

FSL
A Fourier Series Library

Bo Cao

October 10, 2019

1 Revision History

Date	Version	Notes
Oct. 10, 2019	0.9	Draft submitted for review by Dr. Smith
TBD	1.0	Revised in accordance with advice from Dr. Smith

2 Reference Material

This section records information for easy reference.

2.1 Table of Units

This section is not applicable, due to the fact that this library is a pure mathematical computation library.

2.2 Table of Symbols

The table that follows summarizes the symbols used in this document. The choice of symbols is made to be consistent with the Fourier series literature and with existing documentation for Fourier series libraries.

symbol	description
$f(t), g(t), \dots$	$[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ functions of t , where ω is defined as below
\mathbb{R}	Set of real numbers
\mathbb{Z}	Set of integers
$[-\pi/\omega, \pi/\omega]$	Set of real numbers that are neither smaller than $-\pi/\omega$, nor greater than π/ω
\rightarrow	Indicate mapping
\mathbb{A}^*	Non-negative subset of set \mathbb{A} (either \mathbb{R} or \mathbb{Z})
\mathbb{A}^+	Positive subset of set \mathbb{A} (either \mathbb{R} or \mathbb{Z})
$\sum_{i=m_1}^{m_2} a_i$	Summation of $a_i, i = m_1, m_1 + 1, \dots, m_2$
n	Length of cut-off Fourier series
ω	Base frequency of Fourier series
$CFSf, CFSg$	$CFS(f(t), n, \omega)$ and $CFS(g(t), n, \omega)$ respectively.

2.3 Abbreviations and Acronyms

symbol	description
A	Assumption
API	Application Programming Interface
CA	Commonality Analysis
CFS	Cut-off Fourier Series
DD	Data Definition
FS	Fourier Series
GD	General Definition
GS	Goal Statement
IFS	Infinite Fourier Series
IM	Instance Model
LC	Likely Change
PS	Physical System Description
R	Requirement
FSL	Fourier Series Library.
T	Theoretical Model

Contents

1	Revision History	i
2	Reference Material	ii
2.1	Table of Units	ii
2.2	Table of Symbols	ii
2.3	Abbreviations and Acronyms	iii
3	Introduction	1
3.1	Purpose of Document	1
3.2	Scope of the Family	1
3.3	Characteristics of Intended Reader	1
3.4	Organization of Document	1
4	General System Description	2
4.1	Potential System Contexts	2
4.2	Potential User Characteristics	2
4.3	Potential System Constraints	2
5	Commonalities	2
5.1	Background Overview	2
5.2	Terminology and Definitions	3
5.3	Data Definitions	3
5.4	Goal Statements	7
5.5	Theoretical Models	7
6	Variabilities	11
6.1	Instance Models	11
6.2	Assumptions	16
6.3	Calculation	16
6.4	Output	16
7	Requirements	16
7.1	Functional Requirements	17
7.2	Nonfunctional Requirements	17
8	Likely Changes	17
9	Traceability Matrices and Graphs	17
A	Theory for operations on CFS's	20

3 Introduction

3.1 Purpose of Document

This document introduces a library, FSL, mainly designed and implemented for Fourier series related calculations. This document is based on the template introduced in [Smith \(2006\)](#), [Smith and Lai \(2005\)](#), and [Smith et al. \(2017\)](#).

3.2 Scope of the Family

The scope of the family is limited to calculations related to the Fourier series of mathematical functions.

3.3 Characteristics of Intended Reader

The intended readers must have the following knowledge.

- Deep knowledge of Fourier series (found in advanced calculus and real analysis courses);
- Computational error analysis (found in introductory numerical analysis classes); and
- Detailed knowledge of the programming language utilized in the implementation of this library.

3.4 Organization of Document

This document consists of the following sections.

- [section 4](#) generally describes this library, including what it is, who uses it, and what kinds of environment it needs.
- [section 5](#) introduces the common elements of this library. This section overviews its background, defines related data, introduces theories needed for calculation, and states the goals of this library.
- [section 6](#) covers the different functions of this library, and states its variability in assumptions, calculations, and outputs.
- [section 7](#) informs users, developers, maintainers, and reviewers of this library, about its requirements, both functional ones and nonfunctional ones.
- [section 8](#) states the changes of this library that are likely in the future.
- [section 9](#) shows the traceability matrix of this library, to inform the developers of their changes' impacts.
- [Appendix A](#) gives proof of the operation-related theories in [section 5](#).

4 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

4.1 Potential System Contexts

The users provide the data for the library to calculate, and the library returns the calculation results.

- User Responsibilities:
 - Provide the inputs of the functions in the library.
 - When needed, allocate memory space, and pass their location to the library, so that the library can store the output in this user-provided space.
 - For the conversion from mathematical functions to Fourier series, ensure that provided mathematical functions are eligible for transformation.
- FSL Responsibilities:
 - Detect data type mismatch, such as a string of characters instead of a floating point number.
 - Unless otherwise mentioned in this document, detect illegality of input values, such as mismatched length of Fourier series for binary operations.
 - Unless otherwise mentioned, manage the memory space required by this library.

4.2 Potential User Characteristics

The end user of FSL should have an understanding of undergraduate level 2 Calculus and/or Real Analysis, undergraduate level 1 Numerical Analysis, and any one of the programming languages, in which this library provides a set of APIs.

4.3 Potential System Constraints

The potential systems must contain the compilation/interpretation environment for the programming languages in which this library is developed, as well as dependent libraries.

5 Commonalities

5.1 Background Overview

Fourier series are used in lots of analysis, analysis of signals as an example [Papoulis \(1977\)](#). The Fourier series of a function mainly uses a linear combination of 'sin' and 'cos' functions to

represent and approximate this function. In real-life research, we sometimes need to perform Fourier series related operations. For example, when we need to analyze the total effects of two signals, each approximated by its Fourier series, we need to calculate the summation of these two Fourier series. Or, when we need to calculate the strength of a signal from its Fourier series, we need to calculate this Fourier series' amplitude.

5.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- Binary operations: an operation that involves two operands.

[This section needs further expansion. —Author]

5.3 Data Definitions

This section collects and defines all the data needed to build the instance models.

Number	DD1
Label	Infinite Fourier Series (IFS) of mathematical functions
Symbol	<i>IFS</i>
Equation	$IFS(f(t), \omega) = [A_{inf,i}, i = 0, 1, \dots; B_{inf,i}, i = 1, 2, \dots]$, satisfying $f(t) = \sum_{i=0}^{+\infty} A_{inf,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{inf,i} \sin(i\omega t)$
Description	<p>In this equation,</p> <ul style="list-style-type: none"> • $f(t)$ is an $[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ function, whose IFS exists; • ω is the base frequency of this IFS, and it is designated by the user; • $A_{inf,i} \in \mathbb{R}, i = 0, 1, \dots$; and • $B_{inf,i} \in \mathbb{R}, i = 1, 2, \dots$
Sources	Tolstov, G.P. and Silverman, R.A., Fourier Series, Dover Books on Mathematics, 1976, Dover Publications, Tolstov and Silverman (1976)
Ref. By	IM2, A1, C1, C??, O1, and O2

Number	DD2
Label	Cut-off Fourier Series (CFS) of mathematical functions
Symbol	CFS
Equation	$CFS(f(t), n, \omega) = [A_i, i = 0, 1, \dots, n; B_i, i = 1, 2, \dots, n]$, satisfying $A_i = A_{inf,i}, i = 0, 1, \dots, n; B_i = B_{inf,i}, i = 1, 2, \dots, n$, where ω , $A_{inf,i}$'s and $B_{inf,i}$'s come from $IFS(f(t), \omega)$
Description	$n \in \mathbb{Z}^*$ is called cut-off length.
Sources	Defined by author
Ref. By	T1, T2, T3, T4, T5, T6, A1, C1, C??, C??, O1, and O2

From now on, denote A_i 's and B_i 's in $CFSf$ as $A_{f,i}$'s and $B_{f,i}$'s respectively, and same for $A_{g,i}$'s and $B_{g,i}$'s.

Number	DD3
Label	Approximation of function values from CFS
Symbol	App
Equation	$App(CFSf, t_1) = \sum_{i=0}^n A_{f,i} \cos(i\omega t_1) + \sum_{i=1}^n B_{f,i} \sin(i\omega t_1)$
Description	Define approximated function value of a function calculated from its CFS.
Sources	Defined by author
Ref. By	IM2, C1, C??, O1, and O2

Number	DD4
Label	Addition of CFS's
Symbol	+
Equation	$CFSf + CFSg = CFS(f(t) + g(t), n, \omega)$
Description	Define the rule of addition of two CFS's
Sources	Defined by author
Ref. By	IM2, C1, C??, O1, and O2

Number	DD5
Label	Subtraction of CFS's
Symbol	–
Equation	$CFSf - CFSg = CFS(f(t) - g(t), n, \omega)$
Description	Define the rule of subtraction of two CFS's
Sources	Defined by author
Ref. By	IM3, C1, C??, O1, and O2

Number	DD6
Label	Multiplication of CFS's
Symbol	*
Equation	$CFSf * CFSg = CFS(f(t) * g(t), n, \omega)$
Description	Define the rule of multiplication of two CFS's
Sources	Defined by author
Ref. By	IM4, C1, C??, O1, and O2

Number	DD7
Label	Division of CFS's
Symbol	/
Equation	$CFSf / CFSg = CFS(f(t)/g(t), n, \omega)$
Description	Define the rule of division of two CFS's
Sources	Defined by author
Ref. By	IM5, C1, C??, O1, and O2

Number	DD8
Label	Function of CFS's
Symbol	$g(CFS(f(t), n, \omega))$
Equation	$g(CFS(f(t), n, \omega)) = \sum_{i=0}^n a_i CFS^i(f(t), n, \omega)$, in which $a_i, i = 0 : n$ is the first $(n + 1)$ coefficients of the Taylor series of $g(t)$, and the n -th ($n \in \mathbb{Z}^+$) power of CFS is defined as n copies of this CFS multiplied together. (0-th power is defined as the CFS of $f(t) = 1$.)
Description	Define the rule of the function of a CFS
Sources	Defined by author
Ref. By	IM6, A2, C1, and O1

Number	DD9
Label	Amplitude of a CFS
Symbol	$Amp(CFSf)$
Equation	$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} App^2(CSFf, t_1) dt_1} \quad (1)$
Description	Define the amplitude/size of a CFS
Sources	Defined by author
Ref. By	IM7, A1, C1, and O1

Number	DD10
Label	Tolerated equality of two CFS's
Symbol	=
Equation	$CFSf = CFSg$ is the same as $Amp(CFSf - CFSg) \leq err$, where err is an user-given tolerance.
Description	Define whether two CFS's are equal within a given tolerance
Sources	Defined by author
Ref. By	IM8, C1, and O1

5.4 Goal Statements

Given the corresponding inputs, the goal statements are:

- GS1: When given a function $f(t)$, the cut-off length n , and a base frequency ω , return the function's CFS $CFS(f(t), n, \omega)$.
- GS2: When given a CFS $CFS(f(t), n, \omega)$ and a value of t as t_1 , return the approximated value of $f(t_1)$ computed from the given values.
- GS3: When given two CFS's $CFSf$, $CFSg$, and one of the operations addition, subtraction, multiplication, and division, return the result of this operation on these two CFS's.
- GS4: When given a CFS $CFSf$, and a function $g(t)$ from the basic function sets defined by this library, return the CFS $g(CFSf)$. [\[I don't know whether to define the basic function set here or not. I can, and I don't see any possible changes —Author\]](#)
- GS5: When given a set of values including n , ω , A_i 's, and B_i 's, return the CFS built on these values.
- GS6: When given a CFS, store its values of n , ω , A_i 's, and B_i 's in the user-designated spaces.
- GS7: When given a CFS, return its amplitude.
- GS8: When given two CFS's and a tolerance, return whether they are equal within the tolerance.

5.5 Theoretical Models

This section focuses on the general equations and laws that FSL is based on.

Number	T1
Label	Fourier Transformation
Equation	$ \begin{aligned} A_0 &= \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t); \\ A_i &= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \cos(i\omega t), \quad i = 1 : n; \\ B_i &= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \sin(i\omega t), \quad i = 1 : n. \end{aligned} \tag{2} $
Description	The above equation calculates $A_{inf,i}$'s and $B_{inf,i}$'s, thus $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFSf$ from given $f(t)$, n and ω .
Source	Tolstov, G.P. and Silverman, R.A., Fourier Series, Dover Books on Mathematics, 1976, Dover Publications, Tolstov and Silverman (1976)
Ref. By	A1 , C1 , C?? , O1 , and O2

Number	T2
Label	Addition of two CFS's
Equation	$ \begin{aligned} A_{f+g,i} &= A_{f,i} + A_{g,i}, \quad i = 0 : n \\ B_{f+g,i} &= B_{f,i} + B_{g,i}, \quad i = 1 : n \end{aligned} \tag{3} $
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) + g(t), n, \omega)$ (represented by $A_{f+g,i}$ and $B_{f+g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	A3 , C1 , and O1

Number	T3
Label	Subtraction of two CFS's
Equation	$ \begin{aligned} A_{f-g,i} &= A_{f,i} - A_{g,i}, \quad i = 0 : n \\ B_{f-g,i} &= B_{f,i} - B_{g,i}, \quad i = 1 : n \end{aligned} \tag{4} $
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) - g(t), n, \omega)$ (represented by $A_{f-g,i}$ and $B_{f-g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	A3 , C1 , and O1

Number	T4
Label	Multiplication of two CFS's
Equation	$ \begin{aligned} A_{f*g,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f,i} A_{g,i+j} + A_{f,i+j} A_{g,i} + B_{f,i} B_{g,i+j} + B_{f,i+j} B_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f,j} A_{g,i-j} - B_{f,j} B_{g,i-j}), \quad i = 0 : n \\ B_{f*g,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f,i} B_{g,i+j} + A_{f,i+j} B_{g,i} + B_{f,i} A_{g,i+j} + B_{f,i+j} A_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f,j} B_{g,i-j} + B_{f,j} A_{g,i-j}), \quad i = 1 : n \end{aligned} \tag{5} $
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) * g(t), n, \omega)$ (represented by $A_{f*g,i}$ and $B_{f*g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	A3 , C1 , and O1

Number	T5
Label	Division of two CFS's
Equation	<p>Solve the following equations for $A_{f/g,i}$'s and $B_{f/g,i}$'s.</p> $ \begin{aligned} A_{f,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f/g,i} A_{g,i+j} + A_{f/g,i+j} A_{g,i} + B_{f/g,i} B_{g,i+j} + B_{f/g,i+j} B_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f/g,j} A_{g,i-j} - B_{f/g,j} B_{g,i-j}), \quad i = 0 : n \\ B_{f,i} &= \frac{1}{2} \sum_{j=0}^{n-i} (A_{f/g,i} B_{g,i+j} + A_{f/g,i+j} B_{g,i} + B_{f/g,i} A_{g,i+j} + B_{f/g,i+j} A_{g,i}) \\ &\quad + \frac{1}{2} \sum_{j=0}^i (A_{f/g,j} B_{g,i-j} + B_{f/g,j} A_{g,i-j}), \quad i = 1 : n \end{aligned} \tag{6} $
Description	We solve the above equations for $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t)/g(t), n, \omega)$ (represented by $A_{f/g,i}$'s and $B_{f/g,i}$'s respectively) from given $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	A3 , C1 , C?? , O1 , and O2

Number	T6
Label	Amplitude of a CFS
Equation	$Amp(CSFf) = \sqrt{A_0^2 + \frac{1}{2} \sum_{i=1}^n (A_i^2 + B_i^2)} \tag{7}$
Description	The above equation calculates $Amp(CFSf)$ from $CFSf$, especially A_i 's and B_i 's in it.
Source	Developed by author in Appendix A
Ref. By	C1 and O1

6 Variabilities

6.1 Instance Models

Number	IM1
Label	Calculate coefficients of CFS's
Input	$f(t)$, a $[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ function $n \in \mathbb{Z}^*$ $\omega \in \mathbb{R}^+$
Output	An object $CFSf$, containing the following data: n, ω : same as the input $A_i(i = 0 : n), B_i(i = 1 : n)$: using the theory T1
Description	Input: $f(t)$: function to be transformed into Fourier series Input & Output: n : cut-off length ω : base frequency Output: A_i 's and B_i 's: coefficients in $CFSf$
Sources	Same as T1
Ref. By	A1, C1, C??, O1, and O2

Number	IM2
Label	Addition of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t) + g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T2
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the addition</p> <p>Output:</p> <p>$CFS(f(t) + g(t), n, \omega)$: result of the addition</p>
Sources	Same as T2
Ref. By	A3, C1, and O1

Number	IM3
Label	Subtraction of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t) - g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T3
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the subtraction</p> <p>Output:</p> <p>$CFS(f(t) - g(t), n, \omega)$: result of the subtraction</p>
Sources	Same as T3
Ref. By	A3, C1, and O1

Number	IM4
Label	Multiplication of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t) * g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T4
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the multiplication</p> <p>Output:</p> <p>$CFS(f(t) * g(t), n, \omega)$: result of the multiplication</p>
Sources	Same as T4
Ref. By	A3, C1, and O1

Number	IM5
Label	Division of two CFS's
Input	$CFSf, CFSg$
Output	$CFS(f(t)/g(t), n, \omega)$, whose A_i 's and B_i 's are computed using the theory T5
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the division</p> <p>Output:</p> <p>$CFS(f(t) + g(t), n, \omega)$: result of the division</p>
Sources	Same as T5
Ref. By	A3, C1, C??, O1, and O2

Number	IM6
Label	Function of a CFS
Input	$CFSf, g(t)$
Output	A CFS $g(CFSf)$, whose A_i 's and B_i 's are computed using the theory T4 and T2
Description	<p>Input:</p> <p>$g(t)$, a basic function chosen by user from a basic function set.</p> <p>$CFSf$: dependent variable of the function $g(t)$</p> <p>Output:</p> <p>$g(CFSf)$: A CFS being the computed result</p>
Sources	Easily derived from T4 and T2
Ref. By	A2, C1, and O1

Number	IM7
Label	Amplitude of a CFS
Input	$CFSf$
Output	$Amp(CFSf)$, computed using the theory T6
Description	<p>Input:</p> <p>$CFSf$: variable of the amplitude function</p> <p>Output:</p> <p>$Amp(CFSf)$, the amplitude of $CFSf$</p>
Sources	Same as T6
Ref. By	C1, and O1

Number	IM8
Label	Tolerated Equality Comparison of two CFS's
Input	$CFSf, CFSg, tol$
Output	boolean value True if $Amp(CFSf - CFSg) \leq tol$ boolean value False otherwise
Description	<p>Input:</p> <p>$CFSf, CFSg$: operands of the tolerated equality comparison</p> <p>$tol \in \mathbb{R}^*$</p> <p>Output:</p> <p>A boolean value: Whether the two operands are equal within the given error tolerance</p> <p>Note: The calculation of difference's amplitude relies on IM3 and IM7.</p>
Sources	Same as T3 and T6
Ref. By	A3, C1, and O1

Number	IM9
Label	Convert data of other structures (input sources included) to a CFS
Input	$n, \omega, n+1$ real numbers $Ain_i (i = 0 : n)$, and n real numbers $Bin_i (i = 1 : n)$
Output	A CFS object $CFSf$ constructed from the input data
Description	<p>Input: Data needed for construction</p> <p>Output: Constructed CSF object containing the input data</p>
Sources	None
Ref. By	A4, C1, C??, and O1

Number	IM10
Label	Convert CFS to data of structures (output destination included)
Input	$CFSf$
Output	$n, \omega, Aout_i = A_i(i = 0 : n), \text{ and } Bout_i = B_i(i = 1 : n)$
Description	Input: The CFS object to be converted to Output: The data extracted from the input CFS object.
Sources	None
Ref. By	A4, C1, C??, and O1

6.2 Assumptions

- A1: The functions $f(t)$ and $g(t)$ mentioned above must have definitions on $[-\pi/\omega, \pi/\omega]$, in which ω is given by the user, and must have corresponding Fourier series.
- A2: Only the basic functions of CFS's are to be calculated. The aforementioned basic functions are those in a library-defined set.
- A3: For any two-operand operations, the data n and ω of these operands must be the same.
- A4: User shall allocate memory space for any variable other than CFS's in IM9 and IM10.

6.3 Calculation

- C1: Calculate the result based on the called function and input variables.

6.4 Output

- O1: Return the results faithfully.
- O2: Report any detected errors to the user of the library.

7 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

7.1 Functional Requirements

- R1: When applicable, functions of this library accept inputs with the same data type for $f(t)$, A_i 's and/or B_i 's, and ω .
- R2: The error of the output must be decreasing as the n in CFS's is decreasing.
- R3: Output the result with the same floating point data type as that of the input.
- R4: If the called function detects that the input variables do not meet the requirements, generate an error message describing the detected error.
- R5: Manage the memory spaces required by this library (mainly for CFS's), and destroy them the moment they are not needed.

7.2 Nonfunctional Requirements

- All time complexities shall be unrelated to ω .
- The time complexity of IM1 shall be $O(n^2)$ when the input function $f(t)$ is not complex.
- The time complexity of IM2 and IM3 shall be $O(n)$.
- The time complexity of IM4 shall be $O(n^2)$.
- The time complexity of IM5 shall be the same as the best linear equation solver applicable.
- The time complexity of IM7 and IM8 shall be $O(n)$.

8 Likely Changes

LC1: We might add new operations in the future.

LC2: The set of basic functions in IM6 will be decided in the future.

9 Traceability Matrices and Graphs

The following matrices and graphs demonstrates the traceability of this project. The purpose is to provide easy references to the impacts on other components if a certain component is changed. That is, if one component has been changed, other components that share an 'X' with it may need change. Table 1 shows the the dependencies of goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs.

	A1	A2	A3	A4	C1	C??	C??	O1	O2
GS1	X				X	X		X	X
GS2		X			X			X	
GS3			X		X	X		X	X
GS4	X				X	X		X	X
GS5				X	X		X	X	
GS6				X	X		X	X	
GS7				X				X	
GS8			X		X	X		X	X
DD1	X				X	X		X	X
DD2	X				X	X	X	X	X
DD3					X	X		X	X
DD4			X		X	X		X	X
DD5			X		X	X		X	X
DD6			X		X	X		X	X
DD7			X		X	X		X	X
DD8		X			X			X	
DD9					X			X	
T1	X				X	X		X	X
T2			X		X			X	
T3			X		X			X	
T4			X		X			X	
T5			X		X	X		X	X
T6					X			X	
IM1	X				X	X		X	X
IM2			X		X			X	
IM3			X		X			X	
IM4			X		X			X	
IM5			X		X	X		X	X
IM6		X			X			X	
IM7					X			X	
IM8			X		X			X	
IM9				X	X		X	X	
IM10				X	X		X	X	

Table 1: The traceability matrix between goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs

References

- A. Papoulis. *Signal analysis*. McGraw-Hill electrical and electronic engineering series. McGraw-Hill, 1977. ISBN 9780070484603. URL <https://books.google.ca/books?id=Re5SAAAAAAAJ>.
- W. Spencer Smith. Systematic development of requirements documentation for general purpose scientific computing software. In *Proceedings of the 14th IEEE International Requirements Engineering Conference, RE 2006*, pages 209–218, Minneapolis / St. Paul, Minnesota, 2006. URL <http://www.ifi.unizh.ch/req/events/RE06/>.
- W. Spencer Smith and Lei Lai. A new requirements template for scientific computing. In J. Ralyté, P. Ågerfalk, and N. Kraiem, editors, *Proceedings of the First International Workshop on Situational Requirements Engineering Processes – Methods, Techniques and Tools to Support Situation-Specific Requirements Engineering Processes, SREP’05*, pages 107–121, Paris, France, 2005. In conjunction with 13th IEEE International Requirements Engineering Conference.
- W. Spencer Smith, John McCutchan, and Jacques Carette. Commonality analysis for a family of material models. Technical Report CAS-17-01-SS, McMaster University, Department of Computing and Software, 2017.
- G.P. Tolstov and R.A. Silverman. *Fourier Series*. Dover Books on Mathematics. Dover Publications, 1976. ISBN 9780486633176. URL <https://books.google.ca/books?id=XqqNDQeLfAkC>.

A Theory for operations on CFS's

In this appendix, we hereby give the proof of T2, T3, T4, and T6 in corresponding paragraphs. The proof of T5 comes directly from the equation $[f(t)/g(t)] * g(t) = f(t)$.

In the following proofs, suppose we have two functions, $f(t)$ and $g(t)$ with existing IFS and CFS. The n and ω of these IFS's and CFS's are the same, but with different A_i 's and B_i 's (denoted with $A_{f,i}$'s, $B_{f,i}$'s and $A_{g,i}$'s, $B_{g,i}$'s respectively). From the definition of IFS, DD1, we know that

$$f(t) = \sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t), \quad (8)$$

and

$$g(t) = \sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t). \quad (9)$$

Addition and Subtraction Like $f(t)$ and $g(t)$, we also know that

$$f(t) + g(t) = \sum_{i=0}^{+\infty} A_{f+g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i} \sin(i\omega t). \quad (10)$$

By replacing $f(t)$ and $g(t)$ in Equation 10 with Equation 8 and Equation 9, we have

$$\begin{aligned} & \sum_{i=0}^{+\infty} A_{f+g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i} \sin(i\omega t) \\ &= \sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t) + \sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t) \\ &= \sum_{i=0}^{+\infty} (A_{f,i} + A_{g,i}) \cos(i\omega t) + \sum_{i=1}^{+\infty} (B_{f,i} + B_{g,i}) \sin(i\omega t) \end{aligned} \quad (11)$$

By comparing the coefficients in Equation 11, we have

$$\begin{aligned} A_{f+g,i} &= A_{f,i} + A_{g,i} \\ B_{f+g,i} &= B_{f,i} + B_{g,i} \end{aligned} \quad (12)$$

for the IFS and CFS of $f(t) + g(t)$.

Likewise, we have similar conclusions for those of $f(t) - g(t)$.

Multiplication From Equation 8 and Equation 9, we have

$$f(t) * g(t) = \left[\sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t) \right] * \left[\sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t) \right] \quad (13)$$

which is the summation the following 3 terms

$$\begin{aligned}
\text{Term A: } & \sum_{i=0, j=0}^{+\infty} A_{f,i} A_{g,j} \cos(i\omega t) \cos(j\omega t) \\
&= \frac{1}{2} \sum_{i=0, j=0}^{+\infty} A_{f,i} A_{g,j} \cos[(i+j)\omega t] + \frac{1}{2} \sum_{i=0, j=0}^{+\infty} A_{f,i} A_{g,j} \cos[(j-i)\omega t] \\
&= \frac{1}{2} \sum_{i=0}^{+\infty} \left[\sum_{j=0}^i A_{f,j} A_{g,i-j} \right] \cos(i\omega t) + \frac{1}{2} \sum_{j=0}^{+\infty} A_{f,i} A_{g,i} + \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=0}^{+\infty} A_{f,j} A_{g,j+i} + \sum_{j=0}^{+\infty} A_{f,j+i} A_{g,j} \right] \cos(i\omega t) \\
&= A_{f,0} A_{g,0} + \frac{1}{2} \sum_{j=0}^{+\infty} A_{f,i} A_{g,j} + \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=0}^i A_{f,j} + A_{g,i-j} + \sum_{j=0}^{+\infty} (A_{f,i} A_{g,j+i} + A_{f,j+i} A_{g,i}) \right] \cos(i\omega t) \\
\text{Term B: } & \sum_{i=0, j=1}^{+\infty} [A_{f,i} B_{g,j} \cos(i\omega t) \sin(j\omega t) + B_{g,i} A_{f,j} \sin(i\omega t) \cos(j\omega t)] \\
&= \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [A_{f,i} B_{g,j} \sin((i+j)\omega t) - A_{f,i} B_{g,j} \sin((i-j)\omega t)] \\
&+ \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [B_{g,i} A_{f,j} \sin((i+j)\omega t) + B_{g,i} A_{f,j} \sin((i-j)\omega t)] \\
&= \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [A_{f,i} B_{g,j} + B_{g,i} A_{f,j}] \sin((i+j)\omega t) + \frac{1}{2} \sum_{i=0, j=1}^{+\infty} [B_{g,i} A_{f,j} - A_{f,i} B_{g,j}] \sin((i-j)\omega t) \\
&= \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=1}^{+\infty} A_{f,i-j} B_{g,j} + B_{g,i-j} A_{f,j} \right] \sin(i\omega t) - \frac{1}{2} \sum_{i=1}^{+\infty} [B_{g,0} A_{f,i} - A_{f,0} B_{g,i}] \sin(i\omega t) \\
&= \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=1}^{+\infty} (A_{f,i-j} B_{g,j} + B_{g,i-j} A_{f,j}) - B_{g,0} A_{f,i} + A_{f,0} B_{g,i} \right] \sin(i\omega t) \\
\text{Term C: } & \sum_{i=1, j=1}^{+\infty} B_{f,i} B_{g,j} \sin(i\omega t) \sin(j\omega t) \\
&= \frac{1}{2} \sum_{i=1, j=1}^{+\infty} B_{f,i} B_{g,j} [\cos((i-j)\omega t) - \cos((i+j)\omega t)] \\
&= \frac{1}{2} \left[\sum_{j=1}^{+\infty} B_{f,j} B_{g,j} \right] \cos(0\omega t) + \frac{1}{2} \sum_{i=1}^{+\infty} \left[\sum_{j=1}^{+\infty} B_{f,i+j} B_{g,i} + B_{f,i} B_{g,i+j} \right] \cos(i\omega t) \\
&- \frac{1}{2} \sum_{i=2}^{+\infty} \left[\sum_{j=1}^{i-1} B_{f,i-j} B_{g,j} \right] \cos(i\omega t)
\end{aligned} \tag{14}$$

We gather the coefficients of $\cos(i\omega t)$, $i = 0 : n$ and $\sin(i\omega t)$, $i = 1 : n$ respectively, remove any terms containing $A_{f,k}$, $B_{f,k}$, $A_{g,k}$, and $B_{g,k}$ for any $k \geq n$ (k being either i , j , $i - j$, $i + j$, or $j - 1$) as the result of a cut-off, and get the equations in T4.

Amplitude Quoted from DD9, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} App^2(CFSf, t) dt} \quad (15)$$

In Equation 15, replacing $App(CFSf, t)$ with its definition in DD3, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} \left[\sum_{i=0}^n A_{f,i} \cos(i\omega t) + \sum_{i=1}^n B_{f,i} \sin(i\omega t) \right] \left[\sum_{j=0}^n A_{f,j} \cos(j\omega t) + \sum_{j=1}^n B_{f,j} \sin(j\omega t) \right] dt} \quad (16)$$

The part inside the integration in Equation 16 can be expressed as the summation of the following three expressions

$$\begin{aligned} & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=0}^n A_{f,i} A_{f,j} \cos(i\omega t) \cos(j\omega t) dt \\ & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=1}^n A_{f,i} B_{f,j} \sin(i\omega t) \cos(j\omega t) dt \\ & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=0}^n B_{f,i} A_{f,j} \sin(i\omega t) \cos(j\omega t) dt \\ & \int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=1}^n B_{f,i} B_{f,j} \sin(i\omega t) \sin(j\omega t) dt \end{aligned} \quad (17)$$

The integration part of these expressions are $\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \cos(j\omega t)$, $\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \sin(j\omega t)$, and $\int_{-\pi/\omega}^{\pi/\omega} \sin(i\omega t) \sin(j\omega t)$. Calculation shows that the integration results are

$$\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \cos(j\omega t) dt = \begin{cases} 2\pi/\omega, & i = j = 0; \\ \pi/\omega, & i = j \neq 0; \\ 0, & i \neq j. \end{cases} \quad (18)$$

$$\int_{-\pi/\omega}^{\pi/\omega} \cos(i\omega t) \sin(j\omega t) dt = 0 \quad (19)$$

and

$$\int_{-\pi/\omega}^{\pi/\omega} \sin(i\omega t) \sin(j\omega t) dt = \begin{cases} \pi/\omega, & i = j \\ 0, & i \neq j \end{cases} \quad (20)$$

Replacing integration in Equation 17 with integration results Equation 18, Equation 19, and Equation 20, and we have the following results.

$$\begin{aligned}
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=0}^n A_{f,i} A_{f,j} \cos(i\omega t) \cos(j\omega t) dt &= \frac{2\pi}{\omega} A_0^2 + \frac{\pi}{\omega} \sum_{i=1}^n A_i^2 \\
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0, j=1}^n A_{f,i} B_{f,j} \sin(i\omega t) \cos(j\omega t) dt &= 0 \\
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=0}^n B_{f,i} A_{f,j} \sin(i\omega t) \cos(j\omega t) dt &= 0 \\
\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1, j=1}^n B_{f,i} B_{f,j} \sin(i\omega t) \sin(j\omega t) dt &= \frac{\pi}{\omega} \sum_{i=1}^n B_i^2
\end{aligned} \tag{21}$$

Summing them up and putting the result back into Equation 16, we have the expression in T6.