

18.330 Lecture Notes: Fourier Analysis

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1 Fourier Analysis

Recall that the verb *analyze* means “to decompose into constituent pieces.” *Fourier analysis* is the process of decomposing functions into constituent pieces which vary at definite rates – that is, into sinusoids.

Some functions are easy to Fourier-analyze. For example,

$$\begin{aligned} f(t) &= 3 \cos 2\pi t + 19 \sin 4\pi t - 0.14 \cos 7t \\ &= \begin{cases} 3 & \times & \text{(a sinusoid with angular frequency } 2\pi) \\ +19 & \times & \text{(a sinusoid with angular frequency } 4\pi) \\ -0.14 & \times & \text{(a sinusoid with angular frequency } 7) \end{cases} \end{aligned}$$

That’s it! We have Fourier-analyzed the function f .

On the other hand, what about a function like

$$f(t) = e^{-\alpha|t|} \quad \text{or} \quad f(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

How do we break functions like *this* up into pieces that vary with fixed frequencies?

The fourfold way¹

The process of Fourier analysis takes slightly different forms (and goes by slightly different names) depending on **(a)** whether we have access to values of the function $f(t)$ for all values of $t \in [-\infty, \infty]$ or only values within some finite interval $t \in [-\frac{T}{2}, \frac{T}{2}]$ and **(b)** whether we can query $f(t)$ for its value at arbitrary times t (i.e. we have a continuous function $f(t)$) or instead we have just discrete *samples* of the function $f(n\Delta t)$ at evenly spaced time points. The following table summarizes the terminology used in the various cases.

	Continuous function $f(t)$	Discrete samples $f_n \equiv f(n\Delta t), \quad n \in \mathbb{Z}$
Infinite domain $-\infty < t < \infty$	Fourier transform	Semidiscrete Fourier transform
Finite domain $-\frac{T}{2} < t < \frac{T}{2}$	Fourier series	Discrete Fourier transform

Table 1: The fourfold way: The name of the process used to Fourier-analyze a function $f(t)$ depends on whether **(a)** $f(t)$ is defined for all time or only within a finite window, and **(b)** whether we have access to $f(t)$ for all real values of t or just discrete samples at evenly spaced points $n\Delta t$.

You have almost certainly heard before of three of the four table entries here—Fourier transforms, Fourier series, and discrete Fourier transforms—and it is these topics that we will study in depth in 18.330.

The term “semidiscrete Fourier transform” is perhaps less familiar, but you might have encountered it in physics under the name of “Bloch-periodicity”, where it forms the basis of much of crystallography and solid-state physics. In these subjects, the physical variable is the discrete index n (or, equivalently, a discrete lattice vector \mathbf{L}) which ranges over the cells of an infinite periodic lattice, while the Fourier-domain variable is the continuous vector \mathbf{k} (the “Bloch vector”) which ranges over a finite volume of \mathbf{k} -space (the “Brillouin zone” of the lattice.) Although we won’t study this subject in detail, after our discussion of the other three entries in the fourfold way you will be well-equipped to understand the mathematics of Bloch periodicity.

¹I borrowed the idea of a “fourfold way” here from Professor Laurent Demanet.

2 The Fourier transform

The first entry in the fourfold way is the *Fourier transform*. This is what we use when values of the function $f(t)$ we are analyzing are available for arbitrary points t on the entire real line. In this case, the Fourier transform of $f(t)$ is defined to be the function²

$$\tilde{f}(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt. \quad (1)$$

$f(\omega)$ is a function of frequency that tells us how strongly complex exponential with frequency ω is represented in $f(t)$.

Using this formula, we can immediately answer one of the questions we posed at the outset: How do we decompose a function like $E_{\alpha}(t) \equiv e^{-\alpha|t|}$ into sinusoids? The answer is to plug E_{α} into (1):

$$\begin{aligned} \tilde{E}_{\alpha}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} e^{-\alpha|t|} dt \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{(-i\omega+\alpha)t} dt + \int_0^{\infty} e^{(-i\omega-\alpha)t} dt \right] \\ &= \frac{1}{2\pi} \left[\frac{1}{\alpha - i\omega} - \frac{1}{-\alpha - i\omega} \right] \\ &= \frac{\alpha}{\pi(\alpha^2 + \omega^2)}. \end{aligned}$$

What this means is this: The function $e^{-\alpha|t|}$ may be reconstructed by summing sinusoids with all possible frequencies ω . The amplitude of the frequency- ω sinusoid in this sum decays, for large ω , like $1/\omega^2$. Note that the threshold defining “large ω ” is dependent on α : The larger the value of α (i.e. the more rapidly decaying the original exponential) the more sinusoids we have to add with appreciable amplitudes to recover $f(t)$. This is an example of a general phenomenon that we will discuss in the next section.

Linear vs. angular frequency

A quick comment regarding terminology: Given a sinusoid like $\sin \omega t$ or $e^{-i\omega t}$, strictly speaking we should refer to ω as the *angular frequency*. The *frequency* ν is $\nu = \frac{\omega}{2\pi}$. ν is the frequency which with the underlying process repeats itself, while ω is the frequency with which the phase angle in the sinusoid accumulates one radian worth of phase. (The frequency ν is sometimes called the “linear frequency” to distinguish it from the angular frequency.) Linear frequency is

²The theory is nicest if we restrict $f(t)$ to satisfy $\int_{-\infty}^{\infty} |f(t)| dt < \infty$, in which case we say f “is contained in the function space $L^1(\mathbb{R})$.” Fourier transforms can be defined for more general functions, and in some places we will do this without being particularly rigorous about it, but you should be aware that the proper justification of (1) for non- L^1 functions involves a bit of a foray into real analysis.

measured in units of *Hertz* (1 Hertz=once per second), while angular frequency is measured in units of *radians per second*. For example, if we have a pendulum that takes 4 seconds to complete one full cycle, then the linear frequency is $\nu = \frac{1}{4} = 0.25$ Hz and the angular frequency is $\omega = \frac{2\pi}{4} \approx 1.57$ Rad/s.

Terminology of domains

In the above discussion, the function we started out with was $f(t)$, and the Fourier-transformed function was $\tilde{f}(\omega)$. In other words, the original independent variable was the time t , and the transformed variable was the frequency ω . We think of the function $f(t)$ as existing in the *time domain*, while $\tilde{f}(\omega)$ exists in the *frequency domain*.

But Fourier analysis is also useful in situations where the original independent variable is something other than time. For example, it may be the position x , in which case the Fourier-transformed variable is the *spatial frequency* k . (k in this case is sometimes called the “wavenumber.”) In this case it would be a little confusing to say that $f(x)$ exists in the “time domain.” Instead, various alternative terminologies have arisen for labeling the two spaces in which functions live before and after Fourier transformation.

Field	Physical variable	Physical domain	Fourier variable	Fourier domain
Signal processing	time t	time domain	frequency ω	frequency domain
Optics	position x	real space	wavenumber k	k space
Quantum mechanics	position x	position space	momentum p	momentum space
Solid state physics	lattice vector \mathbf{L}	real space	Bloch vector \mathbf{k}	crystal momentum space

As indicated in this table, we will refer collectively to the “before” domain and variable as the “physical domain” and the “physical variable,” and we will refer to the “after” domain and variable as the “Fourier domain” and the “Fourier variable.”³

Dimensional analysis

In physics and engineering problems it’s important to keep in mind that the functions $f(t)$ and $\tilde{f}(\omega)$ have different units. Indeed, looking at (1) we see that

³The term “physical variable” is something of a misnomer since Fourier variables like frequency and spatial wavenumber are certainly physical quantities, but whaddya gonna do.

the RHS contains a dt factor that is not present on the LHS, and therefore

$$\text{units of } \tilde{f} = (\text{units of } f) \cdot (\text{time}) = \frac{\text{units of } f}{\text{frequency}}.$$

Thus, if $f(t)$ has units of volts, then $\tilde{f}(\omega)$ has units of volts \cdot seconds or volts per Hertz.

Properties of Fourier Transforms

There are several properties of the Fourier transform that follow immediately from the definition.

1. *Fourier transform of a derivative.* If, for a given function $f(x)$, we know the Fourier transform $\tilde{f}(k)$, then it's easy to compute the Fourier transform of *derivatives* of $f(x)$. The easiest way to get at this is to write down the Fourier-synthesized version of f :

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

and differentiate both sides with respect to x :

$$\frac{d}{dx} f(x) = \int_{-\infty}^{\infty} ik \tilde{f}(k) e^{ikx} dk$$

The RHS here is the inverse Fourier transform of the Fourier-space function $ik\tilde{f}(k)$, so we identify this function as the Fourier transform of $\frac{df}{dx}$, i.e. we conclude that

$$\text{FT} \left[\frac{df}{dx} \right] = ik \tilde{f}(k) \quad \implies \quad \text{FT} \left[f(x) \right] = \tilde{f}(k). \quad (2)$$

This game can be repeated as many times as we like; for example,

$$\text{FT} \left[\frac{d^2 f}{dx^2} \right] = -k^2 \tilde{f}(k). \quad (3)$$

2. *Derivative of a Fourier transform.* We can also play basically the same trick in the opposite direction. Write down the defining equation of the Fourier transform,

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx,$$

and differentiate both sides with respect to k :

$$\frac{d}{dk} \tilde{f}(k) = \int_{-\infty}^{\infty} -ix f(x) e^{-ikx} dx.$$

What this tells us is that $\frac{d}{dk}\tilde{f}$ is something like the Fourier transform of the function $xf(x)$:

$$\text{FT}\left[f(x)\right] = \tilde{f}(k) \quad \Longrightarrow \quad \text{FT}\left[xf(x)\right] = i\frac{d}{dk}\tilde{f}(k). \quad (4)$$

Note that this statement is the dual of (3).

3. *Fourier transforms of real-valued functions.* If $f(x)$ is a real-valued function, then the information contained in $\tilde{f}(k)$ for $k < 0$ is redundant; values of $\tilde{f}(k)$ on the negative k axis may be recovered from knowledge of \tilde{f} on the positive k axis.

To see the precise relationship, write down the defining equation of the Fourier transform with negative argument k :

$$\tilde{f}(-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{ikx} dx$$

Since $f(x)$ is real-valued, we can write the RHS in the form

$$\begin{aligned} &= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx \right]^* \\ &= \tilde{f}^*(k). \end{aligned}$$

3 Examples of Fourier transforms

Lorentzians

We already saw our first example of a Fourier transform:

$$E_\alpha(x) = e^{-\alpha|x|} \quad \tilde{E}_\alpha(k) = \frac{\alpha}{\pi(\alpha^2 + k^2)}. \quad (5)$$

Both $E_\alpha(x)$ and $\tilde{E}_\alpha(k)$ are “pulse” functions: they are maximal in the vicinity of the origin, and decay to zero as their respective arguments go to infinity. (These functions are known as “Lorentzians.”) The *width* of the two pulses depends on α , in inverse ways: the larger the value of α , the *narrower* the pulse in real space and the *wider* the pulse in Fourier space.

To quantify this a little further, define the *full width at half maximum* (FWHM) of a pulse as the width between the two points at which the pulse has fallen to 1/2 its peak value. The function $E_\alpha(x)$ has peak value of 1 (at $x = 0$) and falls to 1/2 at $x = \pm \ln 2/\alpha$, so

$$\text{FWHM}(E_\alpha) = \frac{2 \ln 2}{\alpha}. \quad (6)$$

On the other hand, the function \tilde{E}_α has peak value $\frac{1}{\pi\alpha}$ (at $k = 0$) and falls to half this value at point $k = \pm\alpha$, so

$$\text{FWHM}(\tilde{E}_\alpha) = 2\alpha. \quad (7)$$

Combining (6) and (7), we have

$$\text{FWHM}(E_\alpha) \cdot \text{FWHM}(\tilde{E}_\alpha) = 4 \ln 2. \quad (8)$$

Notice that equation (8) is *independent of* α ; the same statement holds for *all* functions in the family $\{E_\alpha(t)\}$ regardless of the value of α .

Gaussians

Let $G_\sigma(x)$ be a Gaussian of width σ :

$$G_\sigma(x) \equiv e^{-x^2/\sigma^2}.$$

The Fourier transform of G_σ is

$$\begin{aligned} \tilde{G}_\sigma(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} G_\sigma(x) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - \frac{x^2}{\sigma^2}} dx. \end{aligned}$$

Complete the square: $\frac{x^2}{\sigma^2} + ikx = \frac{1}{\sigma^2} \left(x + \frac{ik\sigma^2}{2} \right)^2 + \frac{k^2\sigma^2}{4}$

$$= \frac{1}{2\pi} e^{-\frac{k^2\sigma^2}{4}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{\sigma^2} \left(x + \frac{ik\sigma^2}{2} \right)^2} dx}_{\sigma \cdot \sqrt{\pi}}$$

$$= \frac{\sigma}{2\sqrt{\pi}} e^{-\frac{k^2\sigma^2}{4}}.$$

Aside from the annoying prefactor⁴, the important point here is that $\tilde{G}_\sigma(k)$ is again a Gaussian in k -space, but with a width *inversely proportional to that of the original Gaussian*:

$$\tilde{G}_\sigma(k) \propto e^{-\frac{k^2}{\tilde{\sigma}^2}} = G_{\tilde{\sigma}}(k)$$

where

$$\tilde{\sigma} \equiv \frac{2}{\sigma}.$$

The FWHM of a Gaussian $G_\sigma(x)$ is $2\sqrt{\ln 2} \cdot \sigma$, and hence for Gaussians the equivalent of (9) is

$$\text{FWHM}(G_\sigma) \cdot \text{FWHM}(\tilde{G}_\sigma) = (2\sqrt{\ln 2} \cdot \sigma) \cdot \left(\frac{4\sqrt{\ln 2}}{\sigma} \right) = 8(\ln 2)^2. \quad (9)$$

Again, this is *independent of σ* : it holds for the entire *family* of Gaussian pulses.

The Heisenberg Uncertainty Principle

Both of the previous examples have illustrated a general phenomenon: The narrower we make a pulse in physical space, the wider that pulse is in Fourier space. The precise details depend on the particular shape of the pulse, so that different families of pulses exhibit different versions of the relations (8) and (9), but the general principle is the same.

Equations like (8) and (9) are sometimes known as “uncertainty relations” due to a phenomenon in quantum mechanics known as the “Heisenberg uncertainty principle.” In quantum mechanics, the position and momentum of a particle are Fourier-conjugate variables, which means that the more precisely we try to pin down the particle’s position in space (i.e. the narrower we make the particle’s wavefunction in real space) the less accurately we can resolve its momentum (i.e. the wider the particle’s wavefunction in momentum space). The precise statement of the uncertainty principle is

$$(\Delta x)(\Delta p) > \frac{\hbar}{2} \approx 10^{-34} \text{ kg m}^2/\text{s}$$

⁴It is possible to play games like multiplying $G_\sigma(x)$ by a certain constant prefactor to ensure that $\tilde{G}_\sigma(k)$ comes out with a nicer prefactor or even a symmetric prefactor (i.e. the same prefactor as G_σ), but we won’t bother.

so, for example, if we have an electron (mass $\approx 10^{-30}$ kg) and we try to resolve its *position* to within $\Delta x \approx 10$ nm, then we can't pin down its *velocity* to any better accuracy than $\Delta_p \approx 10^5$ m/s! This is a huge uncertainty compared to the spatial resolution we are trying to hit.

Fourier transforms of non-pulse functions

Both the Lorentzian and the Gaussian (and their Fourier transforms) are “pulse” functions—they are localized near zero and decay to zero for large arguments. What happens if we try to take the Fourier transform of a non-pulse function—for, example, a function like $f(x) = 1$, or $f(x) = x$, or $f(x) = x^2$?

One way to get at the answer is to consider the Fourier transform of the Lorentzian (5) in the limit $\alpha \rightarrow 0$. In this case, the real-space function approaches simply the constant value 1, i.e.

$$\lim_{\alpha \rightarrow 0} E_\alpha(x) = 1.$$

On the other hand, as $\alpha \rightarrow 0$ the Fourier-space function $\tilde{E}_\alpha(k)$ changes in two ways: **(1)** its *width* gets narrower (recall that its FWHM was 2α) and **(2)** its *height* gets taller [indeed, its value at the origin is $\tilde{E}_\alpha(0) = \frac{1}{\pi\alpha}$.] A limiting process in which a function gets infinitely narrow and infinitely tall sounds like the kind of procedure that defines a Dirac delta function, and indeed it's easy to show that

$$\lim_{\alpha \rightarrow 0} \tilde{E}_\alpha(k) = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\pi(\alpha^2 + k^2)} = \delta(k).$$

Thus we have the Fourier-transform pair

$$f(x) \equiv 1 \quad \implies \quad \tilde{f}(k) = \delta(k). \quad (10)$$

This actually makes sense, if you think about it: The function $f(x) \equiv 1$ already *is* a sinusoid, namely, a sinusoid with zero frequency. To synthesize this function as a sum of sinusoids, we want to set the coefficients of all sinusoids to zero except the single sinusoid with frequency $k = 0$.

Armed with equation (10) and the derivative identity (4), we can now compute the Fourier transform of functions like $f(x) = x$ or $f(x) = x^2$:

$$f(x) = x \quad \implies \quad \tilde{f}(k) = i\delta'(k) \quad (11)$$

$$f(x) = x^2 \quad \implies \quad \tilde{f}(k) = -\delta''(k) \quad (12)$$

where e.g. $\delta'(k)$ is the derivative of the Dirac delta function, which is defined using integration by parts:

$$\int f(u)\delta'(u)du = - \int f'(u)\delta(u)du = -f'(u).$$

In other words, the object δ' should be thought of as a gadget similar to δ , except that when integrated against a function f it pulls out minus the *derivative* of f at the origin, not the value of f like the usual δ function would do.

As anticipated in Footnote 2 above, Fourier transforms like (10), (11), and (12) are not very nice functions—indeed, they are not even *functions* at all, but instead are only *distributions*⁵. This is because the real-space functions $f(x) = \{1, x, x^2\}$ are not contained in the function space L^1 , i.e. they do not satisfy $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. It is nonetheless convenient to use equations like (12) in a sort of operational sense, but you should be aware that these formal manipulations are sweeping some mathematical subtleties under the rug.

⁵What this means in essence is that objects like $\delta(k)$ and $\delta'(k)$ are meaningless in isolation, and only make sense when they appear paired with a nice function under an integral sign.

4 The smoothness of $f(t)$ and the decay of $\tilde{f}(\omega)$

The specific examples in the previous section illustrates a general principle: The more rapidly varying the function $f(t)$, the *less* rapidly the function $f(\omega)$ decays with ω . Contrariwise, if $f(t)$ is *not* rapidly varying (it is “smooth” in a colloquial sense) then its Fourier transform decays rapidly for large ω .

$$\begin{aligned} f(t) \text{ rapidly varying} &\implies \tilde{f}(\omega) \text{ slowly decaying as } \omega \rightarrow \infty \\ f(t) \text{ slowly varying} &\implies \tilde{f}(\omega) \text{ rapidly decaying as } \omega \rightarrow \infty \end{aligned}$$

This makes sense: If $f(t)$ is “slow”, then it doesn’t contain many “fast” frequency components (or the ones it does contain have small amplitudes).

This statement can be quantified by characterizing the smoothness of f in terms of its continuity and that of its derivatives. In particular,

If $f(t)$ and its first $p - 1$ derivatives are continuous, but its p th derivative is discontinuous with bounded variation, then $\tilde{f}(\omega)$ decays at least as rapidly as $|\omega|^{-(p+1)}$ for $|\omega| \rightarrow \infty$.

In particular, if $f(t)$ is C^∞ (it is continuous and all of its derivatives are continuous everywhere, no discontinuities, anytime, anyplace, ever) then $\tilde{f}(\omega)$ decays for large ω *faster than any polynomial*. Functions which decay faster than any polynomial include $e^{-\omega}$, $e^{-\sqrt{\omega}}$, $e^{-\omega^2}$, etc.

Statements like the boxed statements above are generally known as *Paley-Wiener theorems*.

This principle is already illustrated by the particular examples we considered previously. The function $e^{-\alpha|t|}$ is continuous, but its first derivative is not (it has a finite jump at the origin). Thus the statement in the box is satisfied for $p = 1$ and we expect the Fourier transform to decay like ω^2 for large ω , as indeed we found above. On the other hand, the function e^{-t^2/σ^2} is C^∞ , so its Fourier transform should decay faster than any polynomial in ω – and, indeed, the Fourier transform of this function goes like $e^{-\sigma^2\omega^2/4}$, which decays for large ω faster than any polynomial in ω .

Simultaneous compact support of $f(t)$ and $\tilde{f}(\omega)$

Another implication of the smoothness/decay relationship in Fourier analysis, which is also related to the uncertainty-principle ideas of the previous section, has to do with simultaneous compact support of f and \tilde{f} . Recall that a function $f(t)$ is said to have *compact support* if it is only nonzero on a compact subregion of the real line. For example, the function

$$f(t) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

has compact support. On the other hand, the Gaussian e^{-x^2} does not have compact support; for large x it is very small but not exactly zero.

In the same vein as we asked above whether or not we could simultaneously squeeze $f(t)$ and $\tilde{f}(\omega)$ to be narrow pulses, it is interesting to ask if we could find a function $f(t)$ such that both f and \tilde{f} have compact support. The answer is basically *no*, except for the trivial case $f = \tilde{f} = 0$:

If $f(t)$ and $\tilde{f}(\omega)$ both have compact support, then
 $f(t) = \tilde{f}(\omega) = 0$.

5 Fourier series

Next suppose $f(t)$ is a periodic function with period T . This means that $f(t + T) = f(t)$ for all t ; the function f repeats itself every T seconds.⁶ Suppose we try to compute the Fourier transform $\tilde{f}(\omega)$ of this periodic function:

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt. \quad (13)$$

There are two distinct cases we need to analyze:

1. *The frequency ω is an integer multiple of $\frac{2\pi}{T}$.* In this case, the entire integrand of (13) is periodic with period T . Every time interval of width T makes an identical contribution to the integral, and there are an infinite number of such time intervals, so $\tilde{f}(\omega) = \infty$.
2. *The frequency ω is not an integer multiple of $\frac{2\pi}{T}$.* In this case, the integrand of (13) is *not* periodic. [The $f(t)$ factor is periodic with period T , and the $e^{-i\omega t}$ factor is periodic with some period not equal to an integer fraction of T , so the overall integrand is not periodic.] Now what happens is that each time interval of width T makes a contribution to the integral that has essentially the same magnitude, but a random phase factor. These random phase factors cause all the contributions to the integral to *cancel*, and we find $\tilde{f}(\omega) = 0$.

To summarize, if $f(t)$ is periodic with period T , its Fourier transform $\tilde{f}(\omega)$ is zero except when ω is an integer multiple of $\omega_0 \equiv \frac{2\pi}{T}$, at which $\tilde{f}(\omega)$ is infinite. One way to think of this situation is to represent $\tilde{f}(\omega)$ as a train of δ functions:

$$\tilde{f}(\omega) = \sum_{\nu=-\infty}^{\infty} \tilde{f}_{\nu} \delta(\omega - \nu\omega_0) \quad \left(\omega_0 = \frac{2\pi}{T}\right).$$

Another way to think about this is to say that if $f(t)$ is periodic with period T , its Fourier decomposition only contains sinusoids with frequencies $\omega_{\nu} = \nu\omega_0 = \frac{2\nu\pi}{T}$ for $\nu \in \mathbb{Z}$. We can write

$$f(t) = \sum_{\nu \in \mathbb{Z}} \tilde{f}_{\nu} e^{\frac{2i\nu\pi t}{T}} \quad \text{or} \quad f(t) = \sum_{\nu \in \mathbb{Z}} \tilde{f}_{\nu} e^{i\nu\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}. \quad (14)$$

This is the *Fourier series* representation of f . [Here the symbol ν denotes the Greek letter “nu” (it is pronounced like the English word “new”); this symbol is commonly used as an integer-valued index in the Fourier domain, conjugate to the use of the symbol n to index integer-valued quantities in the real-space domain.]

⁶Or every T minutes, or hours, or whatever time units we are using.

Cosines, Sines, and Complex Exponentials

Equation (14) expresses the function $f(t)$ as a linear combination of complex exponentials. As you may have seen in other courses, it is also possible to write the exact same series in terms of cosines and sines:

$$f(t) = \sum_{\nu=0}^{\infty} \tilde{a}_{\nu} \cos(\nu\omega_0 t) + \sum_{\nu=1}^{\infty} \tilde{b}_{\nu} \sin(\nu\omega_0 t) \quad (15)$$

Note that the index ν in (15) runs only over *nonnegative* integers⁷, whereas the sum in (14) runs over both positive and negative integers.

Appearances to the contrary notwithstanding, equations (14) and (15) represent exactly the *same* expansions of exactly the *same* function! To relate the coefficients \tilde{f}_{ν} in (14) to the coefficients $\tilde{a}_{\nu}, \tilde{b}_{\nu}$ in (15), we recall the relationship between cosines, sines, and complex exponentials,

$$e^{i\nu\omega_0 t} = \cos(\nu\omega_0 t) + i \sin(\nu\omega_0 t), \quad (16)$$

together with the facts that cosine and sine are respectively even and odd functions:

$$\cos(-\nu\omega_0 t) = \cos(\nu\omega_0 t), \quad \sin(-\nu\omega_0 t) = -\sin(\nu\omega_0 t). \quad (17)$$

Comparing equations (14), (15), (16), and (17), we see that the coefficients in (14) are related to the coefficients in (15) according to

$$\begin{aligned} \tilde{a}_0 &= \tilde{f}_0 \\ \tilde{a}_{\nu} &= (\tilde{f}_{\nu} + \tilde{f}_{-\nu}) \quad (\nu > 0), \quad \text{or} \quad \tilde{f}_{\nu} = \begin{cases} \frac{1}{2}(\tilde{a}_{\nu} + i\tilde{b}_{\nu}), & \nu < 0 \\ \tilde{a}_0, & \nu = 0 \\ \frac{1}{2}(\tilde{a}_{\nu} - i\tilde{b}_{\nu}), & \nu > 0. \end{cases} \\ \tilde{b}_{\nu} &= i(\tilde{f}_{\nu} - \tilde{f}_{-\nu}) \quad (\nu > 0) \end{aligned} \quad (18)$$

Thus the two forms (14) and (15) are completely equivalent and contain the same information; going back and forth between the two is simply a matter of some arithmetic using (18).

The advantages of the form (15) include the following: **(a)** for real-valued functions $f(t)$, the coefficients \tilde{a}_{ν} and \tilde{b}_{ν} are real-valued, so one never needs to worry about complex numbers; and **(b)** it clearly separates the even and odd parts of the sinusoidal expansion of $f(t)$. For example, if $f(t)$ is an even function [i.e. $f(t) = f(-t)$] then its Fourier series contains only cosine terms, and all the \tilde{b}_{ν} coefficients vanish in (14).

For our purposes, these advantages of (15) are overwhelmed by the greater advantages of (14), which include **(a)** it requires us only to keep track of one set of coefficients $\{\tilde{f}_{\nu}\}$; **(b)** it offers more immediate and transparent connection to the Fourier transform formulas; and **(c)** it is easier to differentiate. Indeed,

⁷Also note that, in the term involving $\sin(\nu\omega_0 t)$, the sum over ν starts at $\nu = 1$, i.e. there is no \tilde{b}_0 coefficient. Do you see why?

differentiating (14) yields a series of precisely the same form as (14), in which the coefficients $\{\tilde{f}_\nu\}$ are simply replaced by $\{i\omega_0\tilde{f}_\nu\}$. (In contrast, differentiating (15) mixes up the sine and cosine terms and introduces annoying minus signs.) For these reasons, we will use the complex-exponential version of the Fourier series [equation (15)] almost exclusively.⁸

⁸The exception is in our treatment of Chebyshev spectral methods; there, we will be working with real-valued even functions, and we will find it convenient to work with the form (15).

Computation of Fourier-series coefficients

To compute the \tilde{f}_ν coefficients, we simply use a finite-time version of the Fourier transform in which we only look at $f(t)$ over one of its periods:

$$\tilde{f}_\nu = \frac{1}{T} \int_0^T f(t) e^{-i\nu\omega_0 t} dt.$$

A simple example

For example, let's Fourier-analyze the function $\cos^2 3t$, which is periodic with period $T = \frac{\pi}{3}$.

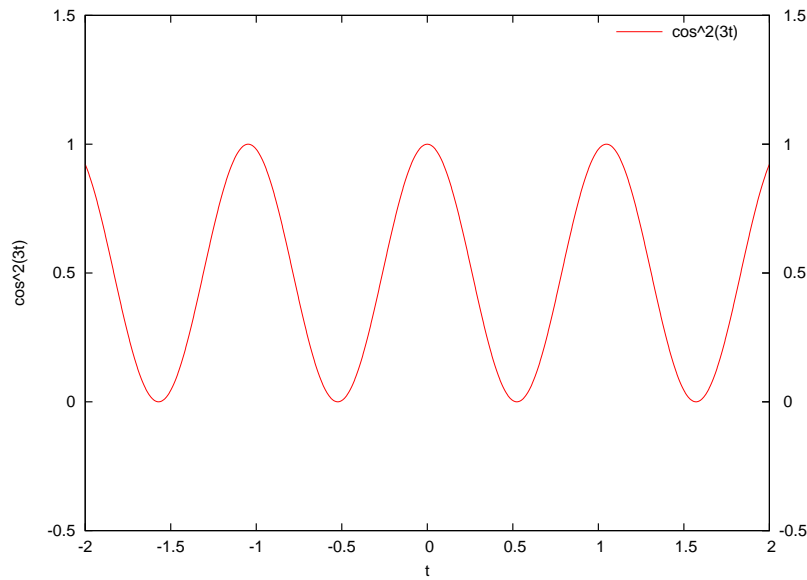


Figure 1: The function $f(t) = \cos^2 3t$.

The base frequency $\omega_0 = \frac{2\pi}{T} = 6$, and the n th Fourier coefficient is

$$\tilde{f}_\nu = \frac{1}{T} \int_0^T e^{-i\nu\omega_0 t} f(t) dt$$

Use $\cos 3t = \frac{1}{2}(e^{3it} + e^{-3it})$, $\cos^2 3t = \frac{1}{4}(e^{6it} + 2 + e^{-6it})$

$$= \frac{1}{4T} \int_0^T e^{-i\nu\omega_0 t} [e^{i\omega_0 t} + 2 + e^{-i\omega_0 t}] dt$$

Now use the orthogonality result stated in the Appendix:

$$= \frac{1}{4} [\delta_{\nu,1} + 2\delta_{\nu,0} + \delta_{\nu,-1}]$$

In other words, the Fourier coefficient \tilde{f}_ν is only nonzero for $\nu = \{-1, 0, 1\}$. The Fourier-synthesized version of $f(t)$ is

$$\begin{aligned} f(t) &= \sum_{\nu} \tilde{f}_\nu e^{-i\nu\omega_0 t} \\ &= \frac{1}{4} e^{i\omega_0 t} + \frac{1}{2} + \frac{1}{4} e^{-i\omega_0 t} \\ &= \frac{1}{2} [1 + \cos \omega_0 t] \\ &= \frac{1}{2} [1 + \cos 6t]. \end{aligned} \tag{19}$$

Of course, we could have used standard trigonometry identities to show that $\cos^2 3t = (1 + \cos 6t)/2$, but it's nice to see this result emerging from the full Fourier analysis procedure.

Fourier Cosine and Sine series

Looking at (19), we see that the function $f(t)$ had only cosine terms⁹ and no sine terms. This is actually a general phenomenon that happens whenever the function we are analyzing is an *even* function, i.e. satisfies $f(t) = f(-t)$: even functions have only cosine terms in their Fourier series. Similarly, odd functions [that is, functions for which $f(t) = -f(-t)$] have only sine terms in their Fourier series. We then speak of a *Fourier cosine series* or a *Fourier sine series*.

In some cases the function you are analyzing is neither even nor odd, but can be *made* into an even or odd function just by shifting the origin of coordinates.¹⁰ More generally, any arbitrary function may be decomposed into even and odd pieces like this:

$$f(t) = f_e(t) + f_o(t), \quad f_e(t) = \frac{1}{2} [f(t) + f(-t)] \quad f_o(t) = \frac{1}{2} [f(t) - f(-t)].$$

⁹Plus a constant term, which may be thought of as a cosine with zero frequency. Note that *sines* with zero frequency are identically zero.

¹⁰For example, the T -periodic function $f(t)$ defined to be 0 for $0 < t < \frac{T}{2}$ and 1 for $\frac{T}{2} < t < T$ is neither even nor odd; but $g(t) = f(t + \frac{T}{4})$ is even.

A more interesting example

As a more interesting example of Fourier series, consider the sawtooth wave depicted below: $f(t)$ is periodic with period T , and for $0 < t < T$ we have $f(t) = t$. [Note that $f(t)$ has the units of t —for example, if we are measuring time in seconds, then $f(t)$ has units of seconds.]

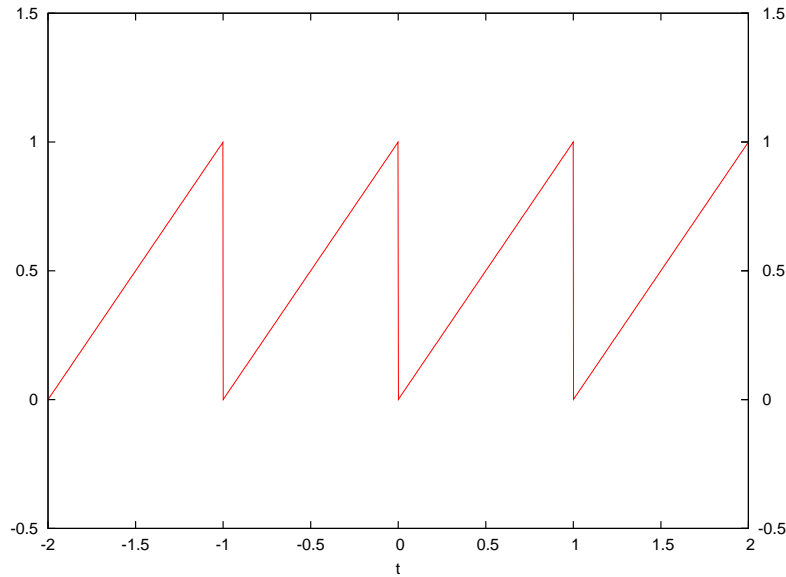


Figure 2: Sawtooth function: $f(t) = t$ for $0 \leq t \leq T$, and $f(t)$ is periodic with period T . (In this plot, the x and y axis labels are measured in units of T .)

The Fourier series of this function is

$$\begin{aligned}
 f(t) &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{\nu} e^{i\nu\omega_0 t} \quad \left(\omega_0 = \frac{2\pi}{T} \right) \\
 \tilde{f}_{\nu} &= \frac{1}{T} \int_0^T f(t) e^{-i\nu\omega_0 t} dt \\
 &= \frac{1}{T} \int_0^T t e^{-i\nu\omega_0 t} dt.
 \end{aligned}$$

The $\nu = 0$ term evaluates to $f_0 = \frac{T}{2}$. For $\nu \neq 0$ we integrate by parts:

$$\begin{aligned} &= \frac{1}{T} \left[-\frac{1}{i\nu\omega_0} \left| t e^{-i\nu\omega_0 t} \right|_0^T + \frac{1}{i\nu\omega_0} \int_0^T e^{-i\nu\omega_0 t} dt \right] \\ &= -\frac{1}{i\nu\omega_0}. \end{aligned}$$

Thus the Fourier series for our function is

$$f(t) = \frac{T}{2} - \frac{1}{i\omega_0} \sum_{\nu \neq 0} \frac{e^{i\nu\omega_0 t}}{\nu}$$

Note that the units are correct: the LHS has units of time, the first term on the RHS has units of time, and the second term on the RHS has units of (angular frequency)⁻¹=time.

Note also that \tilde{f}_ν decays like $1/|\omega|$ for large ω . This is in accordance with our discussion of Paley-Wiener theorems above, since the function $f(t)$ is discontinuous.

We could also rewrite this series in terms of cosines and sines (and eliminate ω_0 in favor of T):

$$f(t) = \frac{T}{2} - \frac{T}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \sin\left(\frac{2\nu\pi t}{T}\right). \quad (20)$$

Note that this is *almost* a Fourier sine series – only the first (constant) term doesn't belong. If we consider the modified function $g(t) = f(t) - \frac{T}{2}$, then this term would go away and the Fourier series for $g(t)$ would be a Fourier sine series – which can only be true if $g(t)$ is an odd function. You should look at the graph of $f(t)$ and convince yourself that shifting the entire curve downward by $T/2$ does indeed yield an odd function.

It seems amazing to think that summing up a bunch of sine functions – each one of which is individually a nice smooth function – can reproduce the jagged, discontinuous behavior of the sawtooth function of Figure 5. But it does!

The Gibbs phenomenon

However, it does with one proviso: If we *truncate* the series by summing only a finite number of terms (that is, if we perform an *incomplete* Fourier synthesis of the function $f(t)$), we encounter the *Gibbs phenomenon*. The Gibbs phenomenon is the appearance of oscillations near discontinuities in the incomplete Fourier synthesis of a discontinuous function. For example, the following plot shows the original sawtooth wave $f(t)$ together with its incomplete Fourier-synthesized versions $f_N(t)$, where $f_N(t) = \sum_{\nu=-N}^N \tilde{f}_\nu e^{i\nu\omega_0 t}$, for $N = 2, 5, 10, 20$.

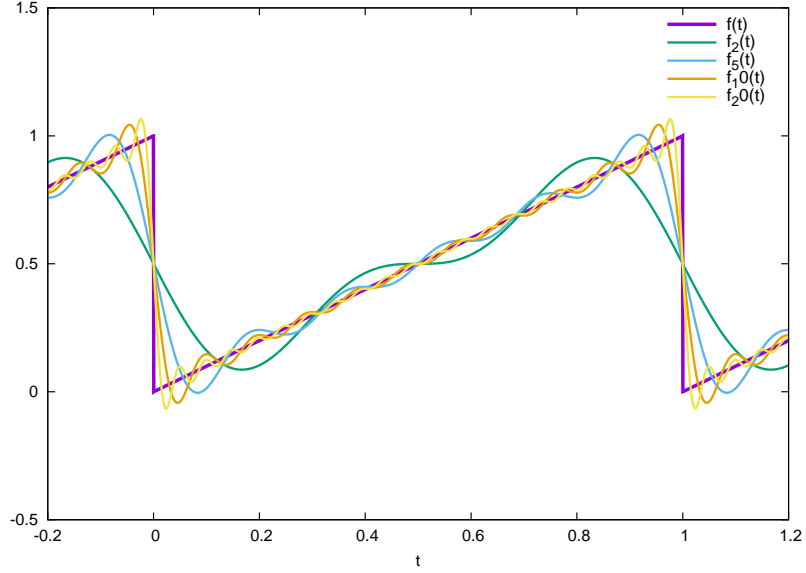


Figure 3: The Gibbs phenomenon. When we truncate the Fourier series (20) at a finite number of terms, we obtain an approximation to the original sawtooth function $f(t)$. Note that, in the regions away from the discontinuity, the approximation more closely hugs the actual function as $N \rightarrow \infty$; however, *near* the discontinuity, the peak error between the function and the approximation *does not decrease* with increasing N . However, the definition of “near the discontinuity” does change with N , and for larger N the errors are confined to a narrower region about the discontinuity.

Convergence of Fourier series at points of discontinuity

Figure 3 also illustrates an important point about Fourier-series representations of discontinuous functions: If the original function $f(t)$ is discontinuous at a point t^* , then the Fourier series $\sum \tilde{f}_\nu e^{i\nu\omega_0 t^*}$ converges to the “midpoint of the discontinuity,” i.e. we have

$$\lim_{N \rightarrow \infty} \sum_{\nu=-N}^N \tilde{f}_\nu e^{i\nu\omega_0 t^*} = \frac{1}{2} [f(t_-^*) + f(t_+^*)]$$

where

$$f(t_-^*) = \lim_{\epsilon \rightarrow 0} f(t^* - \epsilon), \quad f(t_+^*) = \lim_{\epsilon \rightarrow 0} f(t^* + \epsilon).$$

In particular, if we construct a Fourier series to represent the behavior over $[0, T]$ of a function that is not periodic on that interval, then evaluating this Fourier series at $t = 0$ will yield $\frac{1}{2}[f(0) + f(T)]$. This behavior is clearly visible in the figure above.

6 Fourier analysis is a *lossless* process: Parseval's theorem

A very important property of the process of Fourier analysis is that it is *lossless*: after going over to the Fourier domain, we have no less information about f than we started out with. This is true no matter which of the four entries in the fourfold way (Table 1) we are talking about.¹¹

Fourier synthesis

One important consequence of the losslessness of Fourier analysis is that the inverse process – *Fourier synthesis* – exists and may be used to recover the original function from its Fourier-analyzed version. (Again, this is true no matter which of the fourfold-way entries we are talking about.) For example, the inverse of equation (1) reads

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega t} \tilde{f}(\omega) d\omega \quad (21)$$

This is exactly what you expect: we recover $f(t)$ by summing a bunch of sinusoids $e^{i\omega t}$, with the weight of the frequency- ω summand given by $\tilde{f}(\omega)$.

Note that equation (21) is *exact*: there is *no loss in going back and forth between the physical and Fourier domains*. If we didn't have the losslessness property of Fourier analysis, we would have to wonder whether or not the function defined by the RHS of (21) was in some way an inexact representation of our function.

Parseval's Theorem

Another important consequence of the losslessness of Fourier analysis is that it allows us to perform certain computations in the Fourier domain with the confidence that these computations yield the same results as if we had performed them in the physical domain. If we didn't have the losslessness property of Fourier analysis, we would have to wonder whether or not we lost something along the way.

This phenomenon is well illustrated by *Parseval's theorem*. Suppose we have two functions $f(t)$ and $g(t)$ and we want to compute their “inner product,”

$$\langle f|g \rangle = \int_{-\infty}^{\infty} f^*(t)g(t)dt$$

¹¹Note that we may not start out with complete information on the function $f(t)$; for example, we may only have samples of this function at some limited set of time points. In this case, the process of Fourier analysis (which, for a finite set of function samples, would be the discrete Fourier transform) obviously does not magically give us any more information about the original underlying function $f(t)$ than we started out with, but what's important is that it doesn't *lose* any information – after computing the DFT, we can always compute the inverse DFT to recover the original function samples we started with.

Insert the Fourier-synthesized versions of f and g , equation (21):

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{f}^*(\omega) d\omega \right\} \left\{ \int_{-\infty}^{\infty} e^{+i\omega' t} \tilde{g}(\omega') d\omega' \right\} dt$$

Rearrange the order of integration:

$$\begin{aligned} &= \int_{-\infty}^{\infty} \tilde{f}^*(\omega) \int_{-\infty}^{\infty} \tilde{g}(\omega') \underbrace{\left\{ \int_{-\infty}^{\infty} e^{i(\omega' - \omega)t} dt \right\}}_{2\pi\delta(\omega' - \omega)} d\omega' d\omega \\ &= 2\pi \int_{-\infty}^{\infty} \tilde{f}^*(\omega) \tilde{g}(\omega) d\omega. \end{aligned}$$

Thus the inner product of the Fourier transforms of f and g is equal to the inner product of f and g .

Plancherel's theorem

If we take the two functions in Parseval's theorem to be the same function, $g = f$, we obtain *Plancherel's theorem*:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = 2\pi \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega.$$

Fourier-Series Versions of Parseval and Plancherel

The derivations of the Parseval and Plancherel theorems above were for the upper left box of the fourfold way – that is, the case in which we are interested in the behavior of $f(t)$ over all time. If we are instead working in the lower left box, where we are only interested in the behavior of a function over a finite time interval $[0, T]$ (either because the function is periodic with period T , or because we only care about its behavior in an interval of width T), then the corresponding versions of the Parseval and Plancherel theorems are

$$\int_0^T f^*(t)g(t) dt = T \sum_{\nu=-\infty}^{\infty} \tilde{f}_{\nu}^* \tilde{g}_{\nu} \quad (22)$$

$$\int_0^T |f(t)|^2 dt = T \sum_{\nu=-\infty}^{\infty} |\tilde{f}_{\nu}|^2 \quad (23)$$

These are easy to derive by proceeding in exact analogy to the derivation we presented for the infinite-time case.

Computational significance of Parseval and Plancherel

Computationally, the significance of the Parseval and Plancherel theorems is that they allow us to perform computations in either the physical or the Fourier domain depending on which is easier.

7 Poisson summation

The computational impact of Parseval's theorem is that it gives us the option of evaluating certain sums in either the physical domain or the Fourier domain depending on which is easier. If we are trying to compute a physical-domain integral or sum that is more rapidly converging in the Fourier domain, we can just evaluate it in that domain, and Parseval's theorem guarantees that we incur no error in the process.

Poisson summation is a similar technique which, computationally, gives us the choice of evaluating sums in the Fourier or physical domain. More specifically, suppose we have a function $f(t)$, and we want to sum the values of this function at evenly spaced time points separated by Δt . Then Poisson summation tells us we can just as well do the computation by summing samples of $\tilde{f}(\omega)$ at evenly-spaced frequency points separated by $\frac{2\pi}{\Delta t}$.

In equations, the Poisson summation formula reads

$$\sum_{n=-\infty}^{\infty} f(n\Delta t) = \frac{2\pi}{\Delta t} \sum_{\nu=-\infty}^{\infty} \tilde{f}\left(\nu \frac{2\pi}{\Delta t}\right). \quad (24)$$

Note that the units are correct: \tilde{f} has units of

$$\text{units of } \tilde{f} = \frac{\text{units of } f}{\text{frequency}} = (\text{units of } f) \cdot (\text{time})$$

while Δt has units of time; thus

$$\text{units of } \frac{\tilde{f}}{\Delta t} = \text{units of } f.$$

An alternative way to write equation (24) is to define $\Delta\omega \equiv \frac{2\pi}{\Delta t}$; then (24) says

$$\sum_{n=-\infty}^{\infty} f(n\Delta t) = \Delta\omega \sum_{\nu=-\infty}^{\infty} \tilde{f}(\nu\Delta\omega). \quad (25)$$

The important point here is that $\Delta\omega$ and Δt are *inversely* proportional. Thus, if I am summing samples of my time-domain function at very small time intervals [i.e. Δt is small in the LHS of (25)] then the Fourier-domain sum involves samples at very *large* frequency intervals [i.e. $\Delta\omega$ is large in the RHS of (25)]. The more finely I need to sample in the time domain, the more *sparsely* I need to sample in the frequency domain!

It's easy to prove equation (24), and we'll do it below, but first let's investigate some practical applications.

Jacobi θ functions

Recall that the Fourier transform of a Gaussian is a Gaussian, and that, more specifically, the FT of a wide Gaussian is a narrow Gaussian – in particular, the

FT of a Gaussian with width σ in physical space is a Gaussian with width $\frac{2}{\sigma}$ in Fourier space. Thus, if we ever found ourselves wanting to sum the quantity $e^{-n^2\pi x}$ over all integer n , and we were finding our sum slow to converge (because, say, πx might be small and the sum thus slowly convergent) we might be tempted to exploit Poisson summation to evaluate the sum in Fourier space.

To get technical about it, define

$$T_x(t) \equiv e^{-t^2\pi x} \quad (26)$$

where we think of $T_x(t)$ as a function of t parameterized by x . The Fourier transform of (26) is

$$\begin{aligned} \tilde{T}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} T(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t - t^2\pi x} dt \\ &= \frac{1}{2\pi\sqrt{x}} e^{-\frac{\omega^2}{4\pi x}}. \end{aligned}$$

(The integral here is evaluated in the same way as the integral that arose when we computed the Fourier transform of a Gaussian.) Now consider the following function of x , known as the *Jacobi theta function*:¹²

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \sum_{n=-\infty}^{\infty} T_x(n). \quad (27)$$

Applying the Poisson summation formula (24) with $\Delta t = 1$, we immediately find

$$\begin{aligned} \theta(x) &= 2\pi \sum_{\nu=-\infty}^{\infty} \tilde{T}(2\nu\pi) \\ &= \frac{1}{\sqrt{x}} \underbrace{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\pi\nu^2}{x}}}_{\theta(1/x)} \end{aligned} \quad (28)$$

But the sum here is nothing but the original function θ evaluated at the *inverse* of its original argument! We have proven the *functional equation* of the Jacobi θ function:

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right).$$

I find this to be a totally wacky formula. $\theta(x)$ looks like a very complicated function. How could the value of this function at x possibly be related so simply to its value at $1/x$? But it is!

¹²Actually the function defined by equation (27) is only one of several related functions known collectively as Jacobi theta functions.

To demonstrate the computational efficacy of (28), write it in the form

$$\sum_{n=-\infty}^{\infty} e^{-n^2 \pi x} = x^{-1/2} \sum_{\nu=-\infty}^{\infty} e^{-\nu^2 \pi / x}. \quad (29)$$

Suppose we want to compute, to 6-digit accuracy, the value of this sum for $x = 0.04$. Using the LHS to evaluate the sum, we need to sum 11 terms:

LHS sum in (29) with $x = 0.04$:

N	$e^{-N^2 \pi x}$	$\sum_{n=-N}^N e^{-n^2 \pi x}$
0	1.0	1.000000000000000
1	0.8819113782981763	2.763822756596353
2	0.6049225627642709	3.9736678821248947
3	0.322718983267049	4.619105848658993
4	0.133905721399763	4.886917291458519
5	0.04321391826377226	4.973345127986064
6	0.010846710538160161	4.995038549062384
7	0.002117494770632841	4.99927353860365
8	0.00032151151668886733	4.999916561637028
9	3.796825289201935e-5	4.999992498142812
10	3.4873423562089973e-6	4.9999994728275245
11	2.4912565147240595e-7	4.999999971078828

On the other hand, if we use RHS of (29), we only have to sum *one* term to get 6-digit accuracy:

RHS sum in (29) with $x = 0.04$:

N	$e^{-N^2 \pi / x}$	$\sum_{\nu=-N}^N e^{-\nu^2 \pi / x}$
0	1.0	1.000000000000000
1	7.773044498987552e-35	1.000000000000000
2	3.650603079495543e-137	1.000000000000000

The functional equation of the Jacobi θ function is upheld to the accuracy of our calculation:

$$\underbrace{\theta(0.04)}_{4.999999971078828} = \underbrace{\frac{1}{\sqrt{0.04}}}_{5.000000000000000} \cdot \underbrace{\theta\left(\frac{1}{0.04}\right)}_{1.000000000000000}$$

Ewald summation

Finally, Poisson summation is the basis of *Ewald summation*, a wonderful technique for speeding the convergence of real-space sums over particle interactions that is widely used in computational physics and engineering. We will consider this topic in detail in a subsequent set of lecture notes.

Proof of Poisson Summation

This proof is somewhat heuristic, but it captures the essence of the argument.

Start with the LHS of (24) and insert the Fourier-synthesized representation of $f(n\Delta t)$:

$$\sum_{n=-\infty}^{\infty} f(n\Delta t) = \sum_{n=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{in\omega\Delta t} d\omega \right\}$$

Rearrange the summation and integration:

$$= \int_{-\infty}^{\infty} \tilde{f}(\omega) \underbrace{\left\{ \sum_{n=-\infty}^{\infty} e^{in\omega\Delta t} \right\}}_{2\pi \sum_{\nu} \delta(\omega\Delta t - 2\nu\pi)} d\omega$$

The point of this step is that the sum over n inside the curly brackets yields zero (all the terms eventually cancel each other) unless $\omega\Delta t$ is an integer multiple of 2π (this multiple can be *any* integer ν) in which case that sum is infinite. We summarize this situation by describing the quantity in the curly brackets as a δ function which is only nonzero for $\omega\Delta t$ equal to $2\nu\pi$ for arbitrary integers ν , or (in other words) a train of δ functions, one for each integer ν .

$$= 2\pi \sum_{\nu} \int_{-\infty}^{\infty} \tilde{f}(\omega) \delta(\omega\Delta t - 2\nu\pi) d\omega$$

Finally, use the δ -function identity $\delta(ax - b) = \frac{1}{a}\delta(x - b/a)$:

$$\begin{aligned} &= \frac{2\pi}{\Delta t} \sum_{\nu} \int_{-\infty}^{\infty} \tilde{f}(\omega) \delta\left(\omega - \frac{2\nu\pi}{\Delta t}\right) d\omega \\ &= \frac{2\pi}{\Delta t} \sum_{\nu=-\infty}^{\infty} \tilde{f}\left(\frac{2\nu\pi}{\Delta t}\right). \end{aligned}$$

This completes the proof.

8 Fourier analysis and convolution

Another important property of the Fourier-analysis process is that it behaves multiplicatively under convolutions. Again, this is true no matter which of the four entries in the fourfold way (Table 1) we are talking about.

Recall that the convolution of two functions $f(t)$ and $g(t)$ is a sum of copies of $g(t)$, with each copy displaced in time by some time offset τ and weighted in the sum by the value of f at time τ :

$$C(t) \equiv f * g = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau.$$

Let's compute the Fourier transform of $C(t)$:

$$\begin{aligned}\tilde{C}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} C(t)e^{-i\omega t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)g(t - \tau)e^{-i\omega t} d\tau dt\end{aligned}$$

Insert the Fourier-synthesized versions of f and g :

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{i\omega_1 \tau} \tilde{f}(\omega_1) d\omega_1 \right] \left[\int_{-\infty}^{\infty} e^{i\omega_2 (t - \tau)} \tilde{g}(\omega_2) d\omega_2 \right] e^{-i\omega t} d\tau dt$$

Rearrange the order of integration:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} e^{i(\omega_1 - \omega_2)\tau} d\tau \right]}_{2\pi\delta(\omega_1 - \omega_2)} \underbrace{\left[\int_{-\infty}^{\infty} e^{i(\omega_2 - \omega)t} dt \right]}_{2\pi\delta(\omega_2 - \omega)} \tilde{f}(\omega_1)\tilde{g}(\omega_2)d\omega_1 d\omega_2$$

Use the first δ function to evaluate the ω_1 integral, then use the second δ function to evaluate the ω_2 integral:

$$= 2\pi \tilde{f}(\omega)\tilde{g}(\omega).$$

In other words: The frequency- ω Fourier coefficient of the convolution of f and g is just the product of the frequency- ω Fourier coefficients of f and g .

This fact has important implications for signal processing. In particular, it means that the operation of convolution is easier to perform in the frequency domain than the physical domain.

9 Higher-Dimensional Fourier Transforms

The entire theory of Fourier analysis generalizes readily to higher dimensions. For example, let $f(x, y)$ be a function of two variables. By holding x fixed and Fourier-transforming with respect to y , we obtain a mixed physical-space/Fourier-space function $\tilde{f}(x, k_y)$.¹³

$$\tilde{f}(x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik_y y} f(x, y) dy.$$

And now we hold k_y fixed and Fourier-transform $\tilde{f}(x, k_y)$ with respect to x :

$$\begin{aligned} \tilde{f}(k_x, k_y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik_x x} \tilde{f}(x, k_y) dx \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k_x x + k_y y)} f(x, y) dy dx. \end{aligned}$$

It is typical to write this in the form

$$= \frac{1}{(2\pi)^2} \int e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}$$

where the integrations (unless otherwise specified) are generally understood to range over the full range of the \mathbf{x} variable. Written this way, the formula for the D -dimensional Fourier transform actually looks the same, but with a prefactor $\frac{1}{(2\pi)^D}$.

The multidimensional version of Fourier synthesis is

$$f(\mathbf{x}) = \int \tilde{f}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}.$$

Examples of higher-dimensional Fourier transforms

Gaussians Gaussians in D dimensions are easy to Fourier-transform because they are *separable*, i.e. they may be written as a product of D factors each depending on only one variable.

The Coulomb potential A less trivial example is the case of the Coulomb potential in 3 dimensions:

$$\phi^{\text{coulomb}}(\mathbf{r}) = \phi^{\text{coulomb}}(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

¹³Since this function lives half in physical space and half in Fourier space we really should adorn it with a half-tilde instead of the full \sim crown, but I don't know how to typeset that in L^AT_EX.

The Fourier transform of this is

$$\tilde{\phi}^{\text{coulomb}}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int \frac{e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}}{|\mathbf{r}|}$$

A convenient way of evaluating 3D integrals like this is to use polar coordinates in a coordinate system in which \mathbf{k} points in the z direction. In this coordinate system we have $d\mathbf{x} = r^2 \sin \theta dr d\theta d\phi$ and $\mathbf{k} \cdot \mathbf{x} = kr \cos \theta$ (where $k = |\mathbf{k}|$ is the magnitude of \mathbf{k}) so the integral becomes

$$= \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{-ikr \cos \theta}}{r} \cdot r^2 \sin \theta d\phi d\theta dr$$

The ϕ integral can be done immediately to yield 2π . To do the θ integral, change variable to $u = \cos \theta$, $du = -\sin \theta d\theta$:

$$\begin{aligned} &= \frac{1}{(2\pi)^2} \int_0^\infty dr r \int_{-1}^1 e^{-ikru} du \\ &= \frac{1}{(2\pi)^2} \int_0^\infty dr r \underbrace{\left[\frac{1}{-ikr} e^{-ikru} \right]_{-1}^1}_{2 \sin(kr)/kr} \\ &= \frac{1}{2k\pi^2} \int_0^\infty \sin kr dr \end{aligned}$$

Change variables to $t = kr$:

$$= \frac{1}{2\pi^2 k^2} \underbrace{\int_0^\infty \sin t dt}_{=1}$$

and thus¹⁴ we conclude that the 3D Fourier transform of the Coulomb potential is

$$\tilde{\phi}^{\text{coulomb}}(\mathbf{k}) = \frac{1}{2\pi^2 k^2}. \quad (30)$$

A good way to think of (30) is in terms of the Fourier-synthesis picture:

$$\begin{aligned} \frac{1}{\mathbf{r}} &= \phi^{\text{coulomb}}(\mathbf{r}) \\ &= \int \tilde{\phi}^{\text{coulomb}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \\ &= \int \frac{d\mathbf{k}}{2\pi^2} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2}. \end{aligned}$$

Thus, we can recover the Coulomb potential by summing plane waves of all possible wavevectors; the contribution of the plane wave with wavevector \mathbf{k} is weighted in the sum with a factor $1/(2\pi^2|\mathbf{k}|^2)$.

¹⁴ Actually the t integral here doesn't quite make sense as we have written it; the proper justification of the result requires a certain limiting process, which you will work out in your problem set.

Parseval, Plancherel, Poisson in higher dimensions

All of the theorems that we derived above expressing the losslessness property of Fourier analysis extend immediately to the multidimensional case. For example, Parseval's theorem tells us that we can compute the inner product of two D -dimensional functions equally well in real space or in Fourier space:

$$\int f^*(\mathbf{x})g(\mathbf{x})d\mathbf{x} = (2\pi)^D \int \tilde{f}^*(\mathbf{k})\tilde{g}(\mathbf{k})d\mathbf{k}$$

where the integrations on both sides extend over all of \mathbb{R}^D .

For the higher-dimensional generalization of the Poisson summation formula, we have the freedom to choose the sample points with different spacings in the different dimensions. For example, consider a two-dimensional function $f(x, y)$, and suppose we want to evaluate the two-dimensional lattice sum

$$\sum_{n_x, n_y = -\infty}^{\infty} f(n_x \Delta x, n_y \Delta y).$$

In other words, we are sampling f on a grid of points that lie Δx apart in the x direction, and Δy apart in the y direction. All we have to do is apply Poisson summation recursively, first in the y direction and then in x direction (or vice versa, it doesn't matter). The result is

$$\sum_{n_x, n_y = -\infty}^{\infty} f(n_x \Delta x, n_y \Delta y) = \left(\frac{2\pi}{\Delta x}\right) \left(\frac{2\pi}{\Delta y}\right) \sum_{\nu_x, \nu_y} \tilde{f}\left(\frac{2\nu_x \pi}{\Delta x}, \frac{2\nu_y \pi}{\Delta y}\right)$$

where \tilde{f} is the two-dimensional Fourier transform of f .

A Exponential Sums

In several places throughout this document, we have invoked certain sum rules without justification. We'll collect these formulas here just to make sure we have them all in one place and to emphasize that they are all really just slightly different twists on the same basic principle.

Continuous, finite-time version

First suppose we are working over a finite interval of the t axis of width T , i.e. we are in the setting of the Fourier series. Let $\omega_0 = \frac{2\pi}{T}$ be the base frequency (the minimal frequency of any sinusoid in the Fourier-series representation of a function $f(t)$ over our interval). Then our result takes the form

$$\frac{1}{T} \int_0^T e^{i(n_1 - n_2)\omega_0 t} dt = \delta_{n_1, n_2}. \quad (31)$$

(The RHS here is the *Kronecker delta*: it evaluates to 1 if $n_1 = n_2$ and 0 otherwise.) You can easily prove this result by evaluating the integral yourself.

Orthogonality interpretation

A good way to interpret (31) is to say that, for $n_1 \neq n_2$, the functions $f_{n_1}(t) = e^{in_1\omega_0 t}$ and $f_{n_2}(t) = e^{in_2\omega_0 t}$ are *orthogonal* with respect to the inner product $\langle f, g \rangle = \frac{1}{T} \int f^* g dt$. The notion of “inner products” and “orthogonality” are borrowed from geometry, and they mean the same things here: the inner product is an operation that takes two elements and returns a number, and two elements are orthogonal if they have zero inner product.

Continuous, infinite-time version

Next suppose we are working over the entire real line. Then the appropriate version of (31) is

$$\int_{-\infty}^{\infty} e^{i(\omega - \omega')t} dt = 2\pi \delta(\omega - \omega'). \quad (32)$$

Discrete version

The following result was used in our derivation of Poisson summation above, and will be considered further when we discuss discrete Fourier transforms.

$$\sum_{n=-\infty}^{\infty} e^{inkx} = 2\pi \sum_{\nu=-\infty}^{\infty} \delta(xk - 2\nu\pi)$$

What this says is the following: The sum on the LHS yields zero unless x is an integer multiple of $\frac{2\pi}{k}$. (The sum over ν is just allowing for all possible integer multiples.) If x is an integer multiple of $\frac{2\pi}{k}$, then the sum on the LHS is infinite

(all the summands are equal to 1), but infinite in such a way that if we multiply the LHS by some function $f(x)$ and integrate over all x then we get a finite number which depends on the values of $f(x)$ at the points $x = 2\nu\pi/k$.

B Gaussian Integrals

The basic Gaussian integral

The basic Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (33)$$

If we throw a factor α into the exponent, we find instead

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}. \quad (34)$$

To derive this formula, just change variables in the original Gaussian integral (33).

You can use dimensional analysis to remember the α dependence of (34) like this: The entire LHS of (34) has the same units as x because the dx factor in the integral is the only dimensionful quantity in that expression. For example, if x is measured in meters, then the entire LHS of (34) has units of meters. On the other hand, since αx^2 is the argument of an exponential, it must be dimensionless, whereupon $1/\alpha$ must have the same units as x^2 , and thus $1/\sqrt{\alpha}$ must have the same units as x . Since the RHS must have the same units as x , the RHS must be proportional to $1/\sqrt{\alpha}$.

Gaussian integrals with linear and constant terms in the exponent

It may also happen that the exponent contains additional terms of lower order in x , i.e. we may have

$$I(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x + \gamma} dx.$$

The first easy thing to do is to pull a factor of e^γ out of the integral:

$$I(\alpha, \beta, \gamma) = e^\gamma \int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} dx.$$

To make sense of what's left, complete the square:

$$-\alpha x^2 + \beta x = -\alpha \left(x - \frac{\beta}{2\alpha} \right)^2 + \frac{\beta^2}{4\alpha}.$$

Inserting back into the above, we have

$$I(\alpha, \beta, \gamma) = e^{\gamma + \frac{\beta^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha \left(x - \frac{\beta}{2\alpha} \right)^2} dx$$

Now just change variables to $y = x - \frac{\beta}{2\alpha}$:

$$\begin{aligned}
 &= e^{\gamma + \frac{\beta^2}{4\alpha}} \underbrace{\int_{-\infty}^{\infty} e^{-\alpha y^2} dy}_{\sqrt{\pi/\alpha}} \\
 &= \sqrt{\frac{\pi}{\alpha}} e^{\gamma + \frac{\beta^2}{4\alpha}}.
 \end{aligned}$$

Although it's not obvious from this derivation, the formula actually continues to hold for imaginary values of γ and β .¹⁵

¹⁵And even some complex values of α , though not all – for example, it clearly fails for $\alpha = 0$ or $\alpha = -1$, among other values, as the original integral obviously diverges in these cases.