Chapter 10

Supernova Cosmology

10.1 Recent Progress

Recently, rapid progress has been made in measuring the geometry of the Universe. This has in turn lead to an accurate determination of its total mass content and demonstrated that the recent history of the Universe has been dominated by a period of accelerating expansion. These measurements have convinced astronomers of the existence of "dark matter" (required to explain the high mass density of the Universe) and "dark energy" (required to explain the recent acceleration). Before starting work on this project you should read more about these terms, and our understanding of the history of the Universe. A good introduction can be found in Liddle: Introduction to Modern Cosmology.

One of the key pieces of observational evidence is the measurement of the brightness of large samples of extragalactic supernovae. Supernovae have a well defined peak brightness, so that the "luminosity distance" to a supernova can be determined by comparing the flux, f, received with the known peak luminosity,

$$d_L = \left(\frac{L_{\rm peak}}{4\pi f} \right)^{1/2}. \tag{10.1}$$

By measuring d_L for a large sample of supernovae covering different distances, the relation between distance and recession velocity (or redshift) can be measured and compared with theoretical models.

For nearby supernovae, we expect a roughly linear relationship between distance and redshift, known as Hubble's Law. However, if we extend the measurement to more distant objects, the relation becomes curved, and strongly dependent on the matter content of the Universe. Given a set of theoretical models for the matter content of the Universe, we can compare each model with the observed supernova data in order to select the best model.

Some of the first results based on large samples of sufficiently distant supernovae were presented by Schmidt et al., 1998 and Perlmutter et al in 1999. They found the surprising result that the Universe contained a lot of matter ($\Omega_m = 0.28$) but that the dominant contribution to the energy density was in the form of a "Cosmological Constant" or vacuum energy. Their results have been subsequently substantiated by larger surveys and by measurements of the Cosmic Microwave Background (CMB).

10.2 Distances in an Expanding Universe

The aim of this project is to be able to use measurements of supernovae in order to constrain the geometry and mass content of the Universe. Our starting point will be the relation between redshift and distance.

10.2.1 The Friedmann Equation

The expansion rate of the Universe is given by the Friedmann equation. This relates the rate of change of the "scale factor" R (effectively the size of the Universe) to the energy density of the Universe:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3c^2}\left(\rho_{\rm mass}\, c^2 + \rho_{\rm DE}\, c^2\right) - \frac{kc^2}{R^2}, \tag{10.2}$$

where $\rho_{\rm mass}c^2$ and $\rho_{\rm DE}c^2$ are the energy densities in normal matter and "dark energy" respectively (we have omitted the negligible contribution from radiation); k is a constant equal to -1, 0 or 1 for negatively curved, spatially flat and positively curved universes respectively.

The expansion factor of the Universe is defined by

$$a(t) = R(t)/R_0, \tag{10.3}$$

where R_0 is the present day value of the scale factor R, and the redshift, z of a supernova is determined by the expansion factor of the Universe when it occurs

$$a = \frac{1}{1+z}. \tag{10.4}$$

The left hand side of the Friedmann equation can be rewritten in terms of the "Hubble parameter",

$$H = \left(\frac{\dot{R}}{R}\right) \equiv \left(\frac{\dot{a}}{a}\right). \tag{10.5}$$

Note that H is a function of time, and that it changes as the Universe expands. We will make the time dependence explicit below.

An important scale in the Friedmann equation is the "critical density". This is the density the Universe would have if k were zero:

$$\rho_{\rm crit}(t) = \frac{3H^2(t)}{8\pi G}. \tag{10.6}$$

Cosmologists often express the densities appearing in the Friedmann equation in terms of the critical density such that:

$$\Omega_M(t) = \rho_{\rm mass}(t)/\rho_{\rm crit}(t) \tag{10.7}$$

and

$$\Omega_{\rm DE}(t) = \rho_{\rm DE}(t)/\rho_{\rm crit}(t). \tag{10.8}$$

In order to solve the Friedmann equation, we need to know how the matter and dark energy densities depend on time. In general, this is non-trivial, but we can write down simple expressions for their dependence on the expansion factor. Since matter is conserved as the Universe expands,

$$\rho_{\rm mass}(a) = a^{-3}\rho_{\rm mass,0}, \tag{10.9}$$

where the subscript 0 means the value at the present day.

In the simplest models, the dark energy is a fixed property of space (the "cosmological constant") so that it is independent of the expansion factor, i.e.

$$\rho_{\mathrm{DE}}(a) \equiv \rho_\Lambda. \tag{10.10}$$

Combining the above equations, we can rewrite the Friedmann equation as

$$H^2 = \frac{H_0^2 \Omega_{M,0}}{a^3} + H_0^2 \Omega_\Lambda - \frac{kc^2}{R^2}. \tag{10.11}$$

Note that

$$\frac{kc^2}{R_0^2} = H_0^2 \left(\Omega_{M,0} + \Omega_\Lambda - 1 \right). \tag{10.12}$$

10.2.2 The Distance-Redshift Relation

An important distance for this project is the "comoving distance" between the point in space at which the supernova exploded and the observer. This is equivalent to the present-day separation of these points, but it is not the same as the distance traveled by the photons from the supernova. The comoving distance to a supernova with redshift z, corresponding to a comoving coordinate η, is given by

$$R_0 \eta = \int_{a_1}^{1} \frac{c dt}{a} \equiv c \int_0^z \frac{dz'}{H(z')}. \tag{10.13}$$

The peak flux of a supernova with luminosity L is then given by

$$f = \frac{L_{\mathrm{peak}}}{4\pi [R_0 S(\eta)]^2 (1+z)^2}. \tag{10.14}$$

The two factors of $(1+z)$ allow for the reduction of the energy of each photon as it is redshifted by the expansion of the Universe, and the lower frequency with which each photon is emitted. The factor $S(\eta)$ is needed to allow for the effects of General Relativity in distorting space time. If the Universe is "flat", however, (meaning $k = 0$) $S(\eta) = \eta$.

10.3 Comparing Models of the Universe

Using equations 10.11, 10.13 and 10.14, you can now compute the flux expected from a supernova with measured redshift z. Different values of L_{peak}, H_0, $\Omega_{M,0}$, Ω_Λ will give different predictions, and our task is to decide which values are most compatible with the measured supernova data.

The best approach to this problem is to compute the χ^2 statistic for each set of values. This is based on comparing the differences between the observed and model fluxes to the observational errors

$$\chi^2 = \sum \frac{(f_{\mathrm{obs}} - f_{\mathrm{model}})^2}{\sigma_{\mathrm{obs}}^2} \tag{10.15}$$

The smallest value of χ^2 will correspond to the best model. For a one parameter problem, two models are equally good descriptions of the data if $|\chi_1^2 - \chi_2^2| < 1$, and one is a much better fit if $|\chi_1^2 - \chi_2^2| > 9$. Note this use of the χ^2 statistic is distinct from its use to ask whether a model is a good fit to the data. The use of χ^2 to compare to models is closely related to the Bayesian approach to statistics.

In our problem, $L_{\rm peak}$ and H_0 are degenerate (they have the same effect on the predicted flux) and we can only determine their combined value. The best approach is to assume $H_0 = 75\,{\rm km/s/Mpc}$ and to then determine $L_{\rm peak}$ using the local supernova data (eg., $z < 0.1$). For such low redshifts, the comoving distance is accurately approximated by $R_0\eta \approx cz/H_0$. This value of $L_{\rm peak}$ can then be combined with data on distant supernova data to infer the cosmological parameters. Note that if we assume the Universe is flat ($k = 0$), $\Omega_{M,0} = 1 - \Omega_{\Lambda,0}$, so that we only have one parameter to determine from the high redshift data. To determine the best fit value, we must loop over possible values finding the minimum χ^2. To determine the uncertainty, we must find the points on either side that are $\Delta\chi^2 = 1$ greater than the minimum value.

10.4 Your Work Plan

1. **Design Your Program** Your first task is to design a program that you can use to determine the Ω_Λ from the supernova data of Perlmutter et al. *The supernova data are available on the DUO webpage for the computing lab.* You can use the Numpy function “loadtxt” to read the file. The columns of the file give the supernova name, its redshift, the effective peak magnitude and the magnitude error. The “effective” magnitude has already been corrected for the stretching of supernovae light curves, for the band-pass of the original observations and for galactic dust extinction. You can relate the magnitude m to the detected flux, f, using

$$m = m_0 - 2.5\log_{10}(f) \tag{10.16}$$

   where m_0 is a constant (if f is in units of ${\rm erg\,cm^{-2}s^{-1}\AA^{-1}}$, $m_0 = -20.45$, but note that your results do not depend on this value).

   As well as an outline of the code structure, you should include: (a) names and description of the functions you will use: what variables will they take as arguments, what will they return? (b) what data will you input into the program, how will the program output the results? (c) what data structures will the program use to store its internal data, what system of units will you use? Remember to make your program sufficiently flexible to allow you to tackle new problems as well as the Milestone problem below.

2. **Solve the Milestone Problem.** Your milestone program should solve separately for the value of $L_{\rm peak}$ and $\Omega_{\Lambda,0}$ (you may assume $H_0 = 75\,{\rm km/s/Mpc}$). To determine $L_{\rm peak}$, use the supernova data with $z < 0.1$, and the approximation $R_0\eta = cz/H_0$. Then adopt this value to determine $\Omega_{\Lambda,0}$ using supernova data with $z > 0.1$ under the assumption that $k = 0$.

   You should write your own code to determine cosmological distances, but you may find it helpful to check your results using Ned Wright’s web-based calculator. You may wish to use the Scipy integration routines to solve the integral 10.13, but it is best to determine Ω_Λ by evaluating χ^2 for a list of trial values (automated minimisation routines may get trapped in local minima). Always check your results by plotting the observed and predicted supernova magnitudes against redshift.

   At the milestone interview, your program must use the data provide to (i) find the best-fit values of $L_{\rm peak}H_0^2$, then (ii) use this value to determine the best-fit value of Ω_Λ (assuming $k = 0$). Your program should plot the supernova data as a function of redshift and compare this against the best-fit relation you have calculated.

3. **Research with your program.** Now you have a working code, you can investigate supernova cosmology and the geometry of the Universe. High marks will only be given to reports that demonstrate your initiative. Here are some ideas that you may wish to consider. You should not need to address all of these points: these are intended only to provide some possible starting points for your own investigation.

   - Much larger supernova data-sets are now available (eg., http://supernova.lbl.gov). You can improve the accuraccy of the measurement, and investigate the effect of intrinsic variation of SN peak brightness, and/or investigate the systematic uncertainties by using supernova with different intrinsic properties such as colour or host galaxy morphology.
   - In the milestone project, you fitted one parameter at a time. This makes minimisation of χ^2 simple, but this approach is unsatisfactory since the errors in the parameters may be correlated, and it would be better to consider simultaneously minimising χ^2 in a two, three or higher dimensional parameter space (eg. Wall & Jenkins). There are many approaches to multi-dimensional minimisation. Pick one to investigate, but beware of the solution becoming trapped in a local minimum.
   - You do not need to assume that the Universe is flat. However, if $k = \pm 1$, there is a small correction to the distance due to the Universe's geometry, as well as the obvious factor in the Friedmann equation. In general,

     $$S(\eta) = \begin{cases} \sin(\eta), & \text{if k=1,} \\ \eta, & \text{if k=0,} \\ \sinh(\eta), & \text{if k=-1.} \end{cases}$$

     With sufficient supernova, or supernovae at very high redshift, it is possible to determine whether the Universe is flat (an important prediction of "inflation").
   - The χ^2 approach allows you to determine error bars on the model paramaters as well as the best fit values. However, general cosmological models contain many parameters so you must think carefully about how to take into account the correlations between parameters. The Markov-Chain Monte Carlo approach (eg., Ottosen 2012) samples the parameter space with a probability that depends on the likelihood that the model is a good fit to the data. This will lead you into the topic of Bayesian analysis of data.
   - Modern theories suggest that the Dark Energy may not be a simple vaccuum energy. Solutions such as "Qunitessence" (eg., Zlatev et al 1999) allow the apparent value Ω_Λ to be a function of expansion factor. What limits can be placed on the nature of Quintessence? You could also investigate the limits that you can set on alternatives to the theory of General Relativity, for example the class of $F(R)$ models (eg., Li et al., 2012).
   - Other data sets, in particular the power-spectrum of the cosmic microwave background (eg., Ade et al. 2013) play an important role in constraining cosmological paramaters. By adapting the χ^2 analysis to allow for this additional information, we are tackling the problem using Bayesian Statistics. Investigate how the "prior" information from the CMB experiments influences your selection of the best fitting parameters.

References

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