STT43: Multivariate Data Analysis **Term: Spring 2026**

Homework #2 (due 02/08/26)

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All problems on this homework are from the required textbook Applied Multivariate Statistical Analysis (6th edition, by Johnson and Wichern).

Problems from Chapter 2

Problem 2.34

Solution

Consider the vectors

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}.$$

The Cauchy-Schwarz inequality states that

$$(\mathbf{b}'\mathbf{d})^2 \leq (\mathbf{b}'\mathbf{b})(\mathbf{d}'\mathbf{d}).$$

We have:

$$\mathbf{b}'\mathbf{d} = -13, \qquad \mathbf{b}'\mathbf{b} = 21 \quad \text{and} \qquad \mathbf{d}'\mathbf{d} = 15.$$

Hence the inequality holds since,

$$(\mathbf{b}'\mathbf{d})^2 = 169 \quad \text{and} \quad (\mathbf{b}'\mathbf{b})(\mathbf{d}'\mathbf{d}) = 21 \times 15 = 315.$$

Problem 2.35

Solution

Consider the vectors

$$\mathbf{b} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the matrix

$$B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

The extended Cauchy-Schwarz inequality states that

$$(\mathbf{b}'\mathbf{d})^2 \leq (\mathbf{b}'B\mathbf{b})\,(\mathbf{d}'B^{-1}\mathbf{d}).$$

We have:

$$\mathbf{b}'\mathbf{d} = -1, \qquad \mathbf{b}'B\mathbf{b} = 125 \quad \text{and} \qquad \mathbf{d}'B^{-1}\mathbf{d} = 11/6.$$

Therefore the inequality holds since,

$$(\mathbf{b}'\mathbf{d}) = 1 \quad \text{and} \qquad (\mathbf{b}'B\mathbf{b})(\mathbf{d}'B^{-1}\mathbf{d}) = \frac{1375}{6}.$$

Problems from Chapter 3

Problem 3.14

Solution

Consider the data matrix

$$X = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix},$$

with $n = 3$ observations on $p = 2$ variables X_1 and X_2. Form the linear combinations,

$$\mathbf{c}'\mathbf{X} = -X_1 + 2X_2, \quad \mathbf{b}'\mathbf{X} = 2X_1 + 3X_2.$$

the observed values are:

$$\mathbf{c}'\mathbf{X}: \quad -7,\ 1,\ 3$$
$$\mathbf{b}'\mathbf{X}: \quad 21,\ 19,\ 8.$$

a) From first principle, the respective sample means are then

$$\bar{c} = \frac{-7+1+3}{3} = -1, \quad \bar{b} = \frac{21+19+8}{3} = 16.$$

and sample variances are:

$$s_c = \frac{(-7+1)^2 + (1+1)^2 + (3+1)^2}{2} = 28,$$
$$s_b = \frac{(21-16)^2 + (19-16)^2 + (8-16)^2}{2} = 49.$$

The sample covariance is then

$$s_{cb} = \frac{(-7+1)(21-16) + (1+1)(19-16) + (3+1)(8-16)}{2} = -28.$$

b) Using Formulas (3-36) and noting that the sample mean vector and covariance matrix are

$$\bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$$

we get the means as:

$$\mathbf{c}'\bar{\mathbf{x}} = -1, \quad \mathbf{b}'\bar{\mathbf{x}} = 16,$$

and variances and covariance as:

$$\mathbf{c}'S\mathbf{c} = 28, \quad \mathbf{b}'S\mathbf{b} = 49, \quad \mathbf{c}'S\mathbf{b} = -28.$$

The results in (a) and (b) match.

Problem 3.16

Solution

Proof. We have:

$$\begin{aligned} \Sigma &= \mathbf{E}[(\mathbf{V}-\boldsymbol{\mu})(\mathbf{V}-\boldsymbol{\mu})'] \\ &= \mathbf{E}(\mathbf{V}\mathbf{V}') - \mathbf{E}(\mathbf{V})\boldsymbol{\mu}' - \boldsymbol{\mu}\mathbf{E}(\mathbf{V}') + \boldsymbol{\mu}\boldsymbol{\mu}' \\ &= \mathbf{E}(\mathbf{V}\mathbf{V}') - \boldsymbol{\mu}\boldsymbol{\mu}' - \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}' \\ &= \mathbf{E}(\mathbf{V}\mathbf{V}') - \boldsymbol{\mu}\boldsymbol{\mu}'. \end{aligned}$$

Rearranging gives the desired result:

$$\boxed{\mathbf{E}(\mathbf{V}\mathbf{V}') = \Sigma + \boldsymbol{\mu}\boldsymbol{\mu}'}.$$

□

Problems from Chapter 4

Problem 4.3

Solution

Let $\mathbf{X} = (X_1, X_2, X_3)' \sim N_3(\boldsymbol{\mu}, \Sigma)$, where

$$\boldsymbol{\mu} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

a) X_1 and X_2 are **not independent** since

$$Cov(X_1, X_2) = \Sigma_{12} = -2 \neq 0.$$

b) X_2 and X_3 are **independent** since X_2 and X_3 are jointly normal and

$$Cov(X_2, X_3) = \Sigma_{23} = 0.$$

c) $Y = (X_1, X_2)$ and X_3 are **independent** since

$$Cov(Y, X_3) = \Sigma_{Y,X_3} = (0, 0)'.$$

d) $\dfrac{X_1 + X_2}{2}$ and X_3 are **independent** since Y and X_3 are jointly normal and

$$Cov(X_1, X_3) = Cov(X_2, X_3) = 0.$$

e) X_2 and $Y = X_2 - \dfrac{5}{2}X_1 - X_3$ are **not independent** since

$$Cov(X_2, Y) = Var(X_2) - \frac{5}{2}Cov(X_1, X_2) = 10 \neq 0.$$

Problem 4.13

Solution

Let $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$ with $|\Sigma| \neq 0$. Partition $\mathbf{X}$, $\boldsymbol{\mu}$, and Σ conformably as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}} \end{bmatrix},$$

where $\mathbf{X}_1$ is $q \times 1$, $\mathbf{X}_2$ is $(p - q) \times 1$, Σ_{22} is nonsingular, and $\Sigma_{21} = \Sigma_{12}'$.

a) *Proof.* Note that $\boldsymbol{\Sigma}$ can be decomposed as

$$\begin{aligned} \boldsymbol{\Sigma} &= \begin{bmatrix} \boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{0} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \\ &= \mathbf{AB} \end{aligned}$$

Using properties of determinant of block matrices, we get $|A| = |I| = 1$ and then

$$|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma_{22}}||\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}|$$

□

b) *Proof.* Using the hint from the textbook, we write

$$(\mathbf{x}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) = \begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}' \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} & \mathbf{I}\end{bmatrix} \begin{bmatrix}(\mathbf{\Sigma}_{11}-\mathbf{\Sigma}_{12}^{-1}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}^{-1})^{-1} & \mathbf{0}\\ \mathbf{0}' & \mathbf{\Sigma}_{22}^{-1}\end{bmatrix} \begin{bmatrix}\mathbf{I} & -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\\ \mathbf{0} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}' \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} & \mathbf{I}\end{bmatrix} \begin{bmatrix}(\mathbf{\Sigma}_{11}-\mathbf{\Sigma}_{12}^{-1}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}^{-1})^{-1} & \mathbf{0}\\ \mathbf{0}' & \mathbf{\Sigma}_{22}^{-1}\end{bmatrix} \begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$

We arrive at the answer after the above matrix multiplication. □

c) The result in b) shows that the quadratic form in the exponent of the multivariate normal density can be decomposed as

$$(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = (\mathbf{x}_1-\boldsymbol{\mu}_{1|2})'\mathbf{\Sigma}_{1|2}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_{1|2}) + (\mathbf{x}_2-\boldsymbol{\mu}_2)'\mathbf{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2),$$

where

$$\boldsymbol{\mu}_{1|2} = \boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)$$

and

$$\mathbf{\Sigma}_{1|2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$$

are the conditional mean and covariance of $\mathbf{X}_1|\mathbf{X}_2=\mathbf{x}_2$.

From parts (a) and (b), and the factorization of the multivariate normal density, we immediately obtain:

1. **Marginal distribution of $\mathbf{X}_2$:**
$$\mathbf{X}_2 \sim N_{p-q}\big(\boldsymbol{\mu}_2, \Sigma_{22}\big).$$
2. **Conditional distribution of**
$$\mathbf{X}_1|\mathbf{X}_2=\mathbf{x}_2 \quad \sim \quad N_q\big(\boldsymbol{\mu}_{1|2},\ \Sigma_{1|2}\big).$$

Problem 4.14

Solution

It follows from the previous problem that when $\mathbf{\Sigma}_{12}=0$,

$$|\mathbf{\Sigma}| = |\mathbf{\Sigma}_{11}||\mathbf{\Sigma}_{22}| \qquad \text{and} \quad (\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = (\mathbf{x}_1-\boldsymbol{\mu}_1)'\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1) + (\mathbf{x}_2-\boldsymbol{\mu}_2)'\mathbf{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)$$

We can then easily rewrite the distribution of $\mathbf{X}$ as a product of the marginals of $\mathbf{X}_1$ and $\mathbf{X}_2$ following the idea in the previous problem.

Problem 4.15

Solution

Proof. We have $\mathbf{x}_j$ is a $p\times 1$ matrix and hence $\bar{\mathbf{x}}$ is $p\times 1$ and the product $(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})'$ is a $p\times p$. Similarly, $(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})'$ is $p\times p$. Now,

$$\sum_{i=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})' = n(\bar{\mathbf{x}}-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})' = 0$$

and

$$\sum_{i=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})' = (\bar{\mathbf{x}}-\boldsymbol{\mu})n(\bar{\mathbf{x}}-\bar{\mathbf{x}})' = 0$$

□

Problem 4.16

Solution

a) **Marginal distributions for each of $\mathbf{V}_1$ and $\mathbf{V}_2$**
Both $\mathbf{V}_1$ and $\mathbf{V}_2$ are linear combinations of normal random variables and hence are normally distributed. Additionally, $\mathbf{X}_1, \ldots, \mathbf{X}_4$ are independent. We then compute the first and second moments of both distributions.

$$\begin{aligned}\boldsymbol{\mu}_1 &= E[\mathbf{V}_1] = \frac{1}{4}E[\mathbf{X}_1] - \frac{1}{4}E[\mathbf{X}_2] + \frac{1}{4}E[\mathbf{X}_3] - \frac{1}{4}E[\mathbf{X}_4] = 0,\\ \boldsymbol{\Sigma}_1 &= \frac{1}{4}(4\boldsymbol{\Sigma}) = \boldsymbol{\Sigma}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\mu}_2 &= E[\mathbf{V}_1] = \frac{1}{4}E[\mathbf{X}_1] + \frac{1}{4}E[\mathbf{X}_2] - \frac{1}{4}E[\mathbf{X}_3] - \frac{1}{4}E[\mathbf{X}_4] = 0,\\ \boldsymbol{\Sigma}_2 &= \frac{1}{4}(4\boldsymbol{\Sigma}) = \boldsymbol{\Sigma}\end{aligned}$$

b) **Joint distribution**
Note that

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} = \mathbf{A}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \mathbf{X}_3\\ \mathbf{X}_4\end{bmatrix},$$

where $\mathbf{A} = \begin{bmatrix}\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\end{bmatrix}$. We then have

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} \sim N_{2p}\left(\mathbf{A}\mu, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\right) \equiv N_{2p}\left(\mathbf{0}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\right)$$

Problem 4.19

Solution

a) $(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu}) \sim \mathcal{X}_6^2$

b) $\bar{\mathbf{X}} \sim N_6\left(\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{20}\right)$ and $\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N_6(0, \boldsymbol{\Sigma})$

c) $19\mathbf{S} \sim W_6(19, \boldsymbol{\Sigma})$, where $W_6(19, \boldsymbol{\Sigma})$ is a Wishart distribution with 19 d.f.