

FSL
A Fourier Series Library

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December 3, 2019

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Revision History

Date	Version	Notes
Oct. 10, 2019	0.9	Draft submitted for review by Dr. Smith
Dec 3, 2019	1.0	Revised in accordance with advice from Dr. Smith

1 Reference Material

This section records information for easy reference.

1.1 Table of Units

This section is not applicable, due to the fact that this library is a pure mathematical computation library.

1.2 Table of Symbols

The table that follows summarizes the symbols used in this document. The choice of symbols is made to be consistent with the Fourier series literature and with existing documentation for Fourier series libraries.

symbol	description
$f(t), g(t), \dots$	$[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ functions of t , where ω is defined as below
\mathbb{R}	Set of real numbers
\mathbb{Z}	Set of integers
$[-\pi/\omega, \pi/\omega]$	Set of real numbers that are neither smaller than $-\pi/\omega$, nor greater than π/ω
\rightarrow	Indicate mapping
\mathbb{A}^*	Non-negative subset of set \mathbb{A} (either \mathbb{R} or \mathbb{Z})
\mathbb{A}^+	Positive subset of set \mathbb{A} (either \mathbb{R} or \mathbb{Z})
\mathbb{N}	alias of \mathbb{Z}^*
$\sum_{i=m_1}^{m_2} a_i$	Summation of $a_i, i = m_1, m_1 + 1, \dots, m_2$
n	Length of cut-off Fourier series
ω	Base frequency of Fourier series
$CFSf, CFSg$	$CFS(f(t), n, \omega)$ and $CFS(g(t), n, \omega)$, respectively.
A_i, B_i	coefficients of 'sin' and 'cos' functions in a CFS.
App	Approximation operation on a CFS
Amp	Amplitude function of a CFS
tol	Tolerance used for equality of CFS.

1.3 Abbreviations and Acronyms

symbol	description
<hr/>	
A	Assumption
API	Application Programming Interface
CA	Commonality Analysis
CFS	Cut-off Fourier Series
DD	Data Definition
FS	Fourier Series
GD	General Definition
GS	Goal Statement
IFS	Infinite Fourier Series
IM	Instance Model
LC	Likely Change
PS	Physical System Description
R	Requirement
FSL	Fourier Series Library.
T	Theoretical Model

2 Introduction

This document introduces a library, FSL, mainly designed and implemented for Fourier series related calculations. This document is based on the template introduced in [Smith \(2006\)](#), [Smith and Lai \(2005\)](#), and [Smith et al. \(2017\)](#).

2.1 Purpose of Document

The scope of the family is limited to calculations related to the Fourier series of mathematical functions.

2.2 Scope of Requirements

The scope of the libraries is limited to calculations related to the Fourier series of mathematical functions.

2.3 Characteristics of Intended Reader

The intended readers must have the following knowledge.

- Deep knowledge of Fourier series (found in advanced calculus and real analysis courses);
- Computational error analysis (found in introductory numerical analysis classes); and
- Detailed knowledge of the programming language utilized in the implementation of this library.

2.4 Organization of Document

This document consists of the following sections.

- [section 3](#) generally describes this library, including what it is, who uses it, and what kinds of environment it needs.
- [section 4](#) introduces the common elements of this library. This section overviews its background, defines related data, introduces theories needed for calculation, and states the goals of this library.
- [section 5](#) covers the different functions of this library, and states its variability in assumptions, calculations, and outputs.
- [section 6](#) informs users, developers, maintainers, and reviewers of this library, about its requirements, both functional ones and nonfunctional ones.
- [section 7](#) states the changes of this library that are likely in the future.

- [section 8](#) shows the traceability matrix of this library, to inform the developers of their changes' impacts.
- [Appendix A](#) gives proof of the operation-related theories in [section 4](#).
- [Appendix B](#) gives a list of basic functions in this document.

3 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

3.1 System Context

The users provide the data for the library to calculate, and the library returns the calculation results.

- User Responsibilities:
 - Provide the inputs of the functions in the library.
 - When needed, allocate memory space, and pass their location to the library, so that the library can store the output in this user-provided space.
 - For the conversion from mathematical functions to Fourier series, ensure that provided mathematical functions are eligible for transformation.
- FSL Responsibilities:
 - Detect data type mismatch, such as a string of characters instead of a floating point number.
 - Unless otherwise mentioned in this document, detect illegality of input values, such as mismatched length of Fourier series for binary operations.
 - Unless otherwise mentioned, manage the memory space required by this library.

3.2 User Characteristics

The end user of FSL should have an understanding of undergraduate level 2 Calculus and/or Real Analysis, undergraduate level 1 Numerical Analysis, and any one of the programming languages, in which this library provides a set of APIs.

3.3 System Constraints

The potential systems must contain the compilation/interpretation environment for the programming languages in which this library is developed, as well as dependent libraries.

4 Specific System Description

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

4.1 Problem Description and Background Overview

Fourier series are used in lots of analysis, analysis of signals as an example [Papoulis \(1977\)](#). The Fourier series of a function mainly uses a linear combination of 'sin' and 'cos' functions to represent and approximate this function. In real-life research, we sometimes need to perform Fourier series related operations. For example, when we need to analyze the total effects of two signals, each approximated by its Fourier series, we need to calculate the summation of these two Fourier series. Or, when we need to calculate the strength of a signal from its Fourier series, we need to calculate this Fourier series' amplitude.

4.1.1 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- Binary operations: an operation that involves two operands.
- Fourier series: a series of real numbers acting as weights in the weighed sum of 'sin' and 'cos' functions, to represent/approximate a function.
- CFS: short for cut-off Fourier Series, a Fourier Series with finite number of non-zero weights.
- Amplitude: a map $CFS \rightarrow \mathbb{R}^*$ to measure this CFS's strength.

4.1.2 Goal Statements

Given the corresponding inputs, the goal statements are:

- GS1: When given a function $f(t)$, the cut-off length n , and a base frequency ω , return the function's CFS $CFSf = CFS(f(t), n, \omega)$.
- GS2: When given a CFS $CFSf = CFS(f(t), n, \omega)$ and a value of t as t_1 , return the approximated value of $f(t_1)$ computed from the given values.
- GS3: When given two CFS's $CFSf$, $CFSg$, and one of the operations addition, subtraction, multiplication, and division, return the result of this operation on these two CFS's.

- GS4: When given a CFS $CFSf$, and a function $g(t)$ from the basic function sets defined by this library (see [Appendix B](#)), return the CFS $g(CFSf)$.
- GS5: When given a set of values including n , ω , A_i 's, and B_i 's, return the CFS built on these values.
- GS6: When given a CFS, store its values of n , ω , A_i 's, and B_i 's in the user-designated spaces.
- GS7: When given a CFS, return its amplitude.
- GS8: When given two CFS's and a tolerance, return whether they are equal within the tolerance.

4.2 Data Definitions

This section collects and defines all the data needed to build the instance models.

Number	DD1
Label	Infinite Fourier Series (IFS) of mathematical functions
Symbol	IFS
Equation	$IFS(f(t), \omega) = [A_{inf,i}, i = 0, 1, \dots; B_{inf,i}, i = 1, 2, \dots]$, satisfying $f(t) = \sum_{i=0}^{+\infty} A_{inf,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{inf,i} \sin(i\omega t)$
Description	<p>In this equation,</p> <ul style="list-style-type: none"> • $f(t)$ is an $[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ function, whose IFS exists as assumed in A1; • ω is the base frequency of this IFS, and it is designated by the user; • $A_{inf,i} \in \mathbb{R}, i = 0, 1, \dots$; and • $B_{inf,i} \in \mathbb{R}, i = 1, 2, \dots$
Sources	Tolstov, G.P. and Silverman, R.A., Fourier Series, Dover Books on Mathematics, 1976, Dover Publications, Tolstov and Silverman (1976)
Ref. By	IM2, A1 , C1 , O1 , and O2

Number	DD2
Label	Cut-off Fourier Series (CFS) of mathematical functions
Symbol	$CFS : (n \in \mathbb{N}, \omega \in \mathbb{R}^+, [A_0, \dots, A_n, B_1, \dots, B_n] \in \mathbb{R}^{(2n+1)}),$ generated from given $f(t) \in (\mathbb{R} \rightarrow \mathbb{R}), n, \omega$
Equation	$CFS(f(t), n, \omega) = [A_i, i = 0, 1, \dots, n; B_i, i = 1, 2, \dots, n],$ satisfying $A_i = A_{inf,i}, i = 0, 1, \dots, n; B_i = B_{inf,i}, i = 1, 2, \dots, n,$ where $\omega, A_{inf,i}$'s and $B_{inf,i}$'s come from $IFS(f(t), \omega)$
Description	$n \in \mathbb{Z}^*$ is called cut-off length.
Sources	Defined by author
Ref. By	T1, T2, T3, T4, T5, T7, A1, C1, O1, and O2
Note	CFS is a mapping given above, its result is of type $CFST$, and any symbol in this document that consists of CFS followed by lower case letters are objects of type $CFST$.

As said before, we use $CFSf$ and $CFSg$ to represent $CFS(f(t), n, \omega)$ and $CFS(g(t), n, \omega)$ respectively. From now on, denote A_i 's and B_i 's in $CFSf$ as $A_{f,i}$'s and $B_{f,i}$'s respectively, and same for $A_{g,i}$'s and $B_{g,i}$'s.

Number	DD3
Label	Approximation of function values from CFS
Symbol	$App : (CFST, \mathbb{R}) \rightarrow \mathbb{R}$
Equation	$App(CFSf, t_1) = \sum_{i=0}^n A_{f,i} \cos(i\omega t_1) + \sum_{i=1}^n B_{f,i} \sin(i\omega t_1)$
Description	Define approximated function value of a function calculated from its CFS.
Sources	Defined by author
Ref. By	IM3, C1, O1, and O2

From now on, we are going to introduce a series of operators/functions on $CFST$ type objects $CFSf$ (and possibly $CFSg$ in some cases). All data included in these objects are fully decided by the callers of functions/operators. The n and ω used in their DD's are these of the parameters (if there are multiple $CFST$ objects, their n and ω are required to be the same, and will be checked by the function/operator)

Number	DD4
Label	Addition of CFS's
Symbol	$+ : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf + CFSg = CFS(f(t) + g(t), n, \omega)$
Description	Define the rule of addition of two CFS's.
Sources	Defined by author
Ref. By	IM3, C1, O1, and O2

Number	DD5
Label	Subtraction of CFS's
Symbol	$- : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf - CFSg = CFS(f(t) - g(t), n, \omega)$
Description	Define the rule of subtraction of two CFS's
Sources	Defined by author
Ref. By	IM4, C1, O1, and O2

Number	DD6
Label	Multiplication of CFS's
Symbol	$* : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf * CFSg = CFS(f(t) * g(t), n, \omega)$
Description	Define the rule of multiplication of two CFS's
Sources	Defined by author
Ref. By	IM5, C1, O1, and O2

Number	DD7
Label	Division of CFS's
Symbol	$/ : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf / CFSg = CFS(f(t)/g(t), n, \omega)$
Description	Define the rule of division of two CFS's
Sources	Defined by author
Ref. By	IM6, C1, O1, and O2

Number	DD8
Label	Function of CFS's
Symbol	$CFSFunc : (CFST, \mathbb{R} \rightarrow \mathbb{R}) \rightarrow CFST$
Equation	$CFSFunc(CFSf, g) = CFS(g(f(t)), n, \omega).$
Description	Define the rule of the function of a CFS
Sources	Defined by author
Ref. By	IM7, A2, C1, and O1

Number	DD9
Label	Amplitude of a CFS
Symbol	$Amp : CFST \rightarrow \mathbb{R}^*$
Equation	$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} App^2(CSFf, t_1) dt_1} \quad (1)$
Description	Define the amplitude/size of a CFS
Sources	Defined by author
Ref. By	IM8, A1, C1, and O1

Number	DD10
Label	Tolerated equality of two CFS's
Symbol	$TolEq : (CFST, CFST, \mathbb{R}) \rightarrow Bool$
Equation	$TolEq(CFSf, CFSg, err) = (Amp(CFSf - CFSg) \leq err)$
Description	Define whether two CFS's are equal within a given tolerance
Sources	Defined by author
Ref. By	IM9, C1, and O1

4.3 Theoretical Models

This section focuses on the general equations and laws that FSL is based on.

Number	T1
Label	Fourier Transformation
Equation	$ \begin{aligned} A_0 &= \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t); \\ A_i &= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \cos(i\omega t), \quad i = 1 : n; \\ B_i &= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \sin(i\omega t), \quad i = 1 : n. \end{aligned} \tag{2} $
Description	The above equation calculates $A_{inf,i}$'s and $B_{inf,i}$'s, thus $A_i (i = 0 : n)$, $B_i (i = 1 : n)$ in $CFSf$ from given $f(t)$, n and ω . The definitions of these variables can be found in DD1 and DD2.
Source	Tolstov, G.P. and Silverman, R.A., Fourier Series, Dover Books on Mathematics, 1976, Dover Publications, Tolstov and Silverman (1976)
Ref. By	A1, C1, O1, and O2

For the following operations, one of the CFS's can be a constant a . In this case, the corresponding CFS shall be one with $A_0 = a$, $A_i = 0$, $B_i = 0$, ($i = 1 : n$).

Number	T2
Label	Addition of two CFS's
Equation	$ \begin{aligned} A_{f+g,i} &= A_{f,i} + A_{g,i}, \quad i = 0 : n \\ B_{f+g,i} &= B_{f,i} + B_{g,i}, \quad i = 1 : n \end{aligned} \tag{3} $
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) + g(t), n, \omega)$ (represented by $A_{f+g,i}$ and $B_{f+g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1, and O1
Number	T3
Label	Subtraction of two CFS's
Equation	$ \begin{aligned} A_{f-g,i} &= A_{f,i} - A_{g,i}, \quad i = 0 : n \\ B_{f-g,i} &= B_{f,i} - B_{g,i}, \quad i = 1 : n \end{aligned} \tag{4} $
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) - g(t), n, \omega)$ (represented by $A_{f-g,i}$ and $B_{f-g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1, and O1

Number	T4
Label	Multiplication of two CFS's
Equation	$ \begin{aligned} A_{f*g,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f,i}A_{g,i+j} + A_{f,i+j}A_{g,i} + B_{f,i}B_{g,i+j} + B_{f,i+j}B_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f,j}A_{g,i-j} - B_{f,j}B_{g,i-j}), \quad i = 0 : n \\ B_{f*g,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f,i}B_{g,i+j} + A_{f,i+j}B_{g,i} + B_{f,i}A_{g,i+j} + B_{f,i+j}A_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f,j}B_{g,i-j} + B_{f,j}A_{g,i-j}), \quad i = 1 : n \end{aligned} \tag{5} $
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) * g(t), n, \omega)$ (represented by $A_{f*g,i}$ and $B_{f*g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1, and O1

Number	T5
Label	Division of two CFS's
Equation	<p>Solve the following equations for $A_{f/g,i}$'s and $B_{f/g,i}$'s.</p> $ \begin{aligned} A_{f,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f/g,i} A_{g,i+j} + A_{f/g,i+j} A_{g,i} + B_{f/g,i} B_{g,i+j} + B_{f/g,i+j} B_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f/g,j} A_{g,i-j} - B_{f/g,j} B_{g,i-j}), \quad i = 0 : n \\ B_{f,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f/g,i} B_{g,i+j} + A_{f/g,i+j} B_{g,i} + B_{f/g,i} A_{g,i+j} + B_{f/g,i+j} A_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f/g,j} B_{g,i-j} + B_{f/g,j} A_{g,i-j}), \quad i = 1 : n \end{aligned} $ <p style="text-align: right;">(6)</p>
Description	We solve the above equations for $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t)/g(t), n, \omega)$ (represented by $A_{f/g,i}$'s and $B_{f/g,i}$'s respectively) from given $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1 , O1 , and O2

Number	T6
Label	Function of a CFS
Equation	$g(CFS(f(t), n, \omega)) \approx \sum_{i=0}^n a_i CFS^i(f(t), n, \omega)$
Description	The above equation calculates the result of a function being applied to a CFS. $a_i, i = 0 : n$ is the first $(n + 1)$ coefficients of the Taylor series of $g(t)$, and the n -th ($n \in \mathbb{Z}^+$) power of CFS is defined as n copies of this CFS multiplied together. (0-th power is defined as the CFS of $f(t) = 1$.)
Source	Developed by author in Appendix A
Ref. By	C1 and O1

Number	T7
Label	Amplitude of a CFS
Equation	$Amp(CSFf) = \sqrt{A_0^2 + \frac{1}{2} \sum_{i=1}^n (A_i^2 + B_i^2)} \quad (7)$
Description	The above equation calculates $Amp(CFSf)$ from $CFSf$, especially A_i 's and B_i 's in it.
Source	Developed by author in Appendix A
Ref. By	C1 and O1

5 Instance Models

Number	IM1
Label	Calculate coefficients of CFS's
Input	$f(t)$, a $[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ function $n \in \mathbb{Z}^*$ $\omega \in \mathbb{R}^+$
Output	An object $CFSf$, containing the following data: n, ω : same as the input $A_i(i = 0 : n), B_i(i = 1 : n)$: using the theory T1
Description	Input: $f(t)$: function to be transformed into Fourier series Input & Output: n : cut-off length ω : base frequency Output: A_i 's and B_i 's: coefficients in $CFSf$
Sources	Same as T1
Ref. By	A1, C1, O1, and O2

Number	IM2
Label	Calculate approximate function value from CFS
Input	$CFSf = [n, \omega, A_i(i = 0 : n), B_i(i = 1 : n)]$ $t_1 \in \mathbb{R}$
Output	$App(CFSf, t_1)$ using DD 3
Description	Input: $CFSf$: CFS used for function approximation t_1 : variable value Output: $App(CFSf, t_1)$: approximated function value
Sources	Same as T 3
Ref. By	C 1 and O 1

Number	IM3
Label	Addition of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t) + g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T 2
Description	Input: $CFSf, CFSg$: operands of the addition Output: $CFS(f(t) + g(t), n, \omega)$: result of the addition
Sources	Same as T 2
Ref. By	C 1 , and O 1

Number	IM4
Label	Subtraction of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t) - g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T3
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the subtraction</p> <p>Output:</p> <p>$CFS(f(t) - g(t), n, \omega)$: result of the subtraction</p>
Sources	Same as T3
Ref. By	C1, and O1

Number	IM5
Label	Multiplication of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t) * g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T4
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the multiplication</p> <p>Output:</p> <p>$CFS(f(t) * g(t), n, \omega)$: result of the multiplication</p>
Sources	Same as T4
Ref. By	C1, and O1

Number	IM6
Label	Division of two CFS's
Input	$CFSf$, $CFSg$
Output	$CFS(f(t)/g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T5
Description	<p>Input:</p> <p>$CFSf$, $CFSg$: operands of the division</p> <p>Output:</p> <p>$CFS(f(t) + g(t), n, \omega)$: result of the division</p>
Sources	Same as T5
Ref. By	C1, O1, and O2

Number	IM7
Label	Function of a CFS
Input	$CFSf$, $g(t)$
Output	A CFS $g(CFSf)$, whose A_i 's and B_i 's are computed using the theory T4 and T2
Description	<p>Input:</p> <p>$g(t)$, a basic function chosen by user from a basic function set.</p> <p>$CFSf$: dependent variable of the function $g(t)$</p> <p>Output:</p> <p>$g(CFSf)$: A CFS being the computed result</p>
Sources	Easily derived from T4 and T2
Ref. By	A2, C1, and O1

Number	IM8
Label	Amplitude of a CFS
Input	$CFSf$
Output	$Amp(CFSf)$, computed using the theory T7
Description	<p>Input:</p> <p>$CFSf$: variable of the amplitude function</p> <p>Output:</p> <p>$Amp(CFSf)$, the amplitude of $CFSf$</p>
Sources	Same as T7
Ref. By	C1, and O1

Number	IM9
Label	Tolerated Equality Comparison of two CFS's
Input	$CFSf, CFSg, tol$
Output	<p>$TolEq(CFSf, CFSg, tol)$</p> <p>boolean value True if $Amp(CFSf - CFSg) \leq tol$</p> <p>boolean value False otherwise</p>
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the tolerated equality comparison</p> <p>$tol \in \mathbb{R}^*$</p> <p>Output:</p> <p>A boolean value: Whether the two operands are equal within the given error tolerance</p> <p>Note: The calculation of difference's amplitude relies on IM4 and IM8.</p>
Sources	Same as T3 and T7
Ref. By	C1, and O1

Number	IM10
Label	Convert data of other structures (input sources included) to a CFS
Input	$n, \omega, n+1$ real numbers $Ain_i (i = 0 : n)$, and n real numbers $Bin_i (i = 1 : n)$
Output	A CFS object $CFSf$ constructed from the input data
Description	Input: Data needed for construction Output: Constructed CSF object containing the input data
Sources	None
Ref. By	C1, and O1

Number	IM11
Label	Convert CFS to data of structures (output destination included)
Input	$CFSf$
Output	$n, \omega, Aout_i = A_i (i = 0 : n)$, and $Bout_i = B_i (i = 1 : n)$
Description	Input: The CFS object to be converted to Output: The data extracted from the input CFS object.
Sources	None
Ref. By	C1, and O1

5.1 Assumptions

- A1: The functions $f(t)$ and $g(t)$ mentioned above must have definitions on $[-\pi/\omega, \pi/\omega]$, in which ω is given by the user, and must have corresponding Fourier series.
- A2: Only the basic functions of CFS's are to be calculated. The aforementioned basic functions are those in a library-defined set.

5.2 Calculation

- C1: Calculate the result based on the called function and input variables.

5.3 Output

- O1: Return the results faithfully. That is, return the constructed CFS in IM1 and IM10; fill the user-provided space with data from user-designated CFS in IM11; return the

calculated value in IM2, IM8, and IM9; and return the calculated CFS in IM3, IM4, IM5, IM6, and IM7.

O2: Report any detected errors to the user of the library.

6 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

6.1 Functional Requirements

- R1: When applicable, functions of this library accept inputs with the same data type for $f(t)$, A_i 's and/or B_i 's, and ω .
- R2: The operands of binary operations must have the same n and ω .
- R3: The error of the output must be decreasing as the n in CFS's is decreasing.
- R4: Output the result with the same floating point data type as that of the input.
- R5: Manage the memory spaces required by this library (mainly for CFS's), and destroy them the moment they are not needed.

6.2 Nonfunctional Requirements

- All time complexities shall be unrelated to ω .
- The time complexity of IM1 shall be $O(n^2)$ when the input function $f(t)$ is not complex.
- The time complexity of IM3 and IM4 shall be $O(n)$.
- The time complexity of IM5 shall be $O(n^2)$.
- The time complexity of IM6 shall be the same as the best linear equation solver applicable.
- The time complexity of IM8 and IM9 shall be $O(n)$.

7 Likely Changes

LC1: We might add new operations in the future.

LC2: The set of basic functions in IM7 will be decided in the future.

8 Traceability Matrices and Graphs

The following matrices and graphs demonstrates the traceability of this project. The purpose is to provide easy references to the impacts on other components if a certain component is changed. That is, if one component has been changed, other components that share an 'X' with it may need change. Table 1 shows the the dependencies of goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs.

	A1	A2	C1	O1	O2
GS1	X		X	X	X
GS2		X	X	X	
GS3			X	X	X
GS4	X		X	X	X
GS5			X	X	
GS6			X	X	
GS7				X	
GS8			X	X	X
DD1	X		X	X	X
DD2	X		X	X	X
DD3			X	X	X
DD4			X	X	X
DD5			X	X	X
DD6			X	X	X
DD7			X	X	X
DD8		X	X	X	
DD9			X	X	
T1	X		X	X	X
T2			X	X	
T3			X	X	
T4			X	X	
T5			X	X	X
T7			X	X	
IM1	X		X	X	X
IM3			X	X	
IM4			X	X	
IM5			X	X	
IM6			X	X	X
IM7		X	X	X	
IM8			X	X	
IM9			X	X	
IM10			X	X	
IM11			X	X	

Table 1: The traceability matrix between goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs

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A Theory for operations on CFS's

In this appendix, we hereby give the proof of T2, T3, T4, and T7 in corresponding paragraphs. The proof of T5 comes directly from the equation $[f(t)/g(t)] * g(t) = f(t)$.

In the following proofs, suppose we have two functions, $f(t)$ and $g(t)$ with existing IFS and CFS. The n and ω of these IFS's and CFS's are the same, but with different A_i 's and B_i 's (denoted with $A_{f,i}$'s, $B_{f,i}$'s and $A_{g,i}$'s, $B_{g,i}$'s respectively). From the definition of IFS, DD1, we know that

$$f(t) = \sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t), \quad (8)$$

and

$$g(t) = \sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t). \quad (9)$$

Addition and Subtraction Like $f(t)$ and $g(t)$, we also know that

$$f(t) + g(t) = \sum_{i=0}^{+\infty} A_{f+g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i} \sin(i\omega t). \quad (10)$$

By replacing $f(t)$ and $g(t)$ in Equation 10 with Equation 8 and Equation 9, we have

$$\begin{aligned} & \sum_{i=0}^{+\infty} A_{f+g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i} \sin(i\omega t) \\ &= \sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t) + \sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t) \\ &= \sum_{i=0}^{+\infty} (A_{f,i} + A_{g,i}) \cos(i\omega t) + \sum_{i=1}^{+\infty} (B_{f,i} + B_{g,i}) \sin(i\omega t) \end{aligned} \quad (11)$$

By comparing the coefficients in Equation 11, we have

$$\begin{aligned} A_{f+g,i} &= A_{f,i} + A_{g,i} \\ B_{f+g,i} &= B_{f,i} + B_{g,i} \end{aligned} \quad (12)$$

for the IFS and CFS of $f(t) + g(t)$.

Likewise, we have similar conclusions for those of $f(t) - g(t)$.

Multiplication From Equation 8 and Equation 9, we have

$$f(t) * g(t) = \left[\sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t) \right] * \left[\sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t) \right] \quad (13)$$

which is the summation the following 3 terms

$$\begin{aligned}
\text{Term A: } & \sum_{i=0, j=0}^{+\infty} A_{f,i} A_{g,j} \cos(i\omega t) \cos(j\omega t) \\
&= \frac{1}{2} \sum_{i=0, j=0}^{+\infty} A_{f,i} A_{g,j} \cos[(i+j)\omega t] + \frac{1}{2} \sum_{i=0, j=0}^{+\infty} A_{f,i} A_{g,j} \cos[(j-i)\omega t] \\
&= \frac{1}{2} \sum_{i=0}^{+\infty} \left[\sum_{j=0}^i A_{f,j} A_{g,i-j} \right] \cos(i\omega t) + \frac{1}{2} \sum_{j=0}^{+\infty} A_{f,i} A_{g,i} + \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=0}^{+\infty} A_{f,j} A_{g,j+i} + \sum_{j=0}^{+\infty} A_{f,j+i} A_{g,j} \right] \cos(i\omega t) \\
&= A_{f,0} A_{g,0} + \frac{1}{2} \sum_{j=0}^{+\infty} A_{f,i} A_{g,j} + \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=0}^i A_{f,j} + A_{g,i-j} + \sum_{j=0}^{+\infty} (A_{f,i} A_{g,j+i} + A_{f,j+i} A_{g,i}) \right] \cos(i\omega t) \\
\text{Term B: } & \sum_{i=0, j=1}^{+\infty} [A_{f,i} B_{g,j} \cos(i\omega t) \sin(j\omega t) + B_{g,i} A_{f,j} \sin(i\omega t) \cos(j\omega t)] \\
&= \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [A_{f,i} B_{g,j} \sin((i+j)\omega t) - A_{f,i} B_{g,j} \sin((i-j)\omega t)] \\
&+ \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [B_{g,i} A_{f,j} \sin((i+j)\omega t) + B_{g,i} A_{f,j} \sin((i-j)\omega t)] \\
&= \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [A_{f,i} B_{g,j} + B_{g,i} A_{f,j}] \sin((i+j)\omega t) + \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [B_{g,i} A_{f,j} - A_{f,i} B_{g,j}] \sin((i-j)\omega t) \\
&= \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=1}^{+\infty} A_{f,i-j} B_{g,j} + B_{g,i-j} A_{f,j} \right] \sin(i\omega t) - \frac{1}{2} \sum_{i=1}^{+\infty} [B_{g,0} A_{f,i} - A_{f,0} B_{g,i}] \sin(i\omega t) \\
&= \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=1}^{+\infty} (A_{f,i-j} B_{g,j} + B_{g,i-j} A_{f,j}) - B_{g,0} A_{f,i} + A_{f,0} B_{g,i} \right] \sin(i\omega t) \\
\text{Term C: } & \sum_{i=1, j=1}^{+\infty} B_{f,i} B_{g,j} \sin(i\omega t) \sin(j\omega t) \\
&= \frac{1}{2} \sum_{i=1, j=1}^{+\infty} B_{f,i} B_{g,j} [\cos((i-j)\omega t) - \cos((i+j)\omega t)] \\
&= \frac{1}{2} \left[\sum_{j=1}^{+\infty} B_{f,j} B_{g,j} \right] \cos(0\omega t) + \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=1}^{+\infty} B_{f,i+j} B_{g,i} + B_{f,i} B_{g,i+j} \right] \cos(i\omega t) \\
&- \frac{1}{2} \sum_{i=2}^{+\infty} \left[\sum_{j=1}^{i-1} B_{f,i-j} B_{g,j} \right] \cos(i\omega t)
\end{aligned} \tag{14}$$

We gather the coefficients of $\cos(i\omega t)$, $i = 0 : n$ and $\sin(i\omega t)$, $i = 1 : n$ respectively, remove any terms containing $A_{f,k}$, $B_{f,k}$, $A_{g,k}$, and $B_{g,k}$ for any $k \geq n$ (k being either i , j , $i - j$, or $i + j$) as the result of a cut-off, and get the equations in T4.

Amplitude Quoted from DD9, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} App^2(CFSf, t) dt} \quad (15)$$

In Equation 15, replacing $App(CFSf, t)$ with its definition in DD3, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} \left[\sum_{i=0}^n A_{f,i} \cos(i\omega t) + \sum_{i=1}^n B_{f,i} \sin(i\omega t) \right] \left[\sum_{j=0}^n A_{f,j} \cos(j\omega t) + \sum_{j=1}^n B_{f,j} \sin(j\omega t) \right] dt} \quad (16)$$

The part inside the integration in Equation 16 can be expressed as the summation of the following three expressions

$$\begin{aligned} & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=0}^n A_{f,i} A_{f,j} \cos(i\omega t) \cos(j\omega t) dt \\ & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=1}^n A_{f,i} B_{f,j} \sin(i\omega t) \cos(j\omega t) dt + \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=0}^n B_{f,i} A_{f,j} \sin(i\omega t) \cos(j\omega t) dt \\ & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=1}^n B_{f,i} B_{f,j} \sin(i\omega t) \sin(j\omega t) dt \end{aligned} \quad (17)$$

The integration part of these expressions are $\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \cos(j\omega t)$, $\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \sin(j\omega t)$, and $\int_{-\pi/\omega}^{\pi/\omega} \sin(i\omega t) \sin(j\omega t)$. Calculation shows that the integration results are

$$\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \cos(j\omega t) dt = \begin{cases} 2\pi/\omega, & i = j = 0; \\ \pi/\omega, & i = j \neq 0; \\ 0, & i \neq j. \end{cases} \quad (18)$$

$$\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \sin(j\omega t) dt = 0 \quad (19)$$

and

$$\int_{-\pi/\omega}^{\pi/\omega} \sin(i\omega t) \sin(j\omega t) dt = \begin{cases} \pi/\omega, & i = j; \\ 0, & i \neq j. \end{cases} \quad (20)$$

Replacing integration in Equation 17 with integration results Equation 18, Equation 19, and Equation 20, and we have the following results.

$$\begin{aligned}
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=0}^n A_{f,i} A_{f,j} \cos(i\omega t) \cos(j\omega t) dt &= \frac{2\pi}{\omega} A_0^2 + \frac{\pi}{\omega} \sum_{i=1}^n A_i^2 \\
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=1}^n A_{f,i} B_{f,j} \sin(i\omega t) \cos(j\omega t) dt &= 0 \\
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=0}^n B_{f,i} A_{f,j} \sin(i\omega t) \cos(j\omega t) dt &= 0 \\
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=1}^n B_{f,i} B_{f,j} \sin(i\omega t) \sin(j\omega t) dt &= \frac{\pi}{\omega} \sum_{i=1}^n B_i^2
\end{aligned} \tag{21}$$

Summing them up and putting the result back into Equation 16, we have the expression in T7.

Function We will develop the equations of $A_i (i > 0)$'s in the resulted CFS. A_0 and B_i 's can be developed similarly.

$$A_i = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} \pi/\omega g(f(t)) \cos(i\omega t) dt \tag{22}$$

$$= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} \sum_{j=0}^{+\infty} a_j f(t)^j \cos(i\omega t) dt \tag{23}$$

$$= \sum_{j=0}^{+\infty} a_j \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t)^j \cos(i\omega t) dt \tag{24}$$

$$= \sum_{j=0}^{+\infty} a_j (CFS f^j).A_i \tag{25}$$

Intuitively (detailed analysis of introduced error in the future), as an approximation, we do a similar cut-off on the Taylor series. That is, we turn $\sum_{j=0}^{+\infty}$ into $\sum_{j=0}^n$, and we have

$$A_i = \sum_{j=0}^n a_j (CFS f^j).A_i \tag{26}$$

Combining with similar resulted expressions of A_0 and B_i 's, we have the equations in T6.

B List of Basic Functions

The term *Basic Functions* in this library consists of the following functions

- $\sin(x)$
- $\cos(x)$
- $\tan(x)$
- $\arcsin(x)$
- $\arccos(x)$
- $\arctan(x)$
- $\exp(x)$
- $\ln(x + 1)$

We might include more functions if deemed necessary.