

Investigating and understanding Fraunhofer diffraction patterns using Fourier analysis with variations of the 4f Optical System

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Through the study of the Fourier Transforms of blah blah blah. The volume flow rate of a fluid can be expressed functionally as derived by Poiseuille and Hagen. This relationship can be rearranged so that the viscosity of water can be experimentally determined. Using a capillary tube and altering the height of water in a tank, the volume and mass of water which has flowed out in a time period can be measured. Using this data, it is then possible to calculate a value for the viscosity of water. The average value was determined to be 1.00 ± 0.03 mPa s, which does not agree with the literature value, 1.679 mPa s.

I. INTRODUCTION

In the field of optics, a Fraunhofer diffraction pattern can be produced when a coherent beam of radiation falls onto a partially opaque object [1]. The pattern can then be viewed at a “far-field” distance, this length being larger than the initial size of the object, allowing for the spreading of light due to diffraction to dominate in the observation plane [2].

The usage of diffraction patterns includes the studying of crystalline structures. Through X-ray and electron diffraction it is possible to obtain accurate information about the identity of phases present, atomic ordering, and for recognising different metallurgical constituents within a specimen. This allows for potential applications within the pharmaceutical industry for example, where early analysis of different mixtures and compounds can be cost and time saving. [3, 4]

Another application could potentially involve using the diffraction pattern within an illumination system so that x-ray phase contrast microscopy and interferometry can be performed [5]. This could be used to enhance the visibility of fine scale structures, this being especially useful in biology as quantitative information can be obtained about a sample from just phase-contrast images [6].

With these varying applications we find that in order to fully understand diffraction patterns and how they form, we must look at the mathematics behind them.

The main mathematical tool that we must use in building our understanding is Fourier analysis, more specifically, Fourier transforms. A Fourier transform is a way to represent a function in terms of a superposition of sinusoidal functions, with the explicit conditions of the function being defined over an infinite interval and having no particular periodicity [1].

The following theory/mathematics has been adapted from Optics f2f [2].

The Fourier transform $F(u)$ of a single function $f(x)$ can be stated as,

$$F(u) = \mathcal{F}[f(x)](u) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi ux}dx, \quad (1)$$

where (u) is the dependent Fourier variable. This expression tells us the spectrum of frequencies required to form the function $f(x)$.

We can also operate on the initial function then Fourier transform it. In Table I we find the group of functions known as the Fourier transform properties.

The convolution property can be better understood as follows - if we convolve two functions of x , $g(x)$ and $h(x)$,

Property	\mathbf{f}	\mathbf{F}
Linearity	$g(x) + h(x)$	$G(u) + H(u)$
Translation	$f(x - a)$	$F(u)e^{-i2\pi ua}$
Scaling	$f(x/a)$	$aF(ua)$
Convolution	$(g * h)(x)$	GH
Inverse Convolution	gh	$(G * H)(u)$

TABLE I: The Fourier transform properties - different functions \mathbf{f} and their output \mathbf{F} when they have been Fourier transformed using equation 1.

then that is defined as,

$$(g * h)(x) = \int_{-\infty}^{\infty} g(x')h(x - x')dx'. \quad (2)$$

In words, a convolution can be stated as *sliding one function through the other and summing the area which overlaps*. This can be especially useful if we are translating a function and making multiple copies of it.

In the application of optics, the Fourier variable that has been repeatedly used is called the spatial frequency, u . It is defined as the *number of waves per unit length*, and is the real space analogue of frequency. Its mathematical form for the far-field case can be written as a mapping between the Fourier variable and real space position,

$$u = \frac{x}{\lambda z}, \quad (3)$$

where x is the position in real space, λ is the wavelength of radiation being used to illuminate the object, and z is the distance ‘downstream’ from the object.

When considering diffraction gratings, we can describe the physical makeup of one using a mathematical function. This function could then be manipulated using the different Fourier transform properties as found in Table I. The result of which is an expression which can describe the diffraction pattern observed.

When working with optics we must make the distinction between *real* and *Fourier* space. The former indicating the plane in which a diffraction grating would sit, and the latter representing the plane when the diffraction grating (or any radiation) has been Fourier transformed.

There are some basic shapes which can be used as the object which is being illuminated by radiation: a rectangular slit, and a circular aperture. However to consider these cases we must move into 2 dimensions and consider a 2D Fourier transform.

A 2D Fourier transform has the form of a double integral, so equation 1 becomes,

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(ux+vy)} dx dy, \quad (4)$$

where u and v are the spatial frequencies corresponding to the x and y directions respectively.

The mathematics can still be used as a tool, but now an extra set of terms must be considered.

In Table II we find the functional form of the shapes and their output when they are Fourier transformed, the derivations can be found on pp. 63-64 of *Optics f2f* [2].

Shape	f	F
Rectangle	$\text{rect}(x/a)$	$a \text{sinc}(\pi ua)$
Circle	$\text{circ}(\rho/D)$	$(\pi D^2/4) \text{jinc}(\pi \varpi D)$

TABLE II: Functional form of each specified shape and the result when it has been Fourier transformed. Where a is the width of a rectangular pulse, $\rho = \sqrt{x^2 + y^2}$ is the radial distance, $\varpi = \sqrt{u^2 + v^2}$ is the Fourier equivalent of this radial distance, and D is the diameter of the circle.

With the one dimensional rect function, the Fourier function is a *sinc*, this can be defined in the more familiar terms of $\text{sinc}(\pi ua) = \sin(\pi ua)/(\pi ua)$. Similarly, the result for the circ is a *jinc* function, which is a Bessel function of the first order. So rewritten it is: $\text{jinc}(\pi \varpi D) = J_1(\pi \varpi D)/(\pi \varpi D)$.

To repeat one of these functions one can use a *comb* function - a sum of regularly spaced δ -functions. In the context of optics, the real space δ -function contains all spatial frequencies with an equal amplitude. We can Fourier transform the δ -function so $\mathcal{F}[\delta(x)](u) = 1$.

Defining the comb function as a mathematical expression is thus,

$$\text{comb}_N\left(\frac{x}{d}\right) = \sum_{n=-(N-1)/2}^{(N-1)/2} \delta(x - nd), \quad (5)$$

where the variable x has been scaled by the value of d , and N is the number of δ -functions.

If we were to Fourier transform this finite comb, then the outcome is a discrete sum of phasors,

$$\begin{aligned} \mathcal{F}\left[\text{comb}_N\left(\frac{x}{d}\right)\right](u) &= e^{-i(N-1)\pi ud} + e^{-i(N-1)\pi ud} e^{i2\pi ud} \\ &\quad + \dots + e^{i(N-1)\pi ud}. \end{aligned} \quad (6)$$

This is particularly useful in optics when the diffraction gratings can expand from being a single aperture to many apertures e.g. a single slit versus a coarse grating of many slits.

We can relate the mathematics to experimentation through shining radiation through a diffraction grating then observing the diffraction pattern at some distance later.

What is observed can be quantified in terms of the *intensity*, this is found by taking the absolute value of the Fourier transform, squaring it, and applying a scaling factor $I_0/(\lambda^2 z^2)$ - where I_0 is a number, λ is the wavelength of radiation, and z is the distance 'downstream'.

Using an optical system which composes of a laser and two identical lenses, the radiation of the laser passing

through a diffraction grating can be focused to distances equal to multiples of the focal length of the lenses. The resulting pattern can then be observed on a screen or with a CCD. The setup in mind is called the $4f$ optical system - where each component is a focal length, f , apart.

The laser and the lenses have their own mathematical descriptions attached to them.

The profile of the laser beam is of a *gaussian* shape, it can be expressed as $\text{gauss}(x/w_0) = \exp(-x^2/w_0^2)$, with w_0 as the width of the laser beam. Applying a Fourier transform to this we find that the answer is just another gauss, $\mathcal{F}[\text{gauss}(x/w_0)](u) = \sqrt{\pi} w_0 \text{gauss}(\pi u w_0)$.

The lenses on the other hand act as a 'magnifier', the effect they have can be described as: *the field in the focal plane of a lens is proportional to the Fourier transform of the incident field*. This results in the diffraction pattern shifting from being at a far-field distance z , to a focal length distance of f - any terms that included z , are now replaced by f .

Using variations of the $4f$ system we can explore and study different diffraction patterns and attempt to relate them back to their mathematics.

II. METHOD

To create and observe diffraction patterns we needed to set-up the $4f$ system. Figure 1 shows this experimental setup.

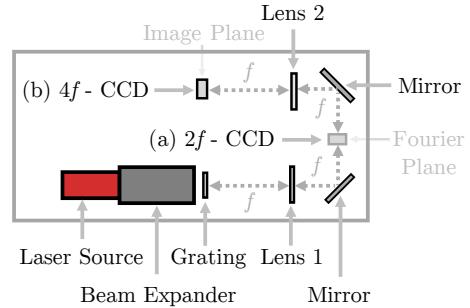


FIG. 1: A plan view schematic of the experimental set-up used to collect data. f are the focal lengths of lenses 1 and 2. Each component was attached to an optical breadboard. (a) $2f$ optical system used in studying the Fourier plane (b) $4f$ optical system used in studying the image plane.

A coherent laser source of wavelength ~ 635 nm was used as our source of radiation. A $\times 10$ beam expander was placed directly after this to allow the beam width to be adjusted during experimentation, this was done in an attempt to focus the majority of the light onto area of the diffraction grating.

To place the lenses at the correct f distances their focal lengths had to be measured. This was done by placing each lens at the beam expander, a screen was then moved closer and further away from the source until the brightest and sharpest spot could be observed. At that distance was thus the focal length, measured with a metre ruler we found that for both lens 1 and 2, their focal lengths were 48 ± 1 cm.

Due to the constraint with the size of the breadboard, mirrors were used to reflect the light so that the set-up would take up less space. Two were positioned so that light could be reflected into a CCD camera, and ensuring that the focal length distances were preserved.

The camera was moved at various stages of experimentation, from being in the Fourier plane at a distance of $2f$ to a distance of $4f$ when we wanted observations made within the image plane.

Our CCD was controlled using the manufacturer's software, images were saved and then later analysed using modules compiled in Python and in MATLAB.

They would convert the image from RGB colour into grayscale, calculate intensity profiles for the x and y directions of the image, and then an attempt at model fitting would be made.

The grayscale images were obtained by converting the three RGB values from a pixel into a singular value defined as the CIE 1931 Linear Luminance,

$$Y_{\text{linear}} = 0.2126R + 0.7152G + 0.0722B, \quad (7)$$

where R, G, B are the respective *red*, *green*, and *blue* values from the pixel [7]. The possible value range for a pixel was fixed between 0 (total black) to 255 (total white).

The intensity profiles were created by locating the rows (and columns) within the images that contained the greatest amount of pixels with the value of 255, those were then averaged together.

A graph could then be plotted, and subsequently a χ^2 value could be calculated when a theoretical model was compared to the intensity graphs.

Throughout our research we varied the gratings that were used, they were placed within a holder just after the beam expander and the pattern that they created was observed at different distances.

Adjustments had to be made so that the pattern observed was as clear as possible. A recurring problem was the over-saturation of the diffraction pattern. This was corrected for by using a neutral density filter, and adjusting the exposure time and the gain of the CCD within the software.

To measure small scale distances of features on the diffraction grating, a travelling microscope with a Vernier scale was used.

III. RESULTS

Figure 2 show multiple plots of the horizontal intensity against the distance along the x direction for different diffraction patterns in the Fourier plane.

Figures 3(a), 3(b), and 3(c) show the horizontal intensity data for the single slit, five slits, and double circle slit plotted with a theoretical model. The reduced χ^2 values for these have been calculated and can be found in Table III.

The horizontal and vertical intensity graphs for the double circular slit have been superimposed and can be found in Figure 3(d).

Grating	χ^2
Single Slit	36.8
Five Slits	8.24×10^6
Double Circle	6.55×10^4

TABLE III: A theoretical model was applied to the single slit, five slits, and double circle diffraction gratings. Only the single slit provided a fit that would allow us to calculate a χ^2 value.

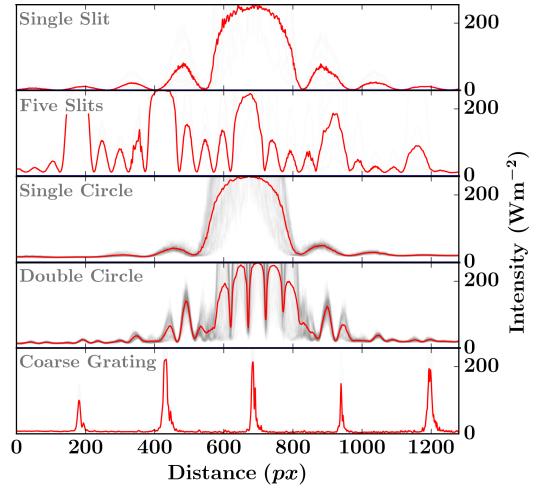


FIG. 2: The horizontal intensity profiles of diffraction patterns within the Fourier plane created with different diffraction gratings. The grey shadow behind the main diagrams are the graphs that were average to create the profile that is shown. The intensity ranges from 0 to 256.

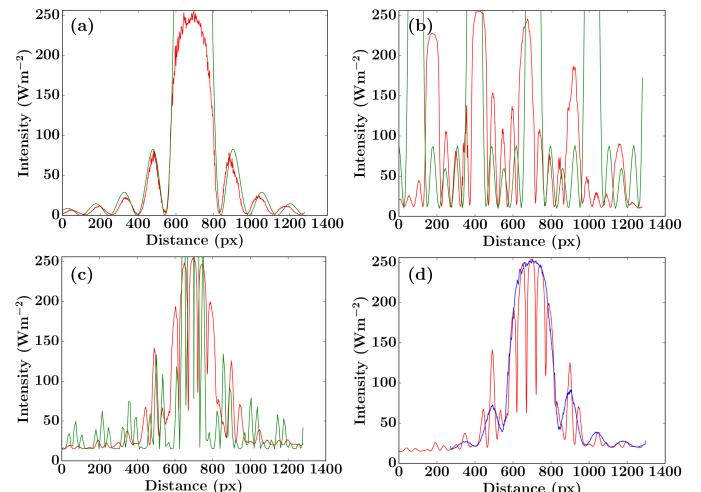


FIG. 3: The red lines are the intensity data from the horizontal direction, the green a theoretical model, and the blue is the vertical intensity data. Data from Fourier plane. The uncertainties within the horizontal and vertical intensities are too small to be seen. (a) Single slit with model. (b) Five slits with model. (c) Double circles with model. (d) Envelope nature of double circle slit.

Grating	Real Space [mm]	Fourier Space [mm]
1	0.20 ± 0.05	0.11 ± 0.03
2	0.12 ± 0.05	0.11 ± 0.01
3	0.08 ± 0.05	0.05 ± 0.01

TABLE IV: For each coarse diffraction grating, the diffraction grating slit width is shown as measured in real space, and as calculated in Fourier space.

IV. DISCUSSION

From an initial inspection of Figure 2 we can see the possible mathematics involved with the diffraction patterns.

We can see the application of the convolution theorem

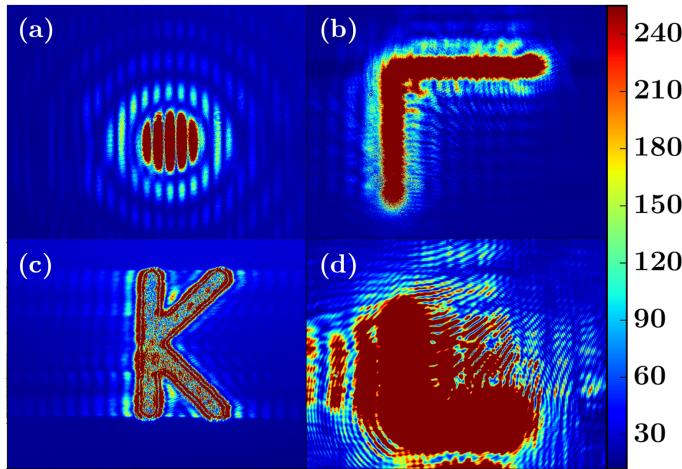


FIG. 4: Intensity maps of four different diffraction patterns taken from different planes: (a) double circle grating in Fourier plane, (b) L grating in image plane, (c) K grating in image plane which has been low-passed filtered, (d) L grating in '3f' plane.

with the five slits and double circle slit - each of their initial functions have been convolved with the comb function, thus producing the semi-repeating pattern that can be seen.

Looking at the calculated values for χ^2_ν we see that they are not near unity at all, a requirement for the theoretical models to be a good fit to the data [8]. For the five and double circles slits they are orders of magnitude out, this tells us that the model used is quite wrong. In the attempt at applying the models for these two gratings (a rect convolved with a comb_5 function, and the Bessel function convolved with a comb_2 respectively), we found that it was difficult to get the parameters correct within the variables. Values would become sensitive to if they were odd or even, producing completely different graphs depending if you either rounded up or down.

Other problems arose such as?

However with the single slit we found that it was possible to achieve a *similar* fit, the implementation of this function was not entirely difficult, but, the same problems with the sensitive parameters did plague us. There is also discrepancies within the data. The principal maxima of the model in Figure 3(a) exceeds the principal maxima of the data.

After exploring patterns within the Fourier plane, we then moved into the image plane for extended research. In Figure 4(b) the pattern is an L shape, the original L grating was the correct way up - if you were a photon passing through the grating you would recognise the shape as an L. We find that the image has been inverted in the y direction.

A curious expansion to our research can be seen in Figure 4(d). This is an image of the diffraction pattern of the L grating as seen in a '3f' system, to create this we moved the source and grating so that it was directly behind the first lens. This pattern has features from both the Fourier and image plane. In the centre there is a clear L shape, something which can be found in the Image plane, but if we also look to the sides we can see what appears to be sinc pattern features. While we do not know the mathematics we can see that it is a mix between Fresnel and Fraunhofer.

Improvements could be made to our investigation to improve the quality of data taken. For example working harder to reduce saturation, during experimentation we had not fully considered the possible effects bias might have on the images that were taking. With future experimentation we would reduce things such as CCD bias by pre-taking images of the dark room so that any stray pixels could be removed during analysis.

Other improvements could be made such as collecting multiple repeats of the diffraction pattern images with different CCD cameras. These images could then be averaged together and a possible uncertainty could be calculated for them.

V. CONCLUSIONS

Conclusion

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Appendix

Appendix