# FSL
# A Fourier Series Library

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# 1 Revision History

| Date | Version | Notes |
|---|---|---|
| Oct. 10, 2019 | 0.9 | Draft submitted for review by Dr. Smith |
| TBD | 1.0 | Revised in accordance with advice from Dr. Smith |

# 2 Reference Material

This section records information for easy reference.

## 2.1 Table of Units

This section is not applicable, due to the fact that this library is a pure mathematical computation library.

## 2.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the Fourier series literature and with existing documentation for Fourier series libraries. The symbols are listed in alphabetical order.

| **symbol** | **description** |
|---|---|
| $f(t)$, $g(t)$, ... | $[-\pi/\omega, \pi/\omega] \to \mathbb{R}$ functions of $t$, where $\omega$ is defined as below |
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{Z}$ | Set of integers |
| $[-\pi/\omega, \pi/\omega]$ | Set of real numbers that are neither smaller than $-\pi/\omega$, nor greater than $\pi/\omega$ |
| $\to$ | Indicate mapping |
| $\mathbb{A}^*$ | Non-negative subset of set $\mathbb{A}$ (either $\mathbb{R}$ or $\mathbb{Z}$) |
| $\mathbb{A}^*$ | Positive subset of set $\mathbb{A}$ (either $\mathbb{R}$ or $\mathbb{Z}$) |
| $\sum_{i=m}^{n} a_i$ | Summation of $a_i, i = m, m+1, ..., n$ |
| $n$ | Length of cut-off Fourier series (definition not applicable to this table) |
| $\omega$ | Base frequency of Fourier series |
| $\mathit{CFSf}$, $\mathit{CFSg}$ | $\mathit{CFS}(f(t), n, \omega)$ and $\mathit{CFS}(g(t), n, \omega)$ respectively. |

## 2.3 Abbreviations and Acronyms

| symbol | description |
|---|---|
| A | Assumption |
| API | Application Programming Interface |
| CA | Commonality Analysis |
| DD | Data Definition |
| FS | Fourier Series |
| GD | General Definition |
| GS | Goal Statement |
| IFS | Infinite Fourier Series |
| IM | Instance Model |
| LC | Likely Change |
| PS | Physical System Description |
| R | Requirement |
| FSL | Fourier Series Library. |
| T | Theoretical Model |

# Contents

# 3 Introduction

## 3.1 Purpose of Document

This document introduces a library, FSL, mainly designed and implemented for Fourier series related calculations.

## 3.2 Scope of the Family

The scope of the family is limited to calculations related to the Fourier series of mathematical functions.

## 3.3 Characteristics of Intended Reader

The intended readers must have the following knowledge.

- Deep knowledge of Fourier series (found in advanced calculus and real analysis courses);
- Computational error analysis (found in introductory numerical analysis classes); and
- Detailed knowledge of the programming language utilized in the implementation of this library.

## 3.4 Organization of Document

TBD

# 4 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

## 4.1 Potential System Contexts

The users provide the data for the library to calculate, and the library returns the calculation results.

- User Responsibilities:
  - Provide the inputs of the functions in the library.
  - Receive the corresponding outputs, and provide and manage the memory space storing them if mentioned in the following sections.
  - For the conversion from mathematical functions to Fourier series, ensure that provided mathematical functions have Fourier series.

- FSL Responsibilities:
  - Detect data type mismatch, such as a string of characters instead of a floating point number.
  - Unless otherwise mentioned in this document, detect legality input value, such as mismatched length of Fourier series included in one calculation.
  - Unless otherwise mentioned, manage the memory space required by this library.

## 4.2 Potential User Characteristics

racteristics⟩? The end user of FSL should have an understanding of undergraduate level 2 Calculus and/or Real Analysis, undergraduate level 1 Numerical Analysis, and any one of the programming languages, in which this library provides a set of APIs.

## 4.3 Potential System Constraints

The potential systems must contain the compilation/interpretation environment for the programming languages in which this library is developed, as well as dependent libraries.

# 5 Commonalities

## 5.1 Background Overview

c_Background⟩?

## 5.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

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## 5.3 Data Definitions

⟨sec_datadef⟩? This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

| Number | DD1 |
|---|---|
| Label ⟨DD:IFS⟩ | **Infinite Fourier Series (IFS) of mathematical functions** |
| Symbol | $IFS$ |
| Equation | $IFS(f(t),\omega) = \left[A_{inf,i}, i = 0,1,...; B_{inf,i}, i = 1,2,...\right]$, satisfying $f(t) = \sum_{i=0}^{+\infty} A_{inf,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{inf,i}\sin(i\omega t)$ |
| Description | In this equation,<br>• $f(t)$ is an $[-\pi/\omega, \pi/\omega]] \to \mathbb{R}$ function, whose IFS exists;<br>• $\omega$ is the base frequency of this IFS, and it is designated by the user;<br>• $A_{inf,i} \in \mathbb{R}, i = 0,1,...$; and<br>• $B_{inf,i} \in \mathbb{R}, i = 1,2,....$ |
| Sources | Arthur L. Schoenstadt, An Introduction to Fourier Analysis, August 18, 2005 |
| Ref. By | IM2, A1, C1, C2, O1, and O2 |

| Number | DD2 |
|---|---|
| Label ⟨DD:CFS⟩ | **Cut-off Fourier Series (CFS) of mathematical functions** |
| Symbol | $CFS$ |
| Equation | $CFS(f(t),n,\omega) = f(t) \to \left[A_i, i = 0,1,...,n; B_i, i = 1,2,...,n\right]$, satisfying $A_i = A_{inf,i}, i = 0,1,...,n; B_i = B_{inf,i}, i = 1,2,...,n$, where $A_{inf,i}$'s and $B_{inf,i}$'s come from $IFS(f(t),\omega)$ |
| Description | $n \in \mathbf{Z}^*$ is called cut-off length. |
| Sources | Defined by author |
| Ref. By | T1, T2, T3, T4, T5, T6, A1, C1, C2, C3, O1, and O2 |

From now on, denote $A_i$'s and $B_i$'s in $CFSf$ as $A_{f,i}$'s and $B_{f,i}$'s respectively, and same for $A_{g,i}$'s and $B_{g,i}$'s.

| Number | DD3 |
|---|---|
| ⟨DD:Approximation⟩ Label | **Approximation of function values from CFS** |
| Symbol | $App(CFSf, t_1)$ |
| Equation | $App(CFSf, t_1) = \sum_{i=0}^{n} A_i \cos(i\omega t) + \sum_{i=1}^{n} B_{inf,i} \sin(i\omega t)$ |
| Description | Define approximated function value of a function calculated from its CFS. |
| Sources | Defined by author |
| Ref. By | IM2, C1, C2, O1, and O2 |

| Number | DD4 |
|---|---|
| ⟨DD:Addition⟩ Label | **Addition of CFS's** |
| Symbol | $CFS(f(t), n, \omega) + CFS(g(t), n, \omega)$ |
| Equation | $CFS(f(t), n, \omega) + CFS(g(t), n, \omega) = CFS(f(t) + g(t), n, \omega)$ |
| Description | Define the rule of addition of two CFS's |
| Sources | Defined by author |
| Ref. By | C1, C2, O1, and O2 |

| Number | DD5 |
|---|---|
| ⟨DD:Subtraction⟩ Label | **Subtraction of CFS's** |
| Symbol | $CFS(f(t), n, \omega) - CFS(g(t), n, \omega)$ |
| Equation | $CFS(f(t), n, \omega) - CFS(g(t), n, \omega) = CFS(f(t) - g(t), n, \omega)$ |
| Description | Define the rule of subtraction of two CFS's |
| Sources | Defined by author |
| Ref. By | C1, C2, O1, and O2 |

| Number | DD6 |
| --- | --- |
| ⟨DD:Multiplication⟩ Label | **Multiplication of CFS's** |
| Symbol | $CFS(f(t), n, \omega) * CFS(g(t), n, \omega)$ |
| Equation | $CFS(f(t), n, \omega) * CFS(g(t), n, \omega) = CFS(f(t) * g(t), n, \omega)$ |
| Description | Define the rule of multiplication of two CFS's |
| Sources | Defined by author |
| Ref. By | C1, C2, O1, and O2 |

| Number | DD7 |
| --- | --- |
| ⟨DD:Division⟩ Label | **Division of CFS's** |
| Symbol | $CFS(f(t), n, \omega)/CFS(g(t), n, \omega)$ |
| Equation | $CFS(f(t), n, \omega)/CFS(g(t), n, \omega) = CFS(f(t)/g(t), n, \omega)$ |
| Description | Define the rule of division of two CFS's |
| Sources | Defined by author |
| Ref. By | C1, C2, O1, and O2 |

| Number | DD8 |
| --- | --- |
| ⟨DD:Function⟩ Label | **Function of CFS's** |
| Symbol | $g(CFS(f(t), n, \omega))$ |
| Equation | $g(CFS(f(t), n, \omega)) = \sum_{i=0}^{n} a_i CFS^i(f(t), n, \omega)$, in which $a_i, i = 0 : n$ is the first $(n+1)$ coefficients of the Taylor series of $g(t)$, and the $n$-th ($n \in \mathbb{Z}^+$) power of CFS is defined as $n$ copies of this CFS multiplied together. (0-th power is defined as the CFS of $f(t) = 1$.) |
| Description | Define the rule of the function of a CFS |
| Sources | Defined by author |
| Ref. By | A1, C1, and O1 |

| Number | DD9 |
|---|---|
| ⟨DD:Amplitude⟩ Label | **Amplitude of a CFS** |
| Symbol | $Amp(CFSf)$ |
| Equation | $$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} App^2(CSFf, t_1)\,dt_1} \quad (1)\ \{?\}$$ . |
| Description | Define the amplitude/size of a CFS |
| Sources | Defined by author |
| Ref. By | A1, C1, and O1 |

| Number | DD10 |
|---|---|
| ?⟨DD:Equality⟩? Label | **Tolerated equality of two CFS's** |
| Symbol | $CFSf = CFSg$ |
| Equation | Same as $Amp(CFSf - CFSg) \leq err$, where $err$ is an user-given tolerance. |
| Description | Define whether two CFS's are equal within a given tolerance |
| Sources | Defined by author |
| Ref. By | C1, and O1 |

## 5.4 Goal Statements

Given the corresponding inputs, the goal statements are:

GS1: When given a function $f(t)$, the cut-off length $n$, and a base frequency $\omega$, return the function's CFS $CFS(f(t), n, \omega)$.

GS2: When given a CFS $CFS(f(t), n, \omega)$ and a value of $t$ as $t_1$, return the approximated value of $f(t_1)$ computed from the given values.

GS3: When given two CFS's $CFSf$, $CFSg$, and an operation in addition, subtraction, multiplication, division, return the result of this operation on these two CFS's.

GS4: When given a CFS $CFSf$, and a function $g(t)$ from the base function sets defined by this library, return the CFS $g(CFSf)$.

GS5: When given values of $n$, $\omega$, $A_i$'s, and $B_i$'s, return the CFS built on these values.

GS6: When given a CFS, store its values of $n$, $\omega$, $A_i$'s, and $B_i$'s in the user-designated space.

GS7: When given a CFS, return its amplitude.

GS8: When given two CFS's and a tolerance, return whether they are equal within the tolerance.

## 5.5 Theoretical Models

_theoretical⟩? This section focuses on the general equations and laws that FSL is based on.

| Number | T1 |
|---|---|
| ⟨T:Transformation⟩ Label | **Fourier Transformation** |
| Equation | $\begin{aligned} A_0 &= \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t); \\ A_i &= \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos(i\omega t),\ \ i = 1:n; \\ B_i &= \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin(i\omega t),\ \ i = 1:n. \end{aligned} \qquad (2)$ ?Eq:DFT? |
| Description | The above equation calculates $A_i(i = 0 : n), B_i(i = 1 : n)$ in *CFSf*. |
| Source | Arthur L. Schoenstadt, An Introduction to Fourier Analysis, August 18, 2005 |
| Ref. By | A1, C1, C2, O1, and O2 |

| | |
|---|---|
| Number | T2 |
| ⟨T:Addition⟩ Label | **Addition of two CFS's** |
| Equation | $$A_{f+g,i} = A_{f,i} + A_{g,i},\ \ i = 0 : n$$ $$B_{f+g,i} = B_{f,i} + B_{g,i},\ \ i = 1 : n$$ (3) {?} |
| Description | The above equation calculates $A_i(i = 0 : n), B_i(i = 1 : n)$ in $CFS(f(t) + g(t), n, \omega$ (represented by $A_{f+g,i}$ and $B_{f+g,i}$ respectively) from $CFSf$ and $CFSg$. |
| Source | Developed by author in Appendix A |
| Ref. By | A2, C1, and O1 |

| | |
|---|---|
| Number | T3 |
| ⟨T:Subtraction⟩ Label | **Subtraction of two CFS's** |
| Equation | $$A_{f-g,i} = A_{f,i} - A_{g,i},\ \ i = 0 : n$$ $$B_{f-g,i} = B_{f,i} - B_{g,i},\ \ i = 1 : n$$ (4) {?} |
| Description | The above equation calculates $A_i(i = 0 : n), B_i(i = 1 : n)$ in $CFS(f(t) - g(t), n, \omega$ (represented by $A_{f-g,i}$ and $B_{f-g,i}$ respectively) from $CFSf$ and $CFSg$. |
| Source | Developed by author in Appendix A |
| Ref. By | A2, C1, and O1 |

| Number | T4 |
|---|---|
| ⟨T:Multiplication⟩ Label | **Multiplication of two CFS's** |
| Equation | $$\begin{aligned} A_{f*g,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f,i}A_{g,i+j} + A_{f,i+j}A_{g,i} + B_{f,i}B_{g,i+j} + B_{f,i+j}B_{g,i}) \\ &+ \frac{1}{2}\sum_{j=0}^{i}(A_{f,j}A_{g,i-j} - B_{f,j}B_{g,i-j}),\ i = 0 : n \\ B_{f*g,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f,i}B_{g,i+j} + Af,i+jB_{g,i} + B_{f,i}A_{g,i+j} + B_{f,i+j}A_{g,i}) \\ &+ \frac{1}{2}\sum_{j=0}^{i}(A_{f,j}B_{g,i-j} + B_{f,j}A_{g,i-j}),\ i = 1 : n \end{aligned} \quad (5)$$ {?} |
| Description | The above equation calculates $A_i(i = 0 : n), B_i(i = 1 : n)$ in $CFS(f(t) * g(t), n, \omega$ (represented by $A_{f*g,i}$ and $B_{f*g,i}$ respectively) from $CFSf$ and $CFSg$. |
| Source | Developed by author in Appendix A |
| Ref. By | A2, C1, and O1 |

| Number | T5 |
|---|---|
| Label | ⟨T:Division⟩ **Division of two CFS's** |
| Equation | Solve the following equations for $A_{f/g,i}$'s and $B_{f/g,i}$'s. $$\begin{aligned} A_{f,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f/g,i}A_{g,i+j} + A_{f/g,i+j}A_{g,i} + B_{f/g,i}B_{g,i+j} + B_{f/g,i+j}B_{g,i}) \\ &\quad + \frac{1}{2}\sum_{j=0}^{i}(A_{f/g,j}A_{g,i-j} - B_{f/g,j}B_{g,i-j}),\ i = 0 : n \\ B_{f,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f/g,i}B_{g,i+j} + Af/g,i+jB_{g,i} + B_{f/g,i}A_{g,i+j} + B_{f/g,i+j}A_{g,i}) \\ &\quad + \frac{1}{2}\sum_{j=0}^{i}(A_{f/g,j}B_{g,i-j} + B_{f/g,j}A_{g,i-j}),\ i = 1 : n \end{aligned} \tag{6}$$ {?} |
| Description | We solve the above equations for $A_i(i = 0 : n), B_i(i = 1 : n)$ in $CFS(f(t)/g(t), n, \omega$ when $CFSf$ and $CFSg$ are given. |
| Source | Developed by author in Appendix A |
| Ref. By | A2, C1, C2, O1, and O2 |

| Number | T6 |
|---|---|
| Label | ⟨T:Amplitude⟩ **Amplitude of a CFS** |
| Equation | $$Amp(CSFf) = \sqrt{A_0^2 + \frac{1}{2}\sum_{i=1}^{n}(A_i^2 + B_i^2)} \tag{7}$$ {?} |
| Description | The above equation calculates $Amp(CFSf)$ from $CFSf$, especially $A_i$'s and $B_i$'s in it. |
| Source | Developed by author in Appendix A |
| Ref. By | C1 and O1 |

# 6 Variabilities

## 6.1 Instance Models

| Number | IM1 |
|---|---|
| Label ⟨IM:CFScoeff⟩ | **Calculate coefficients of CFS's** |
| Input | $f(t)$, a $[-\pi/\omega, \pi/\omega]$ function<br>$n \in \mathbb{Z}^*$, cut-off length<br>$\omega \in \mathbb{R}^+$, base frequency |
| Output | An object *CFSf*, containing the following data:<br>$n$, $\omega$: same as the input<br>$A_i(i = 0 : n)$, $B_i(i = 1 : n)$: using the theory T1 |
| Description | Input:<br>$f(t)$: function to be transformed into Fourier series<br>$n$: cut-off length<br>$\omega$: base frequency<br>Output:<br>$A_i$'s and $B_i$'s: coefficients in *CFSf* |
| Sources | Same as T1 |
| Ref. By | A1, C1, C2, O1, and O2 |

| Number | IM2 |
| --- | --- |
| ⟨IM:Addition⟩ Label | **Addition of two CFS's** |
| Input | $CFSf$, $CFSg$ |
| Output | $CFS(f(t)+g(t),n,\omega)$, whose $A_i$'s and $B_i$'s are computed using the theory T2 |
| Description | Input:<br>$CFSf$, $CFSg$: operands of the addition<br>Output:<br>$CFS(f(t)+g(t),n,\omega)$: result of the addition |
| Sources | Same as T2 |
| Ref. By | A2, C1, and O1 |

| Number | IM3 |
| --- | --- |
| ⟨IM:Subtraction⟩ Label | **Subtraction of two CFS's** |
| Input | $CFSf$, $CFSg$ |
| Output | $CFS(f(t)-g(t),n,\omega)$, whose $A_i$'s and $B_i$'s are computed using the theory T3 |
| Description | Input:<br>$CFSf$, $CFSg$: operands of the subtraction<br>Output:<br>$CFS(f(t)-g(t),n,\omega)$: result of the subtraction |
| Sources | Same as T3 |
| Ref. By | A2, C1, and O1 |

| Number | IM4 |
|---|---|
| ⟨IM:Multiplication⟩ Label | **Multiplication of two CFS's** |
| Input | $CFSf$, $CFSg$ |
| Output | $CFS(f(t) * g(t), n, \omega)$, whose $A_i$'s and $B_i$'s are computed using the theory T4 |
| Description | Input:<br>$CFSf$, $CFSg$: operands of the multiplication<br>Output:<br>$CFS(f(t) * g(t), n, \omega)$: result of the multiplication |
| Sources | Same as T4 |
| Ref. By | A2, C1, and O1 |

| Number | IM5 |
|---|---|
| ⟨IM:Division⟩ Label | **Division of two CFS's** |
| Input | $CFSf$, $CFSg$ |
| Output | $CFS(f(t)/g(t), n, \omega)$, whose $A_i$'s and $B_i$'s are computed using the theory T5 |
| Description | Input:<br>$CFSf$, $CFSg$: operands of the division<br>Output:<br>$CFS(f(t) + g(t), n, \omega)$: result of the division |
| Sources | Same as T5 |
| Ref. By | A2, C1, C2, O1, and O2 |

| Number | IM6 |
|---|---|
| Label ⟨IM:Function⟩ | **Function of a CFS** |
| Input | $CFSf$, $g(t)$ |
| Output | A CFS $g(CFSf)$, whose $A_i$'s and $B_i$'s are computed using the theory T**??** |
| Description | Input:<br>$g(t)$, a basic function chosen by user from a basic fiction set.<br>$CFSf$: dependent variable of the function $g(t)$<br>Output:<br>$g(CFSf)$: An CFS being the computed result |
| Sources | Same as T**??** |
| Ref. By | A1, C1, and O1 |

| Number | IM7 |
|---|---|
| Label ⟨IM:Amplitude⟩ | **Amplitude of a CFS** |
| Input | $CFSf$ |
| Output | $Amp(CFSf)$, computed using the theory T6 |
| Description | Input:<br>$CFSf$: variable of the amplitude function<br>Output:<br>$Amp(CFSf)$, the amplitude of $CFSf$ |
| Sources | Same as T6 |
| Ref. By | C1, and O1 |

| Number | IM8 |
|---|---|
| ⟨IM:ToleratedEquality⟩ Label | **Tolerated Equality Comparison of two CFS's** |
| Input | $CFSf$, $CFSg$, $tol$ |
| Output | boolean value`True` if $Amp(CFSf - CFSg) \leq tol$<br>boolean value `False` otherwise |
| Description | Input:<br>$CFSf$, $CFSg$: operands of the tolerated equality comparison<br>$tol \in \mathbb{R}^*$<br>Output:<br>A boolean value: Whether the two operands are equal within the given error tolerance |
| Sources | Same as T**??** |
| Ref. By | A2, C1, and O1 |

| Number | IM9 |
|---|---|
| ⟨IM:ConvertTo⟩ Label | **Convert data of other structures (input sources included) to a CFS** |
| Input | $n$, $\omega$, $n+1$ real numbers $Ain_i(i = 0 : n)$, and $n$ real numbers $Bin_i(i = 1 : n)$ |
| Output | An CFS object constructed from the input data |
| Description | Input: Data needed for construction<br>Output: Constructed CSF object containing the input data |
| Sources | None |
| Ref. By | A3, C1, C3, and O1 |

| Number | IM10 |
|---|---|
| ⟨IM:ConvertFrom⟩ Label | **Convert CFS to data of structures (output destination included)** |
| Input | *CFSf* |
| Output | $n$, $\omega$, $Aout_i = A_i(i = 0 : n)$, and $Bout_i = B_i(i = 1 : n)$ |
| Description | Input: The CFS object to convert to<br>Output: The data extracted from the input CFS object. |
| Sources | None |
| Ref. By | A3, C1, C3, and O1 |

## 6.2 Assumptions

A1: The functions $f(t)$ and $g(t)$ mentioned above must have definitions on $[-\pi/\omega, \pi/\omega]$, in which $\omega$ is given by the user, and they must be able to be transformed into Fourier series.

A2: For any two-operand operations, the $n$ and $\omega$ of these operands must be the same.

A3: User shall allocate memory space for any variable other than CFS's in IM9 and IM10.

## 6.3 Calculation

_Calculation⟩? C1: Calculate the result based on the called function and input variables.

C2: If the called function detects that the input variables do not meet the requirements, generate an error message describing the detected error.

C3: Manage the memory spaces required by this library, and destroy them the moment they are not needed.

## 6.4 Output

?⟨sec_Output⟩? O1: Return the results faithfully.

O2: Report any detected errors to the user of the library.

# 7 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

## 7.1 Functional Requirements

R1: The data type in the inputs $f(t)$, $A_i$'s and/or $B_i$'s must be same as that of $\omega$.

R2: The floating point data type of the output must be the same one as that of the input.

## 7.2 Nonfunctional Requirements

- All time complexities shall be unrelated to $\omega$.
- The time complexity of IM1 shall be $O(n^2)$ when the input function $f(t)$ is not complex.
- The time complexity of IM2 and IM3 shall be $O(n)$.
- The time complexity of IM4 shall be $O(n^2)$.
- The time complexity of IM5 shall be the same as the best linear equation solver applicable.
- The time complexity of IM7 and IM8 shall be $O(n)$.

# 8 Likely Changes

None currently.

# 9 Traceability Matrices and Graphs

The following matrices and graphs demonstrates the traceability of this project. The purpose is to provide easy references to the impacts on other components if a certain component is changed. That is, if one component has been changed, other components that share an 'X' with it may need to be changed accordingly. Table 1 shows the the dependencies of goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs.

| | A1 | A2 | A3 | C1 | C2 | C3 | O1 | O2 |
|---|---|---|---|---|---|---|---|---|
| GS1 | X | | | X | X | | X | X |
| GS2 | | | | X | | | X | |
| GS3 | | X | | X | X | | X | X |
| GS4 | X | | | X | X | | X | X |
| GS5 | | | X | X | | X | X | |
| GS6 | | | X | X | | X | X | |
| GS7 | | | X | | | | X | |
| GS8 | | X | | X | X | | X | X |
| DD1 | X | | | X | X | | X | X |
| DD2 | X | | | X | X | X | X | X |
| DD3 | | | | X | X | | X | X |
| DD4 | | X | | X | X | | X | X |
| DD5 | | X | | X | X | | X | X |
| DD6 | | X | | X | X | | X | X |
| DD7 | | X | | X | X | | X | X |
| DD8 | X | | | X | | | X | |
| DD9 | | | | X | | | X | |
| T1 | X | | | X | X | | X | X |
| T2 | | X | | X | | | X | |
| T3 | | X | | X | | | X | |
| T4 | | X | | X | | | X | |
| T5 | | X | | X | X | | X | X |
| T6 | | | | X | | | X | |
| IM1 | X | | | X | X | | X | X |
| IM2 | | X | | X | | | X | |
| IM3 | | X | | X | | | X | |
| IM4 | | X | | X | | | X | |
| IM5 | | X | | X | X | | X | X |
| IM6 | X | | | X | | | X | |
| IM7 | | | | X | | | X | |
| IM8 | | X | | X | | | X | |
| IM9 | | | X | X | | X | X | |
| IM10 | | | X | X | | X | X | |

Table 1: The traceability matrix between goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs

:Traceability⟩

# References

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# A Theory for operations on CFS's

x:Operations⟩? In this appendix, we hereby give the proof of T2, T3, T4, and T6 in corresponding paragraphs. The proof of T5 comes directly from the equation $[f(t)/g(t)] * g(t) = f(t)$.

In the following proofs, suppose we have two functions, $f(t)$ and $g(t)$ with existing IFS and CFS. The $n$ and $\omega$ of these IFS's and CFS's are the same, but with different $A_i$'s and $B_i$'s (denoted with $A_{f,i}$'s, $B_{f,i}$'s and $A_{g,i}$'s, $B_{g,i}$'s respectively). From the definition of IFS, DD1, we know that

$$f(t) = \sum_{i=0}^{+\infty} A_{f,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i}\sin(i\omega t), \tag{8}$$ ?Eq:fDef?

and

$$g(t) = \sum_{i=0}^{+\infty} A_{g,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i}\sin(i\omega t). \tag{9}$$ ?Eq:gDef?

&Subtraction⟩? **Addition and Subtraction** Like $f(t)$ and $g(t)$, we also know that

$$f(t) + g(t) = \sum_{i=0}^{+\infty} A_{f+g,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i}\sin(i\omega t). \tag{10}$$ ?Eq:f+gDef?

By replacing $f(t)$ and $g(t)$ in Equation 10 with Equation 8 and Equation 9, we have

$$\begin{aligned}
&\sum_{i=0}^{+\infty} A_{f+g,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i}\sin(i\omega t)\\
=&\sum_{i=0}^{+\infty} A_{f,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i}\sin(i\omega t) + \sum_{i=0}^{+\infty} A_{g,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i}\sin(i\omega t) \qquad (11)\\
=&\sum_{i=0}^{+\infty}(A_{f,i} + A_{g,i})\cos(i\omega t) + \sum_{i=1}^{+\infty}(B_{f,i} + B_{g,i})\sin(i\omega t)
\end{aligned}$$ ?Eq:f+gCoeff?

By comparing the coefficients in Equation 11, we have

$$A_{f+g,i} = A_{f,i} + A_{g,i}B_{f+g,i} = B_{f,i} + B_{g,i} \tag{12}$$ ?Eq:f+gConclus

for the IFS and CFS of $f(t) + g(t)$.

Likewise, we have similar conclusions for those of $f(t) - g(t)$.

ltiplication⟩? **Multiplication** From Equation 8 and Equation 9, we have

$$f(t) * g(t) = [\sum_{i=0}^{+\infty} A_{f,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i}\sin(i\omega t)] * [\sum_{i=0}^{+\infty} A_{g,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i}\sin(i\omega t)] \tag{13}$$ {?}

which consists of the following 3 major terms

$$\begin{aligned}
&\textit{Term A: } \sum_{i=0,j=0}^{+\infty} A_{f,i}A_{g,j}\cos(i\omega t)\cos(j\omega t)\\
&=\frac{1}{2}\sum_{i=0,j=0}^{+\infty} A_{f,i}A_{g,j}\cos[(i+j)\omega t] + \frac{1}{2}\sum_{i=0,j=0}^{+\infty} A_{f,i}A_{g,j}\cos[(j-i)\omega t]\\
&=\frac{1}{2}\sum_{i=0}^{+\infty}[\sum_{j=0}^{i} A_{f,j}A_{g,i-j}]\cos(i\omega t) + \frac{1}{2}\sum_{j=0}^{+\infty} A_{f,i}A_{g,i} + \frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=0}^{+\infty} A_{f,j}A_{g,j+i} + \sum_{j=0}^{+\infty} A_{f,j+i}A_{g,j}]cos(i\omega t)\\
&=A_{f,0}A_{g,0} + \frac{1}{2}\sum_{j=0}^{+\infty} A_{f,}A_{g,j} + \frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=0}^{i} A_{f,j} + A_{g,i-j} + \sum_{j=0}^{+\infty}(A_{f,i}A_{g,j+i} + A_{f,j+i}A_{g,i})]\cos(i\omega t)\\
&\textit{Term B: } \sum_{i=0,j=1}^{+\infty} [A_{f,i}B_{g,j}\cos(i\omega t)\sin(j\omega t) + B_{g,i}A_{f,j}\sin(i\omega t)\cos(j\omega t)]\\
&=\frac{1}{2}\sum_{i=0,j=1}^{+\infty} [A_{f,i}B_{g,j}sin((i+j)\omega t) - A_{f,i}B_{g,j}\sin((i-j)\omega t)]\\
&+\frac{1}{2}\sum_{i=0,j=1}^{+\infty} [B_{g,i}A_{f,j}\sin((i+j)\omega t) + B_{g,i}A_{f,j}\sin((i-j)\omega t)]\\
&=\frac{1}{2}\sum_{i=0,j=1}^{+\infty} [A_{f,i}B_{g,j} + B_{g,i}A_{f,j}]sin((i+j)\omega t) + \frac{1}{2}\sum_{i=0,j=1}^{+\infty} [B_{g,i}A_{f,j} - A_{f,i}B_{g,j}]\sin((i-j)\omega t)\\
&=\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=1}^{+\infty} A_{f,i-j}B_{g,j} + B_{g,i-j}A_{f,j}]\sin(i\omega t) - \frac{1}{2}\sum_{i=1}^{+\infty}[B_{g,0}A_{f,i} - A_{f,0}B_{g,i}]sin(i\omega t)\\
&=\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=1}^{+\infty}(A_{f,i-j}B_{g,j} + B_{g,i-j}A_{f,j}) - B_{g,0}A_{f,i} + A_{f,0}B_{g,i}]\sin(i\omega t)\\
&\textit{Term C: } \sum_{i=1,j=1}^{+\infty} B_{f,i}B_{g,j}\sin(i\omega t)\sin(j\omega t)\\
&=\frac{1}{2}\sum_{i=1,j=1}^{+\infty} B_{f,i}B_{g,j}[\cos((i-j)\omega t) - \cos((i+j)\omega t)]\\
&=\frac{1}{2}[\sum_{j=1}^{+\infty} B_{f,j}B_{g,j}]\cos(0\omega t) + \frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=1}^{+\infty} B_{f,i+j}B_{g,i} + B_{f,i}B_{g,i+j}]\cos(i\omega t)\\
&-\frac{1}{2}\sum_{i=2}^{+\infty}[\sum_{j=1}^{i-1} B_{f,i-j}B_{g,j}]\cos(i\omega t)
\end{aligned} \tag{14}$$

{?}

We gather the coefficients of $\cos(i\omega t), i = 0 : n$ and $\sin(i\omega t), i = 1 : n$ respectively, remove $A_{f,k}$, $B_{f,k}$, $A_{g,k}$, and $B_{g,k}$ for any $k \geq n$ ($k$ being either $i$, $j$, $i-j$, $i+j$, or $j-1$) as the result of a cut-off, and get the equation in T4.

ra:Amplitude⟩? **Amplitude** Quoted from DD9, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} App^2(CFSf, t)\mathrm{d}t} \tag{15}$$ ?Eq:Amp?

In Equation 15, replacing $App^2(CFSf, t)$ with its definition in DD3, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega}[\sum_{i=0}^{n} A_{f,i}\cos(i\omega t) + \sum_{i=1}^{n} B_{f,i}\sin(j\omega t)][\sum_{j=0}^{n} A_{f,j}\cos(j\omega t) + \sum_{j=1}^{n} B_{f,j}\sin(j\omega t)]\mathrm{d}t} \tag{16}$$ ?Eq:Amp1?

The part inside the integration in Equation 16 can be expressed as

$$\begin{aligned} &\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=0,j=0}^{n} A_{f,i}A_{f,j}\cos(i\omega t)\cos(j\omega t)\mathrm{d}t \\ &\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=0,j=1}^{n} A_{f,i}B_{f,j}\sin(i\omega t)\cos(j\omega t)\mathrm{d}t \\ &\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=1,j=0}^{n} B_{f,i}A_{f,j}\sin(i\omega t)\cos(j\omega t)\mathrm{d}t \\ &\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=1,j=1}^{n} B_{f,i}B_{f,j}\sin(i\omega t)\sin(j\omega t)\mathrm{d}t \end{aligned} \tag{17}$$ ?Eq:AmpTerms?

Generally, the terms in Equation 17 can be classified into three categories, $\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\cos(j\omega t)$, $\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t)$, and $\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t)$. Calculation shows that their results are

$$\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\cos(j\omega t) = \begin{cases} 2\pi/\omega, & i = j = 0; \\ \pi/\omega, & i = j \neq 0; \\ 0, & i \neq j. \end{cases} \tag{18}$$ ?Eq:coscos?

$$\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t) = 0 \tag{19}$$ ?Eq:cossin?

and

$$\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t) = \begin{cases} \pi/\omega, & i = j \\ 0, & i \neq j \end{cases} \tag{20}$$ ?Eq:sinsin?

Replacing terms Equation 17 with Equation 18, Equation 19, and Equation 20, and we have $\frac{2\pi}{\omega}A_0^2 + \frac{\pi}{\omega}\sum_{i=1}^{n} A_i^2$, 0, 0 and $\frac{\pi}{\omega}\sum_{i=1}^{n} B_i^2$. Putting them back into Equation 16, we have the expression in T6.