FSL
A Fourier Series Library

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Contents

Revision History

Date	Version	Notes
Oct. 10, 2019	0.9	Draft submitted for review by Dr. Smith
Dec 3, 2019	1.0	Revised in accordance with advice from Dr. Smith

1 Reference Material

This section records information for easy reference.

1.1 Table of Units

This section is not applicable, due to the fact that this library is a pure mathematical computation library.

1.2 Table of Symbols

The table that follows summarizes the symbols used in this document. The choice of symbols is made to be consistent with the Fourier series literature and with existing documentation for Fourier series libraries.

symbol	description
$f(t)$, $g(t)$, ...	$[-\pi/\omega, \pi/\omega] \to \mathbb{R}$ functions of t, where ω is defined as below
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$[-\pi/\omega, \pi/\omega]$	Set of real numbers that are neither smaller than $-\pi/\omega$, nor greater than π/ω
$\to$	Indicate mapping
$\mathbb{A}^*$	Non-negative subset of set $\mathbb{A}$ (either $\mathbb{R}$ or $\mathbb{Z}$)
$\mathbb{A}^+$	Positive subset of set $\mathbb{A}$ (either $\mathbb{R}$ or $\mathbb{Z}$)
$\mathbb{N}$	alias of $\mathbb{Z}^*$
$\sum_{i=m_1}^{m_2} a_i$	Summation of $a_i, i = m_1, m_1 + 1, ..., m_2$
n	Length of cut-off Fourier series
ω	Base frequency of Fourier series
CFSf, CFSg	$\mathit{CFS}(f(t), n, \omega)$ and $\mathit{CFS}(g(t), n, \omega)$, respectively.
A_i, B_i	coeffecients of ‘sin’ and ‘cos’ functions in a CFS.
App	Approximation operation on a CFS
Amp	Amplitude function of a CFS
ϵ	Tolerance used for equality of CFS.

1.3 Abbreviations and Acronyms

symbol	**description**
A	Assumption
API	Application Programming Interface
CA	Commonality Analysis
CFS	Cut-off Fourier Series
DD	Data Definition
FS	Fourier Series
GD	General Definition
GS	Goal Statement
IFS	Infinite Fourier Series
IM	Instance Model
LC	Likely Change
PS	Physical System Description
R	Requirement
FSL	Fourier Series Library.
T	Theoretical Model

2 Introduction

This document introduces a library, FSL, mainly designed and implemented for Fourier series related calculations. This document is based on the template introduced in Smith (2006), Smith and Lai (2005), and Smith et al. (2017).

2.1 Purpose of Document

The scope of the family is limited to calculations related to the Fourier series of mathematical functions.

2.2 Scope of Requirements

The scope of the libraries is limited to calculations related to the Fourier series of mathematical functions.

2.3 Characteristics of Intended Reader

The intended readers must have the following knowledge.

- Deep knowledge of Fourier series (found in advanced calculus and real analysis courses);
- Computational error analysis (found in introductory numerical analysis classes); and
- Detailed knowledge of the programming language utilized in the implementation of this library.

2.4 Organization of Document

This document consists of the following sections.

- section 3 generally describes this library, including what it is, who uses it, and what kinds of environment it needs.
- section 4 introduces the common elements of this library. This section overviews its background, defines related data, introduces theories needed for calculation, and states the goals of this library.
- section 5 covers the different functions of this library, and states its variability in assumptions, calculations, and outputs.
- section 6 informs users, developers, maintainers, and reviewers of this library, about its requirements, both functional ones and nonfunctional ones.
- section 7 states the changes of this library that are likely in the future.

- section 8 shows the traceability matrix of this library, to inform the developers of their changes' impacts.
- Appendix A gives proof of the operation-related theories in section 4.
- Appendix B gives a list of basic functions in this document.

3 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

3.1 System Context

The users provide the data for the library to calculate, and the library returns the calculation results.

- User Responsibilities:
  - Provide the inputs of the functions in the library.
  - When needed, allocate memory space, and pass their location to the library, so that the library can store the output in this user-provided space.
  - For the conversion from mathematical functions to Fourier series, ensure that provided mathematical functions are eligible for transformation.
- FSL Responsibilities:
  - Detect data type mismatch, such as a string of characters instead of a floating point number.
  - Unless otherwise mentioned in this document, detect illegality of input values, such as mismatched length of Fourier series for binary operations.
  - Unless otherwise mentioned, manage the memory space required by this library.

3.2 User Characteristics

The end user of FSL should have an understanding of undergraduate level 2 Calculus and/or Real Analysis, undergraduate level 1 Numerical Analysis, and any one of the programming languages, in which this library provides a set of APIs.

3.3 System Constraints

The potential systems must contain the compilation/interpretation environment for the programming languages in which this library is developed, as well as dependent libraries.

4 Specific System Description

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

4.1 Problem Description and Background Overview

Fourier series are used in lots of analysis, analysis of signals as an example Papoulis (1977). The Fourier series of a function mainly uses a linear combination of 'sin' and 'cos' functions to represent and approximate this function. In real-life research, we sometimes need to perform Fourier series related operations. For example, when we need to analyze the total effects of two signals, each approximated by its Fourier series, we need to calculated the summation of these two Fourier series. Or, when we need to calculate the strength of a signal from its Fourier series, we need to calculate this Fourier series' amplitude.

4.1.1 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- Binary operations: an operation that involves two operands.
- Fourier series: a series of real numbers acting as weights in the weighed sum of 'sin' and 'cos' functions, to represent/approximate a function.
- CFS: short for cut-off Fourier Series, a Fourier Series with finite number of non-zero weights.[Definition of CFS is given here. —Author]
- Amplitude: a map $CFS \to \mathbb{R}^*$ to measure this CFS's strength.

[This section needs further expansion. —Author] [Yes, I agree. In particular, you haven't said what CFS stands for. (Terms should not only be defined in the table of symbols. —SS]

4.1.2 Goal Statements

Given the corresponding inputs, the goal statements are:

GS1: When given a function $f(t)$, the cut-off length n, and a base frequency ω, return the function's CFS $CFSf = CFS(f(t), n, \omega)$.

GS2: When given a CFS $CFSf = CFS(f(t), n, \omega)$ and a value of t as t_1, return the approximated value of $f(t_1)$ computed from the given values.

GS3: When given two CFS's *CFSf*, *CFSg*, and one of the operations addition, subtraction, multiplication, and division, return the result of this operation on these two CFS's.

GS4: When given a CFS *CFSf*, and a function $g(t)$ from the basic function sets defined by this library (see Appendix B), return the CFS $g(CFSf)$.[Basic functions refined in the appendix —Author] [I don't know whether to define the basic function set here or not. I can, and I don't see any possible changes —Author] [The requirements should be complete, so you should define the basic function set. I suggest introducing a data definition for the basic function set. If the set should change, you can always document the change, or "fake it". —SS]

GS5: When given a set of values including n, ω, A_i's, and B_i's, return the CFS built on these values.

GS6: When given a CFS, store its values of n, ω, A_i's, and B_i's in the user-designated spaces.

GS7: When given a CFS, return its amplitude.

GS8: When given two CFS's and a tolerance, return whether they are equal within the tolerance.

4.2 Data Definitions

This section collects and defines all the data needed to build the instance models.

Number	DD1
Label	**Infinite Fourier Series (IFS) of mathematical functions**
Symbol	IFS
Equation	$IFS(f(t),\omega) = \left[A_{inf,i}, i = 0,1,...; B_{inf,i}, i = 1,2,...\right]$, satisfying $f(t) = \sum_{i=0}^{+\infty} A_{inf,i}\cos(i\omega t) + \sum_{i=1}^{+\infty} B_{inf,i}\sin(i\omega t)$
Description	In this equation, • $f(t)$ is an $[-\pi/\omega, \pi/\omega]] \rightarrow \mathbb{R}$ function, whose IFS exists as assumed in A1; [Do you want to add an assumption that the IFS exists? Otherwise, this is something you will need to detect. —SS][Added —Author] • ω is the base frequency of this IFS, and it is designated by the user; • $A_{inf,i} \in \mathbb{R}, i = 0,1,...$; and • $B_{inf,i} \in \mathbb{R}, i = 1,2,....$
Sources	Tolstov, G.P. and Silverman, R.A., Fourier Series, Dover Books on Mathematics, 1976, Dover Publications, Tolstov and Silverman (1976)
Ref. By	IM2, A1, C1, O1, and O2

Number	DD2
Label	**Cut-off Fourier Series (CFS) of mathematical functions**
Symbol	$CFS : (n \in \mathbb{N}, \omega \in \mathbb{R}^+, [A_0, ..., A_n, B_1, ..., B_n] \in \mathbb{R}^{(2n+1)})$, generated from given $f(t) \in (\mathbb{R} \to \mathbb{R}), n, \omega$[What is the type signature for CFS? Is it different than the type for CFSf? —SS][CFSf is the CFS of $f(t)$, as said somewhere in this document. I will reiterate after this DD —Author]
Equation	$CFS(f(t), n, \omega) = [A_i, i = 0, 1, ..., n; B_i, i = 1, 2, ..., n]$, satisfying $A_i = A_{inf,i}, i = 0, 1, ..., n; B_i = B_{inf,i}, i = 1, 2, ..., n$, where ω, $A_{inf,i}$'s and $B_{inf,i}$'s come from $IFS(f(t), \omega)$
Description	$n \in \mathbb{Z}^*$ is called cut-off length.
Sources	Defined by author
Ref. By	T1, T2, T3, T4, T5, T7, A1, C1, O1, and O2
Note	CFS is a mapping given above, its result is of type $CFST$, and any symbol in this document that consists of CFS followed by lower case letters are objects of type $CFST$.

As said before, we use $CFSf$ and $CFSg$ to represent $CFS(f(t), n, \omega)$ and $CFS(g(t), n, \omega)$ respectively. From now on, denote A_i's and B_i's in $CFSf$ as $A_{f,i}$'s and $B_{f,i}$'s respectively, and same for $A_{g,i}$'s and $B_{g,i}$'s.

Number	DD3
Label	**Approximation of function values from CFS**
Symbol	$App : (\text{CFST}, \mathbb{R}) \to \mathbb{R}$ [This function would be clearer if you gave its type signature here. —SS][Addressed here —Author]
Equation	$App(CFSf, t_1) = \sum_{i=0}^{n} A_{f,i} \cos(i\omega t_1) + \sum_{i=1}^{n} B_{f,i} \sin(i\omega t_1)$
Description	Define approximated function value of a function calculated from its CFS.
Sources	Defined by author
Ref. By	IM3, C1, O1, and O2

From now on, we are going to introduce a series of operators/functions on $CFST$ type objects $CFSf$ (and possibly $CFSg$ in some cases). All data included in these objects are fully decided by the callers of functions/operators. The n and ω used in their DD's are these of the parameters (if there are multiple $CSFT$ objects, their n and ω are required to be the same, and will be checked by the function/operator)

Number	DD4
Label	**Addition of CFS's [Adding the type signature would be helpful. —SS][Addressed below —Author]**
Symbol	$+ : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf + CFSg = CFS(f(t) + g(t), n, \omega)$
Description	Define the rule of addition of two CFS's.[Do you want to be able to vary n and ω?, or do you want the input to be the CFSf already calculated for a given n and ω?. The difference in one case is that addition needs to determine the CFS. In the other case the CFS comes in as an "object." —SS][Addressed before this DD —Author]
Sources	Defined by author
Ref. By	IM3, C1, O1, and O2

Number	DD5
Label	**Subtraction of CFS's**
Symbol	$- : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf - CFSg = CFS(f(t) - g(t), n, \omega)$
Description	Define the rule of subtraction of two CFS's
Sources	Defined by author
Ref. By	IM4, C1, O1, and O2

Number	DD6
Label	**Multiplication of CFS's**
Symbol	$* : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf * CFSg = CFS(f(t) * g(t), n, \omega)$
Description	Define the rule of multiplication of two CFS's
Sources	Defined by author
Ref. By	IM5, C1, O1, and O2

Number	DD7
Label	**Division of CFS's**
Symbol	$/ : (CFST, CFST) \rightarrow CFST$
Equation	$CFSf / CFSg = CFS(f(t)/g(t), n, \omega)$
Description	Define the rule of division of two CFS's
Sources	Defined by author
Ref. By	IM6, C1, O1, and O2

Number	DD8
Label	**Function of CFS's**
Symbol	$CFSFunc : (CFST, \mathbb{R} \rightarrow \mathbb{R}) \rightarrow CFST$
Equation	$CFSFunc(CFSf, g) = CFS(g(f(t)), n, \omega)$.
Description	Define the rule of the function of a CFS
Sources	Defined by author
Ref. By	IM7, A2, C1, and O1

Number	DD9
Label	**Amplitude of a CFS**
Symbol	$Amp : CFST \rightarrow \mathbb{R}^*$
Equation	$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} App^2(CSFf, t_1)\, dt_1} \quad (1)$$.
Description	Define the amplitude/size of a CFS
Sources	Defined by author
Ref. By	IM8, A1, C1, and O1

Number	DD10
Label	**Tolerated equality of two CFS's**
Symbol	$TolEq : (CFST, CFST, \mathbb{R}) \rightarrow Bool$
Equation	$TolEq(CFSf, CFSg, err) = (Amp(CFSf - CFSg) \leq \epsilon)$ [Would a relative error make more sense than an absolute error? —SS][Relative error and absolute error can be converted by users easily, with the help of the *Amp* function. I choose to implement an absolute error version because it is more basic, and I can develop a relative error version easily from it if needed. —Author]
Description	Define whether two CFS's are equal within a given tolerance
Sources	Defined by author
Ref. By	IM9, C1, and O1

4.3 Theoretical Models

This section focuses on the general equations and laws that FSL is based on.

Number	T1
Label	**Fourier Transformation**
Equation	$$A_0 = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t); \quad A_i = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos(i\omega t),\ i = 1:n; \quad B_i = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin(i\omega t),\ i = 1:n. \qquad (2)$$
Description	The above equation calculates $A_{inf,i}$'s and $B_{inf,i}$'s, thus $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in *CFSf* from given $f(t)$, n and ω. The definitions of these variables can be found in DD1 and DD2. [There should be a reference to DD1 here. —SS][Added here. —Author]
Source	Tolstov, G.P. and Silverman, R.A., Fourier Series, Dover Books on Mathematics, 1976, Dover Publications, Tolstov and Silverman (1976)
Ref. By	A1, C1, O1, and O2

For the following operations, one of the CFS's can be a constant a. In this case, the corresponding CFS shall be one with $A_0 = a,\ A_i = 0, B_i = 0, (i = 1 : n)$.

Number	T2
Label	**Addition of two CFS's**
Equation	$$\begin{aligned} A_{f+g,i} &= A_{f,i} + A_{g,i},\ \ i = 0 : n \\ B_{f+g,i} &= B_{f,i} + B_{g,i},\ \ i = 1 : n \end{aligned} \tag{3}$$
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) + g(t), n, \omega)$ (represented by $A_{f+g,i}$ and $B_{f+g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A [Great! I was just going to ask how you know the above relation holds, and then I saw your appendix. Good work! —SS]
Ref. By	C1, and O1

Number	T3
Label	**Subtraction of two CFS's**
Equation	$$\begin{aligned} A_{f-g,i} &= A_{f,i} - A_{g,i},\ \ i = 0 : n \\ B_{f-g,i} &= B_{f,i} - B_{g,i},\ \ i = 1 : n \end{aligned} \tag{4}$$
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) - g(t), n, \omega)$ (represented by $A_{f-g,i}$ and $B_{f-g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1, and O1

Number	T4
Label	**Multiplication of two CFS's**
Equation	$$\begin{aligned} A_{f*g,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f,i}A_{g,i+j} + A_{f,i+j}A_{g,i} + B_{f,i}B_{g,i+j} + B_{f,i+j}B_{g,i}) \\ &+ \frac{1}{2}\sum_{j=0}^{i}(A_{f,j}A_{g,i-j} - B_{f,j}B_{g,i-j}),\ i = 0 : n \\ B_{f*g,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f,i}B_{g,i+j} + A_{f,i+j}B_{g,i} + B_{f,i}A_{g,i+j} + B_{f,i+j}A_{g,i}) \\ &+ \frac{1}{2}\sum_{j=0}^{i}(A_{f,j}B_{g,i-j} + B_{f,j}A_{g,i-j}),\ i = 1 : n \end{aligned} \quad (5)$$
Description	The above equation calculates $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t) * g(t), n, \omega)$ (represented by $A_{f*g,i}$ and $B_{f*g,i}$ respectively) from $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1, and O1

Number	T5
Label	**Division of two CFS's**
Equation	Solve the following equations for $A_{f/g,i}$'s and $B_{f/g,i}$'s. $$\begin{aligned} A_{f,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f/g,i}A_{g,i+j} + A_{f/g,i+j}A_{g,i} + B_{f/g,i}B_{g,i+j} + B_{f/g,i+j}B_{g,i}) \\ &\quad + \frac{1}{2}\sum_{j=0}^{i}(A_{f/g,j}A_{g,i-j} - B_{f/g,j}B_{g,i-j}),\ i = 0:n \\ B_{f,i} &= \frac{1}{2}\sum_{j=0}^{n-i}(A_{f/g,i}B_{g,i+j} + Af/g,i+jB_{g,i} + B_{f/g,i}A_{g,i+j} + B_{f/g,i+j}A_{g,i}) \\ &\quad + \frac{1}{2}\sum_{j=0}^{i}(A_{f/g,j}B_{g,i-j} + B_{f/g,j}A_{g,i-j}),\ i = 1:n \end{aligned} \tag{6}$$
Description	We solve the above equations for $A_i(i = 0 : n)$, $B_i(i = 1 : n)$ in $CFS(f(t)/g(t), n, \omega)$ (represented by $A_{f/g,i}$'s and $B_{f/g,i}$'s respectively) from given $CFSf$ and $CFSg$.
Source	Developed by author in Appendix A
Ref. By	C1, O1, and O2

Number	T6
Label	**Function of a CFS**
Equation	$g(CFS(f(t), n, \omega)) \approx \sum_{i=0}^{n} a_i CFS^i(f(t), n, \omega)$
Description	The above equation calculates the result of a function being applied to a CFS. $a_i, i = 0 : n$ is the first $(n+1)$ coefficients of the Taylor series of $g(t)$, and the n-th ($n \in \mathbb{Z}^+$) power of CFS is defined as n copies of this CFS multiplied together. (0-th power is defined as the CFS of $f(t) = 1$.)
Source	Developed by author in Appendix A [Is the proof in the Appendix? I didn't see it? This one definitely isn't intuitive. —SS][Forgot to write down the proof in the first draft, added in this version to the end of this appendix. Also change = to $\approx$ —Author]
Ref. By	C1 and O1

Number	T7
Label	**Amplitude of a CFS**
Equation	$$Amp(CSFf) = \sqrt{A_0^2 + \frac{1}{2}\sum_{i=1}^{n}(A_i^2 + B_i^2)} \quad (7)$$
Description	The above equation calculates $Amp(CFSf)$ from $CFSf$, especially A_i's and B_i's in it.
Source	Developed by author in Appendix A
Ref. By	C1 and O1

5 Instance Models

Number	IM1
Label	**Calculate coefficients of CFS's**
Input	$f(t)$, a $[-\pi/\omega, \pi/\omega] \rightarrow \mathbb{R}$ function $n \in \mathbb{Z}^*$ $\omega \in \mathbb{R}^+$
Output	An object *CFSf*, containing the following data: n, ω: same as the input $A_i (i = 0 : n)$, $B_i (i = 1 : n)$: using the theory T1
Description	Input: $f(t)$: function to be transformed into Fourier series Input & Output: n: cut-off length ω: base frequency Output: A_i's and B_i's: coefficients in *CFSf*
Sources	Same as T1
Ref. By	A1, C1, O1, and O2

Number	IM2
Label	**Calculate approximate function value from CFS**
Input	$CFSf = [n, \omega, A_i(i = 0 : n), B_i(i = 1 : n)]$ $t_1 \in \mathbb{R}$
Output	$App(CFSf, t_1)$ using DD3
Description	Input: $CFSf$: CFS used for function approximation t_1: variable value Output: $App(CFSf, t_1)$: approximated function value
Sources	Same as T3
Ref. By	C1 and O1

Number	IM3
Label	**Addition of two CFS's**
Input	$CFSf$, $CFSg$
Output	$CFS(f(t) + g(t), n, \omega)$, whose A_i's and B_i's are computed using the theory T2
Description	Input: $CFSf$, $CFSg$: operands of the addition Output: $CFS(f(t) + g(t), n, \omega)$: result of the addition
Sources	Same as T2
Ref. By	C1, and O1

Number	IM4
Label	**Subtraction of two CFS's**
Input	$CFSf$, $CFSg$
Output	$CFS(f(t) - g(t), n, \omega)$, whose A_i's and B_i's are computed using the theory T3
Description	Input: $CFSf$, $CFSg$: operands of the subtraction Output: $CFS(f(t) - g(t), n, \omega)$: result of the subtraction
Sources	Same as T3
Ref. By	C1, and O1

Number	IM5
Label	**Multiplication of two CFS's**
Input	$CFSf$, $CFSg$
Output	$CFS(f(t) * g(t), n, \omega)$, whose A_i's and B_i's are computed using the theory T4
Description	Input: $CFSf$, $CFSg$: operands of the multiplication Output: $CFS(f(t) * g(t), n, \omega)$: result of the multiplication
Sources	Same as T4
Ref. By	C1, and O1

Number	IM6
Label	**Division of two CFS's**
Input	$CFSf$, $CFSg$
Output	$CFS(f(t)/g(t), n, \omega)$, whose A_i's and B_i's are computed using the theory T5
Description	Input: $CFSf$, $CFSg$: operands of the division Output: $CFS(f(t) + g(t), n, \omega)$: result of the division
Sources	Same as T5
Ref. By	C1, O1, and O2

Number	IM7
Label	**Function of a CFS**
Input	$CFSf$, $g(t)$
Output	A CFS $g(CFSf)$, whose A_i's and B_i's are computed using the theory T4 and T2
Description	Input: $g(t)$, a basic function chosen by user from a basic function set. $CFSf$: dependent variable of the function $g(t)$ Output: $g(CFSf)$: A CFS being the computed result
Sources	Easily derived from T4 and T2
Ref. By	A2, C1, and O1

Number	IM8
Label	**Amplitude of a CFS**
Input	$CFSf$
Output	$Amp(CFSf)$, computed using the theory T7
Description	Input: $CFSf$: variable of the amplitude function Output: $Amp(CFSf)$, the amplitude of $CFSf$
Sources	Same as T7
Ref. By	C1, and O1

Number	IM9
Label	**Tolerated Equality Comparison of two CFS's**
Input	$CFSf$, $CFSg$, ϵ
Output	$TolEq(CFSf, CFSg, \epsilon)$ boolean value `True` if $Amp(CFSf - CFSg) \leq \epsilon$ boolean value `False` otherwise
Description	Input: $CFSf$, $CFSg$: operands of the tolerated equality comparison $\epsilon \in \mathbb{R}^*$ Output: A boolean value: Whether the two operands are equal within the given error tolerance Note: The calculation of difference's amplitude relies on IM4 and IM8.
Sources	Same as T3 and T7
Ref. By	C1, and O1

Number	IM10
Label	**Convert data of other structures (input sources included) to a CFS**
Input	n, ω, $n+1$ real numbers $Ain_i(i = 0 : n)$, and n real numbers $Bin_i(i = 1 : n)$
Output	A CFS object *CFSf* constructed from the input data
Description	Input: Data needed for construction Output: Constructed CSF object containing the input data
Sources	None
Ref. By	C1, and O1

Number	IM11
Label	**Convert CFS to data of structures (output destination included)**
Input	*CFSf*
Output	n, ω, $Aout_i = A_i(i = 0 : n)$, and $Bout_i = B_i(i = 1 : n)$
Description	Input: The CFS object to be converted to Output: The data extracted from the input CFS object.
Sources	None
Ref. By	C1, and O1

5.1 Assumptions

A1: The functions $f(t)$ and $g(t)$ mentioned above must have definitions on $[-\pi/\omega, \pi/\omega]$, in which ω is given by the user, and must have corresponding Fourier series. [I think you use this assumption in DD1. You should invoke it there. All assumptions need to be invoked somewhere, or else they aren't really assumptions that are being used. This point is part of the SRS checklist. —SS][Done accordingly —Author]

A2: Only the basic functions of CFS's are to be calculated. The aforementioned basic functions are those in a library-defined set. [As discussed elsewhere, you should explicitly define the functions that are basic. —SS]

[Old content: For any two-operand operations, the data n and ω of these operands must be the same. —Author] [Is this an assumption, or is it something that you can test? From your MIS presentation, it looks like this is something you can test. —SS] [When I wrote this down, I intended to assume this because of another design involving the arithmetic of ω, but I later found out that that design is too complex for me, as well as not concrete mathematically, so I drop that, but I failed to delete this. I deleted this in the final version. —Author]

[Old content: User shall allocate memory space for any variable other than CFS's in IM10 and IM11. —Author] [This assumption might be better in the design documentation. —SS] [Noted and deleted. —Author]

5.2 Calculation

C1: Calculate the result based on the called function and input variables.

5.3 Output

O1: Return the results faithfully. That is, return the constructed CFS in IM1 and IM10; fill the user-provided space with data from user-designated CFS in IM11; return the calculated value in IM2, IM8, and IM9; and return the calculated CFS in IM3, IM4, IM5, IM6, and IM7.

O2: Report any detected errors to the user of the library.

6 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

6.1 Functional Requirements

R1: When applicable, functions of this library accept inputs with the same data type for $f(t)$, A_i's and/or B_i's, and ω.

R2: The operands of binary operations must have the same n and ω.

R3: The error of the output must be decreasing as the n in CFS's is decreasing.

R4: Output the result with the same floating point data type as that of the input.

R5: Manage the memory spaces required by this library (mainly for CFS's), and destroy them the moment they are not needed.

6.2 Nonfunctional Requirements

- All time complexities shall be unrelated to ω.
- The time complexity of IM1 shall be $O(n^2)$ when the input function $f(t)$ is not complex.
- The time complexity of IM3 and IM4 shall be $O(n)$.
- The time complexity of IM5 shall be $O(n^2)$.
- The time complexity of IM6 shall be the same as the best linear equation solver applicable.
- The time complexity of IM8 and IM9 shall be $O(n)$.

7 Likely Changes

LC1: We might add new operations in the future.

LC2: The set of basic functions in IM7 will be decided in the future.

8 Traceability Matrices and Graphs

The following matrices and graphs demonstrates the traceability of this project. The purpose is to provide easy references to the impacts on other components if a certain component is changed. That is, if one component has been changed, other components that share an 'X' with it may need change. Table 1 shows the the dependencies of goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs.

[Great work Bo. I have added some comments above, but overall you are definitely on the right track. I have also made a few fixes directly, rather than ask you to make them. You should review the diff so that you can see those fixes. —SS] [My main comment is that you haven't really documented a CA; you have documented an SRS, but called it a CA. In a CA you should identify different family members. In particular, you need to know the parameters of variation for each variability. Rather than cast your problem as a CA, I think it makes more sense to document your problem as an SRS. I don't think it will take much for you to convert your document to follow the SRS template. It will be a useful change though, because, as it is right now, the input, output, variability, etc headings are confusing. —SS] [Done accordingly —Author]

	A1	A2	C1	O1	O2
GS1	X		X	X	X
GS2		X	X	X	
GS3			X	X	X
GS4	X		X	X	X
GS5			X	X	
GS6			X	X	
GS7				X	
GS8			X	X	X
DD1	X		X	X	X
DD2	X		X	X	X
DD3			X	X	X
DD4			X	X	X
DD5			X	X	X
DD6			X	X	X
DD7			X	X	X
DD8		X	X	X	
DD9			X	X	
T1	X		X	X	X
T2			X	X	
T3			X	X	
T4			X	X	
T5			X	X	X
T7			X	X	
IM1	X		X	X	X
IM3			X	X	
IM4			X	X	
IM5			X	X	
IM6			X	X	X
IM7		X	X	X	
IM8			X	X	
IM9			X	X	
IM10			X	X	
IM11			X	X	

Table 1: The traceability matrix between goals, theoretical models, data definitions, and instances models with the assumptions, calculations, and outputs

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A Theory for operations on CFS's

In this appendix, we hereby give the proof of T2, T3, T4, and T7 in corresponding paragraphs. The proof of T5 comes directly from the equation $[f(t)/g(t)] * g(t) = f(t)$.

In the following proofs, suppose we have two functions, $f(t)$ and $g(t)$ with existing IFS and CFS. The n and ω of these IFS's and CFS's are the same, but with different A_i's and B_i's (denoted with $A_{f,i}$'s, $B_{f,i}$'s and $A_{g,i}$'s, $B_{g,i}$'s respectively). From the definition of IFS, DD1, we know that

$$f(t) = \sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t), \tag{8}$$

and

$$g(t) = \sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t). \tag{9}$$

Addition and Subtraction Like $f(t)$ and $g(t)$, we also know that

$$f(t) + g(t) = \sum_{i=0}^{+\infty} A_{f+g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i} \sin(i\omega t). \tag{10}$$

By replacing $f(t)$ and $g(t)$ in Equation 10 with Equation 8 and Equation 9, we have

$$\begin{aligned}
&\sum_{i=0}^{+\infty} A_{f+g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f+g,i} \sin(i\omega t) \\
=&\sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t) + \sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t) \\
=&\sum_{i=0}^{+\infty} (A_{f,i} + A_{g,i}) \cos(i\omega t) + \sum_{i=1}^{+\infty} (B_{f,i} + B_{g,i}) \sin(i\omega t)
\end{aligned} \tag{11}$$

By comparing the coefficients in Equation 11, we have

$$\begin{aligned}
A_{f+g,i} &= A_{f,i} + A_{g,i} \\
B_{f+g,i} &= B_{f,i} + B_{g,i}
\end{aligned} \tag{12}$$

for the IFS and CFS of $f(t) + g(t)$.

Likewise, we have similar conclusions for those of $f(t) - g(t)$.

Multiplication From Equation 8 and Equation 9, we have

$$f(t) * g(t) = [\sum_{i=0}^{+\infty} A_{f,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{f,i} \sin(i\omega t)] * [\sum_{i=0}^{+\infty} A_{g,i} \cos(i\omega t) + \sum_{i=1}^{+\infty} B_{g,i} \sin(i\omega t)] \tag{13}$$

which is the summation the following 3 terms

$$
\begin{aligned}
&\textit{Term A:} \sum_{i=0,j=0}^{+\infty} A_{f,i}A_{g,j}\cos(i\omega t)\cos(j\omega t)\\
&=\frac{1}{2}\sum_{i=0,j=0}^{+\infty} A_{f,i}A_{g,j}\cos[(i+j)\omega t]+\frac{1}{2}\sum_{i=0,j=0}^{+\infty} A_{f,i}A_{g,j}\cos[(j-i)\omega t]\\
&=\frac{1}{2}\sum_{i=0}^{+\infty}[\sum_{j=0}^{i} A_{f,j}A_{g,i-j}]\cos(i\omega t)+\frac{1}{2}\sum_{j=0}^{+\infty} A_{f,i}A_{g,i}+\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=0}^{+\infty} A_{f,j}A_{g,j+i}+\sum_{j=0}^{+\infty} A_{f,j+i}A_{g,j}]\cos(i\omega t)\\
&=A_{f,0}A_{g,0}+\frac{1}{2}\sum_{j=0}^{+\infty} A_{f,}A_{g,j}+\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=0}^{i} A_{f,j}+A_{g,i-j}+\sum_{j=0}^{+\infty}(A_{f,i}A_{g,j+i}+A_{f,j+i}A_{g,i})]\cos(i\omega t)\\
&\textit{Term B:} \sum_{i=0,j=1}^{+\infty}[A_{f,i}B_{g,j}\cos(i\omega t)\sin(j\omega t)+B_{g,i}A_{f,j}\sin(i\omega t)\cos(j\omega t)]\\
&=\frac{1}{2}\sum_{i=0,j=1}^{+\infty}[A_{f,i}B_{g,j}\sin((i+j)\omega t)-A_{f,i}B_{g,j}\sin((i-j)\omega t)]\\
&+\frac{1}{2}\sum_{i=0,j=1}^{+\infty}[B_{g,i}A_{f,j}\sin((i+j)\omega t)+B_{g,i}A_{f,j}\sin((i-j)\omega t)]\\
&=\frac{1}{2}\sum_{i=0,j=1}^{+\infty}[A_{f,i}B_{g,j}+B_{g,i}A_{f,j}]\sin((i+j)\omega t)+\frac{1}{2}\sum_{i=0,j=1}^{+\infty}[B_{g,i}A_{f,j}-A_{f,i}B_{g,j}]\sin((i-j)\omega t)\\
&=\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=1}^{+\infty} A_{f,i-j}B_{g,j}+B_{g,i-j}A_{f,j}]\sin(i\omega t)-\frac{1}{2}\sum_{i=1}^{+\infty}[B_{g,0}A_{f,i}-A_{f,0}B_{g,i}]\sin(i\omega t)\\
&=\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=1}^{+\infty}(A_{f,i-j}B_{g,j}+B_{g,i-j}A_{f,j})-B_{g,0}A_{f,i}+A_{f,0}B_{g,i}]\sin(i\omega t)\\
&\textit{Term C:} \sum_{i=1,j=1}^{+\infty} B_{f,i}B_{g,j}\sin(i\omega t)\sin(j\omega t)\\
&=\frac{1}{2}\sum_{i=1,j=1}^{+\infty} B_{f,i}B_{g,j}[\cos((i-j)\omega t)-\cos((i+j)\omega t)]\\
&=\frac{1}{2}[\sum_{j=1}^{+\infty} B_{f,j}B_{g,j}]\cos(0\omega t)+\frac{1}{2}\sum_{i=1}^{+\infty}[\sum_{j=1}^{+\infty} B_{f,i+j}B_{g,i}+B_{f,i}B_{g,i+j}]\cos(i\omega t)\\
&-\frac{1}{2}\sum_{i=2}^{+\infty}[\sum_{j=1}^{i-1} B_{f,i-j}B_{g,j}]\cos(i\omega t)
\end{aligned} \tag{14}
$$

We gather the coefficients of $\cos(i\omega t), i = 0 : n$ and $\sin(i\omega t), i = 1 : n$ respectively, remove any terms containing $A_{f,k}$, $B_{f,k}$, $A_{g,k}$, and $B_{g,k}$ for any $k \geq n$ (k being either i, j, $i-j$, or $i+j$) as the result of a cut-off, and get the equations in T4.

Amplitude Quoted from DD9, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} App^2(CFSf, t)\mathrm{d}t} \tag{15}$$

In Equation 15, replacing $App(CFSf, t)$ with its definition in DD3, we have

$$Amp(CFSf) = \sqrt{\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega}[\sum_{i=0}^{n} A_{f,i}\cos(i\omega t) + \sum_{i=1}^{n} B_{f,i}\sin(j\omega t)][\sum_{j=0}^{n} A_{f,j}\cos(j\omega t) + \sum_{j=1}^{n} B_{f,j}\sin(j\omega t)]\mathrm{d}t} \tag{16}$$

The part inside the integration in Equation 16 can be expressed as the summation of the following three expressions

$$\begin{aligned}
&\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=0,j=0}^{n} A_{f,i}A_{f,j}\cos(i\omega t)\cos(j\omega t)\mathrm{d}t \\
&\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=0,j=1}^{n} A_{f,i}B_{f,j}\sin(i\omega t)\cos(j\omega t)\mathrm{d}t + \int_{-\pi/\omega}^{\pi/\omega}\sum_{i=1,j=0}^{n} B_{f,i}A_{f,j}\sin(i\omega t)\cos(j\omega t)\mathrm{d}t \\
&\int_{-\pi/\omega}^{\pi/\omega}\sum_{i=1,j=1}^{n} B_{f,i}B_{f,j}\sin(i\omega t)\sin(j\omega t)\mathrm{d}t
\end{aligned} \tag{17}$$

The integration part of these expressions are $\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\cos(j\omega t)$, $\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t)$, and $\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t)$. Calculation shows that the integration results are

$$\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\cos(j\omega t) = \begin{cases} 2\pi/\omega, & i = j = 0; \\ \pi/\omega, & i = j \neq 0; \\ 0, & i \neq j. \end{cases} \tag{18}$$

$$\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t) = 0 \tag{19}$$

and

$$\int_{-\pi/\omega}^{\pi/\omega}\cos(i\omega t)\sin(j\omega t) = \begin{cases} \pi/\omega, & i = j; \\ 0, & i \neq j. \end{cases} \tag{20}$$

Replacing integration in Equation 17 with integration results Equation 18, Equation 19, and Equation 20, and we have the following results.

$$\begin{aligned}
&\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0,j=0}^{n} A_{f,i}A_{f,j}\cos(i\omega t)\cos(j\omega t)\mathrm{d}t = \frac{2\pi}{\omega}A_0^2 + \frac{\pi}{\omega}\sum_{i=1}^{n} A_i^2 \\
&\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=0,j=1}^{n} A_{f,i}B_{f,j}\sin(i\omega t)\cos(j\omega t)\mathrm{d}t = 0 \\
&\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1,j=0}^{n} B_{f,i}A_{f,j}\sin(i\omega t)\cos(j\omega t)\mathrm{d}t = 0 \\
&\int_{-\pi/\omega}^{\pi/\omega} \sum_{i=1,j=1}^{n} B_{f,i}B_{f,j}\sin(i\omega t)\sin(j\omega t)\mathrm{d}t = \frac{\pi}{\omega}\sum_{i=1}^{n} B_i^2
\end{aligned} \tag{21}$$

Summing them up and putting the result back into Equation 16, we have the expression in T7.

Function We will develop the equations of $A_i(i > 0)$'s in the resulted CFS. A_0 and B_i's can be developed similarly.

$$A_i = \frac{\omega}{\pi}\int_{-\pi/\omega} \pi/\omega g(f(t))\cos(i\omega t) \tag{22}$$

$$= \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} \sum_{j=0}^{+\infty} a_j f(t)^j \cos(i\omega t) \tag{23}$$

$$= \sum_{j=0}^{+\infty} a_j \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)^j cos(i\omega t) \tag{24}$$

$$= \sum_{j=0}^{+\infty} a_j (CFSf^j).A_i \tag{25}$$

Intuitively (detailed analysis of introduced error in the future), as an approximation, we do a similar cut-off on the Taylor series. That is, we turn $\sum_{j=0}^{+\infty}$ into $\sum_{j=0}^{n}$, and we have

$$A_i = \sum_{j=0}^{n} a_j (CFSf^j).A_i \tag{26}$$

Combining with similar resulted expressions of A_0 and B_i's, we have the equations in T6.

B List of Basic Functions

The term *Basic Functions* in this library consists of the following functions

- $\sin(x)$
- $\cos(x)$
- $\tan(x)$
- $\arcsin(x)$
- $\arccos(x)$
- $\arctan(x)$
- $\exp(x)$
- $\ln(x+1)$

We might include more functions if deemed necessary.