1.2.26) *and* (1.2.27)*.*

Then, up to extraction of subsequences, we have

$$\mu_0^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}_\ell^2))^*.$$

where $\mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$*. Moreover, we have* $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 1$ *and*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-4/3}|\ln\varepsilon|^{2/3}\mathcal{E}^\varepsilon[u^\varepsilon] \geq E^0[\mu].$$

ii) (Upper Bound) Conversely, given $\mu \in H^{-1}(\mathbb{T}_\ell^2) \cap \mathcal{M}(\mathbb{T}_\ell^2)$*, there exist* $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ *such that* $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 1$ *and for* μ_0^ε *defined by* (1.2.26) *and* (1.2.27) *we have*

$$\mu_0^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}_\ell^2))^*,$$

and

$$\limsup_{\varepsilon\to 0} \varepsilon^{-4/3}|\ln\varepsilon|^{2/3}\mathcal{E}^\varepsilon[u^\varepsilon] \leq E^0[\mu].$$

Based on the result of Theorem 3, we have the following analog of Corollary 1.2.2 for the diffuse interface energy $\mathcal{E}^\varepsilon$.

Corollary 1.2.5. *For given* $\bar{\delta} \in \mathbb{R}$ *and* $\ell > 0$*, let* $\bar{u}^\varepsilon$ *be given by* (4.2.4) *and let* $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ *be minimizers of* $\mathcal{E}^\varepsilon$ *with* $W(u) = \frac{9}{32}(1-u^2)^2$ *and* $\bar{u}^\varepsilon$ *given by* (4.2.4)*. Then, if* u_0^ε *and* μ_0^ε *are defined via* (1.2.26) *and* (1.2.27)*, respectively,* *we have*

$$\mu_0^\varepsilon \rightharpoonup \begin{cases} 0 \\ \frac{1}{2}(\bar{\delta}-\bar{\delta}_c) \end{cases} \quad \textit{in } (C(\mathbb{T}_\ell^2))^*, \qquad \varepsilon^{-4/3}|\ln\varepsilon|^{2/3}\ell^{-2}\min\mathcal{E}^\varepsilon \to \begin{cases} \frac{\bar{\delta}^2}{2\kappa^2} \\ \frac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta}-\bar{\delta}_c), \end{cases} \tag{1.2.29}$$

when $\bar{\delta} \leq \bar{\delta}_c$ *or* $\bar{\delta} > \bar{\delta}_c$*, respectively, with* $\bar{\delta}_c := \frac{1}{2}3^{2/3}\kappa^2$ *and* $\kappa = \frac{2}{3}$*.*

Remark 1.2.6. *The choice of the zero level set of* u^ε *in the definition of the truncated version* u_0^ε *of* u^ε *in* (1.2.26) *was arbitrary. We could equivalently use the level set* $\{u^\varepsilon = c\}$ *for any* $c \in (-1,1)$ *fixed.*

1.3 Proof of Theorem 1

The presentation is clarified by working with the rescaled energy $\bar{E}^\varepsilon$ defined by (2.2.4) rather than E^ε directly. We begin by proving Part i) of Theorem 1, the Lower Bound.

1.3.1 Proof of Lower Bound, Theorem 1 i)

Step 1: Expansion of E^ε *in terms of Green's Potential.*

First we define a *truncated* droplet size:

$$A_i^* := \begin{cases} A_i & A_i > \frac{12\pi\bar{\delta}}{\kappa^2}, \\ \max\left(\left(\frac{12\pi\bar{\delta}}{\kappa^2}\right)^{1/2}, 1\right) A_i^{1/2} & \text{otherwise}, \end{cases} \tag{1.3.1}$$

and the isoperimetric deficit

$$I_{\text{def}}^{\varepsilon} := \frac{1}{|\ln \varepsilon|} \sum_i P_i - 2\sqrt{\pi} A_i^{1/2}, \tag{1.3.2}$$

which will be used throughout the proof. The purpose of defining the truncated droplet size (2.3.2) will become clear. We will make repeated use of the inequality

$$r(\omega) \leq \frac{1}{2} |\partial \omega|, \tag{1.3.3}$$

for any compact set $\omega \subset \mathbb{R}^2$, where r is the radius of ω as defined in [99]:

$$r(\omega) = \inf\{r : \omega \subset B_r\}, \tag{1.3.4}$$

where B_r is any ball of radius r. This statement is proven in [99], Proposition 4.4.

We begin by expanding the term involving v in terms of the Green potential for the operator $-\Delta + \kappa^2 I$ on $\mathbb{T}_\ell^2$:

$$\begin{aligned} \bar{E}^{\varepsilon}(v^{\varepsilon}) \geq & I_{\text{def}}^{\varepsilon} + \frac{1}{|\ln \varepsilon|} \left(\sum_i 2\sqrt{\pi} A_i^{1/2} - \frac{2\bar{\delta}}{\kappa^2} A_i \right) \\ & + 2 \sum_i A_i^2 \iint_{\Omega_i \times \Omega_i} G(x,y) d\mu_i^{\varepsilon}(x) d\mu_i^{\varepsilon}(y) \\ & + 2 \int \int_{|x-y| > \eta_0} G(x,y) d\mu^{\varepsilon}(x) d\mu^{\varepsilon}(y) \end{aligned} \tag{1.3.5}$$

where $\eta_0 > 2\sum_i |\partial \Omega_i| \overset{(1.3.3)}{\geq} 4\sum_i r(\Omega_i) \geq 2\max_{(x,y)\in\Omega_i\times\Omega_i} |x-y|$. Observe that thanks to (1.2.13) we may choose η_0 so that the above holds for all $\varepsilon > 0$. To simplify notation we've set

$$\mu_i^\varepsilon(x) := \frac{\chi_{\Omega_i}}{|\Omega_i||\ln\varepsilon|}. \tag{1.3.6}$$

We recall that the Green's function for $-\Delta + \kappa^2 I$ on $\mathbb{T}_\ell^2$ can be written as $G(x,y) = -\frac{1}{2\pi}\ln|x-y| + O(|x-y|)$. Using this fact we have the following:

$$\begin{aligned}
&2\sum_i A_i^2 \int\int_{\Omega_i\times\Omega_i} G(x,y)d\mu_i^\varepsilon(x)d\mu_i^\varepsilon(y)\\
&\geq -\frac{1}{\pi}\sum_i A_i^2 \int_{\Omega_i\times\Omega_i} \left(\ln|x-y| + C\right) d\mu_i^\varepsilon(x)d\mu_i^\varepsilon(y). && (1.3.7)\\
&\geq -\frac{1}{\pi|\ln\varepsilon|^2}\sum_i A_i^2 \int_{\overline{\Omega}_i\times\overline{\Omega}_i} \left(\ln(|\varepsilon^{1/3}|\ln\varepsilon|^{2/3}|\overline{x}-\overline{y}|) + C\right)\frac{dxdy}{A_i^2}, && (1.3.8)
\end{aligned}$$

where in equation (1.3.8) we have rescaled coordinates $\bar{x} = \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}x$, $\bar{y} = \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}$. Expanding the logarithm (1.3.8), we claim (1.3.8) can be bounded from below by

$$\begin{aligned}
&\sum_i A_i^2\left(\frac{1}{3\pi|\ln\varepsilon|} - O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right) - \frac{1}{|\ln\varepsilon|^2}\int_{\overline{\Omega}_i\times\overline{\Omega}_i}\ln|\overline{x}-\overline{y}|\frac{d\overline{x}d\overline{y}}{|\overline{\Omega}|^2}\right) && (1.3.9)\\
&\overset{(1.3.3)}{\geq} \sum_i A_i^2\left(\frac{1}{3\pi|\ln\varepsilon|} - O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right) - \frac{1}{|\ln\varepsilon|^2}\ln\sum_i|\partial\overline{\Omega}_i|\right) && (1.3.10)\\
&\overset{(1.2.13)}{\geq} \sum_i A_i^2\left(\frac{1}{3\pi|\ln\varepsilon|} - O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right) - \frac{1}{|\ln\varepsilon|^2}\ln|\ln\varepsilon|\right). && (1.3.11)
\end{aligned}$$

Indeed using $|x-y| < r(\Omega_i)$ in (1.3.9) along with (1.3.3) establishes (1.3.10). Now observe the term in parentheses appearing in (1.3.16) is positive for ε sufficiently small. Using this and the fact that $A_i^* \geq A_i$, we may bound

(1.3.11) from below by

$$\sum_i (A_i^*)^2 \left(\frac{1}{3\pi|\ln\varepsilon|} - C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2} \right) \right), \tag{1.3.12}$$

where $C > 0$ is a universal constant. It is clear from the definition of A_i^* that there exists a constant $c > 0$ such that

$$(A_i^*)^2 \leq cA_i.$$

Combining this fact with (1.3.12), choosing $\delta > Cc\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}$ we have

$$\begin{aligned} \bar{E}^\varepsilon(v^\varepsilon) \geq & I^\varepsilon_{\mathrm{def}} + \frac{1}{|\ln\varepsilon|}\left(\sum_i 2\sqrt{\pi}A_i^{1/2} - \left(\frac{2\bar{\delta}}{\kappa^2} + \delta \right) A_i \right) \\ & + \frac{1}{3\pi|\ln\varepsilon|}\sum_i (A_i^*)^2 \\ & + 2\int_{|x-y|>\eta_0} G(x,y)d\mu^\varepsilon(x)d\mu^\varepsilon(y). \end{aligned} \tag{1.3.13}$$

Step 2: Optimization of droplet size.

We have from Step 1 the lower bound

$$\bar{E}^{\varepsilon}(v^{\varepsilon}) \geq \frac{1}{|\ln\varepsilon|}\left(\sum_i (P_i - 2\sqrt{\pi}A_i^{1/2}\right) \tag{1.3.14}$$

$$+ \frac{1}{|\ln\varepsilon|}\sum_i 2\sqrt{\pi}A_i^{1/2} + \frac{1}{3\pi}(A_i^*)^2 - \left(\frac{2\bar{\delta}}{\kappa^2} + \delta\right)A_i \tag{1.3.15}$$

$$+ 2\int_{|x-y|>\eta_0} G(x,y)d\mu^{\varepsilon}(x)d\mu^{\varepsilon}(y) - O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right). \tag{1.3.16}$$

Focusing on (2.3.25) we define

$$f(x) = \frac{2\sqrt{\pi}}{\sqrt{x}} + \frac{1}{3\pi}x,$$

and observe that f is convex and attains its minimum of $3^{2/3}$ at $x = 3^{2/3}\pi$ with

$$f''(x) = \frac{3\sqrt{\pi}}{2x^{5/2}}. \tag{1.3.17}$$

We claim we can now bound (2.3.25) from below by:

$$\frac{1}{|\ln\varepsilon|}\left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \delta\right)\sum_i A_i + \sum_{A_i > \frac{12\pi\bar{\delta}}{\kappa^2}} 2\sqrt{\pi}A_i^{1/2} + \frac{2\bar{\delta}}{\kappa^2}A_i \tag{1.3.18}$$

$$+ \frac{3^{-2/3}}{2\pi^2}\sum_{A_i < \pi 3^{2/3}} A_i(A_i - 3^{2/3}\pi)^2 \tag{1.3.19}$$

$$+ \frac{3\sqrt{\pi}}{2}\frac{1}{|\ln\varepsilon|}\sum_{\frac{12\bar{\delta}\pi}{\kappa^2} \geq A_i \geq 3^{2/3}\pi} A_i^{-3/2}(A_i - 3^{2/3}\pi)^2. \tag{1.3.20}$$

First observe that for $A_i \geq \frac{12\pi\bar{\delta}}{\kappa^2}$, the terms in line (2.3.25) are bounded from below by

$$2\sqrt{\pi}A_i^{1/2} + \frac{2\bar{\delta}}{\kappa^2}A_i \geq 2\sqrt{\pi}A_i^{1/2} + \frac{2\bar{\delta}}{\kappa^2}A_i + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \delta\right)A_i, \tag{1.3.21}$$

since $2\bar{\delta}/\kappa^2 > 3^{2/3}$.

When $A_i < \frac{12\pi\bar{\delta}}{\kappa^2}$ we utilize convexity of f and (1.3.17). Indeed

$$2\sqrt{\pi}A_i^{1/2} + \frac{1}{3\pi}A_i^2 - \left(\frac{2\bar{\delta}}{\kappa^2} + \delta\right)A_i = A_i\left(2\sqrt{\pi}A_i^{-1/2} + \frac{1}{3\pi}A_i - \frac{2\bar{\delta}}{\kappa^2} - \delta\right) \tag{1.3.22}$$

$$\geq \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \delta\right)A_i + I_{\text{vol}}^{\varepsilon}, \tag{1.3.23}$$

the last line following from a second order Taylor expansion of f about $x = 3^{2/3}\pi$ and where we've set $I_{\text{vol}}^{\varepsilon} = (1.3.19) + (1.3.20)$. Combining (2.3.27) and (1.3.23) and summing over i yields (1.3.18)–(1.3.20). We now take the limit $\varepsilon \to 0$ and $\eta_0 \to 0$.

Step 3: The lower bound.

We may in fact conclude from (1.2.13), (2.2.4) and (2.2.19) that

$$\limsup_{\varepsilon\to 0} \int |\nabla v^{\varepsilon}|^2 + \kappa^2|v^{\varepsilon}|^2 < +\infty. \tag{1.3.24}$$

Consequently up to a subsequence

$$v^\varepsilon \rightharpoonup v \text{ in } H^1(\mathbb{T}^2_\ell).$$

$$\mu^\varepsilon \overset{*}{\rightharpoonup} \mu \text{ in } C^*(\mathbb{T}^2_\ell).$$

where

$$-\Delta v + \kappa^2 v = \mu$$

holds in the distributional sense. Now passing to the limit we obtain

$$\liminf_{\varepsilon \to 0} \bar{E}^\varepsilon(v^\varepsilon) \geq \left[3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \delta\right] \int d\mu + 2 \int \int_{|x-y|>\eta_0} G(x,y)d\mu(x)d\mu(y).$$

Recalling $G \geq 0$, a simple application of the Monotone Convergence Theorem yields

$$\liminf_{\varepsilon \to 0} \bar{E}^\varepsilon(v^\varepsilon) \geq \left[3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right] \int d\mu + 2 \int \int G(x,y)d\mu(x)d\mu(y),$$

upon sending $\eta_0 \to 0$ and then sending $\delta \to 0$. $\square$

We now argue in favor of the corresponding upper bound in *ii*) of Theorem 1. The construction resembles quite closely that of the vortex construction in [98] for the two dimensional Ginzburg Landau functional and indeed we borrow certain ideas from their proof.

1.3.2 Proof of Upper Bound, Theorem 1 ii)

As in the proof of the lower bound, we set $\mu_i^\varepsilon = \frac{\chi_{\Omega_i}}{|\ln\varepsilon||\Omega_i|}A_i$ so that $\mu^\varepsilon = \sum \mu_i$. Using the approximation argument of Proposition II.2 in [98] we may assume that $\mu = gdx$ for some density g satisfying

$$\frac{1}{C} \leq g \leq C. \tag{1.3.25}$$

The essential elements of the proof closely parallel the construction in [98], where we occasionally refer the reader to for details.

Step 1: Construction of the configuration.

Given M and μ, divide $\mathbb{T}_\ell^2$ into disjoint squares $\{K_i\}$ of length δ where

$$\varepsilon^{1/3}|\ln\varepsilon|^{-1/3} \ll \delta \ll 1.$$

We claim it is possible to place a total of

$$N(\varepsilon) = \left[\frac{1}{3^{2/3}}\frac{|\ln\varepsilon|}{\pi}\mu(\mathbb{T}_\ell^2)\right],$$

disjoint spherical droplets with centers $\{a_i\}$ in $\mathbb{T}_\ell^2$ each with radii

$$r_i^\varepsilon = 3^{1/3}[\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}]$$

and satisfying for all $i \neq j$

$$|a_i - a_j| \geq \frac{C}{\sqrt{N(\varepsilon)}}, \tag{1.3.26}$$

for some universal constant C depending only on μ. Here $[x]$ denotes the smallest integer $m \leq x$. More precisely, in each K_i we wish to place

$$N_{K_i}(\varepsilon) = \left[\frac{1}{3^{2/3}} \frac{|\ln \varepsilon|}{\pi} \mu(K_i)\right]$$

points satisfying (1.3.26) and in addition

$$d(a_i, \partial K_i) \geq \frac{C}{\sqrt{N(\varepsilon)}}. \tag{1.3.27}$$

Here $[x]$ once again denotes the smallest integer less than or equal to $x \geq 0$. As argued in [98] our ability to do this follows from the estimate:

$$\delta^2/C \leq \mu(K_i) \leq C\delta^2, \tag{1.3.28}$$

which follows from (3.2.2). We finally define our configuration by setting $\Omega_i := B(a_i, r_i)$.

With these choices we have

$$\begin{aligned}
\bar{E}^\varepsilon(v^\varepsilon) &= 2\sqrt{\pi}\left(\frac{\mu(\mathbb{T}^2_\ell)N(\varepsilon)}{|\ln\varepsilon|}\right)^{1/2} - \frac{2\bar{\delta}}{\kappa^2}\mu(\mathbb{T}^2_\ell) + 2\sum_i \int\int_{\Omega_i\times\Omega_i} G(x,y)d\mu_i^\varepsilon(x)d\mu_i^\varepsilon(y) \\
&\quad + 2\sum_{i\neq j}\int\int G(x,y)d\mu_i^\varepsilon(x)d\mu_j^\varepsilon(y) \\
&= 2\left(\frac{1}{3}\right)^{1/3}\mu(\mathbb{T}^2_\ell) - \frac{2\bar{\delta}}{\kappa^2}\mu(\mathbb{T}^2_\ell) + 2\sum_i \int\int_{\Omega_i\times\Omega_i} G(x,y)d\mu_i^\varepsilon(x)d\mu_i^\varepsilon(y) \qquad (1.3.29) \\
&\quad + 2\sum_{i\neq j}\int\int G(x,y)d\mu_i^\varepsilon(x)d\mu_j^\varepsilon(y) \qquad (1.3.30)
\end{aligned}$$

The term on line (1.3.30) will be shown to converge to $\int\int G d\mu d\mu$ while the term involving the Greens potential on line (1.3.29), which represents the self interaction of droplets, will make a first order contribution to the limit. The most technical aspect of this proof is showing that our construction separates the droplets sufficiently so that they do not accumulate near the diagonal.

Step 2: Convergence of the configurations.

Defining μ^ε as before it is clear from construction that

$$\mu^\varepsilon \rightharpoonup \mu \text{ in } C_0^*.$$

Hence it remains to ensure that away from the diagonal $\Delta = \{x = y : (x,y) \subset \mathbb{T}^2_\ell \times \mathbb{T}^2_\ell\}$, the droplets $\{a_i\}$ do not become too concentrated and this is where estimate (1.3.26) comes in. Let Δ_η denote an η neighborhood of the diagonal Δ. Then on $\mathbb{T}^2_\ell \times (\mathbb{T}^2_\ell \backslash \Delta_\eta)$ the continuity of $G(\cdot,\cdot)$ and the weak convergence of μ^ε ensures that

$$\lim_{\varepsilon\to 0}\int\int_{\mathbb{T}^2_\ell\times(\mathbb{T}^2_\ell\setminus\Delta_\eta)} G(x,y)d\mu^\varepsilon(x)d\mu^\varepsilon(y) = \int\int_{\mathbb{T}^2_\ell\times(\mathbb{T}^2_\ell\setminus\Delta_\eta)} G(x,y)d\mu(x)d\mu(y). \tag{1.3.31}$$

Let I_η be the collection of indices (i,j) such that $B_\varepsilon(a_i)\times B_\varepsilon(a_j)\cap\Delta_\eta\neq\emptyset$. Then we have

$$\begin{aligned}\int_{\Delta_\eta} G d\mu^\varepsilon d\mu^\varepsilon &\le \sum_{(i,j)\in I_\eta, i\neq j}\int\int G d\mu_i^\varepsilon d\mu_j^\varepsilon + \sum_{i=1}^{N(\varepsilon)}\int\int G d\mu_i^\varepsilon d\mu_i^\varepsilon \qquad (1.3.32)\\ &\le \sum_{(i,j)\in I_\eta, i\neq j}\int\int G d\mu_i^\varepsilon d\mu_j^\varepsilon + \frac{1}{3\pi|\ln\varepsilon|}\sum_{i=1}^{N(\varepsilon)} A_i^2 + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)\end{aligned}$$

as in (1.3.9). As argued in [98], estimate (1.3.26) allows us to conclude that

$$\sum_{(i,j)\in I_\eta, i\neq j}\int\int G d\mu_i^\varepsilon d\mu_j^\varepsilon \to 0 \text{ as } \eta\to 0.$$

Therefore from (1.3.31) we have

$$\int_{\Delta_\eta} G d\mu^\varepsilon d\mu^\varepsilon \le \frac{1}{3\pi}|\ln\varepsilon|\frac{1}{N(\varepsilon)} + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right) + o(\eta) \text{ as } \eta\to 0. \tag{1.3.33}$$

Finally combining (1.3.33), (1.3.32), (1.3.29)–(1.3.30) along with our choice of $N(\varepsilon)$, we have

$$\lim_{\varepsilon\to 0}\bar{E}^\varepsilon(v^\varepsilon) = \left[3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right]\int d\mu + 2\int\int G(x,y)d\mu(x)d\mu(y),$$

as required. The fact that $v^\varepsilon \rightharpoonup v$ follows from the uniform bounds just demonstrated on the terms involving the Green's potential in lines (1.3.29) and (1.3.30), from which it follows that (1.2.18) is satisfied distributionally. □

Remark 1.3.1. *We have in fact established Theorem 2 which is clear by inspecting (1.4.8) and (1.3.19) and (1.3.20). Indeed we have for an almost minimizing sequence* $\{v^\varepsilon\}_\varepsilon$*,*

$$\lim_{\varepsilon\to 0} E^\varepsilon(v^\varepsilon) - E_0(\bar{\mu}) = 0.$$

Observing that the terms (1.4.8), (1.3.19), (1.3.20) to not contribute to $E_0(\bar{\mu})$*, we establish (1.2.21), (1.2.22) and (1.2.23). From well known [14] principles of* Γ*-convergence,* v^ε *converge to a* $v_{\tilde{\mu}}$*, where* $\tilde{\mu}$ *is a minimizer of* E^0*. By uniqueness of minimizers (re. Corollary 1.2.2) we have that* $\tilde{\mu} = \bar{\mu}$ *and so* $v_{\tilde{\mu}} = v_{\bar{\mu}}$ *by uniqueness of solutions to (1.2.18).*

1.4 Proof of Theorem 3

We now turn to the proof of Theorem 3 extending the result of Theorem 1 for the sharp interface energy E^ε to the diffuse interface energy $\mathcal{E}^\varepsilon$. The proof proceeds by a refinement of the ideas of [80, Sec. 4] to establish matching upper and lower bounds for $\mathcal{E}^\varepsilon$ in terms of E^ε for sequences with bounded energy.

Step 1: Approximate lower bound.

In the following, it is convenient to rewrite the energy from (4.7.1) in an equivalent form

$$\mathcal{E}^\varepsilon[u^\varepsilon] = \int_{\mathbb{T}_\ell^2} \left(\frac{\varepsilon^2}{2} |\nabla u^\varepsilon|^2 + W(u^\varepsilon) + \frac{1}{2} |\nabla v^\varepsilon|^2 \right) dx, \tag{1.4.1}$$
$$-\Delta v^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon, \qquad \int_{\mathbb{T}_\ell^2} v^\varepsilon dx = 0.$$

Fix any $\delta \in (0,1)$ and consider a sequence $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ such that $\limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq 1$ and $\mathcal{E}^\varepsilon[u^\varepsilon] \leq C\varepsilon^{4/3} |\ln \varepsilon|^{2/3}$ for some $C > 0$ independent of ε. Then we claim that

$$\limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 1, \qquad \limsup_{\varepsilon \to 0} \|v^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 0. \tag{1.4.2}$$

Indeed, for the first statement we have from the definition of $\mathcal{E}^\varepsilon$ in (4.7.1) that

$$|\Omega_0^\delta| \leq C\varepsilon^{4/3} |\ln \varepsilon|^{2/3} \delta^{-2}, \qquad \Omega_0^\delta := \{-1 + \delta \leq u^\varepsilon \leq 1 - \delta\}, \tag{1.4.3}$$

for some $C > 0$ independent of ε. Hence, in particular, $\limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \geq 1$, proving the first statement of (1.4.2). To prove the second statement in (1.4.2), we note that by standard elliptic theory [51] we have $\|v^\varepsilon\|_{W^{2,p}(\mathbb{T}_\ell^2)} \leq C'$ for any $p > 2$ and some $C' > 0$ independent of ε and, hence, by Sobolev imbedding $\|\nabla v^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq C''$ for some $C'' > 0$ independent of ε as well. Therefore, applying Poincaré's inequality, we obtain

$$C\varepsilon^{4/3} |\ln \varepsilon|^{2/3} \geq \mathcal{E}^\varepsilon[u^\varepsilon] \geq C' \int_{\mathbb{T}_\ell^2} |v^\varepsilon|^2 dx \geq C'' \|v^\varepsilon\|^4_{L^\infty(\mathbb{T}_\ell^2)}, \tag{1.4.4}$$

for some $C', C'' > 0$ independent of ε, yielding the claim.

In view of (1.4.2), for small enough $\varepsilon > 0$ we have $\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq 1 + \delta^3$ and $\|v^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq \delta^3$, and by the assumption on energy we may further assume that $\mathcal{E}^\varepsilon[u^\varepsilon] \leq \delta^{12}$. Therefore, by [80, Proposition 4.2] there exists a function $\tilde{u}_0^\varepsilon \in \mathcal{A}$ such that

$$\mathcal{E}^\varepsilon[u^\varepsilon] \geq (1 - \delta^{1/2}) E^\varepsilon[\tilde{u}_0^\varepsilon]. \tag{1.4.5}$$

In particular, $(\tilde{u}_0^\varepsilon)$ satisfy the assumptions of Theorem 1, and, therefore, upon extraction of subsequences we have $\tilde{\mu}_0^\varepsilon \rightharpoonup \mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cup H^{-1}(\mathbb{T}_\ell^2)$ in $(C(\mathbb{T}_\ell^2))^*$, where

$$d\tilde{\mu}_0^\varepsilon(x) := \tfrac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1 + \tilde{u}_0^\varepsilon(x))dx. \tag{1.4.6}$$

Furthermore, recalling that by construction the jump set of $\tilde{u}_0^\varepsilon$ coincides with the level set $\{u^\varepsilon = c\}$ for some $c \in (-1+\delta, 1-\delta)$, see the proof of [80, Lemma 4.1], from (1.4.3) we have

$$\|\tilde{u}_0^\varepsilon - u_0^\varepsilon\|_{L^1(\mathbb{T}_\ell^2)} \leq C\varepsilon^{4/3}|\ln\varepsilon|^{2/3}\delta^{-2}, \tag{1.4.7}$$

where u_0^ε is given by (1.2.26), for some $C > 0$ independent of ε (recall that the jump set of u_0^ε is the zero level set $\{u^\varepsilon = 0\}$). Comparing (1.4.7) with (1.4.6), we then see that $\mu_0^\varepsilon \rightharpoonup \mu$ in $(C(\mathbb{T}_\ell^2))^*$ as well. The result of part (i) of Theorem 3 then follows by arbitrariness of $\delta > 0$ via a diagonal process. $\square$

Step 2: Approximate upper bound.

We now construct the approximate upper bounds for a suitable lifting of the recovery sequences in the proof of Theorem 1 to $\mathcal{A}^\varepsilon$. Let $(\tilde{u}_0^\varepsilon) \in \mathcal{A}$ be a recovery sequence from Sec. 1.3.2. These sequences consist of circular droplets of the optimal radius $r = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3} \gg \varepsilon^{1/2}$ and mutual distance $d \geq C|\ln\varepsilon|^{-1/2} \gg \varepsilon^{1/2}$, for some $C > 0$ independent of ε. In addition, since

$$\mathcal{E}^\varepsilon[\tilde{u}^\varepsilon] = \frac{\varepsilon}{2}\int_{\mathbb{T}_\ell^2} |\nabla \tilde{u}_0^\varepsilon|\, dx + \frac{1}{2}\int_{\mathbb{T}_\ell^2} \Big(|\nabla \tilde{v}^\varepsilon|^2 + \kappa^2 |\tilde{v}^\varepsilon|^2\Big) dx, \tag{1.4.8}$$

where $\tilde{v}^\varepsilon(x) = \int_{\mathbb{T}_\ell^2} G(x-y)(\tilde{u}_0^\varepsilon(y) - \bar{u}^\varepsilon)dy$, by the argument of (1.4.4) one can see that $\limsup_{\varepsilon\to 0} \|\tilde{v}^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 0$. Therefore, for any $\delta \in (0,1)$ and $\varepsilon > 0$ sufficiently small we have $\|\tilde{v}^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq \delta$ and $E^\varepsilon[\tilde{u}_0^\varepsilon] \leq \delta^{5/2}$. So we can apply [80, Proposition 4.3] to obtain a function $u^\varepsilon \in \mathcal{A}^\varepsilon$ such that

$$\mathcal{E}^\varepsilon[u^\varepsilon] \leq (1 + \delta^{1/2}) E^\varepsilon[\tilde{u}_0^\varepsilon]. \tag{1.4.9}$$

Furthermore, by construction (see [80, Eq. (4.33)]) and arbitrariness of $\delta > 0$, we also have $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 1$. For the same reason we have

$$\|\tilde{u}_0^\varepsilon - u_0^\varepsilon\|_{L^1(\mathbb{T}_\ell^2)} \leq C\varepsilon^{4/3}|\ln\varepsilon|^{2/3}, \tag{1.4.10}$$

where u_0^ε is given by (1.2.26), for some $C > 0$ independent of ε. Hence $\mu_0^\varepsilon \rightharpoonup \mu = \lim_{\varepsilon\to 0} \tilde{\mu}_0^\varepsilon$ in $(C(\mathbb{T}_\ell^2))^*$. The result of part (ii) of Theorem 3 again follows by arbitrariness of $\delta > 0$ via diagonal process. $\square$

Remark 1.4.1. *It is possible to chose $\delta = \varepsilon^\alpha$ for $\alpha > 0$ sufficiently small in the arguments of the proof of Theorem 3. Therefore, given a sequence of almost minimizers $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ of $\mathcal{E}^\varepsilon$ and the corresponding sequence $(u_0^\varepsilon) \in \mathcal{A}$ of almost minimizers of E^ε, one has*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon[u^\varepsilon] = \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}E^\varepsilon[u_0^\varepsilon] + O(\varepsilon^\alpha), \tag{1.4.11}$$

for some $\alpha \ll 1$, as $\varepsilon \to 0$.

Chapter 2

The Γ-limit of the two-dimensional Ohta-Kawasaki energy. II. Droplet arrangement at the sharp interface level via the renormalized energy.

2.1 Introduction

This is our second paper devoted to the Γ-convergence study of the two-dimensional Ohta-Kawasaki energy functional [87] in two space dimensions in the regime near the onset of non-trivial minimizers. The energy functional has the following form:

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x, y)(u(y) - \bar{u})\, dx\, dy, \tag{2.1.1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $V(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg-Landau potential $V(u) = \frac{9}{32}(1 - u^2)^2$ (for simplicity, the overall coefficient in V is chosen to make the associated surface tension constant to be equal to ε, i.e., we have $\int_{-1}^{1} \sqrt{2V(u)}\, du = 1$), $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, i.e., G_0 solves

$$-\Delta G_0(x, y) = \delta(x - y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x, y)\, dx = 0, \tag{2.1.2}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|} \int_\Omega u\, dx = \bar{u}. \tag{2.1.3}$$

For a discussion of the motivation and the main quantitative features of this model, see our first paper [54], as well as [79, 80]. For specific applications to physical systems, we refer the reader to [34, 58, 73, 78, 79, 86, 87, 110].

In our first paper [54], we established the leading order term in the Γ-expansion of the energy in (4.1.1) in the scaling regime corresponding to the threshold between trivial and non-trivial minimizers. More precisely, we studied the behavior of the energy as $\varepsilon \to 0$ when

$$\bar{u}^{\varepsilon} := -1 + \varepsilon^{2/3} |\ln \varepsilon|^{1/3} \bar{\delta}, \tag{2.1.4}$$

for some fixed $\bar{\delta} > 0$ and when Ω is a flat two-dimensional torus of side length ℓ, i.e., when $\Omega = \mathbb{T}_{\ell}^2 = [0, \ell)^2$, with periodic boundary conditions. As follows from [54, Theorem 2] and the arguments in the proof of [54, Theorem 3], in this regime minimizers of $\mathcal{E}$ consist of many small "droplets" (regions where $u > 0$) and their number blows up as $\varepsilon \to 0$. We showed that, after a suitable rescaling the energy functional in (4.1.1) Γ-converges in the sense of convergence of the (suitably normalized) droplet densities, to the limit functional $E^0[\mu]$ defined for all densities $\mu \in \mathcal{M}(\mathbb{T}_{\ell}^2) \cap H^{-1}(\mathbb{T}_{\ell}^2)$ by:

$$E^0[\mu] = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right) \int_{\mathbb{T}_{\ell}^2} d\mu + 2 \iint_{\mathbb{T}_{\ell}^2 \times \mathbb{T}_{\ell}^2} G(x-y) d\mu(x) d\mu(y), \tag{2.1.5}$$

where $G(x)$ is the *screened* Green's function of the Laplacian, i.e., it solves the periodic problem for the equation

$$-\Delta G + \kappa^2 G = \delta(x) \quad \text{in} \quad \mathbb{T}_{\ell}^2, \tag{2.1.6}$$

and $\kappa = 1/\sqrt{V''(1)} = \frac{2}{3}$. Here we noted that the double integral in (4.1.9) is well defined in the sense $\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y) := \int_{\mathbb{T}_\ell^2} v d\mu$, where the latter is interpreted as the Hahn-Banach extension of the corresponding linear functional defined by the integral on smooth test functions (see also [99, Sec. 7.3.1] and [17] for further discussion). Indeed, $v := G * d\mu$ is the convolution understood distributionally, i.e., $\langle G * d\mu, f\rangle := \langle G * f, d\mu\rangle = \int_{\mathbb{T}_\ell^2}\left(\int_{\mathbb{T}_\ell^2} G(x-y)f(y)dy\right)d\mu(x)$ for every $f \in C^\infty(\mathbb{T}_\ell^2)$ and, hence, by elliptic regularity $\|v\|_{H^1(\mathbb{T}_\ell^2)} \leq C\|f\|_{H^{-1}(\mathbb{T}_\ell^2)}$ for some $C > 0$, so $v \in H^1(\mathbb{T}_\ell^2)$.

In particular, for $\bar\delta > \bar\delta_c$, where

$$\bar\delta_c := \frac{1}{2}3^{2/3}\kappa^2, \tag{2.1.7}$$

the limit energy $E^0[\mu]$ is minimized by $d\mu(x) = \bar\mu\, dx$, where

$$\bar\mu = \tfrac{1}{2}(\bar\delta - \bar\delta_c) \qquad \text{and} \qquad E^0[\bar\mu] = \tfrac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c). \tag{2.1.8}$$

When $\bar\delta \leq \bar\delta_c$, the limit energy is minimized by $\mu = 0$, with $E^0[0] = \bar\delta^2/(2\kappa^2)$. The value of $\bar\delta = \bar\delta_c$ thus serves as the threshold separating the trivial and the non-trivial minimizers of the energy in (4.1.1) together with (4.2.4) for sufficiently small ε. Above that threshold, the droplet density of energy-minimizers converges to the uniform density $\bar\mu$.

The key point that enables the analysis above is a kind of Γ-equivalence between the energy functional in (4.1.1) and its screened sharp interface analog (for general notions of Γ-equivalence or variational equivalence, see [4, 15]):

$$E^\varepsilon[u] = \frac{\varepsilon}{2}\int_{\mathbb{T}_\ell^2} |\nabla u|\, dx + \frac{1}{2}\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} (u(x)-\bar{u}^\varepsilon)G(x-y)(u(y)-\bar{u}^\varepsilon)\, dx\, dy. \quad (2.1.9)$$

Here, G is the screened potential as in (2.1.6), and $u \in \mathcal{A}$, where

$$\mathcal{A} := BV(\mathbb{T}_\ell^2; \{-1, 1\}), \quad (2.1.10)$$

and we note that on the level of E^ε the neutrality condition in (4.1.3) has been removed. As we showed in [54], following the approach of [80], for $\mathcal{E}^\varepsilon$ given by (4.1.1) in which $\bar{u} = \bar{u}^\varepsilon$ and $\bar{u}^\varepsilon$ is defined in (4.2.4), we have

$$\min \mathcal{E}^\varepsilon = \min E^\varepsilon + O(\varepsilon^\alpha \min E^\varepsilon), \quad (2.1.11)$$

for some $\alpha > 0$. Therefore, in order to understand the leading order asymptotic expansion of the minimal energy $\min \mathcal{E}^\varepsilon$ in terms of $|\ln\varepsilon|^{-1}$, it is sufficient to obtain such an expansion for $\min E^\varepsilon$. This is precisely what we will do in the present paper.

In view of the discussion above, in this paper we concentrate our efforts on the analysis of the sharp interface energy E^ε in (4.1.4). An extension of our results to the original diffuse interface energy $\mathcal{E}^\varepsilon$ would lead to further technical complications that lie beyond the scope of the present paper and will be treated elsewhere. Here we wish to extract the next order non-trivial term in the Γ-expansion of the sharp interface energy E^ε after (4.1.9). In contrast to [80], we will not use the Euler-Lagrange equation associated to (4.1.4), so our results about minimizers will also be valid for "almost minimizers" (cf. Theorem 5).

We recall that for $\varepsilon \ll 1$ the energy minimizers for E^ε and $\bar{\delta} > \bar{\delta}_c$ consist of $O(|\ln \varepsilon|)$ nearly circular droplets of radius $r \simeq 3^{1/3}\varepsilon^{1/3}|\ln \varepsilon|^{-1/3}$ uniformly distributed throughout $\mathbb{T}^2_\ell$ [80, Theorem 2.2]. This is in contrast with the study of [28, 29] for a closely related energy, where the number of droplets remains bounded as $\varepsilon \to 0$, and the authors extract a limiting interaction energy for a finite number of points.

By Γ-convergence, we obtained in [54, Theorem 1] the convergence of the droplet density of almost minimizers (u^ε) of E^ε:

$$\mu^\varepsilon(x) := \frac{1}{2}\varepsilon^{-2/3}|\ln \varepsilon|^{-1/3}(1 + u^\varepsilon(x)), \tag{2.1.12}$$

to the uniform density $\bar{\mu}$ defined in (4.1.10). However, this result does not say anything about the microscopic placement of droplets in the limit $\varepsilon \to 0$. In order to understand the asymptotic arrangement of droplets in an energy minimizer, our plan is to blow-up the coordinates by a factor of $\sqrt{|\ln \varepsilon|}$, which is the inverse of the scale of the typical inter-droplet distance, and to extract the next order term in the Γ-expansion of the energy in terms of the limits as $\varepsilon \to 0$ of the blown-up configurations (which will consist of an infinite number of point charges in the plane with identical charge).

We will show that the arrangement of the limit point configurations is governed by the Coulombic renormalized energy W, which was introduced in [99]. That energy W was already derived as a next order Γ-limit for the magnetic Ginzburg-Landau model of superconductivity [99, 100], and also for two-dimensional Coulomb gases [97]. Our results here follow the same method of [100], and yield almost identical conclusions.

The "Coulombic renormalized energy" is a way of computing a total Coulomb interaction between an infinite number of point charges in the plane, neutralized by a uniform background charge (for more details see Section 2.2). It is shown in [100] that its minimum is achieved. It is also shown there that the minimum among simple lattice patterns (of fixed volume) is uniquely achieved by the triangular lattice (for a closely related result, see [20]), and it is conjectured that the triangular lattice is also a global minimizer. This triangular lattice is called "Abrikosov lattice" in the context of superconductivity and is observed in experiments in superconductors [113].

The next order limit of E^ε that we shall derive below is in fact the average of the energy W over all limits of blown-up configurations (i.e. average with respect to the blow up center). Our result says that limits of blow-ups of (almost) minimizers should minimize this average of W. This permits one to distinguish between different patterns at the microscopic scale and it leads, in view of the conjecture above, to expecting to see triangular lattices of droplets (in the limit $\varepsilon \to 0$), around almost every blow-up center (possibly with defects). Note that the selection of triangular lattices was also considered in the context of the Ohta-Kawasaki energy by Chen and Oshita [20], but there they were only obtained as minimizers among simple lattice configurations consisting of non-overlapping ideally circular droplets.

It is somewhat expected that minimizers of the Ohta-Kawasaki energy in the macroscopic setting are periodic patterns in all space dimensions (in fact in the original paper [87] only periodic patterns are considered as candidates for minimizers). This fact has never been proved rigorously, except in one dimension by Müller [77] (see also [92, 118]), and at the moment seems very difficult. For higher-dimensional problems, some recent results in this direction were obtained in [2, 80, 108] establishing equidistribution of energy in various versions of the Ohta-Kawasaki model on macroscopically large domains. Several other results [28, 29, 32, 109] were also obtained to characterize the geometry of minimizers on smaller domains. The results we obtain here, in the regime of small volume fraction and in dimension two, provide more quantitative and qualitative information (since we are able to distinguish between the cost of various patterns, and have an idea of what the minimizers should be like) and a first setting where periodicity can be expected to be proved.

The Ohta-Kawasaki setting differs from that of the magnetic Ginzburg-Landau model in the fact that the droplet "charges" (i.e., their volume) are all positive, in contrast with the vortex degrees in Ginzburg-Landau, which play an analogous role and can be both positive and negative integers. It also differs in the fact that the droplet volumes are not quantized, contrary to the degrees in the Ginzburg-Landau model. This creates difficulties and the major difference in the proofs. In particular we have to account for the possibility of many very small droplets, and we have to show that the isoperimetric terms in the energy suffice to force (almost) all the droplets to be round and of fixed volume. This has to be done at the same time as the lower bound for the other term in the energy, for example an adapted “ball construction” for non-quantized quantities has to be re-implemented, and the interplay between these two effects turns out to be delicate.

Our paper is organized as follows. In Section 2.2 we formulate the problem and state our main results concerning the Γ-limit of the next order term in the energy (4.1.4) after the zeroth order energy derived in [54] is subtracted off. In Section 2.3, we derive a lower bound on this next order energy via an energy expansion as done in [54] however isolating lower order terms obtained via the process. We then proceed via a ball construction as in [64, 99, 103] to obtain lower bounds on this energy in Section 4.5 and consequently obtain an energy density bounded from below with almost the same energy via energy displacement as in [100] in Section 4.7. In Section 2.6 we obtain explicit lower bounds on this density on bounded sets in the plane in terms of the renormalized energy for a finite number of points. We are then in the appropriate setting to apply the multiparameter ergodic theorem as in [100] to extend the lower bounds obtained to global bounds, which we present at the end of Section 2.6. Finally the corresponding upper bound (cf. Part (ii) of Theorem 4) is presented in Section 2.7.

Some notations. We use the notation $(u^\varepsilon) \in \mathcal{A}$ to denote sequences of functions $u^\varepsilon \in \mathcal{A}$ as $\varepsilon = \varepsilon_n \to 0$, where $\mathcal{A}$ is an admissible class. We also use the notation $\mu \in \mathcal{M}(\Omega)$ to denote a positive finite Radon measure $d\mu$ on the domain Ω. With a slight abuse of notation, we will often speak of μ

as the “density” on Ω and set $d\mu(x) = \mu(x)dx$ whenever $\mu \in L^1(\Omega)$. With some more abuse of notation, for a measurable set E we use $|E|$ to denote its Lebesgue measure, $|\partial E|$ to denote its perimeter (in the sense of De Giorgi), and $\mu(E)$ to denote $\int_E d\mu$. The symbols $H^1(\Omega)$, $BV(\Omega)$, $C^k(\Omega)$ and $H^{-1}(\Omega)$ denote the usual Sobolev space, the space of functions of bounded variation, the space of k-times continuously differentiable functions, and the dual of $H^1(\Omega)$, respectively. The symbol $o_\varepsilon(1)$ stands for the quantities that tend to zero as $\varepsilon \to 0$ with the rate of convergence depending only on ℓ, $\bar{\delta}$ and κ.

2.2 Problem formulation and main results

In the following, we fix the parameters $\kappa > 0$, $\bar{\delta} > 0$ and $\ell > 0$, and work with the energy E^ε in (4.1.4), which can be equivalently rewritten in terms of the connected components Ω_i^ε of the family of sets of finite perimeter $\Omega^\varepsilon := \{u^\varepsilon = +1\}$, where $(u^\varepsilon) \in \mathcal{A}$ are almost minimizers of E^ε, for sufficiently small ε (cf. the discussion at the beginning of Sec. 2 in [54]). The sets Ω^ε can be decomposed into countable unions of connected disjoint sets, i.e., $\Omega^\varepsilon = \bigcup_i \Omega_i^\varepsilon$, whose boundaries $\partial\Omega_i^\varepsilon$ are rectifiable and can be decomposed (up to negligible sets) into countable unions of disjoint simple closed curves. Then the density μ^ε in (2.1.12) can be rewritten as

$$\mu^\varepsilon(x) := \varepsilon^{-2/3} |\ln \varepsilon|^{-1/3} \sum_i \chi_{\Omega_i^\varepsilon}(x), \tag{2.2.1}$$

where $\chi_{\Omega_i^\varepsilon}$ are the characteristic functions of Ω_i^ε. Motivated by the scaling analysis in the discussion preceding equation (2.1.12), we define the rescaled areas and perimeters of the droplets:

$$A_i^\varepsilon := \varepsilon^{-2/3}|\ln\varepsilon|^{2/3}|\Omega_i^\varepsilon|, \qquad P_i^\varepsilon := \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}|\partial\Omega_i^\varepsilon|. \tag{2.2.2}$$

Using these definitions, we obtain (see [54, 80]) the following equivalent definition of the energy of the family (u^ε):

$$E^\varepsilon[u^\varepsilon] = \varepsilon^{4/3}|\ln\varepsilon|^{2/3}\left(\frac{\bar\delta^2\ell^2}{2\kappa^2} + \bar E^\varepsilon[u^\varepsilon]\right), \tag{2.2.3}$$

where

$$\bar E^\varepsilon[u^\varepsilon] := \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i^\varepsilon - \frac{2\bar\delta}{\kappa^2}A_i^\varepsilon\right) + 2\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y). \tag{2.2.4}$$

Also note the relation

$$\mu^\varepsilon(\mathbb{T}_\ell^2) = \frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon. \tag{2.2.5}$$

As was shown in [54, 80], in the limit $\varepsilon \to 0$ the minimizers of E^ε are non-trivial if and only if $\bar\delta > \bar\delta_c$, and we have asymptotically

$$\min E^\varepsilon \simeq \frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c)\varepsilon^{4/3}|\ln\varepsilon|^{2/3}\ell^2 \qquad \text{as } \varepsilon\to 0. \tag{2.2.6}$$

Furthermore, if μ^ε is as in (2.2.1) and we let v^ε be the unique solution of

$$-\Delta v^\varepsilon + \kappa^2 v^\varepsilon = \mu^\varepsilon \qquad \text{in} \quad W^{2,p}(\mathbb{T}^2_\ell), \tag{2.2.7}$$

for $p < \infty$, then we have

$$v^\varepsilon \rightharpoonup \bar{v} := \frac{1}{2\kappa^2}(\bar{\delta} - \bar{\delta}_c) \qquad \text{in} \quad H^1(\mathbb{T}^2_\ell). \tag{2.2.8}$$

To extract the next order terms in the Γ-expansion of E^ε we, therefore, subtract this contribution from E^ε to define a new rescaled energy F^ε (per unit area):

$$F^\varepsilon[u] := \varepsilon^{-4/3}|\ln\varepsilon|^{1/3}\ell^{-2}E^\varepsilon[u] - |\ln\varepsilon|\frac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta}-\bar{\delta}_c) + \frac{1}{4\cdot 3^{1/3}}(\bar{\delta}-\bar{\delta}_c)(\ln|\ln\varepsilon| + \ln 9). \tag{2.2.9}$$

Note that we also added the third term into the bracket in the right-hand side of (2.2.9) to subtract the next-to-leading order contribution of the droplet self-energy, and we have scaled F^ε in a way that allows to extract a non-trivial $O(1)$ contribution to the minimal energy (see details in Section 2.3). The main result of this paper in fact is to establish Γ-convergence of F^ε to the renormalized energy W which we now define.

In [100], the renormalized energy W was introduced and defined in terms of the superconducting current j, which is particularly convenient for the studies of the magnetic Ginzburg-Landau model of superconductivity. Here, instead, we give an equivalent definition, which is expressed in terms of the limiting electrostatic potential of the charged droplets, after blow-up, which is the limit of some proper rescaling of v^ε (see below). However, this limiting electrostatic potential will only be known up to additive constants, due to the fact that we will take limits over larger and larger tori. This issue can be dealt with in a natural way by considering *equivalence classes* of potentials, whereby two potentials differing by a constant are not distinguished:

$$[\varphi] := \{\varphi + c \mid c \in \mathbb{R}\}. \tag{2.2.10}$$

This definition turns the homogeneous spaces $\dot{W}^{1,p}(\mathbb{R}^d)$ into Banach spaces of equivalence classes of functions in $W^{1,p}_{loc}(\mathbb{R}^d)$ defined in (2.2.10) (see, e.g., [88]). Here we similarly define the local analog of the homogeneous Sobolev spaces as

$$\dot{W}^{1,p}_{loc}(\mathbb{R}^2) := \left\{[\varphi] \mid \varphi \in W^{1,p}_{loc}(\mathbb{R}^2)\right\}, \tag{2.2.11}$$

with the notion of convergence to be that of the L^p_{loc} convergence of gradients. In the following, we will omit the brackets in $[\cdot]$ to simplify the notation and will write $\varphi \in \dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ to imply that φ is any member of the equivalence class in (2.2.10).

We define the admissible class of the renormalized energy as follows :

Definition 2.2.1. *For given $m > 0$ and $p \in (1, 2)$, we say that φ belongs to* *the admissible class $\mathcal{A}_m$, if $\varphi \in \dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ and φ solves distributionally*

$$-\Delta\varphi = 2\pi \sum_{a\in\Lambda} \delta_a - m, \tag{2.2.12}$$

where $\Lambda \subset \mathbb{R}^2$ is a discrete set and

$$\lim_{R\to\infty} \frac{2}{R^2} \int_{B_R(0)} \sum_{a\in\Lambda} \delta_a(x)dx = m. \tag{2.2.13}$$

Remark 2.2.2. *Observe that if $\varphi \in \mathcal{A}_m$, then for every $x \in B_R(0)$ we have*

$$\varphi(x) = \sum_{a\in\Lambda_R} \ln|x-a|^{-1} + \varphi_R(x), \tag{2.2.14}$$

where $\Lambda_R := \Lambda \cap \bar{B}_R(0)$ is a finite set of distinct points and $\varphi_R \in C^\infty(\mathbb{R}^2)$ is analytic in $B_R(0)$. In particular, the definition of $\mathcal{A}_m$ is independent of p.

We next define the renormalized energy.

Definition 2.2.3. *For a given $\varphi \in \bigcup_{m>0} \mathcal{A}_m$, the renormalized energy W of φ is defined as*

$$W(\varphi) := \limsup_{R\to\infty} \lim_{\eta\to 0} \frac{1}{|K_R|} \left(\int_{\mathbb{R}^2 \setminus \cup_{a\in\Lambda} B_\eta(a)} \frac{1}{2} |\nabla\varphi|^2 \chi_R dx + \pi \ln\eta \sum_{a\in\Lambda} \chi_R(a) \right), \tag{2.2.15}$$

where $K_R = [-R,R]^2$, χ_R is a smooth cutoff function with the properties that $0 < \chi_R < 1$, in $K_R \setminus (\partial K_R \cup K_{R-1})$, $\chi_R(x) = 1$ for all $x \in K_{R-1}$, $\chi_R(x) = 0$ for all $x \in \mathbb{R}^2 \backslash K_R$, and $|\nabla\chi_R| \le C$ for some $C > 0$ independent of R.

Various properties of W are established in [100], we refer the reader to that paper. The most relevant to us here are

1. $\min_{\mathcal{A}_m} W$ is achieved for each $m > 0$.

2. If $\varphi \in \mathcal{A}_m$ and $\varphi'(x) := \varphi(\frac{x}{\sqrt{m}})$, then $\varphi' \in \mathcal{A}_1$ and

$$W(\varphi) = m\left(W(\varphi') - \frac{1}{4}\log m\right), \tag{2.2.16}$$

   hence

$$\min_{\mathcal{A}_m} W = m\left(\min_{\mathcal{A}_1} W - \frac{1}{4}\log m\right).$$

3. W is minimized over potentials in $\mathcal{A}_1$ generated by charge configurations Λ consisting of simple lattices by the potential of a triangular lattice, i.e. [100, Theorem 2 and Remark 1.5],

$$\min_{\substack{\varphi\in\mathcal{A}_1\\ \Lambda \text{ simple lattice}}} W(\varphi) = W(\varphi^{\triangle}) = -\frac{1}{2}\ln(\sqrt{2\pi b}\,|\eta(\tau)|^2) \simeq -0.2011,$$

   where $\tau = a+ib$, $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function, $q = e^{2\pi i\tau}$, a and b are real numbers such that $\Lambda^*_{\triangle} = \frac{1}{\sqrt{2\pi b}}\Big((1,0)\mathbb{Z} \oplus (a,b)\mathbb{Z}\Big)$ is the dual lattice to a triangular lattice $\Lambda^{\triangle}$ whose unit cell has area 2π, and $\varphi^{\triangle}$ solves (2.2.12) with $\Lambda = \Lambda^{\triangle}$.

In particular, from property 2 above it is easy to see that the role of m in the definition of W is inconsequential.

We are now ready to state our main result. Let $\ell^{\varepsilon} := |\ln\varepsilon|^{1/2}\ell$. For a given $u^{\varepsilon} \in \mathcal{A}$, we then introduce the potential (recall that φ^{ε} is a representative in the equivalence class defined in (2.2.10))

$$\varphi^\varepsilon(x) := 2 \cdot 3^{-2/3} |\ln \varepsilon| \; \tilde{v}^\varepsilon(x|\ln \varepsilon|^{-1/2}), \tag{2.2.17}$$

where $\tilde{v}^\varepsilon$ is a periodic extension of v^ε from $\mathbb{T}^2_{\ell^\varepsilon}$ to the whole of $\mathbb{R}^2$. We also define $\mathcal{P}$ to be the family of translation-invariant probability measures on $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ concentrated on $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$.

Theorem 4. **(Γ-convergence of F^ε)** *Fix $\kappa > 0$, $\bar{\delta} > \bar{\delta}_c$, $p \in (1, 2)$ and $\ell > 0$, and let F^ε be defined by (2.2.9). Then, as $\varepsilon \to 0$ we have*

$$F^\varepsilon \xrightarrow{\Gamma} F^0[P] := 3^{4/3} \int W(\varphi) dP(\varphi) + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}, \tag{2.2.18}$$

where $P \in \mathcal{P}$. More precisely:

i) (Lower Bound) Let $(u^\varepsilon) \in \mathcal{A}$ be such that

$$\limsup_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] < +\infty, \tag{2.2.19}$$

and let P^ε be the probability measure on $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ which is the pushforward of the normalized uniform measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by the map $x \mapsto \varphi^\varepsilon(x+\cdot)$, where φ^ε is as in (2.2.17). Then, upon extraction of a subsequence, (P^ε) converges weakly to some $P \in \mathcal{P}$, in the sense of measures on $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ and

$$\liminf_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] \geq F^0[P]. \tag{2.2.20}$$

ii) (Upper Bound) Conversely, for any probability measure $P \in \mathcal{P}$, letting Q be its push-forward under $-\Delta$, there exists $(u^\varepsilon) \in \mathcal{A}$ such that letting Q^ε be the pushforward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x \mapsto -\Delta\varphi^\varepsilon(x+\cdot)$, where φ^ε is as in (2.2.17), we have $Q^\varepsilon \rightharpoonup Q$, in the sense of measures on $W^{-1,p}_{loc}(\mathbb{R}^2)$, and

$$\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \le F^0[P], \tag{2.2.21}$$

as $\varepsilon \to 0$.

We will prove that the minimum of F^0 is achieved. Moreover, it is achieved for any $P \in \mathcal{P}$ which is concentrated on minimizers of $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$.

Remark 2.2.4. *The phrasing of the theorem does not exactly fit the framework of Γ-convergence, since the lower bound result and the upper bound result are not expressed with the same notion of convergence. However, since weak convergence of P_ε to P implies weak convergence of Q_ε to Q, the theorem implies a result of Γ-convergence where the sense of convergence from P_ε to P is taken to be the weak convergence of their push-forwards Q_ε to the corresponding Q.*

The next theorem expresses the consequence of Theorem 4 for almost minimizers:

Theorem 5. *Let $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$ and let $(u^\varepsilon) \in \mathcal{A}$ be a family of almost minimizers of F^0, i.e., let*

$$\lim_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] = \min_{\mathcal{P}} F^0.$$

Then, if P is the limit measure from Theorem 4, P-almost every φ minimizes W over $\mathcal{A}_m$. In addition

$$\min_{\mathcal{P}} F^0 = 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}. \tag{2.2.22}$$

Note that the formula in (2.2.22) is not totally obvious, since the probability measure concentrated on a single minimizer $\varphi \in \mathcal{A}_m$ of W does not belong to $\mathcal{P}$.

The result in Theorem 5 allows us to establish the expansion of the minimal value of the original energy $\mathcal{E}^\varepsilon$ by combining it with (2.2.9) and (2.1.11).

Theorem 6. (Asymptotic expansion of $\min \mathcal{E}^\varepsilon$) *Let $V = \frac{9}{32}(1-u^2)^2$, $\kappa = \frac{2}{3}$ and $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$. Fix $\bar{\delta} > \bar{\delta}_c$ and $\ell > 0$, and let $\mathcal{E}^\varepsilon$ be defined by (4.1.1) with $\bar{u} = \bar{u}^\varepsilon$ from (4.2.4). Then, as $\varepsilon \to 0$ we have*

$$\begin{aligned}\ell^{-2} \min \mathcal{E}^\varepsilon = \frac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c)\varepsilon^{4/3}|\ln\varepsilon|^{2/3} - \frac{1}{4\cdot 3^{1/3}}(\bar{\delta} - \bar{\delta}_c)\varepsilon^{4/3}|\ln\varepsilon|^{-1/3}(\ln|\ln\varepsilon| + \ln 9)\\ + \varepsilon^{4/3}|\ln\varepsilon|^{-1/3}\left(3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}\right) + o(\varepsilon^{4/3}|\ln\varepsilon|^{-1/3}).\end{aligned} \tag{2.2.23}$$

As mentioned above, the Γ-limit in Theorem 4 cannot be expressed in terms of a single limiting function φ, but rather it effectively averages W over all the blown-up limits of φ^ε, with respect to all the possible blow-up centers. Consequently, for almost minimizers of the energy, we cannot guarantee that each blown-up potential φ^ε converges to a minimizer of W, but only that this is true after blow-up except around points that belong to a set with asymptotically vanishing volume fraction. Indeed, one could easily imagine a configuration with some small regions where the configuration does not ressemble any minimizer of W, and this would not contradict the fact of being an almost minimizer since these regions would contribute only a negligible fraction to the energy. Near all the good blow-up centers, we will know some more about the droplets: it will be shown in Theorem 7 that they are asymptotically round and of optimal radii.

We finish this section with a short sketch of the proof. Most of the proof consists in proving the lower bound, i.e. Part (i) of Theorem 4. The first step, accomplished in Section 2.3 is, following the ideas of [80], to extract from F^ε some positive terms involving the sizes and shapes of the droplets and which are minimized by round droplets of fixed appropriate radius. These positive terms, gathered in what will be called M_ε, can be put aside and will serve to control the discrepancy between the droplets and the ideal round droplets of optimal sizes. We then consider what remains when this M_ε is subtracted off from F^ε and express it in blown-up coordinates $x' = x\sqrt{|\ln \varepsilon|}$. It is then an energy functional, expressed in terms of some rescaling of φ^ε which has no sign and which ressembles that studied in [100]. Thus we apply to it the strategy of [100]. The main point is to show that, even though the energy density is not bounded below, it can be transformed into one that is by absorbing the negative terms into positive terms in the energy in the sense of energy displacement [100], while making only a small error. In order to prove that this is possible, we first need to establish sharp lower bounds for the energy carried by the droplets (with an error $o(1)$ per droplet). These lower bounds contain possible errors which will later be controlled via the M_ε term. This is done in Section 4.5 via a ball construction as in [64, 99, 103]. In Section 4.7 we use these lower bounds to perform the energy displacement as in [100]. Once the energy density has been replaced this way by an essentially equivalent energy density which is bounded below, we can apply the abstract scheme of [100] that serves to obtain lower bounds for "two-scale energies" which Γ-converge at the microscopic scale, via the multiparameter ergodic theorem. This is achieved is Section 2.6. Prior to this we obtain explicit lower bounds at the microscopic scale in terms of the renormalized energy for a finite number of points. It is then these lower bounds that get integrated out, or averaged out at the macroscopic scale to provide a global lower bound.

Finally, there remains to obtain the corresponding upper bound. This is done via an explicit construction of a periodic test-configuration, following again the method of [100].

2.3 Derivation of the leading order energy

In preparation for the proof of Theorem 4, we define

$$\rho_\varepsilon := 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{1/6} \quad \text{and} \quad \bar{r}_\varepsilon := \left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3}. \tag{2.3.1}$$

Recall that to leading order the droplets are expected to be circular with radius $3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$. Thus ρ_ε is the expected radius, once we have blown up coordinates by the factor of $\sqrt{|\ln\varepsilon|}$, which will be done below. Also, we know that the expected normalized area A_i is $3^{2/3}\pi$, but this is only true up to lower order terms which were negligible in [54]; as we show below, a more precise estimate is $A_i \simeq \pi\bar{r}_\varepsilon^2$, so $\bar{r}_\varepsilon$ above can be viewed as a "corrected" normalized droplet radius. Since our estimates must be accurate up to $o_\varepsilon(1)$ per droplet and the self-energy of a droplet is of order $A_i^2 \ln\rho_\varepsilon$, we can no longer ignore these corrections.

The goal of the next subsection is to obtain an explicit lower bound for F_ε defined by (2.2.9) in terms of the droplet areas and perimeters, which will then be studied in Sections 4.5 and onward. We follow the analysis of [54], but isolate higher order terms.

2.3.1 Energy extraction

We begin with the original energy $\bar{E}^\varepsilon$ (cf. (2.2.4)) while adding and subtracting the *truncated* self interaction: first we define, for $\gamma \in (0,1)$, truncated droplet volumes by

$$\tilde{A}_i^\varepsilon := \begin{cases} A_i^\varepsilon & \text{if } A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}, \\ (3^{2/3}\pi\gamma^{-1}|A_i^\varepsilon|)^{1/2} & \text{if } A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}, \end{cases} \tag{2.3.2}$$

as in [54]. The motivation for this truncation will become clear in the proof of Proposition 2.5.1, when we obtain lower bounds on the energy on annuli. In [54] the self-interaction energy of each droplet extracted from $\bar{E}^\varepsilon$ was $\frac{|\tilde{A}_i^\varepsilon|^2}{3\pi|\ln\varepsilon|}$, yielding in the end the leading order energy $E^0[\mu]$ in (4.1.9). A more precise calculation of the self-interaction energy corrects the coefficient of $|\tilde{A}_i^\varepsilon|^2$ by an $O(\ln|\ln\varepsilon|/|\ln\varepsilon|)$ term, yielding the following corrected leading order energy for E^ε:

$$E_\varepsilon^0[\mu] := \frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu + 2\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y). \quad (2.3.3)$$

The energy in (2.3.3) is explicitly minimized by $d\mu(x) = \bar{\mu}_\varepsilon\, dx$ (again a correction to the previously known $\bar{\mu}$ from (4.1.10)) where

$$\bar{\mu}_\varepsilon := \frac{1}{2}\left(\bar{\delta} - \frac{3\kappa^2}{2\bar{r}_\varepsilon}\right) \qquad \text{for} \qquad \bar{\delta} > \frac{3\kappa^2}{2\bar{r}_\varepsilon}, \quad (2.3.4)$$

and

$$\min E_\varepsilon^0 = \frac{\bar{\delta}_c\ell^2}{2\kappa^2}\left\{2\bar{\delta}\left(\frac{3}{\bar{r}_\varepsilon^3}\right)^{1/3} - \bar{\delta}_c\left(\frac{3}{\bar{r}_\varepsilon^3}\right)^{2/3}\right\}. \quad (2.3.5)$$

Observing that $\bar{r}_\varepsilon \to 3^{1/3}$ we immediately check that

$$\bar{\mu}_\varepsilon \to \bar{\mu} \quad \text{as } \varepsilon \to 0, \quad (2.3.6)$$

and in addition that (2.3.5) converges to the second expression in (4.1.10). To obtain the next order term, we Taylor-expand the obtained formulas upon substituting the definition of $\bar{r}_\varepsilon$. After some algebra, we obtain

$$\ell^{-2}\min E_\varepsilon^0 = \frac{\bar{\delta}_c}{2\kappa^2}\left(2\bar{\delta} - \bar{\delta}_c\right) - \frac{1}{4\cdot 3^{1/3}}(\bar{\delta} - \bar{\delta}_c)\frac{\ln|\ln\varepsilon| + \ln 9}{|\ln\varepsilon|} + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|^2}\right). \quad (2.3.7)$$

Recalling once again the definition of F^ε from (2.2.9), we then find

$$F^\varepsilon[u^\varepsilon] = |\ln\varepsilon| \left(\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}E^\varepsilon[u^\varepsilon] - \ell^{-2}\min E^0_\varepsilon\right) + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|}\right),$$

and in view of the definition of $\bar{E}^\varepsilon$ from (2.2.3), we thus may write

$$F^\varepsilon[u^\varepsilon] = |\ln\varepsilon|\ell^{-2}\left(\bar{E}^\varepsilon[u^\varepsilon] + \frac{\bar{\delta}^2\ell^2}{2\kappa^2} - \min E^0_\varepsilon\right) + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|}\right). \qquad (2.3.8)$$

Thus obtaining a lower bound for the first term in the right-hand side of (2.3.8) implies, up to $o_\varepsilon(1)$, a lower bound for F^ε. This is how we proceed to prove Lemma 2.3.1 below.

With this in mind, we begin by setting

$$v^\varepsilon = \bar{v}^\varepsilon + \frac{h_\varepsilon}{|\ln\varepsilon|}, \qquad \bar{v}^\varepsilon = \frac{1}{2\kappa^2}\left(\bar{\delta} - \frac{3\kappa^2}{2\bar{r}_\varepsilon}\right), \qquad (2.3.9)$$

where $\bar{v}^\varepsilon$ is the solution to (2.2.7) with right side equal to $\bar{\mu}_\varepsilon$ in (2.3.4).

2.3.2 Blowup of coordinates

We now rescale the domain $\mathbb{T}^2_\ell$ by making the change of variables

$$\begin{aligned} x' &= x\sqrt{|\ln\varepsilon|}, \\ h'_\varepsilon(x') &= h_\varepsilon(x), \\ \Omega'_{i,\varepsilon} &= \Omega^\varepsilon_i\sqrt{|\ln\varepsilon|}, \\ \ell^\varepsilon &= \ell\sqrt{|\ln\varepsilon|}. \end{aligned} \qquad (2.3.10)$$

Observe that

$$\varphi^\varepsilon(x') = 2 \cdot 3^{-2/3} h'_\varepsilon(x') \qquad \forall x' \in \mathbb{T}^2_{\ell^\varepsilon}, \tag{2.3.11}$$

where φ^ε is defined by (2.2.17). It turns out to be more convenient to work with h'_ε and rescale only at the end back to φ^ε.

2.3.3 Main result

We are now ready to state the main result of this section, which provides an explicit lower bound on F^ε. The strategy, in particular for dealing with droplets that are too small or too large is the same as [54], except that we need to go to higher order terms.

Proposition 2.3.1. *There exist universal constants $\gamma \in (0, \frac{1}{6})$, $c_1 > 0, c_2 > 0$, $c_3 > 0$ and $\varepsilon_0 > 0$ such that if $\bar{\delta} > \bar{\delta}_c$ and $(u^\varepsilon) \in \mathcal{A}$ with $\Omega^\varepsilon := \{u^\varepsilon > 0\}$, then for all $\varepsilon < \varepsilon_0$*

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq M_\varepsilon + \frac{2}{|\ln \varepsilon|} \int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' - \frac{1}{\pi \bar{r}_\varepsilon^3} \sum_{A_i^\varepsilon \geq 3^{2/3}\pi\gamma} |\tilde{A}_i^\varepsilon|^2 + o_\varepsilon(1), \tag{2.3.12}$$

where $M_\varepsilon \geq 0$ is defined by

$$M_\varepsilon := \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right) + c_1 \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} A_i^\varepsilon + c_2 \sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 + c_3 \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon. \tag{2.3.13}$$

Remark 2.3.2. *Defining* $\beta := 3^{2/3}\pi\gamma$, *by isoperimetric inequality applied to each connected component of* Ω^ε *separately every term in the first sum in the definition of* M_ε *in* (2.3.13) *is non-negative. In particular,* M_ε *measures the discrepancy between the droplets* Ω_i^ε *with* $A_i^\varepsilon \geq \beta$ *and disks of radius* $\bar{r}_\varepsilon$.

The proposition will be proved below, but before let us examine some of its further consequences. The result of the proposition implies that our a priori assumption $\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] < +\infty$ translates into

$$M_\varepsilon + \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' - \frac{1}{\pi\bar{r}_\varepsilon^3}\sum_{A_i^\varepsilon\geq\beta} |\tilde{A}_i^\varepsilon|^2 \leq C,$$

for some $C > 0$ independent of $\varepsilon \ll 1$, which, in view of (2.3.1) is also

$$M_\varepsilon + \frac{2}{|\ln\varepsilon|}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' - \frac{1}{2\pi}|\ln\rho_\varepsilon|\sum_{A_i^\varepsilon\geq\beta} |\tilde{A}_i^\varepsilon|^2\right) \leq C. \tag{2.3.14}$$

A major goal of the next sections is to obtain the following estimate

$$\frac{1}{|\ln\varepsilon|}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' - \frac{1}{2\pi}|\ln\rho_\varepsilon|\sum_{A_i^\varepsilon\geq\beta} |\tilde{A}_i^\varepsilon|^2\right) \geq -C\ln^2(M_\varepsilon+2), \tag{2.3.15}$$

for some $C > 0$ independent of $\varepsilon \ll 1$, so that the a priori bound (2.3.14) in fact implies that M_ε is uniformly bounded independently of ε for small ε. This will be used crucially in Section 2.6.2.

We note that $h'_\varepsilon(x')$ satisfies the equation

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln \varepsilon|} h'_\varepsilon = \mu'_\varepsilon - \bar{\mu}^\varepsilon \quad \text{in } W^{2,p}(\mathbb{T}^2_{\ell^\varepsilon}) \tag{2.3.16}$$

where we define in $\mathbb{T}^2_{\ell^\varepsilon}$

$$\mu'_\varepsilon(x') := \sum_i A_i^\varepsilon \tilde{\delta}_i^\varepsilon(x'), \tag{2.3.17}$$

and

$$\tilde{\delta}_i^\varepsilon(x') := \frac{\chi_{\Omega'_{i,\varepsilon}}(x')}{|\Omega'_{i,\varepsilon}|}, \tag{2.3.18}$$

which will be used in what follows. Notice that each $\tilde{\delta}_i^\varepsilon(x')$ approximates the Dirac delta concentrated on some point in the support of $\Omega'_{i,\varepsilon}$ and, hence, $\mu'_\varepsilon(x')dx'$ approximates the measure associated with the collection of point charges with magnitude A_i^ε. In particular, the measure $d\mu'_\varepsilon$ evaluated over the whole torus equals the total charge: $\mu'_\varepsilon(\mathbb{T}^2_{\ell^\varepsilon}) = \sum_i A_i^\varepsilon$.

2.3.4 Proof of Proposition 2.3.1

- *Step 1:* We are first going to show that for universally small $\varepsilon > 0$ and all $\gamma \in (0, \frac{1}{6})$ we have

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq T_1 + T_2 + T_3 + T_4 + T_5 + o_\varepsilon(1), \tag{2.3.19}$$

where

$$T_1 = \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right), \tag{2.3.20}$$

$$T_2 = \frac{\gamma^{7/2}}{4\pi} \sum_{3^{2/3}\pi\gamma \le A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2, \tag{2.3.21}$$

$$T_3 = \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2, \tag{2.3.22}$$

$$T_4 = \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} \left(6^{-1}\gamma^{-1} - 1\right) A_i^\varepsilon, \tag{2.3.23}$$

$$T_5 = \frac{2}{|\ln\varepsilon|} \int_{\mathbb{T}_\ell^2} \left(|\nabla h_\varepsilon|^2 + \kappa^2 |h_\varepsilon|^2\right) dx - \frac{1}{\pi\bar{r}_\varepsilon^3} \sum_i |\tilde{A}_i^\varepsilon|^2. \tag{2.3.24}$$

To bound $F^\varepsilon[u^\varepsilon]$ from below, we start from (2.3.8). In particular, in view of (2.2.7) we may rewrite (2.2.4) as

$$\begin{aligned}
\bar{E}^\varepsilon[u^\varepsilon] &= \frac{1}{|\ln\varepsilon|} \sum_i \left(P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon \right) + 2\int_{\mathbb{T}_\ell^2} \left(|\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2\right) dx \\
&= \frac{1}{|\ln\varepsilon|} \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right) \\
&\quad + \frac{1}{|\ln\varepsilon|} \sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon + \frac{1}{\pi\bar{r}_\varepsilon^3} |\tilde{A}_i^\varepsilon|^2 \right) && (2.3.25) \\
&\quad + 2\int_{\mathbb{T}_\ell^2} \left(|\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2\right) dx - \frac{1}{\pi\bar{r}_\varepsilon^3 |\ln\varepsilon|} \sum_i |\tilde{A}_i^\varepsilon|^2. && (2.3.26)
\end{aligned}$$

We start by focusing on (2.3.25). First, in the case $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$ we have $|\tilde{A}_i^\varepsilon|^2 = 3^{2/3}\pi\gamma^{-1} A_i^\varepsilon$ and hence, recalling that $\bar{r}_\varepsilon = 3^{1/3} + o_\varepsilon(1)$, where $o_\varepsilon(1)$

depends only on ε, we have for ε universally small and $\gamma < \frac{1}{6}$:

$$\frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} = \frac{A_i^\varepsilon}{\pi\bar{r}_\varepsilon^3}\left(3^{2/3}\pi\gamma^{-1} - 3\pi\bar{r}_\varepsilon^2 + 3\pi\bar{r}_\varepsilon^2\right) = A_i^\varepsilon\left(\frac{3}{\bar{r}_\varepsilon} + \frac{3^{2/3}}{\bar{r}_\varepsilon^3}\left(\gamma^{-1} - 3\left(\frac{\bar{r}_\varepsilon}{3^{1/3}}\right)^2\right)\right)$$
$$\geq A_i^\varepsilon\left(\frac{3}{\bar{r}_\varepsilon} + \frac{1}{6}\left(\gamma^{-1} - 6\right)\right). \qquad (2.3.27)$$

We conclude that for $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$, we have

$$\left(\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon\right) \geq \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} + \frac{1}{6}\left(\gamma^{-1} - 6\right)\right)A_i^\varepsilon. \qquad (2.3.28)$$

On the other hand, when $A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}$ we have $\tilde{A}_i^\varepsilon = A_i^\varepsilon$ and we proceed as follows. Let us begin by defining, similarly to [54], the function

$$f(x) = \frac{2\sqrt{\pi}}{\sqrt{x}} + \frac{x}{\pi\bar{r}_\varepsilon^3}$$

for $x \in (0, +\infty)$ and observe that f is convex and attains its minimum of $\frac{3}{\bar{r}_\varepsilon}$ at $x = \pi\bar{r}_\varepsilon^2$, with

$$f''(x) = \frac{3\sqrt{\pi}}{2x^{5/2}} > 0.$$

By a second order Taylor expansion of f around $\pi\bar{r}_\varepsilon^2$, using the fact that f'' is decreasing on $(0, +\infty)$, we then have for all $x \leq x_0$

$$\sqrt{4\pi x} + \frac{x^2}{\pi\bar{r}_\varepsilon^3} = xf(x) \geq x\left(\frac{3}{\bar{r}_\varepsilon} + \frac{3\sqrt{\pi}}{4x_0^{5/2}}\left(x - \pi\bar{r}_\varepsilon^2\right)^2\right). \qquad (2.3.29)$$

We, hence, conclude that when $3^{2/3}\pi\gamma \le A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1}$, we have

$$\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi \bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon \ge \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} \right) A_i^\varepsilon + \frac{\gamma^{5/2}}{4\pi^2 \cdot 3^{2/3}} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2, \tag{2.3.30}$$

and when $A_i^\varepsilon < 3^{2/3}\pi\gamma$, we have

$$\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi \bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon \ge \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} \right) A_i^\varepsilon + \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2, \tag{2.3.31}$$

Combining (2.3.28), (2.3.30) and (2.3.31), summing over all i, and distinguishing the different cases, we can now bound (2.3.25) from below as follows:

$$\begin{aligned} \sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon + \frac{1}{\pi \bar{r}_\varepsilon^3} |\tilde{A}_i^\varepsilon|^2 \right) \ge & \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} \right) \sum_i A_i^\varepsilon \\ & + \frac{\gamma^{7/2}}{4\pi} \sum_{3^{2/3}\pi\gamma \le A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 \\ & + \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 \\ & + \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} \left(6^{-1}\gamma^{-1} - 1 \right) A_i^\varepsilon. \end{aligned} \tag{2.3.32}$$

We now focus on the term in (4.2.3). Using (2.3.9), we can write the integral in (4.2.3) as:

$$\begin{aligned} &2\int_{\mathbb{T}^2_\ell}\left(\nabla v^\varepsilon|^2+\kappa^2|v^\varepsilon|^2\right)dx\\ &=\frac{2}{|\ln\varepsilon|^2}\int_{\mathbb{T}^2_\ell}\left(|\nabla h_\varepsilon|^2+\kappa^2h_\varepsilon^2\right)dx \;+\frac{4\kappa^2\bar{v}^\varepsilon}{|\ln\varepsilon|}\int_{\mathbb{T}^2_\ell}h_\varepsilon dx+2\kappa^2|\bar{v}^\varepsilon|^2\ell^2. \end{aligned} \tag{2.3.33}$$

Integrating (2.2.7) over $\mathbb{T}^2_\ell$ and recalling the definition of h_ε in (2.3.9), as well as (2.2.5), leads to

$$\frac{4\kappa^2\bar{v}^\varepsilon}{|\ln\varepsilon|}\int_{\mathbb{T}^2_\ell}h_\varepsilon dx=\frac{4\bar{v}^\varepsilon}{|\ln\varepsilon|}\sum_i A_i^\varepsilon-4\kappa^2|\bar{v}^\varepsilon|^2\ell^2. \tag{2.3.34}$$

Combining (2.3.33) and (2.3.34), we then find

$$\begin{aligned} 2\int_{\mathbb{T}^2_\ell}\left(|\nabla v^\varepsilon|^2+\kappa^2|v^\varepsilon|^2\right)dx&=\frac{2}{|\ln\varepsilon|^2}\int_{\mathbb{T}^2_\ell}\left(|\nabla h_\varepsilon|^2+\kappa^2h_\varepsilon^2\right)dx\\ &\quad-\frac{1}{|\ln\varepsilon|}\left(\frac{3}{\bar{r}_\varepsilon}-\frac{2\bar{\delta}}{\kappa^2}\right)\sum_i A_i^\varepsilon-2\kappa^2|\bar{v}^\varepsilon|^2\ell^2. \end{aligned} \tag{2.3.35}$$

Also, by direct computation using (2.3.5) and (2.3.9) we have

$$2\kappa^2|\bar{v}^\varepsilon|^2\ell^2=\frac{\bar{\delta}^2\ell^2}{2\kappa^2}-\min E_\varepsilon^0. \tag{2.3.36}$$

Therefore, combining this with (2.3.8), (2.3.32) and (2.3.35), after passing to the rescaled coordinates and performing the cancellations we find that

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq T_1 + T_2 + T_3 + T_4 + \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon(x')|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon(x')|^2 \right) dx' \\ - \frac{1}{\pi\bar{r}_\varepsilon^3}\sum_i |\tilde{A}_i^\varepsilon|^2 + o_\varepsilon(1), \tag{2.3.37}$$

which is nothing but (2.3.19).

- *Step 2:* We proceed to absorbing the contributions of the small droplets in (2.3.24) by (2.3.21). To that effect, we observe that, for the function

$$\Phi_\varepsilon(x) := \frac{\gamma^{-5/2}}{4\pi^2\cdot 3^{2/3}} x(x-\pi\bar{r}_\varepsilon^2)^2 - \frac{1}{\bar{r}_\varepsilon^3}x^2 \geq \frac{\gamma^{-5/2}x}{4\pi^2\cdot 3^{2/3}}\left\{\pi^2\bar{r}_\varepsilon^4 - \left(2\pi\bar{r}_\varepsilon^2 + \frac{\gamma^{5/2}}{\bar{r}_\varepsilon^3}\right)x\right\}, \tag{2.3.38}$$

there exists a universal $\gamma \in (0, \frac{1}{6})$ such that $\Phi_\varepsilon(x) \geq x$ whenever $0 \leq x < 3^{2/3}\pi\gamma$ and ε is universally small. Using this observation, we may absorb all the terms with $A_i^\varepsilon < 3^{2/3}\pi\gamma$ appearing in the second term in (2.3.24) into (2.3.22) by suitably reducing the coefficient in front of the latter. This proves the result. $\square$

2.4 Ball construction

The goal of this section is to show (2.3.15) using the abstract framework of Theorem 3 in [100]. The difficulty in doing this, as in the case of the Ginzburg-Landau model treated in [100], is that the energy density $e'_\varepsilon - \frac{1}{\pi}|\ln\rho_\varepsilon|\sum_{A_i^\varepsilon\geq\beta}|\tilde{A}_i^\varepsilon|^2\tilde{\delta}_i^\varepsilon$ is not positive (or bounded below independently of (u^ε)). The next two subsections are meant to go around this difficulty by showing that this energy density can be modified, by displacing a part of the energy from the regions where the energy density is positive into regions where the energy density is negative in order to bound the modified energy density from below while making only a small enough error. This is achieved by obtaining sharp lower bounds on the energy of the droplets. Since their volumes and shapes are a priori unknown, the terms in M_ε are used to control in a quantitative way the deviations from the droplets being balls of fixed volume.

In this section we perform a ball construction which follows the procedure of [100]. The goal is to cover the droplets $\{\Omega'_{i,\varepsilon}\}$ whose volumes are bounded from below by a given $\beta > 0$ with a finite collection of disjoint closed balls whose radii are smaller than 1, on which we have a good lower bound for the energy in the left-hand side of (2.3.15). This is possible for sufficiently small ε in view of the fact that $\ell^\varepsilon \to \infty$ and that the leading order asymptotic behavior of the energy from (2.2.6) yields control on the perimeter and, therefore, the essential diameter of each of $\Omega'_{i,\varepsilon}$. The precise statements are given below. We will also need the following basic result, which holds for sufficiently small ε ensuring that the droplets are smaller than the sidelength of the torus (see the discussion at the beginning of Sec. 2 in [54]).

Lemma 2.4.1. *There exists $\varepsilon_0 > 0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ such that for all $\varepsilon \le \varepsilon_0$ we have*

$$\operatorname{ess\,diam}(\Omega'_{i,\varepsilon}) \le c|\partial\Omega'_{i,\varepsilon}|, \tag{2.4.1}$$

for some universal $c > 0$.

From now on and for the rest of the paper we fix γ to be the constant given in Proposition 2.3.1 and, as in the previous section, we define $\beta = 3^{2/3}\pi\gamma$. We also introduce the following notation which will be used repeatedly below. To index the droplets, we will use the following definitions:

$$I_\beta := \{i \in \mathbb{N} : A_i^\varepsilon \ \geq \ \beta\}, \quad I_E := \{i \in \mathbb{N} : |\Omega'_{i,\varepsilon} \cap (\mathbb{T}^2_{\ell^\varepsilon} \backslash E)| = 0\}, \quad I_{\beta,E} := I_\beta \cap I_E, \tag{2.4.2}$$

where $E \subset \mathbb{T}^2_{\ell^\varepsilon}$. For a collection of balls $\mathcal{B}$, the number $r(\mathcal{B})$ (also called the total radius of the collection) denotes the sum of the radii of the balls in $\mathcal{B}$. For simplicity, we will say that a ball B *covers* $\Omega'_{i,\varepsilon}$, if $i \in I_B$.

The principle of the ball construction introduced by Jerrard [64] and Sandier [103] and adapted to the present situation is to start from an initial set, here $\bigcup_{i \in I_{\beta,U}} \Omega'_{i,\varepsilon}$ for a given $U \subseteq \mathbb{T}^2_{\ell^\varepsilon}$ and cover it by a union of finitely many closed balls with sufficiently small radii. This collection can then be transformed into a collection of *disjoint* closed balls by the procedure, whereby every pair of intersecting balls is replaced by a larger ball whose radius equals the sum of the radii of the smaller balls and which contains the smaller balls. This process is repeated until all the balls are disjoint. The obtained collection will be denoted $\mathcal{B}_0$, its total radius is $r(\mathcal{B}_0)$. Then each ball is dilated by the same factor with respect to its corresponding center. As the dilation factor increases, some balls may touch. If that happens, the above procedure of ball merging is applied again to obtain a new collection of disjoint balls of the same total radius. The construction can be stopped when any desired total radius r is reached, provided that r is universally small compared to ℓ^ε. This yields a collection $\mathcal{B}_r$ covering the initial set and containing a logarithmic energy [64, 103].

We now give the statement of our result concerning the ball construction and the associated lower bounds. Throughout the rest of the paper we use the notation $f^+ := \max(f, 0)$ and $f^- := -\min(f, 0)$.

Proposition 2.4.2. *Let $U \subseteq \mathbb{T}^2_{\ell^\varepsilon}$ be an open set such that $I_{\beta,U} \neq \varnothing$, and assume that* (2.2.19) *holds.*

- *There exists $\varepsilon_0 > 0$, $r_0 \in (0,1)$ and $C > 0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ such that for all $\varepsilon < \varepsilon_0$ there exists a collection of finitely many disjoint closed balls $\mathcal{B}_0$ whose union covers $\bigcup_{i \in I_{\beta,U}} \Omega'_{i,\varepsilon}$ and such that*

$$r(\mathcal{B}_0) \leq c\varepsilon^{1/3} |\ln \varepsilon|^{1/6} \sum_{i \in I_{\beta,U}} P_i^\varepsilon < r_0, \tag{2.4.3}$$

  for some universal $c > 0$. Furthermore, for every $r \in [r(\mathcal{B}_0), r_0]$ there is a family of disjoint closed balls $\mathcal{B}_r$ of total radius r covering $\mathcal{B}_0$.

- *For every $B \in \mathcal{B}_r$ such that $B \subset U$ we have*

$$\int_B \left(|\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2,$$

  for some $c > 0$ depending only on κ and $\bar{\delta}$.

- *If $B \in \mathcal{B}_r$, for any non-negative Lipschitz function χ with support in U,* *we have*

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' - \frac{1}{2\pi} \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2$$
$$\geq -C \|\nabla \chi\|_\infty \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2,$$

where $\chi_i := \int_U \chi \tilde{\delta}_i^\varepsilon dx'$*, with* $\tilde{\delta}_i^\varepsilon(x')$ *defined in* (2.3.18)*, for some* $c > 0$ *depending only on* κ *and* $\bar{\delta}$*, and a universal* $C > 0$*.*

Proof of the first item. Choose an arbitrary $r_0 \in (0, 1)$. As in [54], from the basic lower bound on $\bar{E}^\varepsilon$ (see [54, Equations (2.12) and (2.15)]):

$$\bar{E}^\varepsilon[u^\varepsilon] \geq \frac{1}{|\ln \varepsilon|} \sum_i P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2 |\ln \varepsilon|} \sum_i A_i^\varepsilon + \frac{2}{\kappa^2 \ell^2 |\ln \varepsilon|^2} \left(\sum_i A_i^\varepsilon \right)^2, \qquad (2.4.4)$$

where A_i^ε and P_i^ε are defined in (2.2.2), we obtain with the help of (2.2.19) that

$$\limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i A_i^\varepsilon \leq C, \qquad \limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i P_i^\varepsilon \leq C, \qquad (2.4.5)$$

for some $C > 0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon > 0} F^\varepsilon[u^\varepsilon]$.

As is well-known, the essential diameter of a connected component of a set of finite perimeter on a torus can be bounded by its perimeter, provided that the latter is universally small compared to the size of the torus (see, e.g., [5]). Therefore, in view of the the definition of P_i^ε in (2.2.2) and the second of (2.4.5), for sufficiently small ε it is possible to cover each $\Omega'_{i,\varepsilon}$ with $i \in I_{\beta,U}$

by a closed ball B_i, so that the collection $\tilde{\mathcal{B}}_0$ consisting of all B_i's (possibly intersecting) has total radius

$$r_0(\tilde{\mathcal{B}}_0) \leq C\varepsilon^{1/3}|\ln\varepsilon|^{1/6} \sum_{i\in I_{\beta,U}} P_i^\varepsilon, \tag{2.4.6}$$

for some universal $C > 0$. Furthermore, by the first inequality in (2.4.5) and the fact that $A_i^\varepsilon \geq \beta$ for all $i \in I_{\beta,U}$ the collection $\tilde{\mathcal{B}}_0$ consists of only finitely many balls. Therefore, we can apply the construction à la Jerrard and Sandier outlined at the beginning of this section to obtain the desired family of balls $\mathcal{B}_0$ and $\mathcal{B}_r$, with $r(\mathcal{B}_0) = r(\tilde{\mathcal{B}}_0)$. The estimate on the radii follows by combining the second of (2.4.5) and (2.4.6) and the fact that $\ell^\varepsilon \to \infty$ with the rate depending only on ℓ, for sufficiently small ε depending on ℓ, κ, $\bar{\delta}$, $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ and r_0. $\square$

Proof of the second item. Let $B \subset U$ be a ball in the collection $\mathcal{B}_r$. Denote the radius of B by r_B and set

$$X_\varepsilon := \frac{\kappa^2}{|\ln\varepsilon|}\int_B h'_\varepsilon dx'.$$

Integrating (2.3.16) over B and applying the divergence theorem, we have

$$\int_{\partial B} \frac{\partial h'_\varepsilon}{\partial\nu}\, d\mathcal{H}^1(x') = m_{B,\varepsilon} - X_\varepsilon, \tag{2.4.7}$$

where

$$m_{B,\varepsilon} := \int_B (\mu'_\varepsilon(x') - \bar{\mu}^\varepsilon)dx' = \sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B} \theta_i A_i^\varepsilon - \bar{\mu}^\varepsilon|B|,$$

for some $\theta_i \in [0,1)$ representing the volume fraction in B of those droplets that are not covered completely by B, and ν is the inward normal to ∂B. Using the Cauchy-Schwarz inequality, we then deduce from (2.4.7) that

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') \geq \frac{1}{2\pi r_B}(m_{B,\varepsilon} - X_\varepsilon)^2 \geq \frac{m_{B,\varepsilon}^2 - 2m_{B,\varepsilon}X_\varepsilon}{2\pi r_B}. \tag{2.4.8}$$

By another application of the Cauchy-Schwarz inequality, we may write

$$\frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, dx' \geq \frac{X_\varepsilon^2}{4\pi r_B^2}\frac{|\ln\varepsilon|}{\kappa^2}. \tag{2.4.9}$$

We now add (2.4.8) and (2.4.9) and optimize the right-hand side over X_ε. We obtain

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, dx' \geq \frac{m_{B,\varepsilon}^2}{2\pi r_B}\left(1 - \frac{C r_B}{|\ln\varepsilon|}\right), \tag{2.4.10}$$

for $C = \kappa^4$. Recalling that $r_B \leq r \leq r_0 < 1$, we can choose ε sufficiently small depending only on κ so that the term in parentheses above is positive.

Inserting the definition of $m_{B,\varepsilon}$ into (2.4.10) and discarding some positive terms yields

$$\begin{aligned}\int_{\partial B} |\nabla h_\varepsilon'|^2 d\mathcal{H}^1(x') &+ \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h_\varepsilon'|^2\,dx' \\ &\geq \frac{1}{2\pi r_B}\Big(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B}\theta_i A_i^\varepsilon - \bar\mu^\varepsilon|B|\Big)^2\left(1-\frac{Cr_B}{|\ln\varepsilon|}\right) \\ &\geq \frac{1}{2\pi r_B}\Big(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B}\theta_i A_i^\varepsilon\Big)^2\left(1-2\bar\mu^\varepsilon|B|\Big(\sum_{i\in I_B} A_i^\varepsilon\Big)^{-1}-\frac{Cr_B}{|\ln\varepsilon|}\right).\end{aligned} \tag{2.4.11}$$

We now use the fact that by construction B covers at least one $\Omega'_{i,\varepsilon}$ with $A_i^\varepsilon \geq \beta$. This leads us to

$$\begin{aligned}\int_{\partial B} |\nabla h_\varepsilon'|^2 d\mathcal{H}^1(x') &+ \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h_\varepsilon'|^2\,dx' \\ &\geq \frac{1}{2\pi r_B}\Big(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B}\theta_i A_i^\varepsilon\Big)^2\left(1-\frac{2\pi\bar\mu^\varepsilon r_B^2}{\beta}-\frac{Cr_B}{|\ln\varepsilon|}\right) \\ &\geq \frac{1}{2\pi r_B}\sum_{i\in I_{\beta,B}} |\tilde A_i^\varepsilon|^2\,(1-cr_B)\,,\end{aligned} \tag{2.4.12}$$

for some $c>0$ depending only on κ and $\bar\delta$, where in the last line we used that $A_i^\varepsilon \geq \tilde A_i^\varepsilon$. Hence there exists $r_0\in(0,1)$ depending only on κ, and $\bar\delta$ such that the right-hand side of (2.4.12) is positive.

Finally, let us define $\mathcal{F}(x,r) := \int_{B(x,r)} |\nabla h_\varepsilon'|^2\,dx' + \frac{r\kappa^2}{4|\ln\varepsilon|}\int_{B(x,r)} |h_\varepsilon'|^2\,dx'$, where $B(x,r)$ is the ball centered at x of radius r. The relation (2.4.12) then reads for $B(x,r) = B \in \mathcal{B}_r$ and a.e. $r \in (r(\mathcal{B}_0), r_0]$:

$$\frac{\partial \mathcal{F}}{\partial r} \geq \frac{1}{2\pi r} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 (1 - cr), \tag{2.4.13}$$

with c as before. Then using [99, Proposition 4.1], for every $B \in \mathcal{B}(s) := \mathcal{B}_r$ with $r = e^s r(\mathcal{B}_0)$ (using the notation of [99, Theorem 4.2]) we have

$$\begin{aligned}
\int_{B \setminus \mathcal{B}_0} |\nabla h'_\varepsilon|^2 \, dx' + \frac{r_B \kappa^2}{4|\ln \varepsilon|} \int_B |h'_\varepsilon|^2 \, dx' &\geq \int_0^s \sum_{\substack{B' \in \mathcal{B}(t) \\ B' \subset B}} \frac{1}{2\pi} \sum_{i \in I_{\beta,B'}} |\tilde{A}_i^\varepsilon|^2 \left(1 - cr(\mathcal{B}(t))\right) dt \\
&= \int_0^s \sum_{\substack{B' \in \mathcal{B}(t) \\ B' \subset B}} \frac{1}{2\pi} \sum_{i \in I_{\beta,B'}} |\tilde{A}_i^\varepsilon|^2 \left(1 - ce^t r(\mathcal{B}_0)\right) dt \\
&\geq \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr\right),
\end{aligned} \tag{2.4.14}$$

where we observed that the double summation appearing in the first and second lines is simply the summation over $I_{\beta,B}$. Once again, in view of the fact that $r_B \leq 1$ and that both terms in the integrand of the left-hand side of (2.4.14) are non-negative, this completes the proof of the second item. $\square$

Proof of the third item. This follows [100]. Let χ be a non-negative Lipschitz function with support in U. By the "layer-cake" theorem [72], for any $B \in \mathcal{B}_r$

we have

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' = \int_0^{+\infty} \int_{E_t \cap B} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx'\, dt, \tag{2.4.15}$$

where $E_t := \{\chi > t\}$. If $i \in I_{\beta,B}$, then by construction for any $s \in [r(\mathcal{B}_0), r]$ there exists a unique closed ball $B_{i,s} \in \mathcal{B}_s$ containing $\Omega'_{i,\varepsilon}$. Therefore, for $t > 0$ we can define

$$s(i,t) := \sup \{ s \in [r(\mathcal{B}_0), r] \,:\, B_{i,s} \subset E_t \},$$

with the convention that $s(i,t) = r(\mathcal{B}_0)$ if the set is empty. We also let $B_i^t := B_{i,s(i,t)}$ whenever $s(i,t) > r(\mathcal{B}_0)$. Note that for each $i \in I_{\beta,B}$ we have that $t \mapsto s(i,t)$ is a non-increasing function. In particular, we can define $t_i \geq 0$ to be the supremum of the set of t's at which $s(i,t) = r$ (or zero, if this set is empty).

If $t > t_i$ and $s(i,t) > r(\mathcal{B}_0)$, then for any $x \in \Omega'_{i,\varepsilon}$ and any $y \in B_i^t \setminus E_t$ (which is not empty) we have

$$\chi(x) - t \leq \chi(x) - \chi(y) \leq 2s(i,t)\|\nabla\chi\|_\infty. \tag{2.4.16}$$

Averaging over all $x \in \Omega'_{i,\varepsilon}$, we hence deduce

$$\chi_i - t \leq 2s(i,t)\|\nabla\chi\|_\infty. \tag{2.4.17}$$

Now, for any $t \geq 0$ the collection $\{B_i^t\}_{i \in I_{\beta,B,t}}$, where $I_{\beta,B,t} := \{i \in I_{\beta,B} :$

$s(i,t) > r(\mathcal{B}_0)\}$ is disjoint. Indeed if $i,j \in I_{\beta,B,t}$ and $s(i,t) \geq s(j,t)$ then, since $\mathcal{B}_{s(i,t)}$ is disjoint, the balls $B_{i,s(i,t)}$ and $B_{j,s(i,t)}$ are either equal or disjoint. If they are disjoint we note that $s(i,t) \geq s(j,t)$ implies that $B_{j,s(j,t)} \subseteq B_{j,s(i,t)}$, and, therefore, $B_j^t = B_{j,s(j,t)}$ and $B_i^t = B_{i,s(i,t)}$ are disjoint. If they are equal and $s(i,t) > s(j,t)$, then $B_{j,s(j,t)} \subset E_t$, contradicting the definition of $s(j,t)$. So $s(j,t) = s(i,t)$ and then $B_j^t = B_i^t$.

Now assume that $B' \in \{B_i^t\}_{i \in I_{\beta,B,t}}$ and let s be the common value of $s(i,t)$ for i's in $I_{\beta,B'}$. Then, the previous item of the proposition yields

$$\int_{B'} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{s}{r(\mathcal{B}_0)} - cs \right)^+ \sum_{i \in I_{\beta,B',t}} |\tilde{A}_i^\varepsilon|^2.$$

Summing over $B' \in \{B_i^t\}_{i \in I_{\beta,B,t}}$, we deduce

$$\begin{aligned} \int_{B \cap E_t} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' &\geq \frac{1}{2\pi} \sum_{i \in I_{\beta,B,t}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{s(i,t)}{r(\mathcal{B}_0)} - cs(i,t) \right)^+ \\ &= \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{s(i,t)}{r(\mathcal{B}_0)} - cs(i,t) \right)^+, \end{aligned} \tag{2.4.18}$$

where in the last inequality we took into consideration that all the terms corresponding to $i \in I_{\beta,B} \setminus I_{\beta,B,t}$ give no contribution to the sum in the right-hand side. Integrating the above expression over t and using the fact that

$r_0(\mathcal{B}_0) \leq s(i,t) \leq r$ yields

$$\int_0^{+\infty} \int_{E_t \cap B} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx'\, dt$$
$$\geq \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \int_0^{\chi_i} \left(\ln \frac{s(i,t)}{r(\mathcal{B}_0)} - cr \right)^+ dt$$
$$\geq \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ + \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \int_0^{\chi_i} \ln \frac{s(i,t)}{r} dt. \tag{2.4.19}$$

We now concentrate on the last term in (2.4.19). Using the estimate in (2.4.17) and the definition of t_i, we can bound the integral in this term as follows

$$\int_0^{\chi_i} \ln \frac{s(i,t)}{r} dt \geq \int_{t_i}^{\chi_i} \ln \left(\frac{\chi_i - t}{2r\|\nabla \chi\|_\infty} \right) dt \geq -C\|\nabla \chi\|_\infty, \tag{2.4.20}$$

for some universal $C > 0$, which is obtained by an explicit computation and the fact that $r \leq r_0 < 1$. Finally, combining (2.4.20) with (2.4.19), the statement follows from (2.4.15). $\square$

Remark 2.4.3. *Inspecting the proof, we note that the statements of the proposition are still true with the left-hand sides replaced by* $\int_{B \setminus \mathcal{B}_0} \chi |\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln \varepsilon|} \int_B \chi |h'_\varepsilon|^2 dx'$ *(with* $\chi \equiv 1$ *or* χ *Lipschitz, respectively).*

2.5 Energy displacement

In this section, we follow the idea of [100] of localizing the ball construction and combine it with a "energy displacement" which allows to reduce to the situation where the energy density in (2.3.15) is bounded below. For the proposition below we define for all $x' \in \mathbb{T}^2_{\ell^\varepsilon}$:

$$\nu^\varepsilon(x') := \sum_{i \in I_\beta} |\tilde{A}_i^\varepsilon|^2 \tilde{\delta}_i^\varepsilon(x'), \tag{2.5.1}$$

where $\tilde{\delta}_i^\varepsilon(x')$ is given by (2.3.18). We also recall that ρ_ε defined in (2.3.1) is the expected radius of droplets in a minimizing configuration in the blown up coordinates.

We cover $\mathbb{T}^2_{\ell^\varepsilon}$ by the balls of radius $\frac{1}{4}r_0$ whose centers are in $\frac{r_0}{8}\mathbb{Z}^2$. We call this cover $\{U_\alpha\}_\alpha$ and $\{x_\alpha\}_\alpha$ the centers. We also introduce $D_\alpha := B(x_\alpha, \frac{3r_0}{4})$.

Proposition 2.5.1. *Let h'_ε satisfy* (2.3.16), *assume* (2.2.19) *holds, and set*

$$f_\varepsilon := |\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\,\nu^\varepsilon. \tag{2.5.2}$$

Then there exist $\varepsilon_0 > 0$ as in Proposition 2.4.2 and constants $c, C > 0$ depending only on $\bar{\delta}$ and κ such that for all $\varepsilon < \varepsilon_0$, there exists a family of integers $\{n_\alpha\}_\alpha$ and a density g_ε on $\mathbb{T}^2_{\ell^\varepsilon}$ with the following properties.

- *g_ε is bounded below:*

$$g_\varepsilon \geq -c\ln^2(M_\varepsilon + 2) \quad \textit{on } \mathbb{T}^2_{\ell^\varepsilon}.$$

- *For any α,*

$$n_\alpha^2 \leq C\left(g_\varepsilon(D_\alpha) + c\ln^2(M_\varepsilon+2)\right).$$

- *For any Lipschitz function χ on $\mathbb{T}^2_{\ell^\varepsilon}$ we have*

$$\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(f_\varepsilon - g_\varepsilon)dx'\right| \leq C\sum_\alpha \left(\nu^\varepsilon(U_\alpha) + (n_\alpha + M_\varepsilon)\ln(n_\alpha + M_\varepsilon + 2)\right)\|\nabla\chi\|_{L^\infty(D_\alpha)}. \tag{2.5.3}$$

Proof. The proof follows the method of [101], involving a localization of the ball construction followed by energy displacement. Here we follow [100, Proposition 4.9]. One key difference is the restriction to I_β which means we cover only those $\Omega'_{i,\varepsilon}$ satisfying $A_i^\varepsilon \geq \beta$ as in Proposition (2.4.2).

- *Step 1: Localization of the ball construction.*

We use U_α defined above as the cover on $\mathbb{T}^2_{\ell^\varepsilon}$. For each U_α covering at least one droplet whose volume is greater or equal than β and for any $r \in (r(\mathcal{B}_0), \frac{1}{4}r_0)$ we construct disjoint balls $\mathcal{B}_r^\alpha$ covering all $\Omega'_{i,\varepsilon}$ with $i \in I_{\beta,U_\alpha}$, using Proposition 2.4.2. Then choosing a small enough $\rho \in (r(\mathcal{B}_0), \frac{1}{4}r_0)$ independent of ε (to be specified below), we may extract from $\cup_\alpha \mathcal{B}_\rho^\alpha$ a disjoint family which covers $\cup_{i\in I_\beta}\Omega'_{i,\varepsilon}$ as follows: Denoting by $\mathcal{C}$ a connected component of $\cup_\alpha\mathcal{B}_\rho^\alpha$, we claim that there exists α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. Indeed if $x \in \mathcal{C}$ and letting λ be a Lebesgue number[1] of the covering of $\mathbb{T}^2_{\ell^\varepsilon}$ by $\{U_\alpha\}_\alpha$ (it is easy to see that in our case $\frac{1}{4}r_0 < \lambda < \frac{1}{2}r_0$), there exists α_0 such that $B(x,\lambda) \subset U_{\alpha_0}$. If $\mathcal{C}$ intersected

[1] A Lebesgue number of a covering of a compact set is a number $\lambda > 0$ such that every subset of diameter less than λ is contained in some element of the covering.

the complement of U_{α_0}, there would exist a chain of balls connecting x to $(U_{\alpha_0})^c$, each of which would intersect U_{α_0}. Each of the balls in the chain would belong to some $\mathcal{B}_\rho^{\alpha'}$ with α' such that dist $(U_{\alpha'}, U_{\alpha_0}) \leq 2\rho < \frac{1}{2}r_0$. Thus, calling k the universal maximum number of α''s such that dist $(U_{\alpha'}, U_{\alpha_0}) < \frac{1}{2}r_0$, the length of the chain is at most $2k\rho$ and thus $\lambda \leq 2k\rho$. If we choose $\rho < \lambda/(2k)$, this is impossible and the claim is proved. Let us then choose $\rho = \lambda/(4k)$. By the above, each $\mathcal{C}$ is included in some U_α.

We next obtain a disjoint cover of $\cup_{i\in I_\beta}\Omega'_{i,\varepsilon}$ from $\cup_\alpha \mathcal{B}_\rho^\alpha$. Let $\mathcal{C}$ be a connected component of $\cup_\alpha \mathcal{B}_\rho^\alpha$. By the discussion of the preceding paragraph, there exists an index α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. We then remove from $\mathcal{C}$ all the balls which do not belong to $\mathcal{B}_\rho^{\alpha_0}$ and still denote by $\mathcal{B}_\rho^{\alpha_0}$ the obtained collection. We repeat this process for all the connected components and obtain a disjoint cover $\mathcal{B}_\rho = \cup_\alpha \mathcal{B}_\rho^\alpha$ of $\cup_{i\in I_\beta}\Omega'_{i,\varepsilon}$. Note that this procedure uniquely associates an α to a given $B \in \mathcal{B}_\rho$, as well as to each $\Omega'_{i,\varepsilon}$ for a given $i \in I_\beta$ by assigning to it the ball in $\mathcal{B}_\rho$ that covers it, and then the α of this ball. We will use this repeatedly below. We also slightly abuse the notation by sometimes using $\mathcal{B}_\rho^\alpha$ to denote the union of the balls in the family $\mathcal{B}_\rho^\alpha$.

We now proceed to the energy displacement.

- *Step 2: Energy displacement in the balls.*

Note that by construction every ball in $\mathcal{B}_\rho^\alpha$ is included in U_α. From the last item of Proposition 2.4.2 applied to a ball $B \in \mathcal{B}_\rho^\alpha$, if ε is small enough then, for any Lipschitz non-negative χ we have for some $c > 0$ depending only on κ

and $\bar{\delta}$ and a universal $C > 0$

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' - \frac{1}{2\pi} \left(\ln \frac{\rho}{r(\mathcal{B}_0^\alpha)} - c\rho \right)^+ \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2$$
$$\geq -C\nu^\varepsilon(B) \|\nabla \chi\|_{L^\infty(B)},$$

where ν^ε is defined by (2.5.1). Rewriting the above, recalling the definition (2.3.1) and defining $n_\alpha \geq 1$ to be the number of droplets included in $U_\alpha \supset B$ and satisfying $A_i^\varepsilon \geq \beta$, we have

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' - \frac{1}{2\pi} \left(\ln \frac{\rho}{n_\alpha \rho_\varepsilon} - c \right) \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2 + \int_B \chi \omega_\varepsilon dx'$$
$$\geq -C\nu^\varepsilon(B) \|\nabla \chi\|_{L^\infty(B)},$$

where we set $\bar{r}_\alpha := \frac{r(\mathcal{B}_0^\alpha)}{\rho_\varepsilon}$ and define (recall that α implicitly depends on $i \in I_\beta$)

$$\omega_\varepsilon(x') := \frac{1}{2\pi} \sum_{i \in I_\beta} |\tilde{A}_i^\varepsilon|^2 \ln \left(\frac{r(\mathcal{B}_0^\alpha)}{n_\alpha \rho_\varepsilon} \right) \tilde{\delta}_i^\varepsilon(x') = \frac{1}{2\pi} \sum_{i \in I_\beta} |\tilde{A}_i^\varepsilon|^2 \ln \left(\frac{\bar{r}_\alpha}{n_\alpha} \right) \tilde{\delta}_i^\varepsilon(x'). \tag{2.5.4}$$

The quantity ω_ε in some sense measures the discrepancy between the droplets $\Omega'_{i,\varepsilon}$ and balls of radius ρ_ε. We will thus naturally use M_ε in (2.3.13) to control it. Note also that it is only supported in the droplets, hence in the balls of $\mathcal{B}_\rho$.

Applying Lemma 3.1 of [101] to

$$f_{B,\varepsilon} = \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 - \frac{1}{2\pi} \left(\ln \frac{\rho}{\rho_\varepsilon n_\alpha} - c \right) \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \tilde{\delta}_i^\varepsilon + \omega_\varepsilon \right) \mathbf{1}_B$$

we deduce the existence of a positive measure $g_{B,\varepsilon}$ such that

$$\|f_{B,\varepsilon} - g_{B,\varepsilon}\|_{\mathrm{Lip}^*} \le C\nu^\varepsilon(B), \tag{2.5.5}$$

where Lip^* denotes the dual norm to the space of Lipschitz functions and $C > 0$ is universal.

- *Step 3: Energy displacement on annuli and definition of g_ε.*

We define a set C_α as follows: recall that ρ was assumed equal to $\lambda/(4k)$, where $\lambda \le \frac{1}{4}r_0$ and k bounds the number of α''s such that dist $(U_{\alpha'}, U_\alpha) < \frac{1}{2}r_0$ for any given α. Therefore the total radius of the balls in $\mathcal{B}_\rho$ which are at distance less than r_0 from U_α is at most $k\rho = \frac{1}{16}r_0$. In particular, letting T_α denote the set of $t \in (\frac{r_0}{2}, \frac{3r_0}{4})$ such that the circle of center x_α (where we recall x_α is the center of U_α) and radius t does not intersect $\mathcal{B}_\rho^\alpha$, we have $|T_\alpha| \ge \frac{3}{16}r_0$. We let $C_\alpha = \{x \mid |x - x_\alpha| \in T_\alpha\}$ and recall that $D_\alpha = B(x_\alpha, \frac{3r_0}{4})$.

Let $t \in T_\alpha$. Arguing exactly as in the proof of (2.4.10), we find that

$$\int_{\partial B(x_\alpha,t)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(x_\alpha,t)} |h'_\varepsilon|^2 \, dx' \ge \frac{m_{\varepsilon,t}^2}{2\pi t}\left(1 - \frac{\kappa^2 t}{4|\ln\varepsilon|}\right)$$

with $m_{\varepsilon,t} := \int_{B(x_\alpha,t)} (\mu'_\varepsilon(x') - \bar\mu^\varepsilon)\, dx'$. Arguing as in (2.4.12) and using the fact that $B(x_\alpha, \frac{1}{2}r_0)$ contains all the droplets with $i \in I_{\beta,U_\alpha}$, we find that we can take ε sufficiently small depending on κ, and r_0 sufficiently small depending on κ and $\bar\delta$ such that for all $t \in T_\alpha$,

$$\int_{\partial B(x_\alpha,t)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(x_\alpha,t)} |h'_\varepsilon|^2 \, dx' \ge \frac{1}{4\pi t}\left(\sum_{i\in I_{\beta,U_\alpha}} A_i^\varepsilon\right)^2.$$

Integrating this over $t \in T_\alpha$, using that $|T_\alpha| \geq \frac{3}{16} r_0$, we obtain that

$$\int_{C_\alpha} |\nabla h'_\varepsilon|^2 dx' + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{D_\alpha} |h'_\varepsilon|^2 dx' \geq c \Bigg(\sum_{i \in I_{\beta, U_\alpha}} A_i^\varepsilon \Bigg)^2, \tag{2.5.6}$$

with $c > 0$ depending only on r_0, hence on κ and $\bar{\delta}$.

We now trivially extend the estimate in (2.5.6) to all α's, including those U_α that contain no droplets of size greater or equal than β. The overlap number of the sets $\{C_\alpha\}_\alpha$, defined as the maximum number of sets to which a given $x' \in \mathbb{T}^2_{\ell^\varepsilon}$ belongs is bounded above by the overlap number of the sets $\{D_\alpha\}_\alpha$, call it k'. Since the latter collection of balls covers the entire $\mathbb{T}^2_{\ell^\varepsilon}$, we have $k' \geq 1$. Then, letting

$$\begin{aligned} f'_\varepsilon := f_\varepsilon - \sum_{B \in \mathcal{B}_\rho} f_{B,\varepsilon} = \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho} + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \\ + \frac{1}{2\pi} \sum_{i \in I_\beta} \left(\ln \frac{\rho}{n_\alpha} - c \right) |\tilde{A}_i^\varepsilon|^2 \tilde{\delta}_i^\varepsilon - \omega_\varepsilon, \end{aligned} \tag{2.5.7}$$

and

$$f_{\alpha,\varepsilon} := \frac{1}{2k'} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{C_\alpha} + \frac{1}{2\pi} \sum_{i \in I_{\beta, \mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{\rho}{n_\alpha} - c \right) \tilde{\delta}_i^\varepsilon - \omega_\varepsilon \mathbf{1}_{\mathcal{B}_\rho^\alpha}, \tag{2.5.8}$$

we have

$$
\begin{aligned}
f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon} \geq & \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2\right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\mathcal{B}_\rho} \\
& - \frac{1}{2k'}\sum_\alpha \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) \mathbf{1}_{C_\alpha} + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \\
& \geq \frac{1}{2}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2\right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\mathcal{B}_\rho} + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \geq 0
\end{aligned}
\tag{2.5.9}
$$

and from (2.5.6)

$$
f_{\alpha,\varepsilon}(D_\alpha) \tag{2.5.10}
$$

$$
= \frac{1}{2k'}\int_{C_\alpha}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' + \frac{1}{2\pi}\left(\ln\frac{\rho}{n_\alpha} - c\right)\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2 - \omega_\varepsilon(D_\alpha)
$$

$$
\geq c\left(\sum_{i\in I_{\beta,U_\alpha}} A^\varepsilon_i\right)^2 - \frac{1}{2\pi}\ln n_\alpha \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2 - \omega_\varepsilon(D_\alpha) - C\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2, \tag{2.5.11}
$$

for some $C, c > 0$ depending only on κ and $\bar{\delta}$. Now we combine the middle two terms, using the definition of $\omega_{\varepsilon,\alpha}$ in (2.5.4), to obtain

$$
f_{\alpha,\varepsilon}(D_\alpha) \geq c\left(\sum_{i\in I_{\beta,U_\alpha}} A^\varepsilon_i\right)^2 - \frac{1}{2\pi}\ln\bar{r}_\alpha \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2 - C\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2. \tag{2.5.12}
$$

The next step is to bound $\bar{r}_\alpha$. We separate those $\Omega'_{i,\varepsilon}$ with $A^\varepsilon_i \geq 3^{2/3}\pi\gamma^{-1}$

and those with $A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}$. We denote (with s for "small" and b for "big")

$$
\begin{aligned}
I^s_{\beta,\alpha} &= \left\{ i \in I_{\beta,U_\alpha} : A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1} \right\},\\
I^b_{\beta,\alpha} &= I_{\beta,U_\alpha} \backslash I^s_{\beta,\alpha},\\
n_{\alpha_s} &= \# I^s_{\beta,\alpha}.
\end{aligned}
$$

For the small droplets, we use the obvious bound

$$\sum_{i \in I^s_{\beta,\alpha}} |A_i^\varepsilon|^{1/2} \le \ c n_{\alpha_s}, \tag{2.5.13}$$

with a universal $c > 0$, while for the large droplets we use that in view of the definition of M_ε in (2.3.13) we have

$$\sum_{i \in I^b_{\beta,\alpha}} |A_i^\varepsilon|^{1/2} \le C \sum_{i \in I^b_{\beta,\alpha}} A_i^\varepsilon \le C' M_\varepsilon, \tag{2.5.14}$$

for some universal $C, C' > 0$. We can now proceed to controlling $\bar{r}_\alpha$. By (2.3.1) and (2.4.3), for universally small ε we have

$$\bar{r}_\alpha \le C \sum_{i \in I_{\beta,U_\alpha}} P_i^\varepsilon, \tag{2.5.15}$$

for some universal $C > 0$. In view of (2.3.13), (2.5.14) and (2.5.13), we deduce from Remark 2.3.2 that for universally small ε we have

$$\begin{aligned}\bar{r}_\alpha &\le C\Big(M_\varepsilon + \sqrt{4\pi}\sum_{i\in I_{\beta,U_\alpha}} |A_i^\varepsilon|^{1/2}\Big) \\ &\le C\Big(M_\varepsilon + cn_{\alpha_s} + C'M_\varepsilon\Big) \le C''(n_{\alpha_s}+M_\varepsilon) \le C''(1+n_{\alpha_s}+M_\varepsilon), \quad (2.5.16)\end{aligned}$$

where $c, C, C', C'' > 0$ are universal. Therefore, (2.5.12) becomes

$$\begin{aligned}f_{\alpha,\varepsilon}(D_\alpha) &\ge c\Bigg(\sum_{i\in I^s_{\beta,\alpha}} A_i^\varepsilon\Bigg)^2 + c\Bigg(\sum_{i\in I^b_{\beta,\alpha}} A_i^\varepsilon\Bigg)^2 - C\ln\bar{r}_\alpha \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}_i^\varepsilon|^2 - C''' \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}_i^\varepsilon|^2, \\ &\ge c\beta^2 n_{\alpha_s}^2 + c\Bigg(\sum_{i\in I^b_{\beta,\alpha}} A_i^\varepsilon\Bigg)^2 - C'\ln(C''(1+n_{\alpha_s}+M_\varepsilon))\Bigg(n_{\alpha_s} + \sum_{i\in I^b_{\beta,\alpha}} A_i^\varepsilon\Bigg),\end{aligned}$$

(2.5.17)

where $C, C' > 0$ are universal, $c, C'', C''' > 0$ depend only on κ and $\bar{\delta}$, and C' was chosen so that $C|\tilde{A}_i^\varepsilon|^2 \le C'(A_i^\varepsilon + 1)$.

We now claim that this implies that

$$f_{\alpha,\varepsilon}(D_\alpha) \ge \frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}\Bigg(\sum_{i\in I^b_{\beta,\alpha}} A_i^\varepsilon\Bigg)^2 - C'''\ln^2(M_\varepsilon+2), \qquad (2.5.18)$$

where $C''' > 0$ depends only on κ and $\bar{\delta}$. This is seen by minimization of the right-hand side, as we now detail. For the rest of the proof, all constants will depend only on κ and $\bar{\delta}$. For shortness, we will set $X := \sum_{i\in I^b_{\beta,\alpha}} A_i^\varepsilon$.

First assume $n_{\alpha_s} = 0$. Then (2.5.17) can be rewritten

$$f_{\alpha,\varepsilon}(D_\alpha) \geq cX^2 - C' \ln(C''(1 + M_\varepsilon)))X,$$

By minimization of the quadratic polynomial in the right-hand side, we easily see that an inequality of the form (2.5.18) holds. Second, let us consider the case $n_{\alpha_s} \geq 1$. We may use the obvious inequality $\log(1+x+y) \leq \log(1+x) + \log(1+y)$ that holds for all $x \geq 0$ and $y \geq 0$ to bound from below

$$\frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}X^2 - C' \ln(C''(1 + n_{\alpha_s} + M_\varepsilon))(n_{\alpha_s} + X) \geq \frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}X^2 \\ - C(n_{\alpha_s} + X) - Cn_{\alpha_s} \ln(n_{\alpha_s} + 1) - CX \ln(n_{\alpha_s} + 1) - C \ln(M_\varepsilon + 1)(n_{\alpha_s} + X). \tag{2.5.19}$$

It is clear that the first three negative terms on the right-hand side can be absorbed into the first two positive terms, at the expense of a possible additive constant, which yields

$$\frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}X^2 - C' \ln(C''(n_{\alpha_s} + M_\varepsilon))(n_{\alpha_s} + X) \\ \geq \frac{c}{4}\beta^2 n_{\alpha_s}^2 + \frac{c}{4}X^2 - C \ln(M_\varepsilon + 1)(n_{\alpha_s} + X) - C. \tag{2.5.20}$$

Then by quadratic optimization the right hand side of (2.5.20) is bounded below by $-C \ln^2(M_\varepsilon + 2)$ (after possibly changing the constant). Inserting this into (2.5.17), we obtain (2.5.18).

We then apply [101, Lemma 3.2] over D_α to $f_{\alpha,\varepsilon} + C'''|D_\alpha|^{-1} \ln^2(M_\varepsilon +$

2), where C''' is the constant in the right-hand side of (2.5.18). We then deduce the existence of a measure $g_{\alpha,\varepsilon}$ on $\mathbb{T}^2_{\ell^\varepsilon}$ supported in D_α such that $g_{\alpha,\varepsilon} \geq -C'''|D_\alpha|^{-1}\ln^2(M_\varepsilon+2)$ and such that for every Lipschitz function χ

$$\left|\int_{D_\alpha} \chi(f_{\alpha,\varepsilon} - g_{\alpha,\varepsilon})\,dx'\right| \leq 2\,\mathrm{diam}\ (D_\alpha)\|\nabla\chi\|_{L^\infty(D_\alpha)} f^-_{\alpha,\varepsilon}(D_\alpha)$$
$$\leq C\ln(n_{\alpha_s}+M_\varepsilon+2)\|\nabla\chi\|_{L^\infty(D_\alpha)} \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2, \quad (2.5.21)$$

and we have used the observation that

$$f_{\alpha,\varepsilon} = \frac{1}{2k'}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathbf{1}_{C_\alpha} + \frac{1}{2\pi}\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}}\left(\ln\frac{\rho}{\bar{r}_\alpha} - C\right)|\tilde{A}^\varepsilon_i|^2\delta^\varepsilon_i, \quad (2.5.22)$$

and (2.5.16) to bound the negative part of $f_{\alpha,\varepsilon}$. In particular, taking $\chi = 1$, we deduce, in view of (2.5.18), that

$$g_{\alpha,\varepsilon}(D_\alpha) = f_{\alpha,\varepsilon}(D_\alpha) \geq \frac{c}{2}\beta^2 n^2_{\alpha_s} + \frac{c}{2}\left(\sum_{i\in I^b_{\beta,\alpha}} A^\varepsilon_i\right)^2 - C'''\ln^2(M_\varepsilon+2), \quad (2.5.23)$$

from which it follows that

$$g_{\alpha,\varepsilon}(D_\alpha) \geq c'\left(n^2_{\alpha_s} + (\#I_{\beta,\alpha})^2\right) - C'''\ln^2(M_\varepsilon+2) \geq \frac{1}{2}c' n^2_\alpha - C'''\ln^2(M_\varepsilon+2). \quad (2.5.24)$$

Recalling the positivity of $g_{B,\varepsilon}$ introduced in Step 2, we now let

$$g_\varepsilon := \sum_{B\in\mathcal{B}_\rho} g_{B,\varepsilon} + \sum_\alpha g_{\alpha,\varepsilon} + \left(f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon}\right), \quad (2.5.25)$$

and observe that since $f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon}$ is also non-negative by (2.5.9), and since $\sum_\alpha g_{\alpha,\varepsilon}$ is bounded below by $-k'C'''|D_\alpha|^{-1}\ln^2(M_\varepsilon+2)$, where, as before, k' is the overlap number of $\{D_\alpha\}_\alpha$, we have $g_\varepsilon \geq -c\ln^2(M_\varepsilon+2)$ for some $c>0$ depending only on κ and $\bar\delta$, which proves the first item. The second item follows from (2.5.24), (2.5.25) and the positiveness of $g_{B,\varepsilon}$ and $(f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon})$.

- *Step 4: Proof of the last item.*

Using the definition of g_ε in (2.5.25), for any Lipschitz χ we have

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi g_\varepsilon dx' = \sum_{B\in\mathcal{B}_\rho}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi g_{B,\varepsilon} dx' + \sum_\alpha \int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(g_{\alpha,\varepsilon}-f_{\alpha,\varepsilon})dx' + \int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi f'_\varepsilon dx'.$$

Hence, in view of (2.5.5), (2.5.7) and (2.5.21) we obtain for some $C>0$

$$\begin{aligned}\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(f_\varepsilon - g_\varepsilon)dx'\right| &\leq \sum_{B\in\mathcal{B}_\rho}\left|\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(g_{B,\varepsilon}-f_{B,\varepsilon})dx'\right)\right| + \sum_\alpha\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(g_{\alpha,\varepsilon}-f_{\alpha,\varepsilon})dx'\right| \\ &\leq C\sum_{B\in\mathcal{B}_\rho}\nu^\varepsilon(B)\|\nabla\chi\|_{L^\infty(B)} + C\sum_\alpha \ln(n_{\alpha_s}+M_\varepsilon+2)\|\nabla\chi\|_{L^\infty(D_\alpha)}\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}}|\tilde{A}^\varepsilon_i|^2.\end{aligned} \tag{2.5.26}$$

Using that $|\tilde{A}^\varepsilon_i|^2 \leq C(A^\varepsilon_i+1)$ for a universal $C>0$ and (2.5.14), we have

$$\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}}|\tilde{A}^\varepsilon_i|^2 \leq C(n_{\alpha_s}+M_\varepsilon).$$

Since $n_{\alpha_s} \leq n_\alpha$, the third item follows from (2.5.26). □

We now apply Proposition 2.5.1 to establish uniform bounds on M_ε, which characterizes the deviation of the droplets from the optimal shape.

Proposition 2.5.2. *If* (2.2.19) *holds, then* M_ε *is bounded by a constant depending only on* $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$, κ, $\bar{\delta}$ *and* ℓ.

Proof. From the last item of Proposition 2.5.1 applied with $\chi \equiv 1$ together with the first item, we have

$$\int_{\mathbb{T}^2_{\ell\varepsilon}} f_\varepsilon dx' = \int_{\mathbb{T}^2_{\ell\varepsilon}} g_\varepsilon dx' \geq -C|\ln\varepsilon| \ln^2(M_\varepsilon + 2),$$

with some $C > 0$ depending only on κ, $\bar{\delta}$ and ℓ, while from (2.2.19), (2.3.12) and (2.5.2), we have

$$C' \geq \ell^2 F^\varepsilon[u^\varepsilon] \geq M_\varepsilon + \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell\varepsilon}} f_\varepsilon dx' + o(1) \geq M_\varepsilon - C\ln^2(M_\varepsilon + 2) + o_\varepsilon(1),$$

for some $C' > 0$ depending only on $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$, κ, $\bar{\delta}$ and ℓ. The claimed result easily follows. □

With the help of Proposition 2.5.2, an immediate consequence of Proposition 2.5.1 is the following conclusion.

Corollary 2.5.3. *There exists* $C > 0$ *depending only on* κ, $\bar{\delta}$, ℓ *and* $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ *such that if* g_ε *is as in Proposition 2.5.1 and* (2.2.19) *holds, then* $g_\varepsilon \geq -C$.

In the following, we also define the modified energy density $\bar{g}_\varepsilon$, in which we include back the positive terms of M_ε and a half of $\frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2$ that had been “kept aside” instead of being included in f_ε:

$$\bar{g}_\varepsilon := g_\varepsilon + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 + |\ln\varepsilon|\Bigg\{\sum_i (P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\,\tilde{\delta}_i) + c_1 \sum_{A_i^\varepsilon > \pi 3^{2/3}\gamma^{-1}} A_i^\varepsilon \tilde{\delta}_i$$
$$+ c_2 \sum_{\beta \le A_i^\varepsilon \le \pi 3^{2/3}\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 \tilde{\delta}_i + c_3 \sum_{A_i^\varepsilon < \beta} A_i^\varepsilon \tilde{\delta}_i \Bigg\} \quad (2.5.27)$$

where we recall $\bar{r}_\varepsilon = \left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3}$ and $\tilde{\delta}_i^\varepsilon$ is defined by (2.3.18). These extra terms will be used to control the shapes and sizes of the droplets as well as to control h'_ε. We also point out that in view of (2.5.2), (2.5.3) and (2.3.12), we have

$$\ell^2 F^\varepsilon[u^\varepsilon] \ge \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \bar{g}_\varepsilon dx' + o_\varepsilon(1). \quad (2.5.28)$$

2.6 Convergence

In this section we study the consequences of the hypothesis

$$\forall R > 0,\ \mathcal{C}_R := \limsup_{\varepsilon\to 0} \int_{K_R} \bar{g}_\varepsilon(x + x_\varepsilon^0)dx < +\infty, \quad (2.6.1)$$

where $K_R = [-R, R]^2$ and (x_ε^0) is such that $x_\varepsilon^0 + K_R \subset \mathbb{T}^2_{\ell^\varepsilon}$. This corresponds to “good” blow up centers x_ε^0, and will be satisfied for most of them.

In order to obtain $o_\varepsilon(1)$ estimates on the energetic cost of each droplet under this assumption, we need good quantitative estimates for the deviations of the shape of the droplets from balls of the same volume. A convenient quantity that can be used to characterize these deviations is the *isoperimetric* *deficit*, defined as (in two space dimensions)

$$D(\Omega'_{i,\varepsilon}) := \frac{|\partial\Omega'_{i,\varepsilon}|}{\sqrt{4\pi|\Omega'_{i,\varepsilon}|}} - 1. \tag{2.6.2}$$

The isoperimetric deficit may be used to bound several types of geometric characteristics of $\Omega'_{i,\varepsilon}$ that measure their deviations from balls. The quantitative isoperimetric inequality, which holds for any set of finite perimeter, may be used to estimate the measure of the symmetric difference between $\Omega'_{i,\varepsilon}$ and a ball. More precisely, we have [46]

$$\alpha(\Omega'_{i,\varepsilon}) \leq C\sqrt{D(\Omega'_{i,\varepsilon})}, \tag{2.6.3}$$

where $C > 0$ is a universal constant and $\alpha(\Omega'_{i,\varepsilon})$ is the Fraenkel asymmetry defined as

$$\alpha(\Omega'_{i,\varepsilon}) := \min_{B} \frac{|\Omega'_{i,\varepsilon} \triangle B|}{|\Omega'_{i,\varepsilon}|}, \tag{2.6.4}$$

where $\triangle$ denotes the symmetric difference between the two sets, and the infimum is taken over balls B with $|B| = |\Omega'_{i,\varepsilon}|$. In the following, we will use the notation r_i^ε and a_i^ε for the radii and the centers of the balls that minimize $\alpha(\Omega'_{i,\varepsilon})$, respectively.

On the other hand, in two space dimensions the following inequality due originally to Bonnesen [12] (for a review, see [89]) is applicable to $\Omega'_{i,\varepsilon}$:

$$R_i^\varepsilon \le r_i^\varepsilon \left(1 + c\sqrt{D(\Omega'_{i,\varepsilon})}\right). \tag{2.6.5}$$

Here R_i^ε is the radius of the circumscribed circle of the measure theoretic interior of $\Omega'_{i,\varepsilon}$ and $c > 0$ is universal. Indeed, apply Bonnesen inequality to the saturation of $\Omega'_{i,\varepsilon}$ (i.e., the set with no holes) for each droplet. Then since the set $\Omega'_{i,\varepsilon}$ is connected and, therefore, its saturation has, up to negligible sets, a Jordan boundary [5], Bonnesen inequality applies to it.

2.6.1 Main result

We will obtain local lower bounds in terms of the renormalized energy for a finite number of Dirac masses in the manner of [11]:

Definition 2.6.1. *For any function χ and $\varphi \in \mathcal{A}_m$ (cf. Definition 2.2.1), we denote*

$$W(\varphi, \chi) = \lim_{\eta \to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi |\nabla \varphi|^2 dx + \pi \ln \eta \sum_{p \in \Lambda} \chi(p) \right). \tag{2.6.6}$$

We now state the main result of this section and postpone its proof to Section 2.6.2. Throughout the section, we use the notation of Sec. 4.7. To further simplify the notation, we periodically extend all the measures defined on $\mathbb{T}^2_{\ell^\varepsilon}$ to the whole of $\mathbb{R}^2$, without relabeling them. We also periodically extend the ball constructions to the whole of $\mathbb{R}^2$. This allows us to set, without loss of generality, all $x^0_\varepsilon = 0$.

Theorem 7. *Under assumption* (2.2.19), *the following holds.*

1. *Assume that for any $R > 0$ we have*

$$\limsup_{\varepsilon \to 0} \bar{g}_\varepsilon(K_R) < +\infty, \tag{2.6.7}$$

where $K_R = [-R, R]^2$. Then, up to a subsequence, the measures μ'_ε, defined in (2.3.17), *converge in $(C_0(\mathbb{R}^2))^*$ to a measure of the form $\nu = 3^{2/3}\pi \sum_{a\in\Lambda} \delta_a$ where Λ is a discrete subset of $\mathbb{R}^2$, and $\{\varphi^\varepsilon\}_\varepsilon$ defined in* (2.2.17) *converge weakly in $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ for any $p \in (1, 2)$ to φ which satisfies*

$$-\Delta\varphi = 2\pi \sum_{a\in\Lambda} \delta_a - m \text{ in } \mathbb{R}^2,$$

in the distributional sense, with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$. Moreover, for any sequence $\{\Omega_{i_\varepsilon,\varepsilon}\}_\varepsilon$ which remains in K_R, up to a subsequence, the following two alternatives hold:

   i. *Either $A^\varepsilon_{i_\varepsilon} \le \frac{C_R}{|\ln\varepsilon|}$ and $P^\varepsilon_{i_\varepsilon} \le \frac{C_R}{\sqrt{|\ln\varepsilon|}}$ as $\varepsilon \to 0$,*

   ii. *Or $A^\varepsilon_{i_\varepsilon}$ is bounded below by a positive constant as $\varepsilon \to 0$, and*

$$A^\varepsilon_{i_\varepsilon} \to 3^{2/3}\pi \text{ and } P^\varepsilon_{i_\varepsilon} \to 2 \cdot 3^{1/3}\pi \quad \text{as } \varepsilon \to 0,$$

   with

$$\alpha(\Omega'_{i_\varepsilon,\varepsilon}) \le \frac{C_R}{|\ln\varepsilon|^{1/2}} \quad \text{as } \varepsilon \to 0, \tag{2.6.8}$$

for some $C_R > 0$ independent of ε.

2. If we replace (2.6.7) *by the stronger assumption*

$$\limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R) < CR^2, \tag{2.6.9}$$

where $C > 0$ is independent of R, then we have for any $p \in (1,2)$,

$$\limsup_{R\to+\infty} \left(\frac{1}{|K_R|} \int_{K_R} |\nabla\varphi|^p dx \right) < +\infty. \tag{2.6.10}$$

Moreover, for every family $\{\chi_R\}_{R>0}$ defined in Definition 2.2.3 we have

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R \bar{g}_\varepsilon dx \geq \frac{3^{4/3}}{2} W(\varphi, \chi_R) + \frac{3^{4/3}\pi}{8} \sum_{a\in\Lambda} \chi_R(a) + o(|K_R|). \tag{2.6.11}$$

Remark 2.6.2. *We point out that it is included in Part 1 of Theorem 7 that at most one droplet $\Omega'_{i_\varepsilon,\varepsilon}$ with $A_{i_\varepsilon,\varepsilon}$ bounded from below converges to $a \in \Lambda$. Indeed otherwise in the first item we would have $\mu'_\varepsilon \to 3^{2/3}\pi n_a \sum_{a\in\Lambda} \delta_a$ where $n_a > 1$ is the number of non-vanishing droplets converging to the point a.*

Theorem 7 relies crucially on the following proposition which establishes bounds needed for compactness. Each of the bounds relies on (2.6.7). Throughout the rest of this section, all constants are assumed to implicitly depend on κ, $\bar{\delta}$, ℓ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$.

Lemma 2.6.3. *Let $\bar{g}_\varepsilon$ be as above, assume* (2.6.7) *holds and denote $\mathcal{C}_R = \limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R)$. Then for any R and ε small enough depending on R we* *have*

$$\sum_{\alpha|_{U_\alpha \subset K_R}} n_\alpha^2 \le C(\mathcal{C}_{R+C} + R^2), \tag{2.6.12}$$

$$\sum_{i \in I_{\beta, K_R}} A_i^\varepsilon \le C(\mathcal{C}_{R+C} + R^2), \tag{2.6.13}$$

$$\left| \int_{K_R} \chi_R (f_\varepsilon - g_\varepsilon) dx \right| \le C \sum_{\alpha|_{U_\alpha \subset K_{R+C} \setminus K_{R-C}}} (n_\alpha + 1) \ln(n_\alpha + 2) \le C(\mathcal{C}_{R+C} + R^2), \tag{2.6.14}$$

where $\{\chi_R\}$ *is as in Definition 2.2.3 and* $n_\alpha = \# I_{\beta, U_\alpha}$*, with* U_α *as in the proof of Proposition 2.5.1, for some* $C > 0$ *independent of* ε *or* R*. Furthermore, for any* $p \in (1,2)$ *there exists a* $C_p > 0$ *depending on* p *such that for any* $R > 0$ *and* ε *small enough*

$$\int_{K_R} |\nabla h'_\varepsilon|^p dx \le C_p(\mathcal{C}_{R+C} + R^2). \tag{2.6.15}$$

Proof. First observe that the rescaled droplet volumes and perimeters A_i^ε and P_i^ε are bounded independently of ε, as follows from Proposition 2.5.2 and the definition of M_ε. Then, (2.6.12) and (2.6.13) are a consequence of (2.6.7), the second item in Proposition 2.5.1 together with the upper bound on M_ε. The first inequality appearing in (2.6.14) follows from item 3 of Proposition 2.5.1 with the bound on M_ε, where we took into consideration that only those D_α that are in the $O(1)$ neighborhood of the support of $|\nabla \chi_R|$ contribute to the sum, along with the observation that the mass of ν^ε (of (2.5.1)) is now controlled by n_α (a consequence of the above fact that all droplet volumes are uniformly bounded). The second inequality in (2.6.14) follows from (2.6.12). The bound (2.6.15) is a consequence of Proposition 2.4.2 and follows as in [101] and [100]. We refer the reader to [101], Lemma 4.6 or [100] Lemma 4.6 for the proof in a slightly simpler setting. $\square$

2.6.2 Lower bound by the renormalized energy (Proof of Theorem 7)

We start by proving the first assertions of the theorem.

- *Step 1: All limit droplets have optimal sizes.* From (2.5.27), (2.6.7) and Corollary 2.5.3, for all ε sufficiently small depending on R we have

$$\int_{K_R} \left(\sum_i \left(P_i - \sqrt{4\pi |A_i^\varepsilon|} \right) \tilde{\delta}_i^\varepsilon + c_1 \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} A_i^\varepsilon \tilde{\delta}_i^\varepsilon \right. \\ \left. + c_2 \sum_{\beta \le A_i^\varepsilon \le \pi 3^{2/3}\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 \tilde{\delta}_i^\varepsilon + c_3 \sum_{A_i^\varepsilon < \beta} A_i^\varepsilon \tilde{\delta}_i^\varepsilon \right) dx \le \frac{C_R}{|\ln \varepsilon|}, \quad (2.6.16)$$

where we recall that all the terms in the sums are nonnegative. It then easily follows that for all $i \in I_{K_R}$ the droplets with $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$ do not exist when ε is small enough depending on R, and those with $A_i^\varepsilon < \beta$ satisfy $A_i^\varepsilon = C_R |\ln \varepsilon|^{-1}$ and $P_i^\varepsilon \le C_R |\ln \varepsilon|^{-1/2}$, for some $C_R > 0$ independent of ε. This establishes item (i) of Part 1 of the theorem.

It remains to treat the case of $A_i^\varepsilon \in [\beta, 3^{2/3}\pi\gamma^{-1}]$ when ε is small enough.

It follows from (2.6.16) that

$$D(\Omega'_{i,\varepsilon}) \leq \frac{C_R}{|\ln \varepsilon|}, \tag{2.6.17}$$

for some $C_R > 0$ independent of ε, and since $\bar{r}_\varepsilon = 3^{1/3} + o_\varepsilon(1)$, for all these droplets (or equivalently for all droplets with $A_i^\varepsilon \geq \beta$) we must have

$$A_i^\varepsilon \to 3^{2/3}\pi \quad \text{and} \quad P_i^\varepsilon \to 2 \cdot 3^{1/3}\pi \;\; \text{as } \varepsilon \to 0. \tag{2.6.18}$$

Using (2.6.3), (2.6.8) easily follows from (2.6.18) and (2.6.16).

- *Step 2: Convergence results.*

From boundedness of A_i^ε, (2.6.13) and (2.6.16) we know that $\# I_{\beta,K_R}$ and $\mu'_\varepsilon(K_R)$ are both bounded independently of ε as $\varepsilon \to 0$. We easily deduce from this, the previous step and the definition of μ'_ε that up to extraction, μ'_ε converges in each K_R to at most finitely many point masses which are integer multiples of $3^{2/3}\pi$ and, hence, to a measure of the form $\nu = 3^{2/3}\pi \sum_{a\in\Lambda} d_a \delta_a$, where $d_a \in \mathbb{N}$ and Λ is a discrete set in the whole of $\mathbb{R}^2$. In view of (2.6.15), we also have $h'_\varepsilon \rightharpoonup h \in \dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$, up to extraction (recall that we work with equivalence classes from (2.2.10)). Finally, from the definition of $\bar{g}_\varepsilon$ in (2.5.27) and the bound (2.6.7) we deduce that

$$\frac{\kappa^2}{|\ln \varepsilon|} \int_{K_R} |h'_\varepsilon|^2 \leq C_R$$

from which it follows that $|\ln \varepsilon|^{-1} h'_\varepsilon$ tends to 0 in $L^2_{loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Passing to the limit in the sense of distributions in (2.3.16), we then deduce from the above convergences that we must have

$$-\Delta h = 3^{2/3}\pi \sum_{a\in\Lambda} d_a \delta_a - \bar{\mu} \quad \text{ on } \mathbb{R}^2. \tag{2.6.19}$$

We will show below that $d_a = 1$ for every $a \in \Lambda$, and when this is done, this will complete the proof of the first item after recalling $\varphi^\varepsilon = 2 \cdot 3^{-2/3} h'_\varepsilon$ and $m = 2 \cdot 3^{-2/3}\bar{\mu}$.

- *Step 3: There is only one droplet converging to any limit point a.*

In order to prove this statement, we examine lower bounds for the energy. Fix $R > 1$ such that $\partial K_R \cap \Lambda = \varnothing$ and consider $a \in \Lambda \cap K_R$. From Step 1, (2.2.2) and Lemma 2.4.1, for any $\eta \in (0, \frac{1}{2})$ such that $\eta < \frac{1}{2}\min_{b\in\Lambda\cap K_R\setminus\{a\}} |a - b|$ and for all $r < \eta$, all the droplets converging to a are covered by $B(a, r)$, and $B(a, \eta)$ contains no other droplets with $A_i^\varepsilon \geq \beta$, for ε small enough. There are $d_a \geq 1$ droplets in $B(a, r)$ such that $A_i^\varepsilon \to 3^{2/3}\pi$ as $\varepsilon \to 0$, let us relabel them as $\Omega'_{1,\varepsilon}, \ldots, \Omega'_{d_a,\varepsilon}$.

Let $U = B(a, \eta)$. Arguing as in the proof of the first item of Proposition 2.4.2, by (2.6.18), we may construct a collection $\mathcal{B}_0$ of disjoint closed balls covering $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ and satisfying

$$r(\mathcal{B}_0) \leq C d_a \rho_\varepsilon < \eta, \tag{2.6.20}$$

for some universal $C > 0$, provided ε is small enough, and a collection of disjoint balls $\mathcal{B}_r$ covering $\mathcal{B}_0$ of total radius $r \in [r(\mathcal{B}_0), \eta]$. Choosing $r = \eta^3$, which is always possible for small enough ε, it is clear that $\mathcal{B}_{\eta^3}$ consists of only a single ball contained in $B(a, \frac{3}{2}\eta^3)$ for ε small enough. Applying the second item of Proposition 2.4.2 to that ball, we then obtain

$$\int_{\mathcal{B}_{\eta^3}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{\eta^3}{r(\mathcal{B}_0)} - c\eta^3 \right) \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2. \quad (2.6.21)$$

Therefore, we have

$$\int_{\mathcal{B}_{\eta^3}} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{\eta^3}{r(\mathcal{B}_0)} - c\eta^3 \right) \left(\min_{B(a,\eta)} \chi_R \right) \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2. \quad (2.6.22)$$

On the other hand, we can estimate the contribution of the remaining part of $B(a,\eta)$ as

$$\begin{aligned} &\int_{B(a,\eta)\setminus\mathcal{B}_{\eta^3}} \chi_R |\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,\eta)} \chi_R |h'_\varepsilon|^2 \, dx' \\ &\quad \geq \left(\min_{B(a,\eta)} \chi_R \right) \left(\int_{B(a,\eta)\setminus B(a,2\eta^3)} |\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,\eta)} |h'_\varepsilon|^2 \, dx' \right) \\ &\geq \left(\min_{B(a,\eta)} \chi_R \right) \int_{2\eta^3}^{\eta} \left(\int_{\partial B(a,r_B)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,r_B)} |h'_\varepsilon|^2 \, dx' \right) dr_B. \end{aligned} \quad (2.6.23)$$

Arguing as in (2.4.12) and using the fact that $\eta < \frac{1}{2}$, we obtain

$$\begin{aligned} &\int_{B(a,\eta)\setminus\mathcal{B}_{\eta^3}} \chi_R |\nabla h'_\varepsilon|^2 dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,\eta)} \chi_R |h'_\varepsilon|^2 dx' \\ &\qquad\qquad \geq \frac{1}{2\pi} \left(\min_{B(a,\eta)} \chi_R \right) \ln \frac{1}{2\eta^2} \left(\sum_{i=1}^{d_a} A_i^\varepsilon \right)^2 (1 - C\eta), \end{aligned} \quad (2.6.24)$$

where $C > 0$ is independent of η and ε, for small enough ε.

We will now use crucially the fact shown in Step 1 that all $A_i^\varepsilon \geq \beta$ approach the same limit as $\varepsilon \to 0$. We begin by adding (2.6.21) and (2.6.24) and subtracting $\frac{1}{2\pi}|\ln \rho_\varepsilon| \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2 \chi_R^i$ from both sides. With the help of (2.6.20) we can cancel out the leading order $O(|\ln \rho_\varepsilon|)$ term in the right-hand side of the obtained inequality. Replacing $\tilde{A}_i^\varepsilon$ and A_i^ε with $3^{2/3}\pi + o_\varepsilon(1)$ in the remaining terms and using the fact that $\min_{B(a,\eta)} \chi_R \geq \chi_R(a) - 2\eta \|\nabla \chi_R\|_\infty$ on $B(a,\eta)$, we then find

$$\int_{B(a,\eta)} \chi_R \left(|\nabla h_\varepsilon'|^2 + \frac{\kappa^2}{2|\ln \varepsilon|} |h_\varepsilon'|^2 - \frac{1}{2\pi} |\ln \rho_\varepsilon| \nu^\varepsilon \right) dx' \\ \geq \frac{3^{4/3}\pi}{2} \chi_R(a) \left(d_a^2 \ln \frac{1}{2\eta^2} + d_a \ln \frac{\eta^3}{2} \right) - C, \quad (2.6.25)$$

where $C > 0$ is independent of ε or η.

Now, adding up the contributions of all $a \in \Lambda \cap K_R$ and recalling the definition of f_ε in (2.5.2), we conclude that on the considered sequence

$$\limsup_{\varepsilon \to 0} \int_{K_R} \chi_R f_\varepsilon \, dx' \geq \limsup_{\varepsilon \to 0} \sum_{a \in \Lambda \cap K_R} \int_{B(a,\eta)} \chi_R f_\varepsilon \, dx' \\ \geq \frac{3^{4/3}\pi}{2} |\ln \eta| \sum_{a \in \Lambda \cap K_R} (2d_a^2 - 3d_a) \chi_R(a) - C, \quad (2.6.26)$$

for some $C > 0$ is independent of ε or η. In particular, since $\chi_R(a) > 0$ for all $a \in \Lambda \cap K_R$, the right-hand side of (2.6.26) goes to plus infinity as $\eta \to 0$, unless all $d_a = 1$. But by the estimate (2.6.14) of Proposition 2.6.3, Corollary 2.5.3 and our assumption in (2.6.7) together with (2.5.27), the left-hand side of (2.6.26) is bounded independently of η, which yields the conclusion.

- *Step 4: Energy of each droplet.* Now that we know that for each $a_i \in \Lambda \cap K_R$ there exists exactly one droplet $\Omega'_{i,\varepsilon}$ such that $a_i^\varepsilon \to a_i$ and $A_i^\varepsilon \to 3^{2/3}\pi$, we can extract more precisely the part of energy that concentrates in a small ball around each such droplet. Let B_i be a ball that minimizes Fraenkel asymmetry defined in (2.6.4), i.e., let $B_i = B(a_i^\varepsilon, r_i^\varepsilon)$, and let B be a ball of radius r_B centered at a_i^ε. Arguing as in (2.4.12) in the proof of the second item of Proposition 2.4.2, we can write

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln \varepsilon|} \int_B |h'_\varepsilon|^2 dx' \geq \frac{\varepsilon^{-4/3}|\ln \varepsilon|^{-2/3}|\Omega'_{i,\varepsilon} \cap B|^2}{2\pi r_B} \left(1 - c r_i^\varepsilon\right). \tag{2.6.27}$$

Observe that by the definition of Fraenkel asymmetry we have $|\Omega'_{i,\varepsilon} \cap B| \geq |B| - \frac{1}{2}\alpha(\Omega'_{i,\varepsilon})|B_i|$ for all $r_B < r_i^\varepsilon$. Hence, denoting by $\tilde{r}_i^\varepsilon$ the smallest value of r_B for which the right-hand side of this inequality is non-negative and integrating from $\tilde{r}_i^\varepsilon$ to r_i^ε, we find

$$\begin{aligned} \int_{B_i} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' \\ \geq \frac{\pi}{2}(1 + o_\varepsilon(1))\varepsilon^{-4/3}|\ln \varepsilon|^{-2/3} \int_{\tilde{r}_i^\varepsilon}^{r_i^\varepsilon} r_B^{-1}(r_B^2 - |\tilde{r}_i^\varepsilon|^2)^2 dr_B. \end{aligned} \tag{2.6.28}$$

Since by (2.6.8) and (2.6.18) we have $\tilde{r}_i^\varepsilon / r_i^\varepsilon \to 0$ and $\varepsilon^{-1/3}|\ln \varepsilon|^{-1/6} r_i^\varepsilon \to 3^{1/3}$

as $\varepsilon \to 0$, after an elementary computation we find

$$\int_{\Omega'_{i,\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{3^{4/3}\pi}{8} + o_\varepsilon(1). \tag{2.6.29}$$

On the other hand, by (2.6.5) and (2.6.17) it is possible to choose a collection $\mathcal{B}_0 \subset B(a_i, \eta)$, actually consisting of only a single ball $B(\tilde{a}_i^\varepsilon, R_i^\varepsilon)$ circumscribing $\Omega'_{i,\varepsilon}$, so that

$$r(\mathcal{B}_0) = R_i^\varepsilon \leq r_i^\varepsilon \left(1 + C_R |\ln \varepsilon|^{-1/2}\right) = \rho_\varepsilon + o_\varepsilon(\rho_\varepsilon). \tag{2.6.30}$$

The corresponding ball construction $\mathcal{B}_r$ of the first item of Proposition 2.4.2, with $U = B(a_i, \eta)$ and η as in Step 3 of the proof (again, just a single ball $B(\tilde{a}_i^\varepsilon, r)$), exists and is contained in U for all $r \in [r(\mathcal{B}_0), \eta']$, for any $\eta' \in (r(\mathcal{B}_0), \eta)$, provided ε is sufficiently small depending on η'. In view of the fact that for small enough η' and small enough ε depending on η' we have $\chi_R(x) \geq \chi_R(\tilde{a}_i^\varepsilon) - c|x - \tilde{a}_i^\varepsilon| > 0$, with $c > 0$ independent of ε, η' or R, we obtain that

$$\begin{aligned}
&\int_{B(\tilde a_i^\varepsilon,\eta')\setminus\mathcal{B}_0}\chi_R|\nabla h'_\varepsilon|^2dx'+\frac{\kappa^2}{4|\ln\varepsilon|}\int_{B(\tilde a_i^\varepsilon,\eta')}\chi_R|h'_\varepsilon|^2dx'\\
&\geq\int_{r(\mathcal{B}_0)}^{\eta'}(\chi_R(\tilde a_i^\varepsilon)-cr)\left(\int_{\partial\mathcal{B}_r}|\nabla h'_\varepsilon|^2d\mathcal{H}^1(x)\right)dr+\frac{\kappa^2\chi_R(\tilde a_i^\varepsilon)}{8|\ln\varepsilon|}\int_{B(\tilde a_i^\varepsilon,\eta')}|h'_\varepsilon|^2dx'\\
&\geq\int_{r(\mathcal{B}_0)}^{\eta'}(\chi_R(\tilde a_i^\varepsilon)-cr)\left(\int_{\partial\mathcal{B}_r}|\nabla h'_\varepsilon|^2d\mathcal{H}^1(x)+\frac{\kappa^2}{8\eta'|\ln\varepsilon|}\int_{B(\tilde a_i^\varepsilon,\eta')}|h'_\varepsilon|^2dx'\right)dr\\
&\quad\geq\int_{r(\mathcal{B}_0)}^{\eta'}(\chi_R(\tilde a_i^\varepsilon)-cr)\left(\int_{\partial\mathcal{B}_r}|\nabla h'_\varepsilon|^2d\mathcal{H}^1(x)+\frac{\kappa^2}{4|\ln\varepsilon|}\int_{\mathcal{B}_r}|h'_\varepsilon|^2dx'\right)dr\\
&\qquad\qquad\geq\frac{1}{2\pi}|\tilde A_i^\varepsilon|^2\int_{r(\mathcal{B}_0)}^{\eta'}(\chi_R(\tilde a_i^\varepsilon)-cr)(1-Cr)\frac{dr}{r},\quad(2.6.31)
\end{aligned}$$

for η' and ε sufficiently small, arguing as in (2.4.12) in the proof of Proposition 2.4.2 and taking into account Remark 2.4.3 in deducing the last line. Performing integration in (2.6.31) and using (2.6.30), we then conclude

$$\begin{aligned}
&\int_{B(\tilde a_i^\varepsilon,\eta')\setminus\mathcal{B}_0}\chi_R|\nabla h'_\varepsilon|^2dx'+\frac{\kappa^2}{4|\ln\varepsilon|}\int_{B(\tilde a_i^\varepsilon,\eta')}\chi_R|h'_\varepsilon|^2dx'\\
&\qquad\qquad\geq\frac{1}{2\pi}|\tilde A_i^\varepsilon|^2\chi_R(\tilde a_i^\varepsilon)\ln\left(\frac{\eta'}{\rho_\varepsilon}\right)-C\eta',\quad(2.6.32)
\end{aligned}$$

for ε sufficiently small.

- *Step 5: Convergence.* Using the fact, seen in Step 2, that $h'_\varepsilon\rightharpoonup h$ in $\dot W^{1,p}_{loc}(\mathbb{R}^2)$, we have, by lower semi-continuity,

$$\liminf_{\varepsilon\to0}\int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)}\chi_R|\nabla h'_\varepsilon|^2\,dx'\geq\int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)}\chi_R|\nabla h|^2dx'.\qquad(2.6.33)$$

On the other hand, in view of $\chi_R(\tilde a_i^\varepsilon)=\chi_R^i+O(\rho_\varepsilon)$ by (2.6.30), from (2.6.32) we obtain

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{B(a_i^\varepsilon,\eta)\setminus\mathcal{B}_0}\chi_R|\nabla h'_\varepsilon|^2dx' + \int_{B(a_i^\varepsilon,\eta)}\chi_R\left(\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)dx' \\
&\geq \liminf_{\varepsilon\to 0}\int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0}\chi_R|\nabla h'_\varepsilon|^2dx' + \int_{B(\tilde{a}_i^\varepsilon,\eta')}\chi_R\left(\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)dx' \\
&\qquad\qquad\qquad\qquad\qquad\qquad \geq \frac{3^{4/3}\pi}{2}\chi_R(a_i)\ln\eta' - C\eta', \quad (2.6.34)
\end{aligned}
$$

where we also used that $\chi_R^i \to \chi_R(a)$ as $\varepsilon\to 0$.

We now convert the estimate in (2.6.29) to one over $\mathcal{B}_0$ and involving χ_R as well. Observing that $\Omega'_{i,\varepsilon}\subseteq\mathcal{B}_0$ and that $\chi_R(x')\geq\chi_i^R - 4\rho_\varepsilon\|\nabla\chi_R\|_\infty$ for all $x'\in\Omega'_{i,\varepsilon}$ and ε small enough by (2.6.30), from (2.6.29) and (2.3.1) we obtain

$$
\liminf_{\varepsilon\to 0}\int_{\mathcal{B}_0}\chi_R\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)dx' \geq \frac{3^{4/3}\pi}{8}\chi_R(a_i), \qquad (2.6.35)
$$

where we used the fact that by (2.3.2), (2.3.12) and (2.4.5) the integral in the left-hand side of (2.6.29) may be bounded by $C|\ln\varepsilon|$, for some $C>0$ independent of ε and R. Adding up (2.6.33) with (2.6.34) and (2.6.35) summed over all $a_i\in K_R$, in view of the arbitrariness of $\eta'<\eta$ we then obtain

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)dx' \\
&\quad\geq \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)}\chi_R|\nabla h|^2dx' + \frac{3^{4/3}\pi}{2}\sum_{a\in\Lambda}\chi_R(a)\left(\ln\eta + \frac{1}{4}\right) - C\eta. \quad (2.6.36)
\end{aligned}
$$

Letting now $\eta\to 0$ in (2.6.36), and recalling that $\varphi = 2\cdot 3^{-2/3}h$ and that the definition of $W(\varphi, \chi)$ is given by Definition 2.6.1, we obtain

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|} |h'_\varepsilon|^2 - \frac{1}{2\pi} |\ln \rho_\varepsilon| \nu^\varepsilon \right) dx' \\ \geq \frac{3^{4/3}}{2} W(\varphi, \chi_R) + \frac{3^{4/3}\pi}{8} \sum_{a\in\Lambda} \chi_R(a). \quad (2.6.37)$$

From (2.6.14) we may replace $f_\varepsilon = |\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon$ by g_ε in (2.6.37) with an additional error term:

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R g_\varepsilon dx' \geq \frac{3^{4/3}}{2} W(\varphi, \chi_R) + \frac{3^{4/3}\pi}{8} \sum_{a\in\Lambda} \chi_R(a) - c\Delta(R), \qquad (2.6.38)$$

where

$$\Delta(R) = \limsup_{\varepsilon\to 0} \sum_{\alpha|_{K_{R-C}\subset U_\alpha\subset K_{R+C}}} (n_\alpha + 1)\ln(n_\alpha + 2),$$

for some $c, C > 0$ independent of R. Under hypothesis (2.6.9), from (2.6.12) we have

$$\limsup_{\varepsilon\to 0} \sum_{\alpha|_{U_\alpha\subset K_R}} n_\alpha^2 \leq CR^2,$$

and thus, using Hölder inequality and bounding the number of α's involved in the sum by CR we find

$$\Delta(R) \leq C \limsup_{\varepsilon\to 0} \sum_{\alpha|_{U_\alpha\subset K_{R+C}\setminus K_{R-C}}} (n_\alpha^{3/2} + 1) \\ \leq C' R^{1/4} \limsup_{\varepsilon\to 0} \left(\sum_{\alpha|_{U_\alpha\subset K_{R+C}}} n_\alpha^2 \right)^{3/4} + CR \leq C'' R^{7/4},$$

for some $C, C', C'' > 0$ independent of R. Hence

$$\limsup_{R\to\infty} \limsup_{\varepsilon\to 0} \frac{\Delta(R)}{R^2} = 0,$$

which together with (2.6.38) and the fact that $\bar{g}_\varepsilon \geq g_\varepsilon$ establishes (2.6.11). $\square$

2.6.3 Local to Global bounds via the Ergodic Theorem: proof of Theorem 4, item i.

The proof follows the procedure outlined in [100]. We refer the reader to Sections 4 and 6 of [100] for the proof adapted to the case of the magnetic Ginzburg-Landau energy, which is essentially identical to the present one, with some simplifications due to the fact that we work on the torus. As in [100], we say that $\mu \in \mathcal{M}_0(\mathbb{R}^2)$, if the measure $d\mu + Cdx$ is a positive locally bounded measure on $\mathbb{R}^2$, where C is the constant appearing in Corollary 2.5.3. The measures $d\bar{g}_\varepsilon$ and the functions φ_ε will be alternatively seen as functions on $\mathbb{T}^2_{\ell^\varepsilon}$ or as periodically extended to the whole of $\mathbb{R}^2$, which will be clear from the context. We let χ be a smooth non-negative function on $\mathbb{R}^2$ with support in $B(0,1)$ and with $\int_{\mathbb{R}^2} \chi(x)dx = 1$. We set $X = \dot{W}^{1,p}_{loc}(\mathbb{R}^2) \times \mathcal{M}_0(\mathbb{R}^2)$, and define for every $\mathbf{x} = (\varphi, g) \in X$ the following functional

$$\mathbf{f}(\mathbf{x}) := 2 \int_{\mathbb{R}^2} \chi(y) dg(y). \tag{2.6.39}$$

We note that from (2.5.28) we have for ε sufficiently small

$$F^\varepsilon[u^\varepsilon] + o_\varepsilon(1) \geq \frac{2}{\ell^2|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell\varepsilon}} d\bar{g}_\varepsilon = \fint_{\mathbb{T}^2_{\ell\varepsilon}} \mathbf{f}(\theta_\lambda \mathbf{x}_\varepsilon)d\lambda, \tag{2.6.40}$$

where $\mathbf{x}_\varepsilon := (\varphi^\varepsilon, \bar{g}_\varepsilon)$, θ_λ denotes the translation operator by $\lambda \in \mathbb{R}^2$, i.e., $\theta_\lambda f(x) := f(x+\lambda)$, and $\fint$ stands for the average. Here the last equality follows by an application of Fubini's theorem and the fact that $\int_{\mathbb{R}^2} \chi(x)dx = 1$.

It can be easily shown as in [100] that $\mathbf{f}_\varepsilon = \mathbf{f}$ satisfies the coercivity and Γ-liminf properties required for the application of Theorem 3 in [100] on sequences consisting of $\mathbf{x}_\varepsilon = (\varphi^\varepsilon, \bar{g}_\varepsilon)$ obtained from (u^ε) obeying (2.2.19). This is done by starting with a sequence $\{\mathbf{x}_\varepsilon\}_\varepsilon$ in X such that

$$\limsup_{\varepsilon\to 0} \int_{K_R} \mathbf{f}(\theta_\lambda \mathbf{x}_\varepsilon)d\lambda < +\infty, \tag{2.6.41}$$

for every $R > 0$, which implies that the integral is finite whenever ε is small enough. Consequently $\mathbf{f}_\varepsilon(\theta_\lambda \mathbf{x}_\varepsilon) < +\infty$ for almost every $\lambda \in K_R$. Applying Fubini's theorem again, (2.6.41) becomes

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R(y)d\bar{g}_\varepsilon(y) < +\infty,$$

where $\chi_R = \chi * \mathbf{1}_{K_R}$, and "$*$" denotes convolution. Then since $\chi_R = 1$ in K_{R-1} and $\bar{g}_\varepsilon$ is bounded below by a constant, the assumption (2.6.7) in Part 1 of Theorem 7 is satisfied, and we deduce from that theorem that φ^ε and $\bar{g}_\varepsilon$ converge, upon extraction of a subsequence, weakly in $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ and weakly in the sense of measures, respectively. Furthermore, if $\mathbf{x}_\varepsilon \to \mathbf{x} = (\varphi, g)$ on this subsequence, we have $2\int_{\mathbb{R}^2}\chi(y)d\bar{g}_\varepsilon(y)=\mathbf{f}(\mathbf{x}_\varepsilon)\to\mathbf{f}(\mathbf{x})=2\int_{\mathbb{R}^2}\chi(y)d\bar{g}(y)$.

We may then apply Theorem 3 of [100] to $\mathbf{f}$ on $\mathbb{T}^2_{\ell^\varepsilon}$ and conclude that the measure $\{\widetilde{P}^\varepsilon\}_\varepsilon$ defined as the push-forward of the normalized uniform measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by

$$\lambda\mapsto(\theta_\lambda\varphi^\varepsilon,\theta_\lambda\bar{g}_\varepsilon),$$

converges to a translation-invariant probability measure $\widetilde{P}$ on X with

$$\liminf_{\varepsilon\to0}F^\varepsilon[u^\varepsilon]\geq\int\mathbf{f}(\mathbf{x})d\widetilde{P}(\mathbf{x})=\int\ \mathbf{f}^*(\mathbf{x})d\widetilde{P}(\mathbf{x}),\tag{2.6.42}$$

where

$$\mathbf{f}^*(\varphi,g)=\lim_{R\to\infty}\fint_{K_R}\mathbf{f}(\theta_\lambda\mathbf{x})d\lambda=\lim_{R\to+\infty}\left(\frac{2}{|K_R|}\int_{\mathbb{R}^2}\chi_R(y)dg(y)\right),\tag{2.6.43}$$

provided that $\mathbf{x}$ is in the support of $\widetilde{P}$.

The next step is to show that for $\widetilde{P}$-a.e. $\mathbf{x}$ we have $\varphi\in\mathcal{A}_m$ with $m=3^{-2/3}(\bar{\delta}-\bar{\delta}_c)$, and $\mathbf{f}^*$ can be computed. By [100, Remark 1.6], we have that for $\widetilde{P}$-a.e $\mathbf{x}$, there exists a sequence $\{\lambda_\varepsilon\}_\varepsilon$ such that $\mathbf{x}_\varepsilon=(\theta_{\lambda_\varepsilon}\varphi^\varepsilon,\theta_{\lambda_\varepsilon}\bar{g}_\varepsilon)$ converges to $\mathbf{x}$ in X. In addition, from (2.6.42)–(2.6.43), for $\widetilde{P}$-a.e. $\mathbf{x}$, we have

$$\lim_{R\to+\infty}\fint_{K_R}\mathbf{f}(\theta_\lambda\mathbf{x})d\lambda\ <+\infty,$$

for $\widetilde{P}$-almost every $\mathbf{x}$. Using Fubini's theorem again, together with the definition of $\mathbf{f}$, we then find

$$\lim_{R\to+\infty}\left(\frac{1}{|K_R|}\int_{\mathbb{R}^2}\chi_R(y)dg(y)\right)<+\infty.$$

Therefore, since

$$\int_{\mathbb{R}^2}\chi_R(y)d\bar{g}_\varepsilon(y)\to\int_{\mathbb{R}^2}\chi_R(y)dg(y)\qquad\text{as }\varepsilon\to 0,\tag{2.6.44}$$

a bound of the type (2.6.9) holds, and the results of Part 2 of Theorem 7 hold for $\mathbf{x}_\varepsilon$. In particular, we find that

$$-\Delta\varphi=2\pi\sum_{a\in\Lambda}\delta_a-m,\tag{2.6.45}$$

with $m=3^{-2/3}(\bar{\delta}-\bar{\delta}_c)$, and that

$$\mathbf{f}^*(\varphi,g)=\lim_{R\to\infty}\left(\frac{2}{|K_R|}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R\bar{g}_\varepsilon dx\right)\geq 3^{4/3}W(\varphi)+\frac{3^{4/3}}{8}m.\tag{2.6.46}$$

The result in (2.6.46) follows from the definition of $\mathbf{f}^*$, (2.6.44), (2.6.11), the definition of W, provided we can show that

$$\lim_{R\to+\infty}\frac{1}{|K_R|}\sum_{a\in\Lambda}\chi_R(a)=\lim_{R\to+\infty}\frac{\nu(K_R)}{2\pi|K_R|}=\frac{m}{2\pi}.\tag{2.6.47}$$

The latter can be obtained from (2.6.15), exactly as in Lemma 4.11 of [100], so we omit the proof. Note that with (2.6.45), it proves that $\varphi\in\mathcal{A}_m$, and we thus have the claimed result. Combining (2.6.42) and (2.6.46), we obtain

$$\liminf_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \geq \int \left(3^{4/3}W(\varphi) + \frac{3^{2/3}}{8}(\bar{\delta} - \bar{\delta}_c)\right) d\widetilde{P}(\varphi, g).$$

Letting now P^ε and P be the first marginals of $\widetilde{P}^\varepsilon$ and $\widetilde{P}$ respectively, this proves (2.2.20) and the fact that P-almost every φ is in $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$. □

2.7 Upper bound construction: proof of Part ii) of Theorem 4

We follow closely the construction performed for the magnetic Ginzburg-Landau energy in [100], but our situation is somewhat simpler, since we work on a torus (instead of a domain bounded by a free boundary). The construction given in [100] relies on a result stated as Corollary 4.5 in [100], which we repeat below with slight modifications to adapt it to our setting. These results imply, in particular, that the minimum of W may be approximated by sequences of periodic configurations of larger and larger period. Below for any discrete set of points Λ, $|\Lambda|$ will denote its cardinal.

Proposition 2.7.1 (Corollary 4.5 in [100])**.** *Let $p \in (1,2)$ and let P be a probability measure on $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ which is invariant under the action of translations and concentrated on $\mathcal{A}_1$. Let Q be the push-forward of P under $-\Delta$. Then there exists a sequence $R \to \infty$ with $R^2 \in 2\pi\mathbb{N}$ and a sequence $\{b_R\}_R$ of*

$2R$-periodic vector fields such that:

- *There exists a finite subset Λ_R of the interior of K_R such that*

$$\begin{cases} -\text{div } b_R = 2\pi \sum_{a\in\Lambda_R} \delta_a - 1 & \text{in } \ K_R \\ b_R \cdot \nu = 0 & \text{on } \partial K_R. \end{cases}$$

- *Letting Q_R be the probability measure on $W^{-1,p}_{loc}(\mathbb{R}^2)$, which is defined as the image of the normalized Lebesgue measure on K_R by $x \mapsto -\text{div } b_R(x+\cdot)$, we have $Q_R \to Q$ weakly as $R \to \infty$.*

- $\displaystyle \limsup_{R\to\infty} \frac{1}{|K_R|} \lim_{\eta\to 0} \left(\frac{1}{2} \int_{K_R \setminus \cup_{a\in\Lambda_R} B(a,\eta)} |b_R|^2 dx + \pi |\Lambda_R| \ln \eta \right) \leq \int W(\varphi)\, dP(\varphi).$

Remark 2.7.2. *We would like to make the following observations concerning the vector field b_R constructed in Proposition 2.7.1.*

1. *By construction, the vector fields b_R has no distributional divergence concentrating on ∂K_R and its translated copied since $b_R \cdot \nu$ is continuous across ∂K_R. However, $b_R \cdot \tau$ may not be, and this may create a singular part of the distributional* $\operatorname{curl} b_R$. *This is the difficulty that prevents us from stating the convergence result for P directly in Theorem 4, Part ii).*

2. *We also note that an inspection of the construction in [100] shows that b_R is curl-free in a neighborhood of each point $a \in \Lambda_R$ and that* $\operatorname{curl} b_R$ *belongs to $W^{-1,p}_{\text{loc}}(\mathbb{R}^2)$ for $p < \infty$.*

2.7.1 Definition of the test configuration

We take R the sequence given by Proposition 2.7.1. The first thing to do is to change the density 1 into a suitably chosen density $m_{\varepsilon,R}$, in order to ensure the compatibility of the functions with the torus volume. Recalling that $\bar\mu^\varepsilon > 0$ for $\bar\delta > \bar\delta_c$ and ε small enough, we set

$$m_{\varepsilon,R} = \frac{4R^2}{|\ell^\varepsilon|^2}\left\lfloor \frac{\ell^\varepsilon\sqrt{2\bar\mu^\varepsilon}}{2R\bar r_\varepsilon}\right\rfloor^2 \tag{2.7.1}$$

where, as usual, $\lfloor x\rfloor$ denotes the integer part of a x. We note for later that

$$\left| m_{\varepsilon,R} - \frac{2\bar\mu^\varepsilon}{\bar r_\varepsilon^2}\right| \le \frac{CR}{\ell^\varepsilon} = o_\varepsilon(1). \tag{2.7.2}$$

Recalling also that $\bar r_\varepsilon = 3^{1/3} + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$ and $\bar\mu^\varepsilon - \bar\mu = O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$, we deduce that $m_{\varepsilon,R}\to m$, where $m := 2\cdot 3^{-2/3}\bar\mu$, as $\varepsilon\to 0$, for each R. In particular, $m_{\varepsilon,R}$ is bounded above and below by constants independent of ε and R. The choice of $m_{\varepsilon,R}$ ensures that we can split the torus into an integer number of translates of the square $K_{R'}$ with $R' := \frac{R}{\sqrt{m_{\varepsilon,R}}}$, each of which containing an identical configuration of $\frac{2R^2}{\pi}$ points.

Let $P\in\mathcal{P}$ be given as in the assumption of Part 2 of Theorem 4, i.e., let P be a probability measure concentrated on $\mathcal{A}_m$. Letting $\bar P$ be the push-forward of P by $\varphi\mapsto\varphi(\frac{\cdot}{\sqrt{m}})$, it is clear that $\bar P$ is concentrated on $\mathcal{A}_1$, and by the change of scales formula (2.2.16) we have

$$\int W(\varphi)\,d\bar P(\varphi) = \frac{1}{m}\int W(\varphi)\,dP(\varphi) + \frac14\ln m. \tag{2.7.3}$$

We may then apply Proposition 2.7.1 to $\bar{P}$. It yields a vector field $\bar{b}_R$. We may then rescale it by setting

$$b_{\varepsilon,R}(x) = \sqrt{m_{\varepsilon,R}}\,\bar{b}_R(\sqrt{m_{\varepsilon,R}}x).$$

We note that $b_{\varepsilon,R}$ is a well-defined periodic vector-field on $\mathbb{T}^2_{\ell^\varepsilon}$ because $\frac{\ell^\varepsilon\sqrt{m_{\varepsilon,R}}}{2R}$ is an integer. This new vector field satisfies

$$-\mathrm{div}\ b_{\varepsilon,R} = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} \delta_a - m_{\varepsilon,R} \quad \text{in } \mathbb{T}^2_{\ell^\varepsilon} \tag{2.7.4}$$

for some set of points that we denote $\Lambda_{\varepsilon,R}$, and

$$\frac{1}{|K_R|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{\frac{R}{\sqrt{m_{\varepsilon,R}}}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} |b_{\varepsilon,R}|^2 dx + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln(\eta\sqrt{m_{\varepsilon,R}})\right)$$
$$\leq \int W(\varphi)\,d\bar{P}(\varphi) + o_R(1) \quad \text{as } R\to\infty.$$

Using (2.7.3) and $|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}| = \frac{2R^2}{\pi}$, this can be rewritten as

$$\frac{m_{\varepsilon,R}}{|K_R|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{\frac{R}{\sqrt{m_{\varepsilon,R}}}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} |b_{\varepsilon,R}|^2 dx + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln\eta\right) +$$
$$\frac{m_{\varepsilon,R}}{4}\ln\frac{m_{\varepsilon,R}}{m}$$
$$\leq \frac{m_{\varepsilon,R}}{m}\int W(\varphi)\,dP(\varphi) + o_R(1).$$

But we saw that $m_{\varepsilon,R}\to m$ as $\varepsilon\to 0$ hence $\ln\left(\frac{m_{\varepsilon,R}}{m}\right)\to 0$. Therefore, recalling the definition of R' we have

$$\frac{1}{|K_{R'}|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{R'}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2dx+\pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln\eta\right)$$
$$\leq\int W(\varphi)\,dP(\varphi)+o_R(1)+o_\varepsilon(1). \quad (2.7.5)$$

It thus follows that

$$\frac{1}{(\ell^\varepsilon)^2}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2dx'+\pi|\Lambda_{\varepsilon,R}|\ln\eta\right)\leq\int W(\varphi)\,dP(\varphi)$$
$$+\,o_R(1)+o_\varepsilon(1). \quad (2.7.6)$$

Note that $\Lambda_{\varepsilon,R}$ is a dilation by the factor $1/\sqrt{m_{\varepsilon,R}}$, uniformly bounded above and below, of the set of points Λ_R, hence the minimal distance between the points in $\Lambda_{\varepsilon,R}$ is bounded below by a constant which may depend on R but does not depend on ε. For the same reason, estimates on $b_{R,\varepsilon}$ are uniform with respect to ε.

In addition, we have that $\bar{Q}_{\varepsilon,R}$, the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x\mapsto -\mathrm{div}\ b_{\varepsilon,R}(x+\cdot)$ converges to Q, the push-forward of P by $-\Delta$, as $\varepsilon\to 0$ and $R\to\infty$. The final step is to replace the Dirac masses appearing above by their non-singular approximations:

$$\tilde{\delta}_a:=\frac{\chi_{B(a,r'_\varepsilon)}}{\pi|r'_\varepsilon|^2}\qquad r'_\varepsilon:=\varepsilon^{1/3}|\ln\varepsilon|^{1/6}\bar{r}_\varepsilon, \quad (2.7.7)$$

where $\bar{r}_\varepsilon$ was defined in (2.3.1). Note also that in view of the discussion of

Section 2.3 it is crucial to use droplets with the corrected radius $\varepsilon^{1/3}|\ln\varepsilon|^{1/6}\bar{r}_\varepsilon$ instead of its leading order value $\rho_\varepsilon = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{1/6}$.

Once the set $\Lambda_{\varepsilon,R}$ has been defined, the definition of the test function $u^\varepsilon \in \mathcal{A}$ follows: it suffices to take

$$u^\varepsilon(x) = -1 + 2\sum_{a\in\Lambda_{\varepsilon,R}} \chi_{B(a,r'_\varepsilon)}\left(x|\ln\varepsilon|^{1/2}\right),$$

which means (after blow up) that all droplets are round of identical radii r'_ε and centered at the points of $\Lambda_{\varepsilon,R}$. We now need to compute $F^\varepsilon[u^\varepsilon]$ and check that all the desired properties are satisfied. This is done by working with the associated function h'_ε defined in (2.3.16), i.e. the solution in $\mathbb{T}^2_{\ell^\varepsilon}$ to

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln\varepsilon|}h'_\varepsilon = \pi\bar{r}_\varepsilon^2 \sum_{a\in\Lambda_{\varepsilon,R}} \tilde{\delta}_a - \bar{\mu}^\varepsilon, \tag{2.7.8}$$

obtained from (2.3.16) by explicitly setting all $A_i^\varepsilon = \pi\bar{r}_\varepsilon^2$.

2.7.2 Reduction to auxiliary functions

Let us introduce ϕ_ε, which is the solution with mean zero of

$$-\Delta\phi_\varepsilon = 2\pi\sum_{a\in\Lambda_{\varepsilon,R}} \tilde{\delta}_a - m_{\varepsilon,R} \quad \text{in} \quad \mathbb{T}^2_{\ell^\varepsilon}, \tag{2.7.9}$$

where $m_{\varepsilon,R}$ is as in (2.7.1), and f_ε the solution with mean zero of

$$-\Delta f_\varepsilon = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} \delta_a - m_{\varepsilon,R} \quad \text{in} \quad \mathbb{T}^2_{\ell^\varepsilon}. \tag{2.7.10}$$

We note that f_ε is a rescaling by the factor $m_{\varepsilon,R} \to m$ of a function independent of ε, so all estimates on f_ε can be made uniform with respect to ε.

Lemma 2.7.3. *Let h'_ε and ϕ_ε be as above. We have as $\varepsilon \to 0$*

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} |h'_\varepsilon|^2 dx' \le C_R |\ln \varepsilon| \tag{2.7.11}$$

and for any $1 \le q < \infty$

$$\left\| \nabla \left(h'_\varepsilon - \frac{\bar{r}_\varepsilon^2}{2} \phi_\varepsilon \right) \right\|_{L^q(\mathbb{T}^2_{\ell^\varepsilon})} \le C_{R,q}, \tag{2.7.12}$$

for some constant $C_{R,q} > 0$ independent of ε.

Proof. Since $\Lambda_{\varepsilon,R}$ is $2R'$-periodic, h'_ε is too, and thus

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} |h'_\varepsilon|^2 dx' = \ell^2 |\ln \varepsilon| \fint_{K_{R'}} |h'_\varepsilon|^2 dx' \le C_R |\ln \varepsilon|.$$

For the second assertion, let

$$h_\varepsilon(x) = h'_\varepsilon(x\sqrt{|\ln \varepsilon|}) \qquad \hat{\phi}_\varepsilon(x) = \phi_\varepsilon(x\sqrt{|\ln \varepsilon|})$$

be the rescalings of h'_ε and ϕ_ε onto the torus $\mathbb{T}^2_\ell$. Rescaling (2.7.11) gives

$$\|h_\varepsilon\|_{L^2(\mathbb{T}^2_\ell)} \le C_R. \tag{2.7.13}$$

Furthermore, the function $w_\varepsilon := h_\varepsilon - \frac{1}{2}\bar{r}_\varepsilon^2 \hat{\phi}_\varepsilon$ is easily seen to solve

$$-\Delta w_\varepsilon = -\kappa^2 \left(h_\varepsilon - \fint_{\mathbb{T}^2_\ell} h_\varepsilon dx \right) \quad \text{in } \mathbb{T}^2_\ell.$$

But from elliptic regularity, Cauchy-Schwarz inequality and (2.7.13), we must have

$$\|\nabla w_\varepsilon\|_{L^q(\mathbb{T}^2_\ell)} \le C \left\| h_\varepsilon - \fint_{\mathbb{T}^2_\ell} h_\varepsilon \right\|_{L^2(\mathbb{T}^2_\ell)} \le C_{R,q},$$

which yields (2.7.12). □

The next lemma consists in comparing ϕ_ε and f_ε.

Lemma 2.7.4. *We have*

$$\|\nabla(f_\varepsilon - \phi_\varepsilon)\|_{L^\infty\left(\mathbb{T}^2_{\ell\varepsilon} \setminus \cup_a B(a, r'_\varepsilon)\right)} \le C_R \varepsilon^{1/4}.$$

Proof. We observe that f_ε and ϕ_ε are both $2R'$-periodic. We may thus write

$$\phi_\varepsilon(x) - f(x) = 2\pi \int_{\mathbb{T}^2_{2R'}} G_{2R'}(x-y) \sum_{a \in \Lambda_{\varepsilon,R}} d(\tilde{\delta}_a - \delta_a)(y),$$

where $G_{2R'}$ is the zero mean Green's function for the Laplace's operator on the square torus of size $2R'$ with periodic boundary conditions, i.e. the solution to

$$-\Delta G_{2R'} = \delta_0 - \frac{1}{|\mathbb{T}^2_{2R'}|} \quad \text{in } \mathbb{T}^2_{2R'} \tag{2.7.14}$$

which we may be split as $G_{2R'}(x) = -\frac{1}{2\pi}\log|x| + S_{2R'}(x)$ with $S_{2R'}$ a smooth function. By Newton's theorem (or equivalently by the mean value theorem for harmonic functions applied to the function $\log|\cdot|$ away from the origin), the contribution due to the logarithmic part is zero outside of $\cup_{a\in\Lambda_{\varepsilon,R}} B(a, r'_\varepsilon)$. Differentiating the above we may thus write that for all $x \notin \cup_{a\in\Lambda_{\varepsilon,R}} B(a, r'_\varepsilon)$,

$$\nabla(\phi_\varepsilon - f)(x) = 2\pi \int_{\mathbb{T}^2_{2R'}} \nabla S_{2R'}(x-y) \sum_{a\in\Lambda_{\varepsilon,R}} d(\tilde{\delta}_a - \delta_a)(y). \tag{2.7.15}$$

Using the C^2 character of $S_{2R'}$ we deduce that

$$\|\nabla(f_\varepsilon - \phi_\varepsilon)\|_{L^\infty\left(\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_a B(a,r'_\varepsilon)\right)} \le C_{R'}|\Lambda_{\varepsilon,R}\cap K_{R'}|r'_\varepsilon$$

and the result follows in view of (2.7.7). □

The next step involves a comparison of the energy of ϕ_ε and that of $b_{\varepsilon,R}$ and leads to the following conclusion.

Lemma 2.7.5. *Given $\Lambda_{\varepsilon,R}$ as constructed above, and h'_ε the solution to* (2.7.8), *we have*

$$\frac{1}{(\ell^\varepsilon)^2}\lim_{\eta\to 0}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{2}{\bar{r}_\varepsilon^4}|\nabla h'_\varepsilon|^2 dx' + \pi|\Lambda_{\varepsilon,R}|\ln\eta\right) \le \int W(\varphi)\, dP(\varphi)$$
$$+ o_\varepsilon(1) + o_R(1).$$

Proof. In view of Lemmas 2.7.3 and 2.7.4, it suffices to show the corresponding result for $\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}\frac{1}{2}|\nabla f_\varepsilon|^2\,dx'$ instead of the one for h'_ε. From (2.7.10) and (2.7.4), we have $\operatorname{div}\,(b_{\varepsilon,R}-\nabla f_\varepsilon)=0$ hence by Poincaré's lemma we may write $\nabla f_\varepsilon = b_{\varepsilon,R}+\nabla^\perp\xi_\varepsilon$. We note that $-\Delta\xi_\varepsilon=\operatorname{curl} b_{\varepsilon,R}$, which is in $W^{-1,p}_{loc}$ for any $p<+\infty$ as mentioned in Remark 2.7.2. By elliptic regularity we find that $\nabla\xi_\varepsilon\in L^p_{loc}(\mathbb{R}^2)$ for all $1\le p<+\infty$, uniformly with respect to ε. We may thus write

$$\begin{aligned}\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}&\frac{1}{2}|b_{\varepsilon,R}|^2dx'\\&=\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}\left(\frac{1}{2}|\nabla f_\varepsilon|^2+\frac{1}{2}|\nabla\xi_\varepsilon|^2-\nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon\right)dx',\end{aligned}\tag{2.7.16}$$

where $\nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon$ makes sense in the duality $\nabla\xi_\varepsilon\in L^p$, $p>2$, $\nabla f_\varepsilon\in L^q$, $q<2$. In addition, by the same duality, we have for any $a\in\Lambda_{\varepsilon,R}$,

$$\lim_{\eta\to0}\int_{B(a,\eta)}\nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon=0$$

uniformly with respect to ε. Therefore, we may extend the domain of integration in the last integral in (2.7.16) to the whole of $\mathbb{T}^2_{\ell^\varepsilon}$ at the expense of an error $o_\eta(1)$ multiplied by the number of points, and obtain

$$\begin{aligned}\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}\frac{1}{2}|\nabla f_\varepsilon|^2dx'\le&\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}\frac{1}{2}|b_{\varepsilon,R}|^2dx'\\&+\int_{\mathbb{T}^2_{\ell^\varepsilon}}\nabla f_\varepsilon\cdot\nabla^\perp\xi dx'+o_\eta(|\ln\varepsilon|).\end{aligned}\tag{2.7.17}$$

Noting that the last integral on the right-hand side vanishes by Stokes' theorem (and by approximating ∇f_ε and $\nabla^\perp \xi_\varepsilon$ by smooth functions), adding $\pi|\Lambda_{\varepsilon,R}|\ln \eta$ to both sides, and combining with (2.7.6) we obtain the result. $\square$

In view of (2.7.4) and (2.7.9) we have that $-\mathrm{div}\, b_{\varepsilon,R} + \Delta\phi_\varepsilon = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} (\delta_a - \tilde{\delta}_a) \to 0$ in $W^{-1,p}_{loc}(\mathbb{R}^2)$, so we deduce, since the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x \mapsto -\mathrm{div}\, b_{\varepsilon,R}(x+\cdot)$ converges to Q, that the push-forward of it by $x \mapsto -\Delta\varphi^\varepsilon(x+\cdot)$ also converges to Q. Thus, part ii) of Theorem 4 is established modulo (2.2.21), which remains to be proved.

2.7.3 Calculating the energy

We begin by calculating the exact amount of energy contained in a ball of radius η.

Lemma 2.7.6. *Let h'_ε be as above. Then we have for any $a \in \Lambda_{\varepsilon,R}$,*

$$\int_{B(a,r'_\varepsilon)} |\nabla h'_\varepsilon|^2 dx' = \frac{3^{4/3}\pi}{8} + o_\varepsilon(1) \tag{2.7.18}$$

and

$$\int_{B(a,\eta)\setminus B(a,r'_\varepsilon)} |\nabla h'_\varepsilon|^2 dx' \le \frac{\pi}{2}\bar{r}_\varepsilon^4 \ln\frac{\eta}{\rho_\varepsilon} + o_\varepsilon(1) + o_\eta(1). \tag{2.7.19}$$

Proof. In view of (2.7.12) applied with $q > 2$ and using Hölder's inequality, we have that for all $a \in \Lambda_{\varepsilon,R}$,

$$\int_{B(a,\eta)} \left|\nabla(h'_\varepsilon - \frac{\bar{r}_\varepsilon^2}{2}\phi_\varepsilon)\right|^2 dx' \le o_\eta(1). \tag{2.7.20}$$

Thus it suffices to compute the corresponding integrals for ϕ_ε. Using again the $2R'$-periodicity of ϕ_ε, we may write, with the same notation as in the proof of Lemma 2.7.4

$$\phi_\varepsilon(x) = \int_{\mathbb{T}^2_{2R'}} G_{2R'}(x-y)\left(2\pi \sum_{a\in\bar\Lambda_{\varepsilon,R}} \tilde\delta_a(y) - m_{\varepsilon,R}\right) dy.$$

Since the distances between the points in $\bar\Lambda_{\varepsilon,R}$ are bounded below independently of ε, and the number of points is bounded as well, we may write ϕ_ε in $B(a,\eta)$ as

$$\phi_\varepsilon(x) = \psi_\varepsilon(x) - \int_{\mathbb{T}^2_{2R'}} \ln|x-y|\,\tilde\delta_a(y)\,dy \tag{2.7.21}$$

where $\psi_\varepsilon(x)$ is smooth and its derivative is bounded independently of ε (but depending on R).

Thus the contribution of ψ_ε to the integrals $\int_{B(a,\eta)} |\nabla\phi_\varepsilon|^2$ is $o_\eta(1)$, and its contribution to $\int_{B(a,r'_\varepsilon)} |\nabla\phi_\varepsilon|^2$ is $o_\varepsilon(1)$. There remains to compute the contribution of the logarithmic term in (2.7.21). But this is almost exactly the same computation as in (2.6.27)–(2.6.29), and with (2.7.20) it yields (2.7.18), while it yields as well that

$$\int_{B(a,\eta)\setminus B(a,r'_\varepsilon)} |\nabla\phi_\varepsilon|^2 dx' \le 2\pi \ln\frac{\eta}{r'_\varepsilon} + o_\eta(1). \tag{2.7.22}$$

Now

$$\frac{r'_\varepsilon}{\rho_\varepsilon} = \frac{1}{3^{1/3}}\left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3} = \left(1 + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)\right)^{1/3}.$$

Consequently $\ln\frac{r'_\varepsilon}{\rho_\varepsilon} = o_\varepsilon(1)$, and so we may replace r'_ε with ρ_ε at an extra cost of $o_\varepsilon(1)$ in (2.7.22), and the result follows with (2.7.20). $\square$

We can now combine all the previous results to compute the energy of the test-function u^ε. By following the lower bounds of Proposition 2.3.1, it is easy to see that in our case (all the droplets being balls of radius r'_ε) all the inequalities in that proof become equalities, and thus recalling (2.3.1):

$$F^\varepsilon[u^\varepsilon] = \frac{1}{|\ell^\varepsilon|^2}\left(2\int_{\mathbb{T}^2_{\ell^\varepsilon}}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right)dx' + \pi\bar{r}_\varepsilon^4|\Lambda_{\varepsilon,R}|\ln\rho_\varepsilon\right) + o_\varepsilon(1),$$

with the help of Lemma 2.7.6 we have for every R

$$F^\varepsilon[u^\varepsilon] \le \frac{1}{|\ell^\varepsilon|^2}\left(2\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|\nabla h'_\varepsilon|^2\,dx' + \pi\bar{r}_\varepsilon^4|\Lambda_{\varepsilon,R}|\ln\eta + \frac{3^{4/3}\pi}{4}|\Lambda_{\varepsilon,R}|\right) + o_\varepsilon(1) + o_\eta(1).$$

In view of Lemma 2.7.5, letting $\eta\to 0$, we obtain

$$F^\varepsilon[u^\varepsilon] \le \bar{r}_\varepsilon^4\left(\int W(\varphi)\,dP(\varphi) + o_\varepsilon(1) + o_R(1)\right) + \frac{3^{4/3}\pi}{4|\ell^\varepsilon|^2}|\Lambda_{\varepsilon,R}| + o_\varepsilon(1).$$

Letting $\varepsilon\to 0$, using that $\bar{r}_\varepsilon\to 3^{1/3}$ and the fact that $|\Lambda_{\varepsilon,R}| = \frac{1}{2\pi}m_{\varepsilon,R}|\ell^\varepsilon|^2$ with $m_{\varepsilon,R}\to m$, and then finally letting $R\to\infty$, we conclude that

$$\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \le 3^{4/3}\int W(\varphi)\,dP(\varphi) + \frac{3^{4/3}m}{8}.$$

Since $\frac{1}{8}3^{2/3}m = \frac{1}{8}(\bar{\delta}-\bar{\delta}_c)$, this completes the proof of part ii) of Theorem 4. $\square$

2.7.4 Proof of Theorem 5

In order to prove Theorem 5, it suffices to show that

$$\min_{P\in\mathcal{P}} F^0[P] = 3^{4/3}\min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta}-\bar{\delta}_c)}{8}. \tag{2.7.23}$$

For the proof, we use the following result, adapted from Corollary 4.4 in [100].

Proposition 2.7.7 (Corollary 4.4 in [100]). *Let $\varphi \in \mathcal{A}_1$ be given, such that $W(\varphi) < \infty$. For any R such that $R^2 \in 2\pi\mathbb{N}$, there exists a $2R$-periodic φ_R such that*

$$\begin{cases} -\Delta\varphi_R = 2\pi\sum_{a\in\Lambda_R}\delta_a - 1 & \text{in } K_R,\\ \dfrac{\partial\varphi_R}{\partial\nu} = 0 & \text{on } \partial K_R, \end{cases}$$

where Λ_R is a finite subset of the interior of K_R, and such that

$$\limsup_{R\to\infty}\frac{W(\varphi_R, \mathbf{1}_{K_R})}{|K_R|} \le W(\varphi).$$

Let us take φ to be a minimizer of W over $\mathcal{A}_m$ (which exists from [100]). We may rescale it to be an element of $\mathcal{A}_1$. Then Proposition 2.7.7 yields a φ_R, which can be extended periodically. We can then repeat the same construction as in the beginning of this section, starting from $\nabla\varphi_R$ instead of b_R, and in the end it yields a u^ε with

$$\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \le 3^{4/3}\min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta}-\bar{\delta}_c)}{8}.$$

It follows that

$$\limsup_{\varepsilon\to 0} \min_{\mathcal{A}} F^\varepsilon \le 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}.$$

But by part i) of Theorem 4 applied to a sequence of minimizers of F^ε, we also have

$$\liminf_{\varepsilon\to 0} \min_{\mathcal{A}} F^\varepsilon \ge \inf_{\mathcal{P}} F^0 \ge 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}$$

where the last inequality is an immediate consequence of the definition of F^0. Comparing the inequalities yields that there must be equality and (2.7.23) is proved, which completes the proof of Theorem 5. □

Chapter 3

Uniqueness results for critical points of a non-local isoperimetric problem via curve-shortening

3.1 Introduction

The classical isoperimetric problem has been thoroughly studied, and it is well known since the work of De Giorgi [35] that the unique optimizer to this problem in $\mathbb{R}^d$ is the ball. There has recently been significant interest in the effects of adding a repulsive term to the classic perimeter which favors separation of mass. An example of such an energy is often referred to as the sharp interface Ohta-Kawasaki energy, first introduced in [87], and takes the following form

$$E[u] := \int_{\mathbb{R}^d} |\nabla u| + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)K(x,y)u(y)dxdy, \tag{3.1.1}$$

where $u \in BV(\mathbb{R}^d;\{0,+1\})$ and $BV(\mathbb{R}^d;\{0,+1\})$ denotes the space of functions of bounded variation taking values 0 and +1 in $\mathbb{R}^d$ (see [6] for an introduction to the space BV). The kernel K is generally taken to be the kernel of the Laplacian operator. It is clear that the non-local term favors the separation of mass while the perimeter favors clustering. The above problem describes a number of polymer systems [34, 84, 94] as well as many other physical systems [19, 40, 58, 73, 84] due to the fundamental nature of the Coulombic term. Despite the abundance of physical systems for which (3.1.1) is applicable, rigorous mathematical analysis is fairly recent [2, 21, 23, 26–29, 54, 55, 65, 78–82]. One considers minimizing (3.1.1) over the class

$$\mathcal{A}_m := \{u \in BV(\mathbb{R}^d;\{0,+1\}) : \int_{\mathbb{R}^d} u \, dx = m\}, \qquad u := \chi_\Omega. \tag{3.1.2}$$

For the class of K we consider (see (3.1.9) below), smaller m has the effect of decreasing the strength of the non-local term in (3.1.1), changing the energy by at most a constant. Indeed, by a change of variables $x' = x/\lambda$ and setting $\tilde{u}(x) = u(\lambda x)$ we can write (3.1.1) as

$$E[u] = \lambda^{d-1}\left(\int_{\mathbb{R}^d} |\nabla \tilde{u}| + \lambda^{d+1-p}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(x)K(x,y)\tilde{u}(y)dxdy + C(\lambda)\right), \tag{3.1.3}$$

where $C(\lambda) > 0$ is a constant which arises when K is the logarithmic kernel, and $p \geq 0$ arises from the singularity of the kernel. From this point forward we will often abuse terminology and say $\Omega \in \mathcal{A}_m$ when we mean $\chi_\Omega \in \mathcal{A}_m$ and write $E(\Omega)$ when we mean $E(\chi_\Omega)$. For the remainder of the paper we will study (3.1.1) exclusively for $d = 2$, except for certain results we quote in $\mathbb{R}^d$ for $d \geq 3$.

When $d = 2$, in order for minimizers to exist in $\mathcal{A}_m$ to (3.1.1), it is necessary to modify the logarithmic kernel. The above energy with G replaced by the kernel

$$K(x) = \frac{1}{|x|^\alpha} \text{ for } \alpha \in (0, 2), \tag{3.1.4}$$

was recently studied by Knupfer and Muratov [81, 82]. A simple rescaling, as done above, shows that for small masses, the effect of the non-local term is small compared to that of the perimeter. It is therefore reasonable to expect that for small masses, the unique minimizer to (3.1.1) is the ball. This was shown by Knupfer and Muratov [81, 82] in dimensions $2 \leq d \leq 8$. Moreover they show that for sufficiently large masses, for $2 \leq d$, minimizers of (3.1.1) fail to exist, as it is favorable for mass to split. Very recently a new, somewhat simpler, proof of the uniqueness result which works in any dimension was given by Julin [65] that makes clever use of a new strong form of the quantitative isoperimetric inequality due to Fusco and Julin [46]. In [42], an anisotropic perimeter functional with gravitational field is studied. There it is shown that minimizers are uniformly close to Wulff shapes (ie. minimizers of the anisotropic perimeter) when the mass is sufficiently small. All of these results concern the global minimality of (3.1.1) in the small mass regime.

There is also much interest in critical points to (3.1.1) which are not necessarily locally minimizing. Such questions are not addressed in the above results and are the primary focus of this paper. Here a critical point to (3.1.1) is a set $\Omega \in \mathcal{A}_m$ for which the first variation with respect to volume preserving diffeomorphisms vanish.

Definition 3.1.1. *A set* $\Omega \in \mathcal{A}_m$ *is a* critical point *of* (3.1.1) *if for all volume preserving diffeomorphisms* $\phi_t : \Omega \to \phi_t(\Omega) =: \Omega_t$ *it holds that*

$$\frac{dE(\Omega_t)}{dt}\Big|_{t=0} = 0. \tag{3.1.5}$$

In particular a simple calculation [31] shows that Ω is a smooth critical point if and only if it solves the following Euler-Lagrange equation

$$\kappa(y) + \phi_\Omega(y) = \lambda \text{ on } \partial\Omega \tag{3.1.6}$$

where ϕ_Ω is the potential generated by Ω, ie.

$$\phi_\Omega(y) = \int_\Omega K(x,y)dx, \tag{3.1.7}$$

κ is the curvature of $\partial\Omega$ and λ is the Lagrange multiplier arising from the volume constraint. We however do not wish to assume a-priori regularity of the boundary as critical points may in general not be smooth. An important example demonstrating this is the coordinate axes in $\mathbb{R}^2$ minus the origin. In this case the generalized mean curvature is constant on the reduced boundary $\partial^*\Omega = \Omega$, $m = 0$ and hence (3.1.6) is satisfied everywhere on $\partial^*\Omega$ (see [107] for a reference on generalized mean curvature and the theory of varifolds). The example of a figure 8 with center at O also shows that while the curvature of a closed curve can be smooth and uniformly bounded on $\partial\Omega\backslash\{O\} = \partial^*\Omega$, there is not necessarily a natural way to make sense of the curvature or variations of (3.1.1) near O. In $\mathbb{R}^2$ however we can continue to make sense of the curvature at O if there exists a parametrization of the boundary by a closed, rectifiable (ie. has finite length) curve. By the results of [5], any connected set with finite perimeter has a boundary $\partial\Omega$ which can be decomposed into a countable union of Jordan curves $\{\gamma_k\}_k$ with disjoint interiors. In this case however, as the figure 8 example demonstrates, one cannot make sense of variations near points on the curve which are not locally homeomorphic to $[0, 1]$, and can thus only expect to extract information from the Euler-Lagrange information on the reduced boundary. However, when the boundary $\partial\Omega$ can be decomposed into a countable *disjoint* union of closed, rectifiable curves, there is a natural way to consider variations of the domain, even on the compliment of the reduced boundary, by considering variations of the curve in the normal direction induced by the parametrization. We thus make the following definition.

Definition 3.1.2. *(Admissible curves) A connected set Ω with finite perimeter will be called* admissible *if its boundary $\partial\Omega$ can be decomposed into a countable number of closed, disjoint curves γ_k each admitting a $W^{1,1}$ parametrization $\gamma_k : [0, 1] \to \mathbb{R}^2$ with $|\gamma_k'(t)| = L_k$ for $t \in [0, 1]$. In particular we may write*

$\gamma_k(t) = \gamma_k(0) + L_k \int_0^t e^{i\theta_k(r)} dr.$

Note that the above class includes all connected, rectifiable 1-manifolds. Indeed any manifold has a boundary which is locally homeomorphic to $[0, 1]$ and thus does not intersect any other segments of the boundary. In particular, each γ_k is therefore simple and disjoint from every other γ_j for $j \neq k$, and never intersects itself transversally. Working within the class of admissible curves, we are able to rigorously compute the Euler-Lagrange equation and extract sufficient information from it to conclude that admissible critical points are convex, in the small mass/energy regime. The details will be presented in Section 3.2.

The classification of critical points corresponding to either the perimeter term or non-local term, considered separately, has been well studied [3, 43]. In particular it is a well known result of Alexandrov [3] that in dimensions $d \geq 2$ the only simply connected, compact, constant mean curvature surface is the ball. For the non-local term, Fraenkel showed somewhat recently [43] that if $\phi_\Omega \equiv$ constant on $\partial\Omega$ then Ω must be the ball. This was recently extended to general Riesz kernels by Reichel [91], however restricted to the class of convex sets. The question then naturally arises of knowing how one can classify the solutions to (3.1.6). In $\mathbb{R}^2$ one can easily construct annuli which satisfy (3.1.6) for particular choices of radii (see Counter Example 1). Even in $\mathbb{R}^3$ examples of tori and double tori solutions to (3.1.6) exist [95], showing that compact, connected solutions to (3.1.6) exist other than the ball. Since a smooth set is a critical point in the sense of Definition 4.1.1 if and only if it satisfies 3.1.6

(see [31]), this equivalently shows a lack of uniqueness for critical points of (3.1.1). We provide a partial answer to this question (see Theorem 17 below) for a range of values of $(m, E) \in \mathbb{R}^+ \times \mathbb{R}^+$ sufficiently small, by showing that the only connected critical point is the ball in the class of admissible sets in that range. More precisely, when the parameter defined by (3.1.10) below is sufficiently small.

The way that we characterize critical points is by showing that any set Ω as described above which is not a constant curvature surface satisfies

$$\frac{dE(\Omega_t)}{dt}\Big|_{t=0} \neq 0, \tag{3.1.8}$$

where Ω_t is the evolution of Ω under area-preserving curve shortening flow, which is admissible under Definition 4.1.1. Details will follow in Sections 3.3 and 3.4.

Our main results (Theorems 17 and 9 below) will hold for

$$K(x,y) = -\frac{1}{2\pi}\log|x-y| \text{ or } K(x,y) = \frac{1}{|x-y|^\alpha} \text{ when } \alpha \in (0,1), \tag{3.1.9}$$

with minor modifications to the proofs. It will always be made clear below which kernel is being used. When a constant depends on α, this will mean exclusively for the kernel K for $\alpha \in (0,1)$. In all such cases the dependence of the constant on α may be dropped for the logarithmic kernel.

We begin by defining the following parameters

$$\bar{\eta} := \begin{cases} m^{1/2}L^2(1+|\log L|) & \text{for} \quad K(x,y) = -\frac{1}{2\pi}\log|x-y|, \\ m^{1/2}L^{2-\alpha} & \text{for} \quad K(x,y) = \frac{1}{|x-y|^\alpha} \quad \alpha \in (0,1), \end{cases} \tag{3.1.10}$$

where $L = |\partial\Omega|$. Our results will be stated in terms of these rescaled parameters. A simple scaling analysis of (3.1.1) reveals

$$\left\langle \frac{d(E-L)}{dL}, \zeta \right\rangle = \gamma \frac{\int_{\partial\Omega} \phi_\Omega(y)\zeta(y)dS(y)}{\int_{\partial\Omega} \kappa(y)\zeta(y)dS(y)} \sim \bar{\eta}, \tag{3.1.11}$$

where E is defined by (3.1.1), dS is surface measure on $\partial\Omega$ and with some abuse of notation $\left\langle \frac{d(E-L)}{dL}, \zeta \right\rangle$ denotes the variation in the sense of Definition 4.1.1 induced by the normal velocity $\zeta : \partial\Omega \to \mathbb{R}$. Thus $\bar{\eta}$ represents the rate of change of the non-local term in the energy with respect to the length of the boundary. Our result can thus be stated formally as saying that when the the change of the non-local term is small compared to a change in length, the critical points can be classified entirely in terms of those of the length term in (3.1.1), and thus are constant curvature curves. In other words, not only does the non-local term dominate the perimeter, but the rate of change of the non-local term slaves to the rate of change of the perimeter.

For minimizers we have a natural a priori bound on L coming from the positivity of both terms in the energy (3.1.1), which we don't have for non-minimizing critical points. This explains the need for introducing (3.1.10). The terms $\bar{\eta}_{cr}$ below are critical values of $\bar{\eta}$ which can be made explicit and are described in more detail in Section 3.4.

Theorem 8. *There exists* $\bar{\eta}_{cr} = \bar{\eta}_{cr}(\alpha) > 0$ *such that whenever* $\bar{\eta} < \bar{\eta}_{cr}$*, the only admissible critical point of* (3.1.1) *in* $\mathcal{A}_m$ *in the sense of Definition 3.1.2 is the ball.*

The Theorem is false when the mass is larger, as annuli satisfying (3.1.6) can easily be constructed.

Counter Example 1. *There exists a smooth, compact, connected set* Ω *solving* (3.1.6) *with* $\bar{\eta} > \bar{\eta}_{cr}$ *which is not the ball.*

Remark 3.1.3. *It is easy to see that there cannot exist critical points with multiple disjoint components which are separable by a hyperplane. Indeed if* Ω_1 *and* Ω_2 *are two disjoint components of* Ω *which can be separated, then let* c *be a vector so that* $(x-y)\cdot c > 0$ *for* $(x,y) \in \Omega_1 \times \Omega_2$*. Then we have*

$$\frac{d}{dt}\iint_{\Omega_1\times\Omega_2} \log|x+ct-y|dxdy = -\iint_{\Omega_1\times\Omega_2} \frac{(x-y)}{|x-y|^2}\cdot cdxdy < 0. \quad (3.1.12)$$

Consequently Ω *cannot be critical in the sense of Definition 4.1.1. Theorem 17 however leaves open the possibility of more intricate solutions to* (3.1.6)*, with multiple connected components. A similar calculation shows the same result for the kernel* $K(x) = 1/|x|^\alpha$*.*

Using the same techniques we also obtain the following stability results. The above theorems all rely on Theorem 12 in Section 3.4 which provides an explicit estimate for the rate of decrease of the energy (3.1.1) along area-preserving curve shortening flow. The following stability result for minimizers in $\mathbb{R}^2$ is a simple corollary of Theorem 12.

Corollary 3.1.4. *Let Ω be any convex set in $\mathbb{R}^2$. Then there exists an $m_{cr} > 0$ such that whenever $m < m_{cr}$ there is a constant $C = C(m_{cr}) > 0$ such that*

$$E(\Omega) \geq E(B) + C(L - 2\sqrt{\pi} m^{1/2}),$$

where $L = |\partial\Omega|$.

The main geometric inequality which we prove in this paper in order to control the non-local term along the flow is the following.

Theorem 9. *Let $\Omega \subset \mathbb{R}^2$ be a convex set with $\kappa \in L^2(\partial\Omega)$. Then there is an explicit constant $C = C(|\partial\Omega|, \alpha) > 0$ such that*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|^2_{L^\infty(\partial\Omega)} \leq C \int_{\partial\Omega} |\kappa - \bar{\kappa}|^2 dS,$$

where the bar denotes the average over $\partial\Omega$ and dS is 1-dimensional Hausdorff measure.

The above theorem provides a quantitative estimate of the closeness to an equipotential surface in terms of the distance to a constant curvature surface. The inequality relies on an isoperimetric inequality due to Gage [48] applied to curve shortening for convex sets. We hope that the above inequality will be of interest even outside the context of Ohta-Kawasaki.

Remark 3.1.5. *We remark that if Ω is any connected set with $\kappa \in L^2(\partial\Omega)$ we can prove the weaker inequality*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|^2_{L^\infty(\partial\Omega)} \leq C\sqrt{\int_{\partial\Omega} |\kappa - \bar{\kappa}|^2 dS}. \tag{3.1.13}$$

Indeed if one follows the proof of Theorem 9, one can apply Cauchy-Schwarz on line (3.5.7) *and bound $\int_{\partial\Omega} p^2 dS$ by CL^2, thus establishing* (3.1.13) *with $C \sim L^3$ when combined with Proposition 3.5.2. This inequality turns out not to be sufficient to show that the energy* (3.1.1) *decreases along the flow however. Observe that neither inequality can hold without the assumption of connectedness, as the example of two disjoint balls demonstrates.*

Remark 3.1.6. *In dimensions $d \geq 3$, we expect the above inequality to continue to hold, but are unable to demonstrate it without an assumption that the sets Ω satisfy a positive uniform lower bound on the principal curvatures of the surface $\partial\Omega$. In this case the constant C also depends on this lower bound. Proving Theorem 9 is the only obstacle in extending our results to the case $\mathbb{R}^3$. The proof presented in Section 3.5.1 fails in $\mathbb{R}^d$ for $d \geq 3$ since the Gaussian curvature and mean curvature do not agree.*

Our paper is organized as follows. In Section 3.2 we set up the appropriate framework for critical points, defining precisely in what sense (3.1.6) is satisfied and showing that critical points are convex when $\bar{\eta}$ is sufficiently small. In Section 3.3 we introduce the area-preserving curve shortening flow and state some of the main results concerning the flow that we will need. In Section

3.4 we state precisely the result showing (3.1.8), Theorem 12. In Section 3.5 we establish the necessary inequalities needed to control the behavior of the non-local term in terms of the decay of perimeter along the flow (cf. Theorem 9). Finally we use the geometric inequalities established in Section 3.5 to prove the above theorems in Section 3.6 by differentiating the energy (3.1.1) along the flow. The counter example (cf. Counter Example 1) will appear at the end of Section 3.5.

3.2 The weak Euler-Lagrange equation

In this section we rigorously compute the Euler-Lagrange equation for the class of curves admissible in the sense of Definition 3.1.2. We work in $\mathbb{R}^2$ for simplicity of presentation and hence set $\gamma \equiv 1$. The calculation of the Euler-Lagrange equation.

In the class of admissible curves (cf. Definition 3.1.2) the energy (3.1.1) may be written as

$$E(u) = \sum_k \int_0^1 |\gamma_k'(s)| ds + \iint_{\Omega \times \Omega} G(x - y) dx dy, \tag{3.2.1}$$

where γ_k is as in Definition 3.1.2, and $|\gamma_k'(s)| = L_k$ for a.e $s \in [0, 1]$. Since $\gamma_k \cap \gamma_j = \emptyset$ when $j \neq k$, the variations $\gamma_k \mapsto \gamma_k + tv$ such that $\int_0^{L_k} v(s) \cdot (\gamma_k'(s))^{\perp} ds = 0$ are admissible for $t > 0$ sufficiently small in Definition 4.1.1 by letting $\Omega_t =$ Int $(\gamma_k + tv)$, where Int(γ) denotes the interior of the closed curve γ.

Proposition 3.2.1. *(Weak Euler-Lagrange equation) Let Ω be a critical point to* (3.2.1) *in the sense of Definition 4.1.1, which is admissible in the sense of Definition 3.1.2. Then for every k it holds that*

$$\kappa(\gamma_k(s)) + \phi_\Omega(\gamma_k(s)) = \lambda \tag{3.2.2}$$

where $\kappa(\gamma_k(s)) = L_k^{-2}\gamma_k''(s) \cdot (\gamma_k'(s))^\perp$ is the curvature and λ is the Lagrange multiplier from the volume constraint. Moreover $\gamma_k \in C^{3,\alpha}([0,1])$ for all k.

Proof. By taking variations $t \mapsto \gamma_k + tv$ as described above and differentiating (3.2.1) with respect to t, we have

$$\int_0^1 \gamma_k'(s) \cdot v'(s)ds + L_k \int_0^1 \phi_\Omega(\gamma_k(s))v(s) \cdot (\gamma_k'(s))^\perp ds = 0, \tag{3.2.3}$$

for all $v \in W^{1,\infty}([0,1];\mathbb{R}^2)$. Equation (3.2.3) is the weak Euler-Lagrange equation for (3.2.1). Observe that then

$$v \mapsto \int_0^1 \gamma_k'(s) \cdot v'(s)ds,$$

is a bounded linear functional on $W^{1,\infty}([0,1])$ which extends continuously to a bounded linear functional on $C^0([0,1])$. Indeed this follows from (3.2.3), since $\phi_\Omega(\gamma_k(s)) \in C^{1,\alpha}([0,1])$ ([51]) and $\gamma_k \in W^{1,1}([0,1])$. Thus by the Riesz representation theorem, γ_k'' is a finite, vector valued Radon measure on $[0,1]$. In fact, since $L_k\gamma_k''(s) = (\lambda - L_k\phi_\Omega(\gamma_k(s)))(\gamma_k'(s))^\perp$ as a measure, it follows that $\gamma_k' \in W^{1,1}([0,1])$. Recalling that $\gamma_k(s) = \gamma_k(0) + L_k \int_0^s e^{i\theta(r)}dr$, we have

$\gamma_k''(s) = L_k\theta'(s)e^{i\theta(s)}$ for a.e $s \in [0,1]$ since $\gamma_k' \in W^{1,1}([0,1])$ and hence $|\gamma_k''(s)| = L_k|\theta'(s)|$ a.e. This implies $\theta' \in L^1([0,1])$. Then $L_k^2\kappa(\gamma_k(s)) := \gamma_k''(s)\cdot(\gamma_k'(s))^\perp = L_k^2\theta'(s)$ is defined a.e $s \in [0,1]$ and is in $L^1([0,1])$ with $\kappa(\gamma_k(s))+\phi_\Omega(\gamma_k(s)) = \lambda$ holding for a.e $s \in [0,1]$. Then by standard elliptic theory [51], it follows that $\gamma_k \in C^{3,\alpha}([0,1])$ for $\alpha \in (0,1)$, implying $C^{3,\alpha}$ regularity of the boundary and that (3.2.2) holds strongly for all $s \in [0,1]$. $\square$

We then have the following approximation Theorem.

Proposition 3.2.2. *Let γ be a closed rectifiable curve with $\theta \in BV([0,1])$. Then there exists a sequence of C^2 curves γ_n such that*

$$\gamma_n \to \gamma \in W^{1,1}([0,1]) \tag{3.2.4}$$

$$\theta_n \to \theta \in L^1([0,1]) \tag{3.2.5}$$

$$\theta_n' \rightharpoonup \theta' \in (C([0,1])^*, \tag{3.2.6}$$

where $(C[0,1])^*$ is the dual of the space of continuous functions on $[0,1]$.

Proof. Let $\theta_n := \eta^{1/n} * \theta$ where $\eta^{1/n} := \eta(nx)$ and η is the standard mollifier. Then since $\|\theta\|_{BV([0,1])} < +\infty$ we have

$$\limsup_{n\to+\infty} \|\theta_n\|_{BV([0,1])} < +\infty. \tag{3.2.7}$$

Thus we have, up to a subsequence, $\theta_n' \rightharpoonup \theta'$ in the weak sense of measures and $\theta_n \to \theta$ in $L^1([0,1])$ by the embedding $BV([0,1]) \subset\subset L^1([0,1])$. The convergence $\gamma_n \to \gamma$ in $W^{1,1}([0,1])$ follows immediately. $\square$

Using the above approximation Theorem we prove that the Gauss-Bonnet theorem continues to hold for closed, rectifiable curves. This is not technically necessary in this section as we have proven that $\partial\Omega$ is always parameterizable by $C^{3,\alpha}$ curves by Proposition 3.2.1. However we will use this result in Section 3.5 to prove the inequalities hold without the assumption of smoothness of the boundary.

Proposition 3.2.3. *Let γ be a closed, rectifiable curve with $\theta \in BV([0,1])$. Then there exists $N \in \mathbb{Z}$ such that*

$$\int_0^1 \theta'(s)ds = \int_{\partial\Omega} \kappa(y)dS(y) = 2\pi N,$$

where N is called the winding number.

Proof. Let γ_n be as in Proposition 3.2.2. It follows from the Gauss-Bonnet Theorem for C^2 curves that

$$\int_0^1 \theta_n'(s)ds = 2\pi N_n,$$

for all n where $N_n \in \mathbb{Z}$ must be bounded uniformly in n, since θ_n is uniformly bounded in $BV([0,1])$. Using Proposition 3.2.2 we have $\theta_n' \rightharpoonup \theta'$ weakly in $(C([0,1])^*$ allowing us to pass to the limit in the above, implying that $N_n = N$ for some $N \in \mathbb{Z}$ for sufficiently large n. Thus $\int_0^1 \theta'(s)ds = \int_{\partial\Omega} \kappa(y)dS(y) = 2\pi N$. $\square$

Proposition 3.2.4. *Let Ω be a critical point of* (3.1.1) *in the sense of Definition 4.1.1, admissible in the sense of Definition 3.1.2. Then there exists*

$\bar{\eta}_{cr} > 0$ *such that whenever* $\bar{\eta} < \bar{\eta}_{cr}$, Ω *is convex.*

Proof. Each $\gamma_k \in C^{3,\alpha}([0,1])$ by Proposition 3.2.1. Let γ_k be any interior curve parameterizing $\partial\Omega_k$, ie. the interior of γ_k is contained in the interior of some other curve γ_j for $j \neq k$. By Gauss-Bonnet (cf. Proposition 3.2.3) $⨍_{\partial\Omega_k} \kappa(y)dS(y) = \frac{2\pi N}{L_k}$ where N is the winding number of the curve γ_k, and (3.2.2) we have

$$\kappa(y) + \phi_\Omega(y) = \frac{2\pi N}{L_k} + \bar{\phi}_\Omega \tag{3.2.8}$$

holds for $y \in \partial\Omega_k$ where the bar denotes average over $\partial\Omega_k$ and where $N \in \mathbb{Z}$. We wish to show that γ_k is a simple curve, ie. $N = -1$. There is a universal constant $C > 0$ such that

$$\begin{aligned} &|\phi_\Omega(y)| \leq Cm(1 + |\log L|) \text{ when } K(x,y) = -\frac{1}{2\pi}\log|x-y| \\ &|\phi_\Omega(y)| \leq C\frac{m}{L^\alpha} \text{ when } K(x,y) = \frac{1}{|x-y|^\alpha}. \end{aligned} \tag{3.2.9}$$

Assume first that $N > 0$. Then for $y \in \partial\Omega_k$ we deduce from (3.2.8)–(3.2.9) and $L_k \leq L$

$$\begin{aligned} \kappa(y) \geq \frac{1}{L}\left(2\pi N - CmL(|\log L| + 1)\right) \geq \frac{1}{L}(2\pi N - (2\sqrt{\pi})^{-1}C\bar{\eta}) \\ \text{when } K(x,y) = -\frac{1}{2\pi}\log|x-y| \end{aligned} \tag{3.2.10}$$

and

$$\kappa(y) \geq \frac{1}{L}\left(2\pi N - CmL^{1-\alpha}\right) \geq \frac{1}{L}(2\pi N - (2\sqrt{\pi}C)^{-1}\bar{\eta}) \qquad (3.2.11)$$
$$\text{when } K(x,y) = \frac{1}{|x-y|^{\alpha}}.$$

where the second inequalities follow from the isoperimetric inequality $2\sqrt{\pi}m^{1/2} \leq L$. It is clear that when $\bar{\eta}$ is sufficiently small, $\kappa > 0$ for all points on $\partial\Omega_k$, which is a clear contradiction since γ_k was assumed to be an interior curve. When $N = 0$ then we have once again from (3.2.8) and (3.2.9)

$$|L\kappa(y)| \leq C\bar{\eta}, \qquad (3.2.12)$$

for $C > 0$ and all $y \in \partial\Omega_k$. Clearly there is always some $y \in \partial\Omega_k$ such that $\kappa(y) \geq \frac{\pi}{L_k}$. Indeed letting γ_k be a unit speed parametrization of $\partial\Omega_k$, we restrict to $s \in [0, s_0] \subset [0, L_k]$ so that $0 \leq \theta_k(s) \leq 2\pi$. Then $\int_0^{s_1} \theta_k'(s)ds = 2\pi$ and thus by the mean value theorem, there exists an $s \in [0, s_0]$ such that $\theta_k'(s) = \frac{2\pi}{s_0} \geq \frac{\pi}{L_k}$ since $s_0 \in [0, L_k]$. This contradicts (3.2.12) for $\bar{\eta}$ sufficiently small. Arguing similarly when $N < 0$, we conclude that in this case, $\kappa < 0$ everywhere when $\bar{\eta}$ is sufficiently small and hence γ_k is simple. Thus we have shown that each γ_k is simple when $\bar{\eta}$ sufficiently small since our estimates do not depend on γ_k. We now show that Ω must in fact be convex.
As before we have

$$\kappa(y) + \phi_{\Omega}(y) = \frac{2\pi N}{L} + \bar{\phi}_{\Omega}, \qquad (3.2.13)$$

where now the average is taken over all of $\partial\Omega$. Since each γ_k is simple, $N \leq 1$

and we claim that in fact $N = 1$. First we show that $N \neq 0$. In this case we have $\bar{\kappa} = 0$ and thus from (3.2.8) and (3.2.9)

$$|L\kappa(y)| \leq C\bar{\eta}, \tag{3.2.14}$$

for $C > 0$ and all $y \in \partial\Omega$. As before, there is always some $y \in \partial\Omega$ such that $\kappa(y) \geq \frac{\pi}{L_1}$, where L_1 denotes the length of outer component of $\partial\Omega$, denoted as $\partial\Omega_1$ (ie. the interior of γ_k is contained in the interior of γ_1 for all k). This is a contradiction of (3.2.14) however since $L_1 \leq L$. To see that $N \geq 0$, assume that $N < 0$. Then once again from (3.2.8) and (3.2.9)

$$\begin{aligned} \kappa(y) \leq \frac{1}{L}\left(2\pi N + CmL(|\log L| + 1)\right) \leq \frac{1}{L}(2\pi N + (2\sqrt{\pi})^{-1}C\bar{\eta}) \\ \text{when } K(x,y) = -\frac{1}{2\pi}\log|x-y| \end{aligned} \tag{3.2.15}$$

$$\begin{aligned} \kappa(y) \leq \frac{1}{L}\left(2\pi N + CmL^{1-\alpha}\right) \leq \frac{1}{L}(2\pi N + (2\sqrt{\pi})^{-1}C\bar{\eta}) \\ \text{when } K(x,y) = \frac{1}{|x-y|^{\alpha}}. \end{aligned} \tag{3.2.16}$$

By choosing $\bar{\eta}$ sufficiently small, then we would have $\kappa < 0$ everywhere on $\partial\Omega_1$ which is a contradiction, since γ_1 was assumed to be the exterior curve. Thus $N = 1$ and Ω is simply connected, ie. γ_1 is simple. Then we have by

Proposition 3.2.3 and (3.2.2) again that

$$\kappa(y) \geq \frac{1}{L}\left(2\pi - C\bar{\eta}\right) \text{ when } K(x,y) = -\frac{1}{2\pi}\log|x-y| \tag{3.2.17}$$

$$\kappa(y) \geq \frac{1}{L}\left(2\pi - C\bar{\eta}\right) \text{ when } K(x,y) = \frac{1}{|x-y|^{\alpha}}, \tag{3.2.18}$$

for all $y \in \partial\Omega$, showing that $\kappa > 0$ whenever $\bar{\eta}$ is chosen small enough. Thus Ω is convex when $\bar{\eta}$ is chosen sufficiently small. □

The assumption that $\bar{\eta}$ be sufficiently small is not simply a technical assumption, as Counter Example 1 demonstrates.

3.3 Area preserving mean curvature flow

We let Ω be a smooth, compact subset of $\mathbb{R}^2$ with boundary $\partial\Omega$. Letting X_0 be a local chart of Ω so that

$$X_0 : E \subset \mathbb{R}^2 \to X_0(E) \subset \partial\Omega \subset \mathbb{R}^2.$$

Then we let $X(x,t)$ be the solution to the evolution problem

$$\begin{aligned} \frac{\partial}{\partial t}X(x,t) &= -(\kappa(t,x) - \bar{\kappa}(t)) \cdot \nu(x,t), \quad x \in E, t \geq 0 \\ X(\cdot,0) &= X_0, \end{aligned} \tag{3.3.1}$$

where $\kappa(t,x)$ is the mean curvature of $\partial\Omega_t$ at the point x, $\bar{\kappa}(t)$ is the average of the mean curvature on $\partial\Omega_t$:

$$\bar{\kappa}(t) = \fint_{\partial\Omega_t} \kappa(y)dS(y), \tag{3.3.2}$$

$\nu(x,t)$ is the normal to $\partial\Omega_t$ at the point x and dS is the 1-dimensional Hausdorff measure. The flow (3.3.1) in dimensions $d \geq 3$ was first introduced by Huisken [63] who established existence and asymptotic convergence to round spheres for initially convex domains. The planar version for curves was introduced by Gage [47].

For convenience of notation we define

$$L(t) := |\partial\Omega_t| \tag{3.3.3}$$

$$A(t) := |\Omega_t|. \tag{3.3.4}$$

It is easy to see that the surface area of $\partial\Omega$ is decreasing along the flow. Indeed differentiating the perimeter we have

$$\frac{dL}{dt} = -\int_{\partial\Omega_t} (\kappa - \bar{\kappa})^2 dS. \tag{3.3.5}$$

The introduction of the non-local term in (3.1.1) will create a term which competes with (3.3.5) along the flow (3.3.1) as the non-locality favors the spreading of mass. The main element of the proof of Theorem 12 stated in

Section 3.4 will therefore be to show that when the mass is small, the decay in perimeter is sufficient to compensate for the increase in energy of the non-local term in (3.1.1) along the flow (3.3.1). This is where Theorem 9 will play a crucial role. Before we proceed we must recall some now well known results about the flow (3.3.1).

The main result of [47] due to Gage is the following:

Theorem 10. *(Global existence) If Ω is convex, then the evolution equation* (3.3.1) *has a smooth solution Ω_t for all times $0 \leq t < \infty$ and the sets Ω_t converge in the C^∞ topology to a round sphere, enclosing the same volume as Ω, exponentially fast, as $t \to +\infty$.*

In addition there is a local in time existence result in [47]

Theorem 11. *(Local existence) If Ω is any smooth embedded set, then there exists a $T > 0$ such that the evolution equation* (3.3.1) *has a smooth solution Ω_t for $t \in [0, T)$.*

Finally we are able to state the main result concerning the flow (3.3.1) and the energy (3.1.1).

3.4 Main Results

Our main result, from which the other results follow, is the following.

Theorem 12. *(Energy decrease along the flow) Let Ω be admissible in the sense of Definition 3.1.2 with $\kappa \in L^2(\partial\Omega)$ and denote Ω_t the evolution of Ω under* (3.3.1).

- *If Ω is convex, then there exists $\bar{\eta}_{cr} > 0$ such that whenever $\bar{\eta} < \bar{\eta}_{cr}$ it holds that*

$$\frac{dE(\Omega_t)}{dt}\Big|_{t=0} < 0, \tag{3.4.1}$$

with E defined by (3.1.1).

Moreover a lower bounds for $\bar{\eta}_{cr}$ is given by

$$\max_{\Omega\subset\mathbb{R}^2}\left(\bar{\eta}^2\frac{\int_{\partial\Omega}(\kappa-\bar{\kappa})^2 dS}{L\|\phi-\bar{\phi}\|_{L^\infty(\partial\Omega)}}\right)^{\frac{1}{2}}$$

where the maximum is taken over convex sets.

- *If Ω is in addition assumed to be smooth, then there exists a constant a $T > 0$ and $C > 0$ depending only on $\bar{\eta}_{cr}$ (or $\bar{\gamma}_{cr}$) such that*

$$\frac{dE(\Omega_t)}{dt} \leq -C\int_{\partial\Omega_t}(\kappa-\bar{\kappa})^2 dS_{\Omega_t} \tag{3.4.2}$$

for all $t \in [0, T)$. When Ω is convex, $T = +\infty$.

Remark 3.4.1. *From Theorem 13 below, we will see that a lower bound for $\bar{\eta}_{cr}$ is given by the minimum of $\frac{32}{\pi}$ for $K(x, y) = -\frac{1}{2\pi}\log|x-y|$ and $\frac{8}{\pi}\left(1+\frac{1}{\pi}\right)^{1-\alpha}$ for $K(x, y) = \frac{1}{|x-y|^\alpha}$ and the value of $\bar{\eta}_{cr}$ given by Proposition 3.2.4 (which is also explicitly computable).*

Using this result we can immediately prove Corollary 3.1.4.

Proof of Corollary 3.1.4 Using Proposition 3.2.1, we can invoke (3.4.2) along with (3.3.5), applying Theorem 10 to conclude the desired inequality. $\square$

3.5 Geometric inequalities

The main result of this section is a geometric inequality which relates the closeness of connected constant curvature curves and equipotential curves. This will be used in Section 3.6 to control the non-local term along the flow (3.3.1).

Theorem 13. *Let $\Omega \subset \mathbb{R}^2$ be convex with $\kappa \in L^2(\partial\Omega)$. Then there is a constant $C = C(L, \alpha) > 0$ such that*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|^2_{L^\infty(\partial\Omega)} \leq C \int_{\partial\Omega} |\kappa - \bar{\kappa}|^2 dS. \tag{3.5.1}$$

i) When $K(x,y) = -\frac{1}{2\pi}\log|x-y|$, the constant can be chosen to be $C = \frac{32AL^3}{\pi}(1 + |\log L|)^2 = \frac{32}{\pi L}\bar{\eta}^2$.

ii) When $K(x,y) = \frac{1}{|x-y|^\alpha}$ for $\alpha \in (0,1)$, the constant can be chosen to be $C = \frac{8A}{\pi}\left(1 + \frac{1}{\pi}\right)^{2(1-\alpha)} L^{3-2\alpha} = \frac{8}{\pi L}\bar{\eta}^2$.

We first control the isoperimetric deficit in terms of the right hand side of (3.5.1), then control the left hand side of (3.5.1) by the isoperimetric deficit. This relies crucially on the fact that the ball is the only compact, connected CMC surface in $\mathbb{R}^2$. We will make repeated use of Bonnesen's inequality [13], which states that given any set simply connected $\Omega \subset \mathbb{R}^2$, it holds that

$$L^2 - 4\pi A \geq \pi^2(R_{\text{out}}(\Omega) - R_{\text{in}}(\Omega))^2, \tag{3.5.2}$$

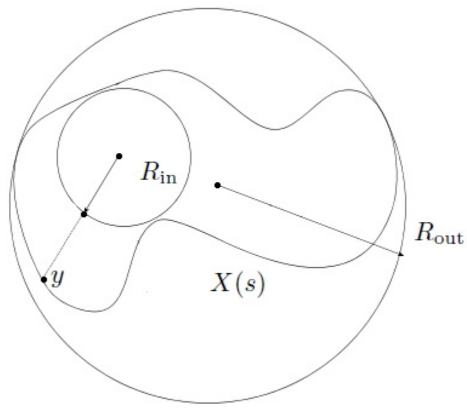

Figure 3.1: The boundary of Ω is parametrized by $X(s)$ and has inner and outer radii R_{in} and R_{out}.

where R_{in} and R_{out} are

$$R_{\text{in}}(\Omega) = \sup_{B_r \subset \Omega} r \tag{3.5.3}$$

$$R_{\text{out}}(\Omega) = \inf_{\Omega \subset B_R} R.$$

See Figure 1 for a diagram showing the various quantities.

3.5.1 A reverse quantitative isoperimetric inequality

Let $X(s) = (x(s), y(s)) \in \mathbb{R}^2$ be a unit speed parametrization of the curve enclosing Ω, ν the normal to the surface at the point $X(s)$ and $p(s) := \langle X(s), -\nu(s) \rangle$ the support function. We will also use A instead of m for the area to emphasize the geometric nature of the inequalities, although the two are equivalent.

Proposition 3.5.1. *Let $\Omega \subset \mathbb{R}^2$ be convex with $\kappa \in L^2(\partial\Omega)$. Then it holds that*

$$L - 2\sqrt{\pi}A^{1/2} \le \frac{A}{\pi}\int_{\partial\Omega}(\kappa - \bar{\kappa})^2. \tag{3.5.4}$$

Proof. Using the generalized Gauss-Green theorem we have

$$A = \int_\Omega \det DX(x,y)dxdy = \frac{1}{2}\int_{\partial\Omega}\langle X, -\nu\rangle dS = \frac{1}{2}\int_0^L p(s)ds, \tag{3.5.5}$$

where we set $p(s) = \langle X(s), -\nu(s)\rangle$ and $\kappa(s) = X''(s)\cdot\nu(s)$. In addition we have

$$\begin{aligned}\int_{\partial\Omega} p\kappa dS = \int_0^L p(s)\kappa(s)ds &= -\int_0^L \langle X(s), \kappa\nu(s)\rangle ds = -\int_0^L \langle X(s), X''(s)\rangle ds \\ &= -\langle X, X'\rangle\Big|_0^L + \int_0^L \langle X'(s), X'(s)\rangle ds \\ &= L. \end{aligned} \tag{3.5.6}$$

Adding and subtracting $\bar{\kappa}$ to κ in the integrand on the left side of (3.5.6), and using the Gauss-Bonnet theorem for curves (cf. Proposition 3.2.3), we find

$$L - \frac{4\pi A}{L} = \int_{\partial\Omega} p(\kappa - \bar{\kappa})dS. \tag{3.5.7}$$

We first prove the inequality when Ω is symmetric about the origin of p.

Adding and subtracting $\bar{p}$ from p in (3.5.7) we have

$$\begin{aligned}\frac{L^2 - 4\pi A}{L} &\leq \int_{\partial\Omega} (p - \bar{p})(\kappa - \bar{\kappa}) dS \\ &\leq \sqrt{\int_{\partial\Omega} (p - \bar{p})^2 dS} \sqrt{\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS},\end{aligned} \tag{3.5.8}$$

where we've applied Cauchy-Schwarz on the last line. A simple calculation yields

$$\int (p - \bar{p})^2 = \int p^2 - \frac{4A^2}{L} \tag{3.5.9}$$

Using an inequality due to Gage [48] for convex sets symmetric about the origin, we have

$$\int p^2 dS \leq \frac{LA}{\pi}. \tag{3.5.10}$$

Inserting (3.5.10) into (3.5.9) we have

$$\int (p - \bar{p})^2 dS \leq \frac{A}{\pi L} \left(L^2 - 4\pi A \right). \tag{3.5.11}$$

Inserting the above into (3.5.8) we have

$$\sqrt{L^2 - 4\pi A} \leq \sqrt{\frac{AL}{\pi}} \sqrt{\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS}. \tag{3.5.12}$$

Squaring both sides, dividing both sides by L we have

$$L - 2\sqrt{\pi} A^{1/2} \leq \frac{A}{\pi} \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS. \tag{3.5.13}$$

When Ω is not symmetric about the origin, then choose a point $O \in \Omega$ and any straight line passing through O. Then this line will divide the set Ω into two segments, Ω_1 and Ω_2. We claim it is always possible to choose this line so that Ω_1 and Ω_2 have the same area. Indeed let $F(\Omega, \theta) := |\Omega_1| - |\Omega_2|$ where θ denotes the angle of the line with respect to some fixed axis. If $F(\Omega, 0) = 0$ then we are done. Otherwise $F(\Omega, 0) > 0$ without loss of generality. But then $F(\Omega, 2\pi) < 0$ and hence by the intermediate value theorem we conclude there exists a $\theta \in (0, 2\pi)$ such that $F(\Omega, \theta) = 0$. Without loss of generality we may orient this line to be parallel to the x axis. Let Ω_1' and Ω_2' be the sets formed by reflection across the origin. Then we can apply (3.5.4) and we obtain for $i = 1, 2$

$$L_i' - 2\sqrt{\pi}A^{1/2} \leq \frac{A}{\pi}\int_{\partial\Omega_i'} (\kappa_i' - \bar{\kappa}_i')^2 dS = \frac{A}{\pi}\left(\int_{\partial\Omega_i'} (\kappa_i')^2 - \frac{4\pi^2}{L_i'}\right).$$

Adding the two inequalities over $i = 1, 2$ and using

$$\frac{1}{L_1'} + \frac{1}{L_2'} \geq \frac{2}{L_1' + L_2'},$$

we obtain after division by 2

$$L - 2\sqrt{\pi}A^{1/2} \leq \frac{A}{\pi}\int_{\partial\Omega} \kappa^2 dS - \frac{4\pi^2}{L} = \frac{A}{\pi}\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS,$$

the desired inequality. $\square$

Next we obtain a quantitative estimate for the closeness to an equipotential surface in terms of the isoperimetric deficit. We will present the proof below for $K(x,y) = -\frac{1}{2\pi}\log|x-y|$ and show how the proof is adapted to the case $K(x,y) = \frac{1}{|x-y|^\alpha}$ in the remark following the proof.

Proposition 3.5.2. *Let $\Omega \subset \mathbb{R}^2$ be simply connected. Then there exists a constant $C = C(L,\alpha) > 0$ such that*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|^2_{L^\infty(\partial\Omega)} \le C(L^2 - 4\pi A), \tag{3.5.14}$$

where ϕ_Ω is defined by (3.1.7) *for G defined by* (3.1.9).

- *When $K(x,y) = -\frac{1}{2\pi}\log|x-y|$ the constant can be chosen to be $C = 16L^2(1+|\log L|)^2$.*
- *When $K(x,y) = \frac{1}{|x-y|^\alpha}$ for $\alpha \in (0,1)$, the constant can be chosen to be $C = 4\left(1+\frac{1}{\pi}\right)^{2-2\alpha} L^{2-2\alpha}$.*

Proof of Proposition 3.5.2

Consider any two points $y, z \in \partial\Omega$ and assume first $\phi_\Omega(y) > \phi_\Omega(z)$. Then we have

$$\begin{aligned}\phi_\Omega(y) - \phi_\Omega(z) &= \int_\Omega (G(x,y)+C) - (G(x,z)+C)dx &(3.5.15)\\ &\le \int_{B_{R_{\text{out}}}} (G(x,y)+C)dx - \int_{B_{R_{\text{in}}}} (G(x,z)+C)dx, &(3.5.16)\end{aligned}$$

where $C = C(R_{\text{out}}) > 0$ is a constant chosen so that $G + C$ is positive on

$B_{R_{\text{out}}}$. In particular C can be chosen to be

$$C = C_1 := \max(0, \frac{1}{2\pi} \log 2R_{\text{out}}).$$

Using radial symmetry of the Laplacian equation (3.5.16) is in fact equal to

$$\frac{1}{2}(R_{\text{out}}^2 - |y|^2) - \frac{1}{2}(|z|^2 - R_{\text{in}}^2) + C_1(R_{\text{out}}^2 - R_{\text{in}}^2) - \frac{R_{\text{out}}^2}{2} \log R_{\text{out}} + \frac{R_{\text{in}}^2}{2} \log R_{\text{in}} \tag{3.5.17}$$

Arguing similarly when $\phi_\Omega(y) \leq \phi_\Omega(z)$ we conclude for all $y, z \in \partial\Omega$ that

$$\begin{aligned} |\phi_\Omega(y) - \phi_\Omega(z)| &\leq (C_1 + 2R_{\text{out}})(R_{\text{out}} - R_{\text{in}}) + \frac{1}{2}\left|R_{\text{out}}^2 \log R_{\text{out}} - R_{\text{in}}^2 \log R_{\text{in}}\right| \\ &\leq \left(2 + \frac{1}{\pi}\right) R_{\text{out}} (R_{\text{out}} - R_{\text{in}}) + \frac{1}{2}\left|R_{\text{out}}^2 \log R_{\text{out}} - R_{\text{in}}^2 \log R_{\text{in}}\right|. \end{aligned} \tag{3.5.18}$$

Observe that the function $f(x) = x^2 \log x$ with $f(0) = 0$ is C^1 and so assuming $x > y$ we have

$$|f(x) - f(y)| \leq |f'|_{L^\infty(0,x)}|x - y| \leq 2|x|(1 + |\log x|)|x - y|, \tag{3.5.19}$$

where in the last inequality we've used the fact that $x \mapsto 2|x|(1 + |\log x|)$ is monotone increasing on $(0, +\infty)$. Using Bonnesen's inequality (cf. equation (3.5.2)) we have

$$R_{\text{out}} \leq R_{\text{in}} + \frac{1}{\pi}\sqrt{L^2 - 4\pi A} \leq \frac{2L}{\pi} \tag{3.5.20}$$

and thus

$$\left|R_{\text{out}}^2 \log R_{\text{out}} - R_{\text{in}}^2 \log R_{\text{in}}\right| \leq 2L(|\log L|+1)(R_{\text{out}}-R_{\text{in}}) \leq 2L(|\log L|+1)\sqrt{L^2 - 4\pi A} \tag{3.5.21}$$

Inserting this into (3.5.18) and using (3.5.2), we obtain

$$|\phi_\Omega(y) - \phi_\Omega(z)| \leq 4L(1 + |\log L|)\sqrt{L^2 - 4\pi A}. \tag{3.5.22}$$

Choosing z so that $|\phi_\Omega(y) - \phi_\Omega(z)| \leq |\phi_\Omega(y) - \bar{\phi}|$ and maximizing over $y \in \partial\Omega$ yields the desired inequality. □

Remark 3.5.3. *The above proof easily adapts to the case of the kernel* $K(x) = \frac{1}{|x|^\alpha}$. *Indeed the constant* C_1 *above can be taken to be zero, since* $K > 0$ *everywhere and line* (3.5.17) *can simply be replaced by*

$$\left(\int_{B(0,1)} \frac{dx}{|x-y|^\alpha}\right)\left(R_{out}^{2-\alpha} - R_{in}^{2-\alpha}\right) \leq 2\pi R_{out}^{1-\alpha}(R_{out} - R_{in}) \text{ for } x \in \partial B(0,1),$$

where we've performed a first order Taylor expansion of the function $f(x) = x^{2-\alpha}$ *about the point* $x = R_{in}$. *Using Bonnesen's inequality once again we obtain*

$$|\phi_\Omega(y) - \phi_\Omega(z)| \leq 2\left(1 + \frac{1}{\pi}\right)^{1-\alpha} L^{1-\alpha}\sqrt{(L^2 - 4\pi A)}.$$

The rest of the proof is argued identically to the logarithmic case.

Proof of Theorem 13 item i) Now the proof of Theorem 13, items *i*) and *ii*) immediately follows from combining Proposition 3.5.2 and Proposition 3.5.1 and using $L^2 - 4\pi A \leq 2L(L - 2\sqrt{\pi}A^{1/2})$.

3.6 Proof of Theorems

Proof of Theorem 12: Let $C(L,\alpha)$ denote the constant appearing in Theorem 13. Observe that variations $\Omega \to \Omega_t$ induced by the normal velocity $\kappa - \bar{\kappa}$ are admissible in the sense of Definition 4.1.1. Then for any convex set Ω with $\kappa \in L^2(\partial\Omega)$ we have

$$\frac{dE(\Omega_t)}{dt}\Big|_{t=0} = -\int_{\partial\Omega}(\kappa-\bar{\kappa})^2 dS - \int_{\partial\Omega}(\kappa-\bar{\kappa})(\phi_\Omega - \bar{\phi}_\Omega)dS \tag{3.6.1}$$

$$\leq -\int_{\partial\Omega}(\kappa-\bar{\kappa})^2 dS + \sqrt{C(L,\alpha)}L^{1/2}\int_{\partial\Omega}(\kappa-\bar{\kappa})^2 dS \tag{3.6.2}$$

$$\leq -(1-\bar{\eta}\tilde{C})\int_{\partial\Omega}(\kappa-\bar{\kappa})^2 dS, \tag{3.6.3}$$

whenever $\bar{\eta} < \bar{\eta}_{cr}$ with $\bar{\eta}_{cr}$ defined as in Theorem 12 and where $\tilde{C} > 0$ is universal . The first line is simply the computation of the derivative of E along the flow (3.3.1) at $t = 0$. Line (3.6.2) follows from Theorem 13 along with Cauchy Schwarz. This establishes the first part of Theorem 12, observing that (3.6.3) implies the lower bound on $\bar{\eta}_{cr}$. When Ω is smooth, there exists $t \in [0,T)$ such that the above holds for all $t \in [0,T)$ by Theorem 11 with $T = +\infty$ when Ω is convex (cf. Theorem 10), establishing the second part of Theorem 12.

Proof of Theorem 17 We have in fact established Theorem 17 from the above calculations. Indeed we have shown that

$$\frac{dE(\Omega_t)}{dt}\Big|_{t=0} < 0,$$

where the map $\Omega \mapsto \Omega_t$ is an admissible variation in Definition 4.1.1. Therefore by definition Ω is not a critical point if it does not have constant curvature. Using the characterization of compact, connected, constant curvature curves in $\mathbb{R}^2$ establishes the claim. $\square$

Counter Example 1

We consider annuli of radii $r, R > 0$ with $r < R$ and let $\Omega = \{x : r \leq |x| \leq R\}$. Then we can explicitly calculate ϕ_Ω, $\bar{\phi}_\Omega$, κ and $\bar{\kappa}$ on $\partial\Omega$ for both the kernels K and $-\frac{1}{2\pi}\log|x|$. We consider first the logarithmic case.

Case 1: $K(x,y) = -\frac{1}{2\pi}\log|x-y|$

Solving Poisson's equation $-\Delta\phi_\Omega = 1_\Omega$ explicitly in radial coordinates we obtain

$$\phi_\Omega(x) = \frac{1}{2}(R^2 - |x|^2) - \frac{1}{2}(|x|^2 - r^2) - \frac{R^2}{2}\log R + \frac{r^2}{2}\log r \text{ for } x \in [r, R]. \tag{3.6.4}$$

It is easily seen that $\bar{\kappa} = 0$ with $\kappa = \frac{-1}{r}$ on ∂B_r and $\kappa = \frac{1}{R}$ on ∂B_R. We first demonstrate Counter Example 1. We wish to find $R > r > 0$ such that

$$\kappa(x) + \phi_\Omega(x) = \kappa(y) + \phi_\Omega(y),$$

for $x \in \partial B_r$, $y \in \partial B_R$. We claim this is the case. Indeed setting $R = 2r$ with $r = \left(\frac{1}{2}\right)^{1/3}$ we have $\bar{\eta} := \left(\frac{1}{2}\right)^{1/3}(3\pi)^{1/2}(1+\log(6\pi 2^{-1/3}))$. Then with these choices of r and R,

$$-\frac{1}{r} + \frac{1}{2}(R^2 - r^2) = \frac{1}{R} + \frac{1}{2}(r^2 - R^2). \tag{3.6.5}$$

holds. Thus Ω is a solution to (3.1.6) for these choices of r and R.

Case 2: $K(x,y) = \frac{1}{|x-y|^\alpha}$

In this case we have

$$\begin{aligned} \phi_\Omega(x) &= R^{2-\alpha}\int_{B(0,1)} \frac{d\bar{y}}{|x/R - \bar{y}|^\alpha} - r^{2-\alpha}\int_{B(0,1)} \frac{d\bar{y}}{|x/r - \bar{y}|^\alpha} && (3.6.6) \\ &= \left(\int_{B(0,1)} \frac{dy}{|\tilde{x} - y|^\alpha}\right)(R^{2-\alpha} - r^{2-\alpha}) \text{ for } \tilde{x} \in \partial B(0,1). && (3.6.7) \end{aligned}$$

As in the previous case, it is seen via direct computation that Ω is a solution to (3.1.6) when $R = 2r$ and $\bar{\eta} = m^{1/2}L^{2-\alpha} = C_0$ where C_0 is an explicit constant, thus establishing Counter Example 1 for $G = K$.

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Chapter 4

Asymptotics of non-minimizing stationary points of the Ohta-Kawasaki energy and its sharp interface analogue

4.1 Introduction

This paper is devoted to the convergence of stationary points of the Ohta-Kawasaki energy functional [87] in the small volume regime. The energy functional has the following form:

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x,y)(u(y) - \bar{u})\, dx\, dy, \tag{4.1.1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $V(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg-Landau potential $V(u) = \frac{1}{4}(1 - u^2)^2$, $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, i.e., G_0 solves

$$-\Delta G_0(x,y) = \delta(x - y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x,y)\, dx = 0, \tag{4.1.2}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|} \int_\Omega u\, dx = \bar{u}. \tag{4.1.3}$$

For a discussion of the motivation and the main quantitative features of this model, see [54], as well as [79,80]. For specific applications to physical systems, we refer the reader to [34, 58, 73, 78, 79, 86, 87, 110]. For the remainder of this paper we focus on the case where Ω is the flat n-dimensional torus $\mathbb{T}^n = [0, 1)^n$ with periodic boundary conditions, unless otherwise specified.

We focus most of our attention on the following "sharp interface" version of (4.1.1):

$$E^{\epsilon}[u] = \frac{\varepsilon}{2}\int_{\mathbb{T}^n} |\nabla u| + \frac{1}{2}\iint_{\mathbb{T}^n\times\mathbb{T}^n} (u(x)-\bar{u})G(x-y)(u(y)-\bar{u})dxdy, \quad (4.1.4)$$

reserving most of our analysis of the diffuse interface energy (4.1.1) to the final section of this paper (Section 4.7). In (4.1.4), G is the *screened* Poisson kernel solving

$$-\Delta G + \kappa^2 G = \delta(x-y) \text{ in } \mathbb{T}^n, \quad (4.1.5)$$

for $\kappa = 1/\sqrt{V''(1)} > 0$ and $u \in \mathcal{A}$ where

$$\mathcal{A} := \{u \in BV(\mathbb{T}^n; \{-1, 1\})\}. \quad (4.1.6)$$

The charge neutrality condition (cf. equation (4.1.3)) is no longer imposed, i.e. $\int_{\mathbb{T}} u \neq \bar{u}$. This is related to the fact that the charge of the minority phase is expected to partially redistribute itself into the majority phase to ensure screening of the induced non-local field (see [54] for a more detailed discussion). The energy (4.1.4) was first studied in [80] where the connection between (4.1.1) and (4.1.4) is made for exact minimizers. Moreover, when $n = 2$, it is shown that when $\bar{u}$ is close to -1, minimizers of (4.1.4) form almost spherical "droplets" of the minority phase $\{u = +1\}$ with the same radius, distributed uniformly throughout the domain. In [54]– [55] the full Γ-limit of (4.1.4) was computed to first and second order near the onset of non-trivial minimizers (see [14] for an introduction to Γ-convergence), with [54] addressing the Γ limit of (4.1.1) as well. There it is shown that, in addition, almost minimizers form (on average) almost spherical droplets of the phase $\{u = +1\}$, with almost the same radius and which are once again distributed uniformly throughout the domain. An important observation in these works is that, as $\varepsilon \to 0$, the number of disjoint connected components of $\{u^\varepsilon = +1\}$ may be unbounded [54, 55, 80], and the results can thus be seen as generalizations of the work of Choksi and Peletier who study a suitably rescaled version of (4.1.1) and (4.1.4) in the absence of screening (ie. $\kappa = 0$) and when the number of droplets constrained to be finite [28, 29]. More precisely, they compute the Γ-limit in this setting of (4.1.1) and (4.1.4) in [29] and [28] respectively, showing, in particular, that the droplets of the minority phase $\{u = +1\}$ shrink to points whose magnitudes and locations are determined via a limiting Coulombic interaction energy. A related result concerning minimizers is the work of Alberti-Choksi-Otto and Spadaro [2,108], wherein it is shown that the energy of minimizers of (4.1.1) and (4.1.4) respectively is uniformly distributed throughout the domain.

All of the above results are concerned with *minimizing* stationary points of the energies (4.1.1) and (4.1.4). Moreover, all of the results regarding the asymptotics of minimizers when the number of droplets is unbounded work only in dimension $n = 2$. In this paper we address a question left open in the above analysis which is that of the asymptotic behavior of a priori *non-minimizing* stationary points of the energies (4.1.1) and (4.1.4) which, moreover, applies to any dimension $n \geq 2$. There has been some work in this context by Röger and Tonegawa [96]. They show that when the number of droplets is constrained to be finite in a bounded domain Ω with a fixed volume fraction, that any sequence of critical points $(u^\varepsilon)_\varepsilon$ of (4.1.1), i.e. solutions to

$$-\varepsilon^2 \Delta u^\varepsilon + V'(u^\varepsilon) + \phi_\varepsilon = \lambda_\varepsilon,$$

where $\phi_\varepsilon(x) = \left(G_0(x - \cdot) * (u^\varepsilon - \bar{u}^\varepsilon)\right)(x)$ and λ_ε is a Lagrange multiplier arising from (4.1.3), satisfying mild bounds on the energy, converge in an appropriate sense to the Gibbs-Thompson law:

$$\sigma H = \begin{cases} -\phi + \lambda & \text{for} \quad x \in \partial^*\{u = +1\} \\ 0 & \text{for} \quad x \in \partial\{u = +1\} \backslash \partial^*\{u = +1\}. \end{cases} \tag{4.1.7}$$

Here H is the mean curvature of $\{u = +1\}$ where $u \in BV(\Omega; \{-1, +1\})$ and ϕ are both appropriately rescaled limits of u^ε and ϕ_ε respectively, σ is an integer which arises from the 'folding' of the interfaces, $\partial^*\{u = +1\}$ denotes the reduced boundary of $\{u = +1\}$ (see Section 4.3) and λ is the limiting Lagrange multiplier constant. This establishes the connection between critical points of the diffuse interface energy (4.1.1) and its sharp interface analogue (replacing the first Cahn-Hilliard term with perimeter). Our goal differs from that of [96], as we wish to establish the distribution of the small droplets in the regime where the volume of the minority phase vanishes, and the number of droplets is not constrained to be finite a priori for (4.1.1) and (4.1.4).

To understand our goal more precisely, we recall some of the main results of [54] for almost minimizers of (4.1.4). We begin by setting

$$\bar{u}^\varepsilon = -1 + \delta(\varepsilon),$$

in (4.1.4) and show for almost minimizers of (4.1.4), when $\delta(\varepsilon) = \varepsilon^{2/3}|\ln\varepsilon|^{1/3}\bar{\delta}$ with $\bar{\delta} > 0$, that the number of droplets of $\{u^\varepsilon = +1\}$ is $O(|\ln\varepsilon|)$ as $\varepsilon \to 0$ and, moreover, that

$$\omega_\varepsilon := \bar{\delta}\delta(\varepsilon)^{-1}(1 + u^\varepsilon) \rightharpoonup \bar{\omega} \text{ in } C(\mathbb{T}^2)^*, \tag{4.1.8}$$

where ω_ε is the "normalized droplet density" of the phase $\{u^\varepsilon = +1\}$ and where $\bar{\omega}$ is the unique constant density minimizer to

$$E^0[\omega] = \frac{\bar{\delta}^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}^2} d\omega + 2\iint_{\mathbb{T}^2\times\mathbb{T}^2} G(x-y)d\omega(x)d\omega(y), \tag{4.1.9}$$

over all Radon measures $\omega \in H^{-1}(\mathbb{T}^2)$. Moreover, $\bar{\omega}$ is given explicitly by

$$\bar{\omega} = \max\left(\tfrac{1}{2}(\bar{\delta} - \bar{\delta}_c), 0\right) \qquad \text{with} \qquad E^0[\bar{\omega}] = \tfrac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c), \tag{4.1.10}$$

where $\bar{\delta}_c > 0$ is the critical volume fraction for the onset of non-trivial minimizers (ie. $\bar{\omega} \neq 0$). In addition, setting v_ε to be the solution to

$$-\Delta v_\varepsilon + \kappa^2 v_\varepsilon = \omega_\varepsilon,$$

we conclude that

$$\nabla v_\varepsilon \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{T}^2). \tag{4.1.11}$$

The convergence in equations (4.1.8) and (4.1.11) show that v_ε and ω_ε are asymptotically constant in an averaged sense as $\varepsilon \to 0$, which physically suggests the droplets are uniformly distributed throughout the domain. One way of phrasing the goal of this paper, is to ask the following question: Do the normalized droplet densities still converge weakly to a constant when we drop the assumption of minimality? Moreover, does this fact continue to hold in higher dimensions? We answer these questions under the single assumption that the perimeter of the set $\{u^\varepsilon = +1\}$ vanishes as $\varepsilon \to 0$. We make similar conclusions for the diffuse interface energy (4.1.1), but reserve this discussion for a separate section (Section 7).

Before we proceed, we give a precise definition of a stationary point of (4.1.4). In addition to the class $\mathcal{A}$ defined above, we occasionally consider stationary points in $\mathcal{A}$ with mass constraint m:

$$\mathcal{A}_m := \left\{ u \in \mathcal{A} : \int_{\mathbb{T}^n} u = m \right\}. \tag{4.1.12}$$

Definition 4.1.1. *A function $u \in \mathcal{A}$ is said to be a* stationary point of (4.1.4) in $\mathcal{A}$ *if for any C^1 vector field $X : \mathbb{T}^n \to \mathbb{R}^n$ we have, setting $\phi_t(x) = x + tX(x)$, that*

$$\frac{d}{dt}\Big|_{t=0} E^\varepsilon(u \circ \phi_t) = 0. \tag{4.1.13}$$

If (4.1.13) *holds only for all* ϕ_t *such that* $u \circ \phi_t \in \mathcal{A}_m$ *for all* $t \in (-\varepsilon, \varepsilon)$ *and some* $\varepsilon > 0$*, then we call* u *a* stationary point of (4.1.4) in $\mathcal{A}_m$.

We proceed by showing that, away from a very small set on which the droplets are concentrated, we obtain a limiting condition on the measure ω_ε which takes the form

$$\omega_\varepsilon \nabla v^\varepsilon \to \omega \nabla v = 0, \tag{4.1.14}$$

in a suitably weak sense. The convergence above clearly does not follow from the weak convergence of ω_ε (cf. equation (4.1.8)) and the weak convergence of the potential (cf. equation (4.1.11)). This is similar to the problem which arises when studying weak limits of solutions to the Euler equations in vorticity form as in [10, 18, 33, 36, 37, 120] in dimension $n = 2$. It was originally Delort [33] who first recognized the phenomenon of 'vorticity concentration cancellation', which allows one to nonetheless pass to the limit in (4.1.14) in a distributional sense when ω_ε has a distinguished sign. Similar analysis was done by DiPerna and Majda [10, 36, 37] which which allows for ω to have mixed signs under additional assumptions. There are, of course, natural regularity issues with the above equation, and we will see in Theorem 14 that the regularity we assume on ω allows us to obtain more precise information from (4.1.14). When ω is a smooth density for instance, it is clear that (4.1.14) implies that ω is constant on $\mathbb{T}^n$, so that the normalized droplet densities converge weakly to a constant. We obtain two characterizations of this condition, both of which imply that the droplets of the minority phase satisfy a kind of "force balance" condition. Moreover, unlike the analysis of the Euler equations, our approach applies to all dimensions $n \geq 2$.

We have the *additional difficulty*, however, that we have contributions from the local terms in (4.1.1) and (4.1.4) which measure the perimeter of the level sets of u when we take variations of the energy. Here we adopt, and generalize, the techniques in [102] and [99, Chapter 13] which were used to prove similar results in the context of Ginzburg-Landau. There it is shown that it suffices to establish (4.1.14) away from a very small set where the contributions of the surface terms are concentrated. Thus this framework can be seen as a generalization of the method of 'vorticity concentration cancellation' introduced by Delort [33] for measures with distinguished sign, which allows for additional contributions to (4.1.14) that are concentrated on small sets, and which also allows for the measures to take on mixed signs, making it somewhat more similar to the work of DiPerna and Majda [10, 36, 37].

In order to make sense of (4.1.4) and its first variation, we must use extensively the theory of sets of finite perimeter (see [41, 74] for nice expositions, or [107] for a more general treatment which includes varifolds, which may have higher co-dimension). Unlike the analysis of the corresponding Euler-Lagrange equation which corresponds to minimizers in [80], here we will assume no minimality, and thus cannot expect global smoothness of the boundary. While it is known that local minimizers have boundaries which are of class $C^{3,\alpha}$ for $\alpha > 0$ [80,109], to the best of the author's knowledge, the question of regularity of the reduced boundary of stationary points of (4.1.4) has not been addressed, although this is conjectured in [31], where it is shown that if $\partial\{u = +1\}$ is

C^2 and u is a stationary point of (4.1.4) in $\mathcal{A}_m$ with $\varepsilon = 1$, then (4.1.7) holds strongly on $\{u = +1\}$ with $\sigma = 1$ and $v(x) = (G(x - \cdot) * (u(\cdot) - \bar{u}))(x)$. We provide a simple proof that the reduced boundary of any stationary point of (4.1.4) is of class $C^{3,\alpha}$, utilizing Allard's regularity theorem [**?**], and present a rigorous derivation of the Euler-Lagrange equation satisfied by stationary points of (4.1.4). The additional regularity allows us to make stronger statements concerning the limiting behavior of critical points in dimension $n = 2$; in particular, we show that, in the case of a bounded number of droplets which have bounded isoperimetric deficit, the generalized mean curvature of each connected component of $\{u^\varepsilon = +1\}$ (appropriately renormalized) is asymptotically constant.

Our paper is organized as follows. In Section 4.2 we set up certain notation which will be used throughout the paper, and present our three main results in Sections 4.2.1, 4.2.3 and 4.2.4 respectively. In Section 4.3 we provide a brief introduction to the theory of sets of finite perimeter and weak mean curvature, along with Allard's regularity theorem. In Section 4.4 we prove the main result of Section 4.2.3, which states that stationary points of (4.1.4) satisfy the Euler-Lagrange equation strongly on the reduced boundary, which is of class $C^{3,\alpha}$ for some $\alpha > 0$. In Section 4.5 we prove the main result of Section 4.2.1 for stationary points of the sharp interface energy (4.1.4). We then address the case of the diffuse interface energy (cf. equation (4.1.1)) in Section 4.7, where we prove the main result of Section 4.2.4.

Notation: We will denote $\mathcal{D}'(\Omega)$ as the space of distributions on Ω and $H^k(\Omega)$ and $W^{k,p}(\Omega)$ will, as usual, denote the standard Sobolev spaces. We denote as $\mathcal{H}^k$ the standard k-dimensional Hausdorff measure. For a measurable set $E \subset \Omega$, $\mathrm{Per}(E \cap \Omega)$ will denote its relative perimeter, which is the $\mathcal{H}^{n-1}$ dimensional Hausdorff measure of its relative boundary $\partial(E \cap \Omega)$, and $|E|$ will denote its standard n-dimensional Lebesgue measure. We write as $\mathbb{T}^n = [0,1)^n$, the standard flat n-dimensional torus. With some abuse of notation, we will sometimes say $E \subset \mathcal{A}$ (or $\mathcal{A}_m$) when we mean χ_E, the indicator function of E, belongs to $\mathcal{A}$ (respectively $\mathcal{A}_m$). Finally we denote α_{n-1} as the volume of the unit ball in $\mathbb{R}^{n-1}$.

4.2 Problem formulation and main results

In this section, we first rewrite the energy (4.1.4) in a way which is more convenient for the subsequent presentation and analysis. We begin with the result of Ambrosio et al. [5] which allows us to decompose (up to $\mathcal{H}^{n-1}$ negligible sets) $\{u = +1\}$ into a countable collection of connected components $\{\Omega_i\}$ contained in a single cell of $\mathbb{T}^n$ when $\mathrm{Per}(\{u = +1\})$ is sufficiently small:

$$u(x) = -1 + 2\sum_i \chi_{\Omega_i}(x), \tag{4.2.1}$$

and we set

$$\bar{u} = -1 + \delta(\varepsilon), \tag{4.2.2}$$

where we assume $\delta(\varepsilon)$ is bounded as $\varepsilon \to 0$. We define the "normalized droplet density"

$$\omega_\varepsilon := \frac{\sum_i \chi_{\Omega_i}}{\sum_i |\Omega_i|},$$

so that ω_ε is a probability measure on $\mathbb{T}^n$ for all $\varepsilon > 0$. If we insert (4.2.1) and (4.2.2) into the sharp interface energy (4.1.4) we obtain

$$\begin{aligned} E^\varepsilon[u^\varepsilon] = \varepsilon \sum_i \mathrm{Per}(\Omega_i) - \frac{2\delta(\varepsilon)}{\kappa^2} \sum_i |\Omega_i| \\ + 2 \iint_{\mathbb{T}^n \times \mathbb{T}^n} G(x-y) \sum_i \chi_{\Omega_i}(x) \sum_i \chi_{\Omega_i}(y) dx\, dy + \frac{\delta(\varepsilon)^2}{2\kappa^2}, \quad (4.2.3) \end{aligned}$$

where we set

$$-\Delta v_\varepsilon + \kappa^2 v_\varepsilon = \frac{\sum_i \chi_{\Omega_i}}{\sum_i |\Omega_i|} =: \omega_\varepsilon. \qquad (4.2.4)$$

The rewriting of (4.1.4) expressed by (4.2.3) will turn out to be more convenient for our purposes, as it allows us to focus on a non-local energy which depends only on the renormalized droplet density ω_ε (and not $\bar{u}^\varepsilon$). Our goal is to derive a suitably weak form of (4.1.14). We proceed by computing the Euler-Lagrange equation of (4.1.4) and show that this is equivalent to a certain 2 tensor $\{S_{ij}\} = S^\varepsilon$ having zero divergence. The idea is then to pass to the limit in the condition

$$\mathrm{div}\ S^\varepsilon = 0$$

as $\varepsilon \to 0$, and obtain a weak form of (4.1.14) as the limiting condition. This may at first appear surprising, as there will be contributions (in the form of curvature) from the perimeter term in (4.2.3), and (4.1.14) seems to depend only on the non-local terms. As alluded to before, we show that the contributions from these local terms occur in a very small set so that we are still able to conclude (4.1.14) in an appropriately weak sense outside of this set, and this turns out to be enough to make our main conclusions. More precisely, we show that the set where the local terms are concentrated in the Euler-Lagrange equations have arbitrarily small 1-capacity.

We recall from Evans-Gariepy [41] the definition of p-capacity of a set $E \subset \mathbb{R}^n$:

$$\mathrm{Cap}_p(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla\varphi|^p ; \varphi \in L^{p^*}(\mathbb{R}^n), \nabla\varphi \in L^p(\mathbb{R}^n), E \subset \ \mathrm{int}(\varphi \geq 1) \right\},$$

where $\mathrm{int}(A)$ denotes the interior of A and $p^* = 2p/(2-p)$. We will show that up to a set of very small 1-capacity, the tensor S^ε is close to the tensor T^ε in $L^1(\mathbb{T}^n)$ defined by

$$T^\varepsilon_{ij} = -\partial_i v_\varepsilon \partial_j v_\varepsilon + \frac{1}{2}\delta_{ij}(|\nabla v_\varepsilon|^2 + \kappa^2 v_\varepsilon^2),$$

where the condition

$$\mathrm{div}\ T^\varepsilon = 0$$

implies that

$$\omega_\varepsilon \nabla v_\varepsilon = 0 \text{ in } L^1_{loc}. \tag{4.2.5}$$

Our goal is to pass to the limit in this condition and obtain the weak form of

(4.1.14):

$$\operatorname{div} T = 0, \tag{4.2.6}$$

up to a set of arbitrarily small 1-capacity, where T is the 2-tensor with components T_{ij} given by

$$T_{ij} = -\partial_i v \partial_j v + \frac{1}{2}\delta_{ij}(|\nabla v|^2 + \kappa^2 v^2), \tag{4.2.7}$$

and v is the distributional limit of v^ε (cf. equation (4.2.4)) obtained from the weak convergence of ω_ε to ω. The condition (4.2.5) is in fact obtained by taking variations of the non-local term in (4.2.3) of the form $v_t(x) = v(x + tX(x))$, often called "inner variations". More precisely, condition (4.2.5) arises from the vanishing of

$$\frac{d}{dt}\Big|_{t=0} \int_{\mathbb{T}^n} |\nabla v_t|^2 + \kappa^2 v_t^2 dx.$$

The vanishing of the divergence of this tensor (cf. equation (4.2.6) implies, in particular, that v is constant on the support of ω if $\omega \in L^p(\mathbb{T}^n)$ for large enough p and a 'vanishing gradient property' if $\omega = \sum_{i=1}^d b_i \delta_{a_i}$ (see Theorem 1), which formally states that the force on each particle is balanced by the others. We now make some of these notions precise in order to state our main result, and begin with the following definition, taken from [99].

Definition 4.2.1. (Divergence-free in finite part) *Assume X is a vector field in $\mathbb{T}^n$. We say X is divergence-free in finite part if there exists a family of sets $\{E_\delta\}_{\delta>0}$ such that*

1. *We have* $\lim_{\delta\to 0} \operatorname{Cap}_1(E_\delta) = 0$.

2. *For every* $\delta > 0$, $X \in L^1(\mathbb{T}^n \backslash E_\delta)$.

3. *For every* $\zeta \in C^\infty(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n \backslash F_\delta} X \cdot \nabla \zeta = 0,$$

where $F_\delta = \zeta^{-1}(\zeta(E_\delta))$.

If T *is a 2-tensor with coefficients* $\{T_{ij}\}_{1 \leq i,j \leq n}$ *we say* T *is divergence-free in finite part if the vectors* $T_i = (T_{i1}, T_{i2}, \cdots, T_{in})$ *are, for* $i = 1, 2, \cdots, n$.

To see that the above definition is consistent with the ordinary notion of divergence free, we borrow the following proposition from [99].

Proposition 4.2.2. *Assume that* X *is divergence free in finite part in* $\mathbb{T}^n$ *and that* $X \in L^1(\mathbb{T}^n \backslash E)$. *Then for every* $\zeta \in C_c^\infty(\mathbb{T}^n)$ *we have*

$$\int_{\mathbb{T}^n \backslash F} X \cdot \nabla \zeta = 0,$$

where $F = \zeta^{-1}(\zeta(E))$. *In particular, if* X *is in* $L^1(\mathbb{T}^n)$, *then* $F = \emptyset$ *in the above and therefore* div $X = 0$ *in* $\mathcal{D}'(\mathbb{T}^n)$.

4.2.1 Main result I: The sharp interface energy (4.1.4)

Our first main result concerning stationary points of (4.1.4) is the following.

Theorem 14. (Equidistribution of droplets) *Let* $u^\varepsilon \in \mathcal{A}$ *be a sequence* *of stationary points of* (4.1.4) *in* $\mathcal{A}$ *in the sense of Definition 1 and assume*

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{T}^n} |\nabla u^\varepsilon| = 0. \tag{4.2.8}$$

Then for any $p \in (1, n/(n-1))$, ω_ε *converges in* $W^{-1,p}$ *to a probability measure* ω *and* v_ε *converges in* $W^{1,p}$ *to* v, *where* v *and* ω *are related via*

$$-\Delta v + \kappa^2 v = \omega. \tag{4.2.9}$$

Moreover, the symmetric 2-tensor T_ω *with coefficients* T_{ij} *given by* (4.2.7) *is divergence free in finite part. In addition, we have the following characterizations of the divergence free condition on* T_ω.

0. *If* $\int d\omega_\varepsilon = 0$ *for all* $\varepsilon > 0$ *sufficiently small, then*

$$\omega \equiv 0. \tag{4.2.10}$$

1. *If* $\omega \in H^{-1}(\mathbb{T}^n)$ *then*

$$\mathrm{div}\ T = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n). \tag{4.2.11}$$

2. *If* $\omega \in L^p$ *for* $p > 1$ *when* $n = 2$ *and* $p \geq 2n/(n+1)$ *otherwise, then in fact*

$$\omega = \mathbf{1}\, dx,$$

   the uniform Lebesgue measure on $\mathbb{T}^n$.

3. *If* $\omega = \sum_{i=1}^d b_i \delta_{a_i}$ *then setting* $v(x) = -\frac{1}{\alpha_{n-1}} \Phi(|x - a_i|) + H_i(x)$ *where*

Φ is the fundamental solution to the Laplace equation in $\mathbb{R}^n$ and H_i is smooth in a neighborhood of a_i, we have

$$\nabla H_i(a_i) = 0, \tag{4.2.12}$$

for $i = 1, \cdots, d$.

Theorem 14 is analogous to the results obtained for Ginzburg-Landau [99, 102], with the droplets playing the role of the vortices in the magnetic Ginzburg-Landau model. The main difference in our case is that we are dealing with sharp interface version of (4.1.1) so that u^ε takes on only the values $+1$ and -1. We must therefore be careful concerning regularity issues on the boundary of the set $\{u^\varepsilon = +1\}$, and consequently use the theory of finite perimeter sets (Section 4.3). Our proof, however, is in some ways simpler as we will have no contributions from the local terms outside the support of ω_ε. This is no longer true for the analysis of (4.1.1) in Section 4.7, and some additional analysis is needed. In addition, the vortices in the Ginzburg-Landau model are quantized, and we do not a priori know the shape or volume of the droplets in this model. Theorem 16 in Section 4.2.3 provides some information about the shape of these droplets; in particular, they are asymptotically round as $\varepsilon \to 0$ when $n = 2$ under assumptions on the number of droplets and their isoperimetric deficit ratio.

4.2.2 Interpretation of Theorem 14

The hypothesis (4.2.8) is essential to our proofs, as it will be seen to imply that $\mathrm{Cap}_1(\{u^\varepsilon = +1\}) = o_\varepsilon(1)$ as $\varepsilon \to 0$. This allows us to show that $\operatorname{div} T^\varepsilon$ converges, in a distributional sense, outside of the set $\{u^\varepsilon = +1\}$ to $\operatorname{div} T_\omega$. The smallness of the set $\{u^\varepsilon = +1\}$ allows us to demonstrate that the limiting tensor T_ω is divergence free in finite parts.

The conditions of Cases 2 and 3 are simply consequences of the divergence free condition on T_ω (see Section 4.5). The condition (4.2.12) is called the "vanishing gradient property", first established in the context of Ginzburg-Landau in [11] where $\{(a_i, b_i)\}_i$ is a critical point of the "renormalized energy" associated to the problem. The condition (4.2.12) can be interpreted as saying the sum of the Coulombic forces from the neighboring droplets balance each other.

When ω is regular enough (Case 2), then in fact it is equal to the uniform Lebesgue measure on $\mathbb{T}^n$, meaning the droplets are uniformly distributed throughout the domain. When we only know that $\omega \in H^{-1}(\mathbb{T}^n)$ as in Case 1 above, the measure ω can be concentrated on lower dimensional hypersurfaces [8, 85, 102]. In [8] it is shown that the limiting vortices of solutions of the two-dimensional Ginzburg-Landau equations, which bears much resemblance with the droplets in our case, can concentrate on lines. In [102] an explicit solution to (4.2.9) when $\Omega = \mathbb{R}^2$ and ω is supported on $\partial B(0, R)$ for some $R > 0$ is constructed. Analysis concerning precisely when there exists solutions to (4.2.9) in a bounded domain Ω with ω concentrated on a smooth, closed $n-1$ dimensional hypersurface in Ω is studied in [85]. In all cases, the above analysis shows that we can have $\omega \in H^{-1}(\Omega)$, while it is not in general true that $\omega << dx$. Here we demonstrate a simple example on $\mathbb{T}^n$, although ω does not have a distinguished sign in this case.

Example 1. *Define the function*

$$v(x_1, \cdots, x_n) = \begin{cases} x_1 + 1 & \text{for} \quad -1 \leq x \leq -1/2 \\ -x_1 & \text{for} \quad -1/2 \leq x \leq 1/2 \\ x_1 - 1 & \text{for} \quad 1/2 \leq x \leq 1 \end{cases} . \tag{4.2.13}$$

Then v is a distributional solution to

$$-\Delta v(x_1, \cdots, x_n) = -\delta(x_1 - 1/2) + \delta(x_1 + 1/2) + 1 - 1 \text{ for } (x_1, \cdots, x_n) \in [-1, 1)^n,$$

where $\delta(x_1)$ is the Dirac measure at $x_1 = 0$. The divergence free condition div $T_\omega = 0$ *in this case is equivalent to requiring that*

$$\int_{-1}^{1} v_{x_1}^2(s)\phi'(s)ds = 0 \text{ for all } \phi \in C^1([-1, 1)) \text{ periodic },$$

which is clearly satisfied since $|v_{x_1}| = 1$ a.e. In addition, the distribution $-\Delta v$ on $[-1, 1)^n$ is in $H^{-1}([-1, 1)^2)$ since a direct computation yields

$$\int_{[-1,1)^n} |\nabla v|^2 dx_1 \cdots dx_n < +\infty.$$

In the following section we recall that we say $E \subset \mathcal{A}$ (respectively $E \subset \mathcal{A}_m$)

if the characteristic function of E, χ_E, belongs to $\mathcal{A}$ (respectively $\mathcal{A}_m$).

4.2.3 Main Result II: Regularity of stationary points of (4.1.4) and asymptotic roundness of droplets

For $\gamma \in \mathbb{R}$ we consider the more general functional $I_\gamma : \mathcal{A} \to \mathbb{R}$ given by

$$I_\gamma(E) := \mathrm{Per}(E \cap \Omega) + \gamma \int_E \int_E G(x,y)\, dy\, dx + \int_E f(x) dx, \tag{4.2.14}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $f \in C^2(\Omega)$, $\gamma \in \mathbb{R}$ is a constant parameter and $G \in L^1(\Omega \times \Omega)$ is a kernel of Ω such that, setting $v_E(x) := (G(y - \cdot) * \chi_E(y))(x)$ we have

$$\|v_E\|_{W^{2,p}(K)} \leq C(n, K, E), \tag{4.2.15}$$

for $p > n$ and each $E \subset \Omega$ and $K \subset\subset \Omega$. For instance, the Green's function of any uniformly elliptic, linear operator with Dirichlet, Neumann or periodic boundary conditions satisfies (4.2.15) (see [**?**] for instance).

The reduced boundary of a set E is said to be of class $C^{k,\alpha}$ if each point in $\partial^* E$ is locally contained in the graph of a function which is $C^{k,\alpha}$. Our main result for the regularity of the reduced boundary is the following.

Theorem 15. *Let E be a stationary point of the functional* (4.2.14) *in $\mathcal{A}$ or $\mathcal{A}_m$. Then the reduced boundary $\partial^* E$ belongs to the class $C^{3,1-n/p}$. In* *particular, the equation*

$$H(x) + 2\gamma v_E + f(x) = \lambda,$$

holds strongly on $\partial^* E$ where H is the mean curvature of $\partial^* E$, and λ is a Lagrange multiplier. When E is a stationary point in the class $\mathcal{A}$, then $\lambda = 0$.

Theorem 15 is, to the best of the author's knowledge, the first rigorous derivation of the Euler-Lagrange equation of (4.2.14) and first result concerning the regularity of the reduced boundary. The proof follows from an almost immediate application of Allard's regularity theorem and De Giorgi's structure theorem (Section 4.3). Theorem 15 applied to (4.2.3) with $\Omega = \mathbb{T}^2$, $E = \{u^\varepsilon = +1\}$, $f = -\frac{2\delta(\varepsilon)}{\kappa^2}$ and Green's function G of $\mathbb{T}^2$ yields the equation

$$\varepsilon H_\varepsilon - \frac{2\delta(\varepsilon)}{\kappa^2} + v_\varepsilon \sum_j |\Omega_j| = 0 \text{ on } \partial^*\{u^\varepsilon = +1\}. \tag{4.2.16}$$

We will use (4.2.16) to show that when the number of droplets is finite and they have bounded isoperimetric deficit, they become asymptotically round as $\varepsilon \to 0$ in $n = 2$.

We recall that the Green's function on $\mathbb{T}^2$ can be written as

$$G(x-y) = -\frac{1}{2\pi} \log|x-y| + S(x-y) \text{ for } x, y \in \mathbb{T}^2, \tag{4.2.17}$$

where S is a continuous function. If we consider a single round droplet so that

$u^\varepsilon = -1 + \chi_{B(x,r_\varepsilon)}$, then formally we expect from (4.2.16) and (4.2.17) that

$$H_\varepsilon \simeq \frac{\log r_\varepsilon}{\varepsilon} r_\varepsilon^2 + \varepsilon^{-1}\delta(\varepsilon). \tag{4.2.18}$$

When $\varepsilon^{-1}\delta(\varepsilon) = O(r_\varepsilon^2 \log r_\varepsilon)$, as is the case for minimizers [54, 55, 80], then we have

$$H_\varepsilon = O\left(\frac{\log r_\varepsilon}{\varepsilon} r_\varepsilon^2\right) \text{ as } \varepsilon \to 0. \tag{4.2.19}$$

Equation (4.2.19) provides us with a hint of what the correct scaling of H_ε should be as the droplets shrink to points.

We now make the assumption that $u^\varepsilon = -1 + \sum_{j=1}^{N(\varepsilon)} \chi_{\Omega_j}$ for $N(\varepsilon) = O(1)$ as $\varepsilon \to 0$ so that the number of droplets is constrained to be finite. In the case that u^ε is minimizing, it is shown in [54, 55, 80] that any two droplets stay sufficiently far apart. This is no longer true in our case, and we must account for the situation where multiple droplets converge to the same point in $\mathbb{T}^2$, while still finding an appropriate normalization of H_ε as the droplets shrink to points. Motivated from the above discussion, we define

$$\rho_\varepsilon := \frac{-\varepsilon}{\sum_{j=1}^{N(\varepsilon)} \log \operatorname{Per}(\Omega_{j_i}) \sum_{j=1}^{N(\varepsilon)} |\Omega_j|}, \tag{4.2.20}$$

to be the "normalized radius" and

$$\bar{\delta} := \liminf_{\varepsilon \to 0} \frac{-\delta(\varepsilon)}{\sum_{j=1}^{N(\varepsilon)} \log \operatorname{Per}(\Omega_{j_i}) \sum_{j=1}^{N(\varepsilon)} |\Omega_j|}, \tag{4.2.21}$$

to be the "normalized volume fraction". When we work in the scaling regime of minimizers as in [54,55,80] then it is shown that there exists a $\bar{\delta}_{cr} > 0$ such that whenever $\bar{\delta} > \bar{\delta}_{cr}$ we have $\mathrm{Per}(\Omega_j) = O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3})$, $|\Omega_j| = O(\varepsilon^{2/3}|\ln\varepsilon|^{-2/3}|)$ and thus, when $N_\varepsilon = O(1)$ as $\varepsilon \to 0$,

$$\rho_\varepsilon = O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) = O\left(r_\varepsilon\right) \text{ as } \varepsilon \to 0,$$

where $r_\varepsilon = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$ is the energetically preferred radius of a single droplet as shown in [54, 55, 80]. We have the following Theorem concerning the asymptotic roundness of droplets when $n = 2$ as $\varepsilon \to 0$.

Theorem 16. (Asymptotic roundness of droplets when $n = 2$) *Assume the hypotheses of Theorem 14 and, in addition, that $u^\varepsilon = -1 + 2\sum_{i=1}^{N(\varepsilon)} \chi_{\Omega_i}$ for $N(\varepsilon) = O(1)$ as $\varepsilon \to 0$ with bounded isoperimetric deficit:*

$$\limsup_{\varepsilon\to 0} \frac{\sum_{j=1}^{N(\varepsilon)} \mathrm{Per}(\Omega_j)^2}{\sum_{j=1}^{N(\varepsilon)} |\Omega_j|} < +\infty, \tag{4.2.22}$$

and $\bar{\delta} \in (0, +\infty)$. Then there exists a $\bar{\delta}_{cr}$ such that for $\bar{\delta} > \bar{\delta}_{cr}$ the following holds. Let Ω_{j_i} have center of mass converging (subsequentially) to a_i for $j_i = 1, \cdots, d_i$. Then there exists a constant $c_i > 0$ such that such that

$$\|\rho_\varepsilon H_\varepsilon - c_i\|_{L^\infty\left(\cup_{j_i=1}^{d_i} \partial\Omega_{j_i}\right)} \to 0 \text{ as } \varepsilon \to 0, \tag{4.2.23}$$

up to subsequences, where H_ε is the mean curvature of $\{u^\varepsilon = +1\}$ and ρ_ε is given by (4.2.20).

Remark 4.2.3. *The reason our result holds only in dimension $n = 2$ is due to the specific scaling of the logarithmic potential, as can be seen by* (4.2.17) *. Indeed, for very small droplets, the leading order contribution from the potential v_ε is independent of the shape of the droplet. The assumption* (4.2.22) *is required in order to ensure the next order term in the expansion of the potential v_ε is controlled. In the case of minimizers as in [54, 55, 80], bounds on the energy imply the condition* (4.2.22)*.*

4.2.4 Main result III: The diffuse interface energy equation (4.1.1)

For the diffuse interface energy (4.1.1), the analysis is very similar to that of the sharp interface energy (4.1.4), however we must use the unscreened kernel for the Laplace operator and thus define

$$\tilde{T}_{ij} = -\partial_i v_\varepsilon \partial_j v_\varepsilon + \frac{1}{2}\delta_{ij}|\nabla v_\varepsilon|^2,$$

where

$$v_\varepsilon(x) = \int_{\mathbb{T}^n} G(x-y)\frac{1+u^\varepsilon(y)}{\delta(\varepsilon)}dy,$$

and we make the particular choice $V(u) = \frac{1}{4}(1-u^2)^2$. We must now work in the class $\mathcal{A}_{\bar{u}}$ given by

$$\mathcal{A}_{\bar{u}} := \left\{ u \in H^1(\mathbb{T}^n) : \int_{\mathbb{T}^n} u = \bar{u} \right\},$$

due to (4.1.3). For the energy (4.1.1), we define a critical point as follows.

Definition 4.2.4. *A function $u \in \mathcal{A}_{\bar{u}}$ is said to be a critical point of* (4.1.1) *if for any $v \in H^1(\mathbb{T}^n)$ satisfying $\int_{\mathbb{T}^n} v = 0$ we have*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}^\varepsilon(u+tv) = 0.$$

A simple calculation along with standard elliptic theory reveals that u^ε is $C^{3,\alpha}$ and solves the elliptic equation

$$-\varepsilon^2 \Delta u^\varepsilon + u^\varepsilon(1-(u^\varepsilon)^2) + \delta(\varepsilon) v_\varepsilon = \lambda_\varepsilon \text{ in } \mathbb{T}^n, \tag{4.2.24}$$

where λ_ε is the Lagrange multiplier corresponding to the volume constraint when taking variations in Definition 4.2.4.

We show that if $u^\varepsilon \in \mathcal{A}_{\bar{u}^\varepsilon}$, with $\bar{u}^\varepsilon = -1 + \delta(\varepsilon)$, is a sequence of critical points of $\mathcal{E}^\varepsilon$ with the perimeter of the minority phase vanishing, then $\tilde{T}^\varepsilon$ converges up to a small set to the tensor $\tilde{T}_\omega$ with coefficients defined by

$$\tilde{T}_{ij} = -\partial_i v \partial_j v + \frac{1}{2}\delta_{ij}|\nabla v|^2, \tag{4.2.25}$$

where now

$$-\Delta v = \omega - 1 \text{ on } \mathbb{T}^n,$$

and ω is a probability measure on $\mathbb{T}^n$. More precisely, we prove the following.

Theorem 17. (Diffuse interface energy) *Let $u^\varepsilon \in \mathcal{A}_{\bar{u}^\varepsilon}$ be a sequence of critical points of* (4.1.1) *in the sense of Definition 4.2.4 which satisfy*

$\limsup_\varepsilon |\lambda_\varepsilon| < +\infty$ *and*

$$\limsup_{\varepsilon\to 0} \operatorname{Per}\left(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\}\right) = 0 \text{ for } \alpha > 0, \tag{4.2.26}$$

with

$$\bar{u}^\varepsilon = -1 + \delta(\varepsilon) \text{ and } \delta(\varepsilon) = o_\varepsilon(1) \text{ as } \varepsilon \to 0. \tag{4.2.27}$$

Then for any $p \in (1, n/(n-1))$, $\omega_\varepsilon := \frac{1+u^\varepsilon}{\delta(\varepsilon)}$ *converges in* $W^{-1,p}$ *to a probability measure* ω *and* v_ε *converges in* $W^{1,p}$ *to* v *where*

$$-\Delta v = \omega - 1 \text{ on } \mathbb{T}^n.$$

Moreover, the symmetric 2-tensor T_ω *with coefficients* T_{ij} *given by* (4.2.25) *is divergence free in finite part. In particular, cases 0., 1., 2. and 3. of Theorem 14 continue to hold for* ω.

Remark 4.2.5. *The specific choice of* $\delta(\varepsilon)^{1+\alpha}$ *in* (4.2.26) *is a technical limitation which is required in the proofs.*

4.3 Mathematical preliminaries: Sets of finite perimeter

Here we introduce the basic notions of sets of finite perimeter. A detailed exposition on these topics can be found in [74]. For a more general treatment of varifolds, we refer the reader to [107]. Let $E \in \mathbb{R}^n$ be a Lebesgue measurable set. We say that E has finite perimeter if

$$\sup_{\substack{\varphi \in C_c^1(\mathbb{R}^n) \\ \|\varphi\|_{L^\infty} \leq 1}} \int_E \operatorname{div} \varphi < +\infty. \tag{4.3.1}$$

By the Riesz-Representation theorem, the above implies the existence of a vector valued Radon measure μ_E such that generalized Gauss-Green formula holds true

$$\int_E \nabla \varphi = \int_{\mathbb{R}^n} \varphi d\mu_E \text{ for all } \varphi \in C_c^1(\mathbb{R}^n).$$

The measure μ_E is referred to as the Gauss-Green measure of E and the total perimeter of the set E is defined as

$$\operatorname{Per}(E) = |\mu_E|(\mathbb{R}^n).$$

In the case that E has a C^1 boundary, then we have

$$\begin{aligned} \mu_E &= \nu_E \mathcal{H}^{n-1} \llcorner \partial E \\ \operatorname{Per}(E) &= \mathcal{H}^{n-1}(\partial E), \end{aligned}$$

and, in particular, we have

$$\nu_E(x) = \lim_{r \to 0^+} \fint_{B(x,r) \cap \partial E} \nu_E d\mathcal{H}^{n-1} = \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r)}.$$

For a generic set E of finite perimeter, we therefore define the *reduced boundary*, denoted $\partial^* E$, as those $x \in \partial E$ such that the above limit exists and belongs to S^{n-1}. The Borel vector field $\nu_E : \partial^* E \to S^{n-1}$ is called the *measure theoretic unit normal of* E. When ∂E is C^1, then the measure-theoretic outer unit normal agrees with the classical definition.

We will call a set E *countably n-1-rectifiable* if there exist countably many C^1-hypersurfaces U_h and compact sets $K_h \subset U_h$ and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$ such that

$$\partial^* E = F \cup \bigcup_{h \in \mathbb{N}} K_h.$$

We now recall a simplified statement of De Giorgi's structure theorem (see [74, 107]) which suffices for our purposes.

Theorem 18. (De Giorgi's structure theorem) *Suppose* E *has locally finite perimeter. Then* $\partial^* E$ *is countably n-1-rectifiable. In addition for all* $x \in \partial^* E$

$$\theta(x) := \lim_{r \to 0^+} \frac{|\mu_E|(B(x,r))}{\alpha_{n-1} r^{n-1}} = 1, \tag{4.3.2}$$

where α_{n-1} *is the volume of the unit ball in* $\mathbb{R}^{n-1}$*. Moreover* $\mathcal{H}^{n-1}(\partial E \backslash \partial^* E) = 0$.

The formal interpretation of Theorem 18 is that for each point on the reduced boundary of ∂E (which is a set of $\mathcal{H}^{n-1}$ full measure), the tangent plane exists and is a single hyperplane. One can consider the example of a figure 8 which is symmetric about the origin in $\mathbb{R}^2$. Here we have $\theta(0) = 2$ while $\theta(x) = 1$ for each other x on the boundary. We will crucially use Theorem 18 to prove Theorem 15. Allard's regularity theorem (Section 3.2) states that for any $x \in \partial^* E$, if we also have certain bounds on the mean curvature of the surface at the point x, then x is locally contained in the graph of a Lipschitz function. In this setting, one can thus reduce the question of regularity of the reduced boundary to one answered by standard estimates for second order elliptic equations (see [**?**, 50, 52, 115] for instance).

4.3.1 The first variation of perimeter

We wish to define a one-parameter family of diffeomorphisms with initial velocity $X \in C_c^1(\Omega; \mathbb{R}^n)$ which is a collection $\{\phi_t\}_{t\in(-\varepsilon,\varepsilon)}$for $\varepsilon > 0$ defined as

$$\phi_t(x) = x + tX(x), \ x \in \Omega. \tag{4.3.3}$$

We call $\{\phi_t\}_{-\varepsilon<t<\varepsilon}$ a *local variation in* Ω associated with X if in addition

$$\phi_t(\Omega) \subset\subset \Omega. \tag{4.3.4}$$

The *first variation of perimeter* is then easily computed as (see [41, 74, 107])

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Per}(\phi_t(E)) = \int \mathrm{div}_E X d\mathcal{H}^{n-1}, \quad X \in C_0^1(\Omega; \mathbb{R}^n), \tag{4.3.5}$$

where $\mathrm{div}_E X$ is the tangential divergence of the vector field X with respect to E:

$$\mathrm{div}_E X = \mathrm{div} X - \nu_E(x) \cdot \nabla X(x) \nu_E(x).$$

Observe that the first variation is a linear functional on $C_0^1(\Omega; \mathbb{R}^n)$. In the special case that it has a continuous extension to $C_0^0(\Omega; \mathbb{R}^n)$ it can be represented by a vector valued Radon measure, which has a singular part with respect to μ_E and a non-singular part, using the Radon-Nikodym theorem.

We thus have

$$\int \mathrm{div}_E X d\mathcal{H}^{n-1} = -\int X \cdot \vec{H} d\mathcal{H}^{n-1} - \int X \cdot \nu_E d\sigma_E, \tag{4.3.6}$$

where $|\vec{H}| \in L^p_{loc}(\partial^* E)$ and σ_E denotes the singular part of the measure. We call $\vec{H}$ the *vector valued generalized mean curvature*. When we can write $\vec{H} = H\nu_E$, we call H the *generalized mean curvature*.

4.3.2 Allard's Regularity Theorem

Here we assume that E is a set of finite perimeter. If we set $\delta \in (0, 1/2)$ to be a constant determined below, we make the following hypotheses for $r > 0$ sufficiently small:

$$\begin{cases} \theta \geq 1 \ \mu_E - \ \text{a.e}, \quad 0 \in \mathrm{spt}\mu_E \\ \alpha^{-1}_{(n-1)} r^{-(n-1)} |\mu_E|(B_r(0)) \leq 1 + \delta, \quad \left(\int_{B_r(0)} |H|^p d|\mu_E| \right)^{1/p} r^{1-(n-1)/p} \leq \delta. \end{cases} \tag{4.3.7}$$

The conditions on the ratios of of $|\mu_E|(B_r(0))\alpha^{-1}_{n-1} r^{-(n-1)}$, as alluded to in the discussion following De Giorgi's theorem (cf. Theorem 18), exclude the possibility of self-intersections of multiple tangent planes to the surface ∂E at the point 0. If, in addition, we have a local estimate on the mean curvature near 0, then Allard's regularity theorem tells us that locally the point 0 is contained in the graph of some function φ. Once again the symmetric figure

8 in $\mathbb{R}^2$ is a simple example, where the center point can clearly not be written in terms of such a graph, and the first condition on the second line of (4.3.7) fails since $\theta(0) = 2$.

The following theorem has been adapted from [107], where the theorem is presented for general n-varifolds, which allow for higher co-dimensional surfaces. Below $B_r^{(n-1)}(0)$ (or $B_\rho^n(0)$) denote the balls centered at 0 in $\mathbb{R}^{n-1}$ (respectively $\mathbb{R}^n$) with radius ρ.

Theorem 19. (Allard's Regularity Theorem) *Assume that* (4.3.7) *is satisfied for $p > n-1$ and $r > 0$ sufficiently small. Then there exists $\delta = \delta(n,p)$, $\rho = \rho(n,p) \in (0,1)$ such that there is a linear isometry q of $\mathbb{R}^n$ and a $\varphi \in C^{1,1-(n-1)/p}(B_\rho^{(n-1)}(0);\mathbb{R})$ with $\varphi(0) = 0$, spt $\mu_E \cap B_\rho^n(0) = q(\text{graph } (\varphi)) \cap B_\rho^n(0)$, and*

$$r^{-1}\sup|\varphi| + \sup|D\varphi| + r^{1-(n-1)/p} \sup_{\substack{x,y\in B_\rho^{n-1}(0)\\ x\neq y}} |x-y|^{-(1-(n-1)/p)}|D\varphi(x) - D\varphi(y)| \leq c\delta^{1/4(n-1)}. \quad (4.3.8)$$

Remark 4.3.1. *Notice we only need to know that $\theta \geq 1$ μ_E-a.e, which is true on the reduced boundary by Theorem 18, in order to apply Theorem 19. While it generally holds that $\theta \geq 1$* everywhere *on ∂E (and not just $\partial^* E$) for stationary sets of the perimeter functional (see Theorem 17.6 in [74]), and we expect only minor changes to the proof are needed when the non-locality is added, this is not required for the regularity result we state.*

We provide some explanation of Theorem 19. The function φ in Theorem

19 is the local Lipschitz graph which contains the point $0 \in \partial E$. Thus up to rotations and translations (which are governed by the linear isometry q), the point 0 can be written in local coordinates in the ball $B_\rho^{n-1}(0)$ as the graph of a Lipschitz function $\varphi : B_\rho^{n-1}(0) \to \mathbb{R}$. Moreover explicit estimates are given via (4.3.8) which control the $C^{1,1-(n-1)/p}$ norm of the graph φ.

One key aspect of the above theorem is that no minimality assumptions are made on the perimeter of the set, and we may thus use it in the context of non-minimizing critical points of the non-local isoperimetric problem (4.1.4). Using the above we are ready to present the first variation of (4.1.4).

4.4 Proof of Theorem 15

We proceed to prove Theorem 15 as follows. We first compute the first variation of the non-local isoperimetric functional (4.2.14) in Proposition 4.4.1, and then derive the Euler-Lagrange equation for the vector valued mean curvature in Proposition 4.4.2. The proof of Proposition 4.4.2 follows from Proposition 4.4.1 after constructing a local variation in $\mathcal{A}_m$, which is the only technical aspect of the proof. One of the technical difficulties is going from the vector valued Euler-Lagrange equation (cf. Proposition 4.4.2) to the scalar valued Euler-Lagrange equation (cf. Theorem 15). The way we proceed is to derive estimates from the vector valued Euler-Lagrange equation which we use in conjunction with Allard's regularity theorem and De Giorgi's structure theorem to conclude each point of the reduced boundary is locally contained in the graph of a Lipschitz function. We then obtain additional regularity by using the well established regularity theory for non-linear second order elliptic equations [**?**, 50, 52].

4.4.1 The first variation of the non-local perimeter functional

Our first result is a simple computation after using (4.3.5).

Proposition 4.4.1. (First variation of non-local perimeter) *Let E have locally finite perimeter and $\{\phi_t\}_{-\varepsilon<t<\varepsilon}$ be a family of local variations in Ω. Then it holds:*

$$\frac{d}{dt} I_\gamma(\phi_t(E))\Big|_{t=0} = \int_{\partial E} \mathrm{div}\,_E X \, d\mathcal{H}^{n-1} + 2\gamma \int_{\partial E} v_E X \cdot \nu_E \, d\mathcal{H}^{n-1} + \int_{\partial E} f(x) X \cdot \nu_E d\mathcal{H}^{n-1}.$$

Proof. The first variation of perimeter is equation (4.3.5). It remains to compute the first variation of the non-local term; the computation of the first variation of the third term is similar. By the change of variables formula it holds that

$$\int_{\phi_t(E)} \int_{\phi_t(E)} G(x,y)\, dx\, dy = \int_E \int_E G(\phi_t(x), \phi_t(y))\, |\det D\phi_t(x)| |\det D\phi_t(y)|\, dx\, dy. \tag{4.4.1}$$

Hence, we compute using the symmetry of G and (4.4.1)

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \int_{\phi_t(E)} \int_{\phi_t(E)} G(x,y)\,dx\,dy \\
&\qquad = 2\int_E \int_E \nabla_x G(x,y) \cdot X(x) dx\,dy + 2\int_E \int_E G(x,y)\,\mathrm{div} X(x)\,dx\,dy \\
&\qquad = 2\int_E \int_E \mathrm{div}\ (G(\cdot,y)X)(x)\,dx\,dy
\end{aligned}
$$

Now applying the divergence theorem on E and Fubini's theorem, the above becomes

$$2\int_E \int_{\partial E} G(x,y) X \cdot \nu_E d\mathcal{H}^{n-1}(x)\,dy = 2\int_{\partial E} v_E X \cdot \nu_E\, d\mathcal{H}^{n-1}.$$

□

Before we proceed to prove the main result, we need the following basic lemma, which allows us to construct a local variation in $\mathcal{A}_m$.

Lemma 1. *There exists a vector field $Y \in C_c^1(\Omega;\mathbb{R}^n)$ such that $\int_E \mathrm{div}\, Y dx =$ 1.*

Proof. Assume by contradiction that for every vector field $X \in C_c^1(\Omega,\mathbb{R}^n)$: $\int_E \mathrm{div} X dx = 0$. Then by Du Bois-Reymond's lemma [104] we conclude that

$$\chi_E = 0 \quad \text{or} \quad \chi_E = 1 \quad \text{a.e. on } \Omega,$$

where we used that Ω is connected. Hence,

$$E = \Omega \quad \text{or} \quad E = \emptyset \quad \text{in the measure theoretic sense.}$$

This contradicts the assumption that $0 < |E| < |\Omega|$, proving the claim. $\square$

We now can derive the vector valued Euler-Lagrange equation from (4.4.1). It remains to construct the local variation in $\mathcal{A}_m$ which is the main aspect of the proof of Proposition 4.4.2 below.

Proposition 4.4.2. (Euler-Lagrange equation of non-local perimeter) *Let E be a stationary point of* (4.2.14) *in $\mathcal{A}_m$ or $\mathcal{A}$. Then there exists a real number λ such that the set E has vector valued generalized mean curvature*

$$\vec{H} := (\lambda - 2\gamma v_E - f(x))\nu_E. \tag{4.4.2}$$

That is, for every vector field $X \in C_c^1(\Omega; \mathbb{R}^n)$ the following variational equation is true:

$$\int_{\partial E} \operatorname{div}_E X d\mathcal{H}^{n-1} = -\int_{\partial E} (\lambda - 2\gamma v_E - f(x))\nu_E \cdot X d\mathcal{H}^{n-1}. \tag{4.4.3}$$

For stationary points in $\mathcal{A}$, we have $\lambda = 0$.

Proof. Step 1: Construction of the local variation.

The case of variations in $\mathcal{A}$ is an immediate consequence of Proposition 4.4.1. For the case of $\mathcal{A}_m$, we must construct a volume constrained local variation in Ω with initial velocity X, i.e. $\phi_t(E) \subset \mathcal{A}_m$ for all $t \in (-\varepsilon, \varepsilon)$ for some ε

sufficiently small. Let $Y \in C_c^1(\Omega, \mathbb{R}^n)$ be a vector field such that $\int_E \operatorname{div} Y dx = 1$. The existence of such a vector field is guaranteed by Lemma 1. Let $\{\phi_t\}$ be the flow of X and $\{\psi_s\}$ the flow of Y. For $(t, s) \in \mathbb{R}^2$ set

$$A(t, s) := \operatorname{Per}(\psi_s(\phi_t(E))) \cap \Omega)$$

and

$$\mathcal{V}(t, s) := |\psi_s(\phi_t(E))| - |E|.$$

Then $\mathcal{V} \in C^1(\mathbb{R}^2)$, $\mathcal{V}(0, 0) = 0$ and $\partial_s \mathcal{V}(0, 0) = \int_E \operatorname{div} Y dx = 1$. The implicit function theorem ensures the existence of an open interval I containing 0 and a function $\sigma \in C^1(I)$ such that

$$\mathcal{V}(t, \sigma(t)) = 0 \quad \text{for all } t \in I \text{ and } \quad \sigma'(0) = -\frac{\partial_t \mathcal{V}(0, 0)}{\partial_s \mathcal{V}(0, 0)}.$$

Hence,

$$t \mapsto \psi_{\sigma(t)} \circ \phi_t$$

is a 1-parameter family of C^1-diffeomorphisms of Ω and thus a volume constrained local variation in Ω with initial velocity X.

Step 2: Computing the first variation.

The fact that E is a stationary point in the class $\mathcal{A}_m$ then implies from Proposition 4.4.1

$$\frac{d}{dt}\Big|_{t=0} A(t,\sigma(t)) = \int_{\partial E} \mathrm{div}_E X = -2\gamma \int_{\partial E} v_E X\cdot\nu_E \, d\mathcal{H}^{n-1} - \int_{\partial E} f(x) X\cdot\nu_E \, d\mathcal{H}^{n-1},$$

which again yields

$$\begin{aligned}
-2\gamma \int_{\partial E} v_E X\cdot\nu_E \, d\mathcal{H}^{n-1} - \int_{\partial E} f(x) X\cdot\nu_E \, d\mathcal{H}^{n-1} &= \partial_t \mathcal{A}(0,0) + \sigma'(0)\partial_s\mathcal{A}(0,0) \\
&= \int_{\partial E} \mathrm{div}_E X \, d\mathcal{H}^{n-1} + \sigma'(0)\int_{\partial E} \mathrm{div}_E Y \, d\mathcal{H}^{n-1} \\
&= \int_{\partial E} \mathrm{div}_E X \, d\mathcal{H}^{n-1} - \frac{\int_E \mathrm{div}\ X dx}{\int_E \mathrm{div}\ Y dx}\int_{\partial E} \mathrm{div}_E Y \, d\mathcal{H}^{n-1} \\
&= \int_{\partial E} \mathrm{div}_E X \, d\mathcal{H}^{n-1} - \lambda \int_{\partial E} X\cdot\nu_E \, d\mathcal{H}^{n-1},
\end{aligned}$$

where $\lambda := \int_{\partial E} \mathrm{div}_E Y d\mathcal{H}^{n-1}$, and we've used the divergence theorem on the last line. Therefore, setting $\vec{H} := (\lambda - 2\gamma v_E - f(x))\nu_E$, we have

$$\int_{\partial E} \mathrm{div}\ {}_E X \, d\mathcal{H}^{n-1} = -\int_{\partial E} X\cdot\vec{H} \, d\mathcal{H}^{n-1}$$

for every vector field $X \in C^1_c(\Omega,\mathbb{R}^n)$. □

Remark 4.4.3. *We point out that we cannot directly apply elliptic theory to* (4.4.3) *as the equation involves the vector valued mean curvature. To prove Theorem 15 we therefore use Allard's regularity theorem and De Diorgi's structure theorem to conclude that the corresponding Euler-Lagrange equation for the* scalar *mean curvature holds, and then we are able to use elliptic estimates as in [50, 52].*

We may now use Allard's regularity theorem and De Giorgi's structure theorem to prove Theorem 15.

Proof of Theorem 15:

By De Giorgi's theorem (Theorem 18) we know that for each point $x \in \partial^* E$ we have $\theta(x) = 1$. Assume that x is the origin without loss of generality. By definition of θ we have for fixed $\delta \in (0, 1/2)$ that for r sufficiently small

$$\alpha_{n-1}^{-1} r^{-n-1} |\mu_E|(B_r(0)) \leq 1 + \delta,$$

and that $\theta \geq 1$ μ_E a.e. From (4.2.15) and Sobolev embeddings, we have

$$\|v_E\|_{C^{1-n/p}(K)} \leq C(E, K), \tag{4.4.4}$$

for $K \subset\subset \Omega$. Using Proposition 4.4.2, (4.4.4) and the fact that f is continuous we then conclude (recalling $\vec{H} = H\nu_E$)

$$\|H\|_{L^\infty(\partial E)} \leq C(E, \|f\|_{L^\infty}, \lambda, K). \tag{4.4.5}$$

Then from (4.4.5) we therefore have for $r > 0$ sufficiently small and fixed $\delta > 0$

$$\left(\int_{B_r(0)} |H|^p d|\mu_E| \right)^{1/p} r^{1-(n-1)/p} \leq C(E, \|f\|_{L^\infty}, \lambda) |\mu_E|^{1/p}(\Omega) r^{1-(n-1)/p} \leq \delta.$$

Thus the hypotheses of Theorem 19 (cf. equation (4.3.7)) are satisfied and we conclude that 0 is locally contained in a $C^{1,1-n/p}$ graph $\varphi : B_\rho^{n-1}(0) \to \mathbb{R}$.

From Proposition 4.4.2 we claim we may write in local coordinates $B' := B^{n-1}_{\rho/2}(0)$ that φ weakly solves the elliptic equation

$$\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) = \lambda - 2\gamma v_E - f(x) \text{ for } x \in B'. \tag{4.4.6}$$

Indeed, this can be seen by choosing $X = \eta e_n$ where η is $C^1_c(B')$ and e_n is the standard basis vector which is 0 except for the nth entry where it is 1. Then recalling that $\operatorname{div}_E X = \operatorname{div} X - \nu_E \cdot \nabla X \nu_E$, we have $\operatorname{div}_E X = -\nabla\eta \cdot \nu_E \nu_E^n$ where ν_E^n is the nth component of the normal vector. Since ∂E is locally a graph, we may re-scale so that $\nu_E^n = \frac{1}{\sqrt{1+|\nabla'\varphi|^2}}$ where ∇' is the gradient in $\mathbb{R}^{n-1}$. Then equation (4.4.3) becomes

$$\begin{aligned}\int_{B'} -\nabla\eta \cdot \nu_E \frac{1}{\sqrt{1+|\nabla'\varphi|^2}} d\mathcal{H}^{n-1}(x,\varphi(x)) &= \int_{B'} -\nabla\eta \cdot \nu_E dx' \\ &= \int_{B'} (\lambda - 2\gamma v_E - f(x))\eta dx', \quad (4.4.7)\end{aligned}$$

which is the weak form of (4.4.6). Then $\varphi \in C^{3,1-n/p}(B^{n-1}_{\rho/4}(0))$ from (4.4.6) and elliptic regularity for the minimal surface operator (see Theorem 3.5 in [52] or Theorem 7.2 in [115] for instance). Thus (4.4.6) holds strongly in $B^{n-1}_{\rho/4}(0)$. Since the point $x \in \partial^* E$ was arbitrary, we have

$$H(x) + 2\gamma v_E + f(x) = \lambda \text{ on } \partial^* E.$$

□

4.5 Proof of Theorem 14

As seen previously in Section 2 (cf. equation (4.2.5)), a direct computation yields

$$\text{div } T^\varepsilon = \nabla v_\varepsilon \omega_\varepsilon \text{ in } L^1_{loc}(\mathbb{T}^n), \tag{4.5.1}$$

where T^ε is the 2-tensor with coefficients T_{ij} given by

$$T_{ij} = -\partial_i v_\varepsilon \partial_j v_\varepsilon + \frac{1}{2}\left(|\nabla v_\varepsilon|^2 + \kappa^2 v_\varepsilon^2\right)\delta_{ij}. \tag{4.5.2}$$

As discussed in the beginning of Section 4.2, we proceed by showing that the Euler-Lagrange equation obtained in Theorem 15 is equivalent to the vanishing of a certain 2-tensor S_ε. The part of S^ε which does not include T^ε will be shown to be concentrated on $\{u^\varepsilon = +1\}$, which will be shown to have vanishing 1-capacity as $\varepsilon \to 0$, as a result of our assumption that $\text{Per}(\{u^\varepsilon = +1\})$ vanishes as $\varepsilon \to 0$. The first step is the following proposition, which has been adapted from [99] and generalized to dimensions $n \geq 2$. The purpose of it will become clear in the proof of Theorem 14, where we will cover the set $\{u^\varepsilon = +1\}$ by small balls and use the fact that the 1-capacity of a ball $B(x,r)$ is $\alpha_{n-1}r^{n-1}$ [41].

Proposition 4.5.1. *Assume K is a compact subset of $\mathbb{R}^n$. Then there exists a finite covering of K by closed balls $B_1, \cdots, B_k$ such that*

$$\sum_k r(B_k)^{n-1} \leq C\mathcal{H}^{n-1}(\partial K).$$

Proof. Since ∂K is compact it suffices to work with a finite covering, and then taking closures and using Lemma 4.1 of [99], we may assume the balls are closed and disjoint, by possibly increasing the constant C in the proposition. Indeed if B_1 and B_2 are two balls which intersect, then there exists a ball B containing $B_1 \cup B_2$ such that $r(B) \leq r(B_1) + r(B_2)$ and thus $r(B)^{n-1} \leq C(r(B_1)^{n-1} + r(B_2)^{n-1})$.

In particular $A = \mathbb{R}^n \backslash \bigcup_{i=1}^k B_i$ is connected. Now if $B_1, \cdots, B_k$ cover ∂K, we claim they cover K. The claim follows by noting that A, which is connected, intersects the compliment of K since K is bounded. Thus if A intersected K it would also intersect ∂K, which is impossible from the definition of A. Thus $K \subset \mathbb{R}^n \backslash A = \bigcup_{i=1}^k B_i$. The result then follows by the definition of $n-1$ dimensional Hausdorff measure. $\square$

We now finally define precisely what we mean by L^1 convergence 'up to a small set'. This definition is taken from [99].

Definition 4.5.2. *We say a sequence $\{X_k\}_k$ in $L^1(\Omega)$ converges in $L^1_\delta(\Omega)$ to X if $X_k \to X$ in $L^1_{loc}(\Omega)$ except on a set of arbitrarily small 1-capacity, or precisely if there exists a family of sets $\{E_\delta\}_{\delta>0}$ such that for any compact $K \subset \Omega$,*

$$\lim_{\delta \to 0} \mathrm{Cap}_1(K \cap E_\delta) = 0, \quad \forall \delta > 0 \lim_{k \to +\infty} \int_{K \backslash E_\delta} |X_k - X| = 0.$$

We define similarly the convergence in L^2_δ by replacing L^1 by L^2 in the above.

It is clear that ∇v^ε cannot converge to ∇v strongly in L^2 in general, even if we have a uniform bound in $H^1(\mathbb{T}^n)$. However the fundamental observation is that away from a set of very small 1-capacity, we do in fact have strong L^2 convergence as long as the measures converge weakly in $(C(\mathbb{T}^n))^*$. The following result is adapted from [99] to work in higher dimensions.

Proposition 4.5.3. *Assume $\{\alpha_k\}_k$ is a sequence of measures such that for some $p \in (1, n/(n-1))$*

$$\lim_{k\to+\infty} \|\alpha_k\|_{W^{-1,p}(\Omega)} \|\alpha_n\|_{C^0(\Omega)^*} = 0,$$

for $\Omega \subset \mathbb{R}^n$ bounded and open where $\|\alpha_k\|_{C^0(\Omega)^*}$ denotes the total variation of α_k, $\int_\Omega |\alpha_k|$. Then letting h_k be the solution of

$$-\Delta h_k + \kappa^2 h_k = \alpha_k \text{ in } \Omega,$$

it holds that h_k and ∇h_k converge to 0 in $L^2_\delta(\Omega)$.

Proof. We begin by noticing that $W^{1,q}$ embeds into C^0 for $q > n$, and thus the $(C^0)^*$ norm dominates the $W^{-1,p}$ norm for $p \in (1, n/(n-1))$. Thus the hypothesis implies that $\|\alpha_k\|_{W^{-1,p}}$ tends to zero as $k \to +\infty$. We let

$$\delta_k = \left(\frac{\|\alpha_k\|_{W^{-1,p}}}{\|\alpha_k\|_{C^0(\Omega)^*} + 1} \right)^{1/2}, \quad F_n = \{x \in \Omega | |h_k| \geq \delta_k\}. \tag{4.5.3}$$

Then we use the well known bound on p-capacity of F_n (see [41, Lemma 1])

$$\mathrm{Cap}_p(F_k) \leq C \frac{\|h_k\|^p_{W^{1,p}}}{\delta_k^p}. \tag{4.5.4}$$

Then by elliptic regularity we have $\|h_k\|_{W^{1,p}} \leq C\|\alpha_k\|_{W^{-1,p}}$ and so from

(4.5.3)–(4.5.4) we have

$$\mathrm{Cap}_p(F_k) \le C\|\alpha_k\|_{W^{-1,p}}^{p/2}(\|\alpha_k\|_{C^0(\Omega)^*}+1)^{p/2},$$

which therefore tends to 0 as $k \to +\infty$. This implies that $\mathrm{Cap}_1(F_k) \to 0$ as $n \to +\infty$. From a well known property of Sobolev functions, the truncated function $\bar{h}_k = \max(-\delta_k, \min(h_k, \delta_k))$ satisfies $\nabla \bar{h}_k = 0$ a.e in F_n, hence

$$\int_{\Omega\setminus F_k} |\nabla h_k|^2 = \int_\Omega \nabla h_k \cdot \nabla \bar{h}_k.$$

It follows that

$$\int_{\Omega\setminus F_k} |\nabla h_k|^2 + h_k^2 \le \int_\Omega \nabla h_k \cdot \nabla \bar{h}_k + h_k \bar{h}_k = \int_\Omega \bar{h}_k d\alpha_k,$$

where the last equality follows from $-\Delta h_k + h_k = \alpha_k$. The right hand side is bounded above by $\delta_k \|\alpha_k\|_{C^0(\Omega)^*}$, hence by $\left(|\alpha_k\|_{W^{-1,p}}\|\alpha_k\|_{C^0(\Omega)^*}\right)^{1/2}$ and therefore tends to zero as $k \to +\infty$. Thus

$$\lim_{k\to+\infty} \|h_k\|_{L^2(\Omega\setminus F_k)} = \lim_{k\to+\infty} \|\nabla h_k\|_{L^2(\Omega\setminus F_k)} = 0. \tag{4.5.5}$$

To conclude, since $\lim_{k\to+\infty} \mathrm{Cap}_1(F_k) = 0$ there is a subsequence still denoted by $\{k\}$ so that $\sum_k \mathrm{Cap}_1(F_k) < +\infty$. We define

$$E_\delta = \bigcup_{k>\frac{1}{\delta}} F_k.$$

Then $\mathrm{Cap}_1(E_\delta)$ tends to zero as $\delta \to 0$ since it is bounded above by the tail of a convergent series. Moreover, for any $\delta > 0$ we have $F_k \subset E_\delta$ when k is large enough and therefore (4.5.5) implies that $\lim_{k\to+\infty} \|h_k\|_{L^2(\Omega\setminus E_\delta)} = \lim_{k\to+\infty} \|\nabla h_k\|_{L^2(\Omega\setminus E_\delta)} = 0$. $\square$

We will see in the proof of Theorem 14 that Proposition 4.5.3 implies that T^ε converges to T in $L^1_\delta(\mathbb{T}^n)$. The proof of Theorem 14 then follows after applying the following proposition contained in [99].

Proposition 4.5.4. *Assume $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of divergence-free vector fields which converge to T in $L^1_\delta(\mathbb{T}^n)$. Then T is divergence-free in finite part.*

We are now ready to present the proof of Theorem 14. The characterizations of ω in items 0,1,2,3 will be contained in Propositions 4.5.5 and 4.5.6 below.

Proof of first part of Theorem 14: We begin by observing that if we define J^ε to be the 2-tensor with coefficients $J_{ij} = (\delta_{ij} - \nu_i\nu_j)|\mu_\varepsilon|$, where μ_ε is the Gauss-Green measure of $\{u^\varepsilon = +1\}$ as in Section 4.3, we have

$$\int_{\partial\{u^\varepsilon=+1\}} \mathrm{div}\ _E X d\mathcal{H}^{n-1} = \int_{\mathbb{T}^n} (\mathrm{div}\ _E X - \partial_i X^j \nu_i \nu_j)\, d|\mu_\varepsilon| = \int_{\mathbb{T}^n} J_{ij}\partial_i X^j. \tag{4.5.6}$$

By Theorem 15 applied to (4.2.3), (4.5.1) and (4.5.6), we claim the criticality condition for E^ε can be written as

$$\mathrm{div}\ S^\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n),$$

in where S_ε is the 2-tensor given by

$$S_{ij}^{\varepsilon} = T_{ij}^{\varepsilon} - \frac{\varepsilon}{b_\varepsilon^2}\left(\delta_{ij} - \nu_i\nu_j\right)|\mu_\varepsilon| + \frac{1}{b_\varepsilon^2}\delta_{ij}\frac{(u_\varepsilon+1)\delta(\varepsilon)}{\kappa^2} - \delta_{ij}v_\varepsilon\omega_\varepsilon, \tag{4.5.7}$$

where we've set $b_\varepsilon = \sum_j |\Omega_j|$. Indeed, applying Theorem 15 to (4.2.3) with $E = \{u^\varepsilon = +1\}$, $f = -\frac{2\delta(\varepsilon)}{\kappa^2}$, $\Omega = \mathbb{T}^n$ with Green's potential G of $\mathbb{T}^n$ we have

$$\frac{\varepsilon}{b_\varepsilon^2}H_\varepsilon\mu_\varepsilon - \frac{\delta(\varepsilon)}{b_\varepsilon^2\kappa^2}\mu_\varepsilon + \frac{1}{b_\varepsilon}v_\varepsilon\mu_\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n).$$

Using (4.5.1) and (4.5.6), a direct computation yields

$$\operatorname{div} S^\varepsilon = \nabla v_\varepsilon\omega_\varepsilon - \frac{\varepsilon}{b_\varepsilon^2}H_\varepsilon\mu_\varepsilon + \frac{\delta(\varepsilon)}{b_\varepsilon^2\kappa^2}\mu_\varepsilon - \nabla v_\varepsilon\omega_\varepsilon - \frac{1}{b_\varepsilon}v_\varepsilon\mu_\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n). \tag{4.5.8}$$

From Proposition 4.5.1, there exists a collection of balls $B_1, \cdots, B_k$ which cover $\{u^\varepsilon = +1\}$ with $\sum_{i=1}^k r(B_i)^{n-1} \le C\mathcal{H}^{n-1}(\{u^\varepsilon = 1\})$. Define Z_ε to be the union of these balls. Then we have

$$S_\varepsilon = T_\varepsilon \text{ in } Z_\varepsilon^c.$$

By subadditivity of the 1-capacity [41] and the fact that the 1-capacity of a ball $B(x,r)$ is $\alpha_{n-1}r^{n-1}$ [41] we have via the vanishing of $\operatorname{Per}(\{u^\varepsilon = +1\})$ (cf. equation (4.2.8)) that

$$\operatorname{Cap}_1(Z_\varepsilon) \to 0, \quad \int_{\mathbb{T}^n\setminus Z_\varepsilon} |S_\varepsilon - T_\varepsilon| = 0.$$

Now choose a decreasing subsequence $\{\varepsilon_k\}$ tending to zero such that

$$\sum_k \mathrm{Cap}_1(Z_{\varepsilon_k}) < +\infty$$

and let

$$E_\delta = \bigcup_{k>\frac{1}{\delta}} Z_{\varepsilon_k}.$$

Finally we define

$$F_\delta := E_\delta \cup \tilde{E}_\delta, \tag{4.5.9}$$

where, in view of the defintion of L^2_δ convergence (cf. Definition 4.5.2), $\tilde{E}_\delta$ are the sets given by Proposition 4.5.3. Then once again by subadditivity of capacity we have

$$\lim_{\delta\to 0} \mathrm{Cap}_1(F_\delta) = 0.$$

Since ω_ε is a family of probability measures on $\mathbb{T}^n$, we have $\omega_\varepsilon \to \omega$ weakly in $(C^0(\mathbb{T}^n))^*$ up to a subsequence, and thus $\omega_\varepsilon \to \omega$ strongly in $W^{-1,p}$ for $p \in (1, n/(n-1))$ via the compact embedding $(C^0(\mathbb{T}^n))^* \subset\subset W^{-1,p}$ which follows from the compact embedding $W^{1,q}(\mathbb{T}^n) \subset\subset C^0(\mathbb{T}^n)$ for $q > n$. From Proposition 4.5.3 we therefore conclude

$$\nabla v_\varepsilon \to \nabla v \text{ in } L^2_\delta(\mathbb{T}^n).$$

Thus, recalling $S_\varepsilon = T_\varepsilon$ in F^c_δ we have

$$S_\varepsilon - T_\omega \text{ converges to 0 in } L^1_\delta(\mathbb{T}^n),$$

where the sets F_δ in Definition 4.2.1 are given by (4.5.9). Thus T_ω is divergence free in finite part from Proposition 4.5.7. $\square$

It now remains to prove the characterizations of Theorem 14, ie. items 0, 1, 2 and 3, which we divide into Propositions 4.5.5 and 4.5.6 below.

Proposition 4.5.5. *Let $-\Delta v + v = \omega \in H^{-1}(\mathbb{T}^n)$ and that T_ω is divergence free in finite parts. Then it holds distributionally that*

$$\text{div } T_\omega = 0.$$

Moreover we have the following

- *If $n = 2$ then $v \in W^{1,\infty}$.*
- *If, in addition, $\omega \in L^p$ for $p \geq \frac{2n}{n+1}$ when $n > 2$ and $p \geq 1$ for $n = 2$, then*

$$\omega = \boldsymbol{1} dx.$$

Proof. When $n = 2$ for both cases, see [99]. The proofs are very similar to [99] but we generalize them for arbitrary dimension. First observe that div $T_\omega = 0$ is an immediate consequence of Proposition 4.2.2. When $n > 2$, if $\omega \in L^p$ for $p \geq \frac{2n}{n+1}$ then $\nabla v \in L^q$ for $q \leq \frac{p}{p-1}$ by standard elliptic theory. Let $\omega_n = \omega * \rho_n$ where $\{\rho_n\}_n$ is a regularizing kernel and define $v_k = v * \rho_k$ and let T_k be the tensor with coefficients $-\partial_i v_k \partial_j v_k + \frac{1}{2}(|\nabla v_k|^2 + v_k^2)\delta_{ij}$. Then ω_k tends to ω in L^p and since $\nabla v \in L^{p/(p-1)}$, ∇v_n tends to ∇v in $L^{p/(p-1)}$. By Hölder's inequality we obtain

$$\omega_k \nabla v_k \to \omega \nabla v, \quad T_n \to T_\omega \text{ in } L^1_{loc}(\mathbb{T}^n).$$

It follows that div $T_k \to$ div $T_\omega = 0$ and that $\omega_k \nabla v_k \to \omega \nabla v$ in $\mathcal{D}'(\mathbb{T}^n)$. Since div $T_k = \omega_k \nabla v_k$ we conclude $\omega \nabla v = \lim_k \text{div } T_k = 0$ in $L^1_{loc}(\mathbb{T}^n)$ and thus a.e. Then since $\Delta v = 0$ a.e on the set $F = \{\nabla v = 0\}$, we have $\omega = \kappa^2 v$ a.e on the set F, and $\omega = 0$ a.e on the complement of F from $\omega \nabla v = 0$. Thus we obtain

$$\omega = \kappa^2 v \mathbf{1}_{|\nabla v| = 0}.$$

Multiply by v and integrating by parts, using the periodic boundary conditions on $\mathbb{T}^n$ we obtain that $\nabla v = 0$ a.e and thus v, and therefore ω is constant. Since $\int \omega = 1$ it follows that $\omega = \mathbf{1} dx$. $\square$

We have the following interpretation of the divergence free condition when ω is a finite linear combination of Dirac masses.

Proposition 4.5.6. *Let $-\Delta v + \kappa^2 v = \omega = \sum_{i=1}^d b_i \delta_{a_i}$ and assume that T_ω is divergence free in finite parts. Then setting $v(x) = \Phi(|x - a_i|) + H_i(x)$ it holds that*

$$\nabla H_i(a_i) = 0.$$

Before we continue with the proof of Proposition 4.5.6, we need the following Proposition which follows almost immediately from Proposition 4.2.2. The proof is simple and contained in [99].

Proposition 4.5.7. *If X is divergence free in finite parts and is continuous in a neighborhood U of the boundary of a smooth, compact set K in Ω, then*

$$\int_{\partial K} X \cdot \nu_K dS = 0.$$

Proof of Proposition 4.5.6: We present the proof for $n = 3$; the general case is similar. Assume that ω is a single Dirac mass at the origin with mass 4π without loss of generality. Then in spherical coordinates we have

$$\nu = \frac{\partial}{\partial r} \quad \tau = \frac{1}{r}\frac{\partial}{\partial \theta} \quad \eta = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}. \tag{4.5.10}$$

We compute $T_\omega \cdot \nu$ in the (ν, τ, η) basis to find

$$T_\omega \cdot \nu = \frac{1}{2}((\partial_\tau v)^2 + (\partial_\eta v)^2 - (\partial_\nu v)^2 + v^2)\nu - (\partial_\nu v \partial_\tau v)\tau - (\partial_\nu v \partial_\eta v)\eta \tag{4.5.11}$$

Then we write $h = \Phi + H$ where Φ is the positive solution to $-\Delta\Phi = 4\pi\delta(x)$ and H is smooth in a neighborhood of 0. Then we have $\partial_\nu \Phi = -\frac{1}{r^2} + o_r(1)$ as $r \to 0$ and $\partial_\tau \Phi = \partial_\eta \Phi = 0$. Thus as $r \to 0$ we have

$$T_\omega \cdot \nu = \frac{1}{2}\left(-\frac{1}{r^4} + 2\frac{\partial_\nu H}{r^2}\right)\nu + \left(\frac{\partial_\tau H}{r^2}\right)\tau + \left(\frac{\partial_\eta H}{r^2}\right)\eta. \tag{4.5.12}$$

Now using the fact that the integral of $\vec{I}(r)$ of $T_\omega \cdot \nu$ over $\partial B(0, r)$ is zero by Proposition 4.5.7, we have as $r \to 0$ that

$$0 = \nabla H(0) \cdot \vec{I}(r) = 4\pi|\nabla H(0)|^2 + o_r(1).$$

This implies $\nabla H(0) = 0$. $\square$

4.6 Proof of Theorem 16

We are now ready to prove Theorem 16. The main idea of the proof is simple. We use Theorem 15 to write down the Euler-Lagrange equation satisfied on the reduced boundary of $\{u^\varepsilon = +1\}$. To leading order, the potential v_ε is constant on the boundary of an isolated droplet Ω_{j_i} whose center of mass converges to a_i (up to a subsequence), due to the logarithmic scaling of G on $\mathbb{T}^2$ (cf. equation (4.2.17)). The control of the isoperimetric deficit (4.2.22) controls the size of the error in making this approximation, and allows us to conclude the curvature is asymptotically constant on the reduced boundary of droplets converging to a_i.

Proof of Theorem 16: By assumption, we have

$$u^\varepsilon(x) = -1 + 2\sum_{J=1}^{N(\varepsilon)} \chi_{\Omega_j}, \tag{4.6.1}$$

where $N(\varepsilon) = O(1)$ as $\varepsilon \to 0$. We then apply Theorem 15 to (4.2.3) to conclude that

$$\frac{\varepsilon}{\sum_j |\Omega_j|} H_\varepsilon - \frac{2\delta(\varepsilon)}{\kappa^2 \sum_j |\Omega_j|} + v_\varepsilon = 0 \text{ on } \partial^*\{u^\varepsilon = +1\}, \tag{4.6.2}$$

holds for all $\varepsilon > 0$. Since $N(\varepsilon) = O(1)$ as $\varepsilon \to 0$ and $\mathrm{Per}(\Omega_i) \to 0$ for each i, we conclude from compactness of $\mathbb{T}^2$ that the center of mass of each Ω_i converges up to a subsequence to some a_i. Let J_i be the set of indices so that the center of mass of Ω_j converges to a_i. We now expand the potential near

a_i, first recalling that

$$v_\varepsilon(x) = \int_{\mathbb{T}^2} G(x-y)\frac{\sum_j \chi_{\Omega_j}(y)}{\sum_j |\Omega_j|}dy. \tag{4.6.3}$$

Then, using (4.2.17), we have for ε sufficiently small, in a neighborhood of a_i

$$v_\varepsilon(x) = \int_{\mathbb{T}^2} -\frac{1}{2\pi}\log|x-y|\frac{\sum_{j\in J_i} \chi_{\Omega_j}(y)}{\sum_j |\Omega_j|}dy + S_i^\varepsilon(x), \tag{4.6.4}$$

where S_i^ε is uniformly bounded in ε. Then letting $\bar{x}_j = \mathrm{Per}(\Omega_j)^{-1}x$, $\bar{y}_j = \mathrm{Per}(\Omega_j)^{-1}y$, v_ε in these variables becomes

$$\begin{aligned} v_\varepsilon(x) &= \int_{\mathbb{T}^2} -\frac{1}{2\pi}\log|x-y|\frac{\sum_{j\in J_i} \chi_{\Omega_j}(y)}{\sum_j |\Omega_j|}dy + S_i^\varepsilon(x) \\ &= \frac{\sum_{j\in J_i} -\frac{1}{2\pi}\log \mathrm{Per}(\Omega_j)|\Omega_j|}{\sum_j |\Omega_j|} + \frac{\sum_j \mathrm{Per}(\Omega_j)^2}{\sum_j |\Omega_j|}\int_{\bar{\Omega}_j} -\frac{1}{2\pi}\log|\bar{x}_j - \bar{y}_j|d\bar{y}_j \\ &+ S_i^\varepsilon(x). \end{aligned} \tag{4.6.5}$$

We then use the inequality

$$\mathrm{essdiam}(\bar{\Omega}_j) \le \frac{1}{2}\mathrm{Per}(\bar{\Omega}_j) = \frac{1}{2}, \tag{4.6.6}$$

which follows (for instance) from [5, Theorem 7 and Lemma 4] noting that in view of [5, Proposition 6(ii)] it suffices to consider only simple sets [5, Definition 3]. Thus we have from (4.6.6) and the defintion of $\bar{x}_j$, $\bar{y}_j$

$$\left|\int_{\bar{\Omega}_j} -\frac{1}{2\pi}\log|\bar{x}_j - \bar{y}_j|d\bar{y}_j\right| \le \frac{1}{2\pi}\int_{B(0,2)} |\log|x||dx \le C. \tag{4.6.7}$$

where $C > 0$ is independent of ε. Inserting (4.6.7) into (4.6.5) and using the bound on the isoperimetric defecit (4.2.22) we have using the fact that $\mathrm{Per}(\{u^\varepsilon = +1\}) \to 0$ as $\varepsilon \to 0$ (cf. equation (4.2.8)) that for any $k \in J_i$ and $x_i^\varepsilon \in \bigcup_{j \in J_i} \partial^* \Omega_{j_i}$

$$\frac{v_\varepsilon(x_i^\varepsilon)}{\sum_j \log \mathrm{Per}(\Omega_j)} = -\frac{1}{2\pi} \frac{\sum_{j \in J_i} \log \mathrm{Per}(\Omega_j)|\Omega_j|}{\sum_j \log \mathrm{Per}(\Omega_j) \sum_j |\Omega_j|} + o_\varepsilon(1).$$

Rewriting the Euler-Lagrange equation (4.6.2) we have

$$\left\| \frac{-\varepsilon H_\varepsilon}{\sum_j \log \mathrm{Per}(\Omega_j) \sum_j |\Omega_j|} + c_i^\varepsilon \right\|_{L^\infty(\bigcup_{j \in J_i} \partial^* \Omega_j)} \to 0 \text{ as } \varepsilon \to 0, \tag{4.6.8}$$

where

$$c_i^\varepsilon = -\frac{1}{2\pi} \frac{\sum_{j \in J_i} \log \mathrm{Per}(\Omega_j)|\Omega_j|}{\sum_j \log \mathrm{Per}(\Omega_j) \sum_j |\Omega_j|} + \frac{2}{\kappa^2} \frac{\delta(\varepsilon)}{\sum_j \log \mathrm{Per}(\Omega_j) \sum_j |\Omega_j|}. \tag{4.6.9}$$

Now choose a subsequence ε_k so that the $\liminf$ in the definition of $\bar{\delta}$ (cf. (4.2.21)) is achieved as $\varepsilon_k \to 0$. It is clear that the first term in the definition of c_i^ε is bounded uniformly and positive as $\varepsilon \to 0$, and therefore converges subsequentially to some $c_i^0 \geq 0$. Therefore we have (possibly taking a further subsequence) that

$$c_i^{\varepsilon_k} \to c_i^0 - \frac{2}{\kappa^2} \bar{\delta},$$

as $\varepsilon_k \to 0$. Choosing $\bar{\delta}_{cr} = \frac{\kappa^2 c_i^0}{2}$, we obtain the result. $\square$

4.7 The diffuse interface energy

In this section we study

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + V(u)\right) dx + \frac{1}{2}\int_\Omega\int_\Omega (u(x)-\bar{u})G_0(x,y)(u(y)-\bar{u})\,dx\,dy. \tag{4.7.1}$$

We make the particular choice of $V(u) = \frac{1}{4}(1-u^2)^2$, but our results will hold, with minor adjustments to the proofs, under general assumptions on V. Recalling the discussion in Section 4.2.4 we know that any stationary point u^ε of (4.7.1) in the sense of Definition 4.2.4 is a critical point, defined as a solution to

$$-\frac{\varepsilon^2}{\delta(\varepsilon)}\Delta u^\varepsilon - \frac{1}{\delta(\varepsilon)}u^\varepsilon(1-(u^\varepsilon)^2) + v_\varepsilon = \lambda_\varepsilon, \tag{4.7.2}$$

where λ_ε is the Lagrange multiplier arising from the volume constraint and

$$v_\varepsilon(x) = \int_{\mathbb{T}^n} G(x-y)\frac{1+u^\varepsilon(y)}{\delta(\varepsilon)}dy.$$

We recall our main assumption that

$$\limsup_{\varepsilon\to 0} \operatorname{Per}\left(\{u^\varepsilon \geq -1+\delta(\varepsilon)^{1+\alpha}\}\right) = 0 \text{ for some } \alpha > 0. \tag{4.7.3}$$

Our methods will be very similar to those of the sharp interface energy (4.1.4), and follow closely the methods of [99,102] for Ginzburg-Landau. In particular, we first show that (4.7.2) is equivalent to a certain 2-tensor $S_\varepsilon = \{S_{ij}\}$ having zero divergence (cf. Proposition 4.7.1 below). We then use (4.7.3) to cover the set where u^ε is close to $+1$ by balls whose boundaries have very small $\mathcal{H}^{n-1}$ measure (cf. Proposition 4.5.1). Finally we show that away from the set where u^ε is close to $+1$, S_ε is close in L^1 to the tensor $T_\varepsilon = \{T_{ij}\}$ defined by

$$T_{ij} = -\partial_i v_\varepsilon \partial v_\varepsilon + \frac{1}{2}\delta_{ij}|\nabla v_\varepsilon|^2. \tag{4.7.4}$$

We begin by observing that if u^ε solves (4.7.2) then it holds by direct computation that

$$\text{div } S^\varepsilon = 0, \tag{4.7.5}$$

where

$$\begin{aligned} S^\varepsilon_{ij} &= \partial_i v^\varepsilon \partial_j v^\varepsilon - \frac{\varepsilon^2}{\delta(\varepsilon)^2}\partial_i u^\varepsilon \partial_j u^\varepsilon \\ &+ \frac{\delta_{ij}}{2}\left(\frac{\varepsilon^2}{\delta(\varepsilon)^2}|\nabla u^\varepsilon|^2 + \frac{1}{4\delta(\varepsilon)^2}(1-|u^\varepsilon|^2)^2 - |\nabla v^\varepsilon|^2 - \frac{(u^\varepsilon+1)}{\delta(\varepsilon)}\lambda_\varepsilon\right) + \delta_{ij}\frac{v^\varepsilon(u^\varepsilon+1)}{\delta(\varepsilon)}. \end{aligned} \tag{4.7.6}$$

This is summarized in the following proposition.

Proposition 4.7.1. *Let u^ε be a solution to* (4.7.2). *Then*

$$\text{div } S^\varepsilon = 0 \ \textit{in} \ \mathcal{D}'(\mathbb{T}^n),$$

where S_ε is given by (4.7.6).

Proof. A direct computation using the fact that $u^\varepsilon \in C^2(K)$ for any $K \subset\subset \mathbb{T}^n$

yields

$$\operatorname{div} S^\varepsilon = \nabla u^\varepsilon \left(-\frac{\varepsilon^2}{\delta(\varepsilon)} \Delta u^\varepsilon + \frac{1}{\delta(\varepsilon)} u^\varepsilon (1 - (u^\varepsilon)^2) + v_\varepsilon - \lambda_\varepsilon \right) = 0.$$

□

Proposition 4.7.2. *Let* $\{u_\varepsilon\}_\varepsilon$ *be a sequence of solutions to* (4.7.2) *satisfying* (4.7.3) *and*

$$\limsup_{\varepsilon \to 0} |\lambda_\varepsilon| < +\infty. \tag{4.7.7}$$

For any $\varepsilon > 0$ *define the 2-tensors* S_ε *as above and* T_ε *by*

$$T^\varepsilon_{ij} = \partial_i v^\varepsilon \partial_j v^\varepsilon - \frac{\delta_{ij}}{2} |\nabla v^\varepsilon|^2. \tag{4.7.8}$$

Then $T_\varepsilon - S_\varepsilon$ *tends to* 0 *in* $L^1_\delta(\mathbb{T}^n)$.

Proof. We once again argue as in [99] for Ginzburg-Landau. From (4.7.3) and Proposition 4.5.1, the set of x in $\mathbb{T}^n$ such that $u(x) \geq -1 + \delta(\varepsilon)^{1+\alpha}$ can be covered by a collection of balls $B_1, \cdots, B_k$ such that

$$\sum_{i=1}^{k} r(B_i)^{n-1} \leq C\mathcal{H}^{n-1}(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\}).$$

We denote Z_ε as the union of these balls and and observe that

$$\lim_{\varepsilon \to 0} \operatorname{Cap}_1(Z_\varepsilon) = 0.$$

This follows from the fact that the 1-capacity of a ball $B(x, r)$ is $\alpha_{n-1} r^{n-1}$

and the capacity is subadditive so $\mathrm{Cap}_1(Z_\varepsilon) \leq C\mathcal{H}^{n-1}(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\})$, which tends to zero by assumption. The difference between S^ε and T^ε is

$$
\begin{aligned}
&S^\varepsilon - T^\varepsilon = S^\varepsilon_{ij} = -\frac{\varepsilon^2}{\delta(\varepsilon)^2}\partial_i u^\varepsilon \partial_j u^\varepsilon \\
&+ \frac{\delta_{ij}}{2}\left(\frac{\varepsilon^2}{\delta(\varepsilon)^2}|\nabla u^\varepsilon|^2 + \frac{1}{4\delta(\varepsilon)^2}(1-|u^\varepsilon|^2)^2 - \frac{1}{\delta(\varepsilon)}(u^\varepsilon+1)\lambda_\varepsilon\right) + \delta_{ij}\frac{v_\varepsilon(u^\varepsilon+1)}{\delta(\varepsilon)}.
\end{aligned}
\tag{4.7.9}
$$

Thus it is easily seen that

$$
|S^\varepsilon - T^\varepsilon| \leq C\left(\frac{\varepsilon^2}{\delta(\varepsilon)^2}|\nabla u^\varepsilon|^2 + \frac{1}{2\delta(\varepsilon)^2}(1-|u^\varepsilon|^2)^2 + \frac{2}{\delta(\varepsilon)}(|v_\varepsilon| + |\lambda_\varepsilon|)|u^\varepsilon+1|\right).
\tag{4.7.10}
$$

Now define the function $\chi : [0,1] \to [0,1]$ to be the affine interpolation between the values $\chi(-1) = 1$, $\chi(-1+\delta(\varepsilon)^{1+\alpha}) = 1$ and $\chi(-1/2) = 1/2$ and $\chi(0) = 0$ and $\chi(1) = -1$. Multiply (4.7.2) by $\chi(u^\varepsilon) + u^\varepsilon$ and integrating by parts we have

$$
\begin{aligned}
\frac{1}{\delta(\varepsilon)^2}\int_{\mathbb{T}^n} \varepsilon^2|\nabla u^\varepsilon|^2(\chi'(u^\varepsilon)+1) - u^\varepsilon(\chi(u^\varepsilon)+u^\varepsilon)(1-(u^\varepsilon)^2) \\
= -\frac{1}{\delta(\varepsilon)}\int_{\mathbb{T}^n}(\chi(u^\varepsilon)+u^\varepsilon)(v^\varepsilon - \lambda_\varepsilon)dx.
\end{aligned}
\tag{4.7.11}
$$

The set $\{\chi(u^\varepsilon) = 1\}$ contains the set $u^\varepsilon(x) \leq -1 + \delta(\varepsilon)^{1+\alpha}$ and therefore Z^c_ε. When $|u^\varepsilon| \geq 1/2$, which is true on $\{\chi(u^\varepsilon) = 1\}$, the left side of (4.7.11) can be bounded from below by

$$
\frac{1}{2\delta(\varepsilon)^2}\int_{Z^c_\varepsilon} \varepsilon^2|\nabla u^\varepsilon|^2 + \frac{1}{4}(1-(u^\varepsilon)^2)^2.
$$

Indeed on $\{\chi(u^\varepsilon) = 1\}$ we have

$$|u^\varepsilon|(1 - |u^\varepsilon|) \geq (1 - |u^\varepsilon|^2),$$

since $|u^\varepsilon| \geq (1 + |u^\varepsilon|)$ when $u^\varepsilon \in (-1, -1/2)$. Since $(u^\varepsilon + 1)/\delta(\varepsilon)$ is bounded in $(C^0(\mathbb{T}^n))^*$ and therefore in $W^{-1,p}$ for $p \in (1, n/(n-1))$ by standard embeddings, we conclude that v_ε is bounded uniformly in L^1. Then by the definition of χ and the fact that $u^\varepsilon \in (-1, -1 + \delta(\varepsilon)^{1+\alpha})$ where $\chi(u^\varepsilon) = 1$ we have

$$\frac{1}{\delta(\varepsilon)} \left| \int_{\mathbb{T}^n} (\chi(u^\varepsilon) + u^\varepsilon) v_\varepsilon \right| = \frac{1}{\delta(\varepsilon)} \left| \int_{Z_\varepsilon^c} (1 + u^\varepsilon) v_\varepsilon \right| \leq C\delta(\varepsilon)^\alpha \|v_\varepsilon\|_{L^1}. \quad (4.7.12)$$

Combining the above we conclude

$$\frac{1}{\delta(\varepsilon)^2} \int_{Z_\varepsilon^c} \varepsilon^2 |\nabla u^\varepsilon|^2 + (1 - (u^\varepsilon)^2)^2 = o_\varepsilon(1) \text{ as } \varepsilon \to 0. \quad (4.7.13)$$

Focusing on the remaining terms in (4.7.10), it remains to show that

$$\frac{1}{\delta(\varepsilon)} \int_{Z_\varepsilon^c} |1 + u^\varepsilon|(|v_\varepsilon| + |\lambda_\varepsilon|) = o_\varepsilon(1) \text{ as } \varepsilon \to 0. \quad (4.7.14)$$

This however follows from Hölder's inequality and the definition of Z_ε^c:

$$\frac{1}{\delta(\varepsilon)} \int_{Z_\varepsilon^c} |u^\varepsilon + 1|((|v_\varepsilon + |\lambda_\varepsilon|) \leq C(\|v_\varepsilon\|_{L^1} + |\lambda_\varepsilon|)\delta(\varepsilon)^\alpha \leq C\delta(\varepsilon)^\alpha, \quad (4.7.15)$$

where we've used the fact that $\limsup_\varepsilon |\lambda_\varepsilon| < +\infty$. Finally combining (4.7.15) and (4.7.13) and using (4.7.10) we conclude that

$$\int_{\mathbb{T}^n \setminus Z_\varepsilon} |T_\varepsilon - S_\varepsilon| \to 0 \text{ as } \varepsilon \to 0,$$

the desired result. $\square$

We now complete the proof of Theorem 17.

Proof. Choose a decreasing subsequence $\{\varepsilon_k\}$ tending to zero such that

$$\sum_k \mathrm{Cap}_1(Z_{\varepsilon_k}) < +\infty$$

and let

$$E_\delta = \bigcup_{k > \frac{1}{\delta}} Z_{\varepsilon_k}.$$

Since $\omega_\varepsilon := (1 + u^\varepsilon)/\delta(\varepsilon)$ is a family of probability measures on $\mathbb{T}^n$, we have $\omega_\varepsilon \to \omega$ weakly in $(C^0(\mathbb{T}^n))^*$ up to a subsequence, and thus $\omega_\varepsilon \to \omega$ strongly in $W^{-1,p}$ for $p \in (1, n/(n-1))$ via the compact embedding $(C^0(\mathbb{T}^n))^* \subset\subset W^{-1,p}$ which follows from the compact embedding $W^{1,q}(\mathbb{T}^n) \subset\subset C^0(\mathbb{T}^n)$ for $q > n/(n-1)$. Now define

$$F_\delta := E_\delta \cup \tilde{E}_\delta, \tag{4.7.16}$$

where $\tilde{E}_\delta$ are the sets given by Proposition 4.5.3 with $\kappa = 0$. Then by subadditivity of capacity [41] we have

$$\lim_{\delta \to 0} \mathrm{Cap}_1(F_\delta) = 0.$$

From Proposition 4.5.3 we therefore conclude

$$\nabla v_\varepsilon \to \nabla v \text{ in } L^2_\delta(\mathbb{T}^n).$$

Thus, combining the above, we have

$$S_\varepsilon - T_\omega \text{ converges to } 0 \text{ in } L^1_\delta(\mathbb{T}^n),$$

where the sets F_δ in Definition 4.2.1 are given by (4.7.16). Thus T_ω is divergence free in finite part from Proposition 4.5.7. $\square$

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