STT 843: Multivariate Analysis

4. The Multivariate Normal Distribution (Chapter 4.1-4.2)

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Outline

Introduction

- A generalization of the familiar bell shaped normal density to several dimensions plays a fundamental role in multivariate analysis
- While real data are never exactly multivariate normal, the normal density is often a useful approximation to the "true" population distribution because of a central limit effect.
- One advantage of the multivariate normal distribution stems from the fact that it is mathematically tractable and "nice" results can be obtained.

To summarize, many real-world problems fall naturally within the framework of normal theory. The importance of the normal distribution rests on its dual role as both population model for certain natural phenomena and approximate sampling distribution for many statistics.

Univariate normal distribution

- Recall that the univariate normal distribution, with mean μ and variance σ^2, has the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-[(x-\mu)/\sigma]^2/2} \qquad -\infty < x < \infty$$

- The term

$$\left(\frac{x-\mu}{\sigma}\right)^2 = (x-\mu)\left(\sigma^2\right)^{-1}(x-\mu)$$

can be generalized for $p \times 1$ vector $\mathbf{x}$ of observations on several variables as

$$(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})$$

The $p \times 1$ vector $\boldsymbol{\mu}$ represents the expected value of the random vector $\boldsymbol{X}$, and the $p \times p$ matrix Σ is the variance-covariance matrix of $\boldsymbol{X}$.

- A p-dimensional normal density for the random vector $\boldsymbol{X}^{\mathrm{T}} = [X_1, X_2, \ldots, X_p]$ has the form

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-(\mathbf{X}-\boldsymbol{\mu})^{\mathrm{T}}\Sigma^{-1}(\mathbf{X}-\boldsymbol{\mu})/2}$$

where $-\infty < x_i < \infty, i = 1, 2, \ldots, p$. We should denote this p-dimensional normal density by $N_p(\boldsymbol{\mu}, \Sigma)$.

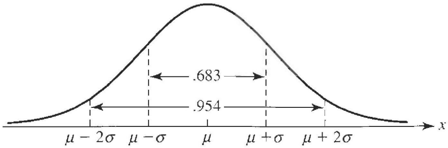


Figure 4.1 A normal density with mean μ and variance σ^2 and selected areas under the curve.

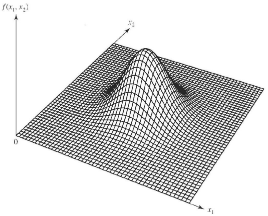
Example 4.1 (Bivariate normal density)

- Let us evaluate the $p = 2$ variate normal density in terms of the individual parameters $\mu_1 = \mathrm{E}(X_1)$, $\mu_2 = \mathrm{E}(X_2)$, $\sigma_{11} = \mathrm{Var}(X_1)$, $\sigma_{22} = \mathrm{Var}(X_2)$, and $\rho_{12} = \sigma_{12}/\left(\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\right) = \mathrm{Corr}(X_1, X_2)$.
- If Σ is positive definite, so that Σ^{-1} exists, then

$$\Sigma \mathbf{e} = \lambda \mathbf{e} \quad \text{implies} \quad \Sigma^{-1}\mathbf{e} = \frac{1}{\lambda}\mathbf{e}$$

  so $(\lambda, \mathbf{e})$ is an eigenvalue-eigenvector pair for Σ corresponding to the pair $(1/\lambda, \mathbf{e})$ for Σ^{-1}. Also Σ^{-1} is positive definite.

(a) $\sigma_{11} = \sigma_{22}$ and $\rho_{12} = 0$.



(b) $\sigma_{11} = \sigma_{22}$ and $\rho_{12} = .75$.

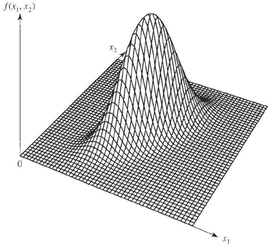


Figure 4.2 Two bivariate normal distributions.

- Constant probability density contour
$= \{$ all $\mathbf{x}$ such that $(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) = c^2\}$
$=$ surface of an ellipsoid centered at $\boldsymbol{\mu}$.

- Contours of constant density for the p-dimensional normal distribution are ellipsoids defined by $\mathbf{x}$ such the that

$$(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) = c^2$$

  These ellipsoids are centered at $\boldsymbol{\mu}$ and have axes $\pm c\sqrt{\lambda_i}\mathbf{e}_i$, where $\Sigma\mathbf{e}_i = \lambda_i\mathbf{e}_i$ for $i = 1, 2, \ldots, p$.

Example 4.2 (Contours of the bivariate normal density)Obtain the axes of constant probability density contours for a bivariate normal distribution when $\sigma_{11} = \sigma_{22}$

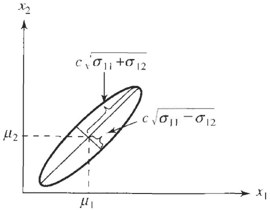


Figure 4.3 A constant-density contour for a bivariate normal distribution with $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} > 0$ (or $\rho_{12} > 0$).

The solid ellipsoid of $\mathbf{x}$ values satisfying

$$(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) \le \chi_p^2(\alpha)$$

has probability $1-\alpha$ where $\chi_p^2(\alpha)$ is the upper (100α) th percentile of a chi-square distribution with p degrees of freedom.

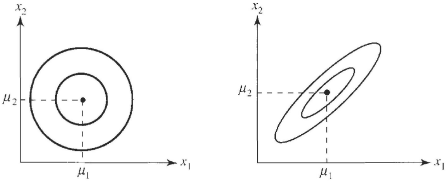


Figure 4.4 The 50% and 90% contours for the bivariate normal distributions in Figure 4.2.