Last Week

- Pseudo-code
- Counting Inversions Problem
  - Problem definition
  - A brute force algorithm
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Polynomial Multiplication Problem
  - Problem definition
  - A brute force algorithm
  - A first divide-and-conquer algorithm
  - An improved divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

Quicksort Problem

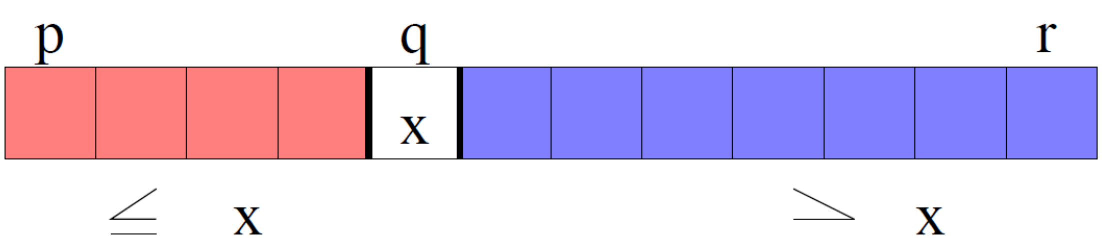
Outline

- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

Partition

- Partition
  - Given: An array of numbers
  - Partition: Rearrange the array A[p..r] in place into two (possibly empty) subarrays A[p..q-1] and A[q+1..r] such that

$$A[u] < A[q] < A[v] \;\; for\ any\ p \le u \le q-1\ and\ q+1 \le v \le r$$



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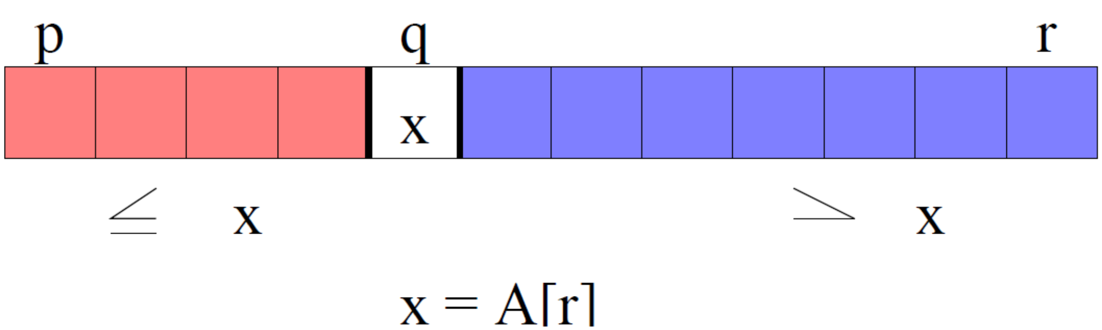

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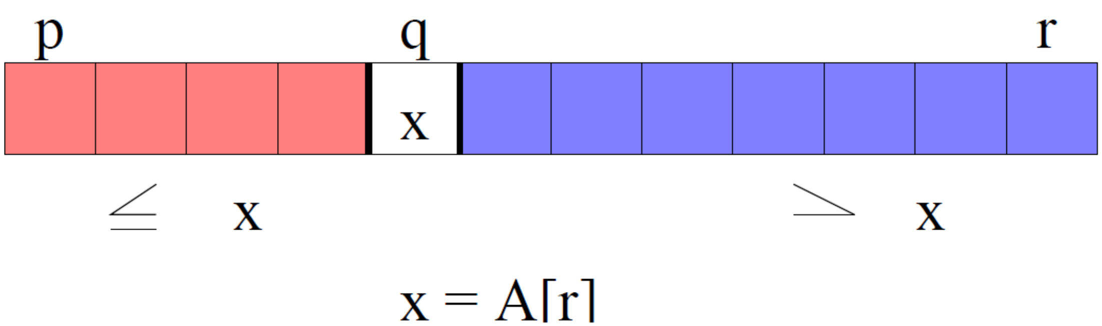

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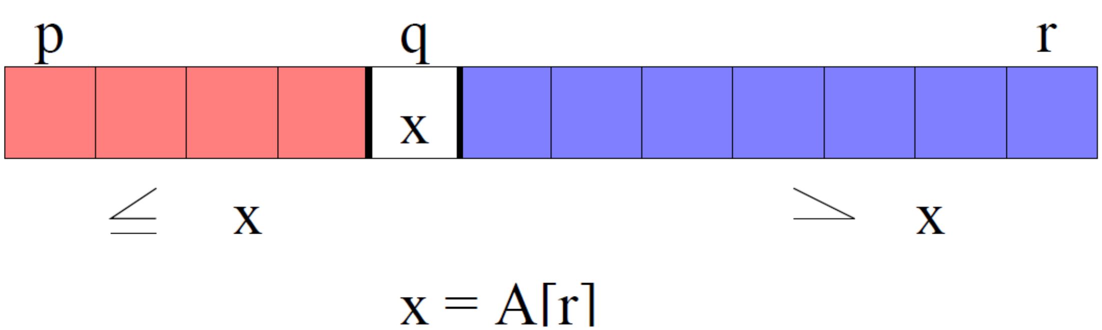

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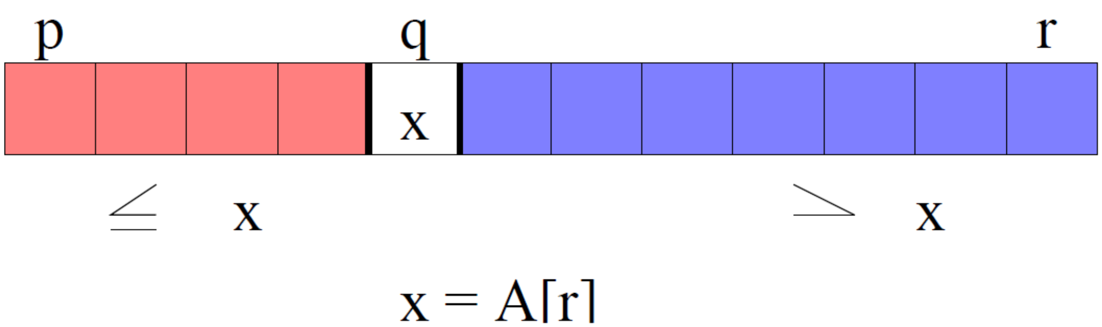

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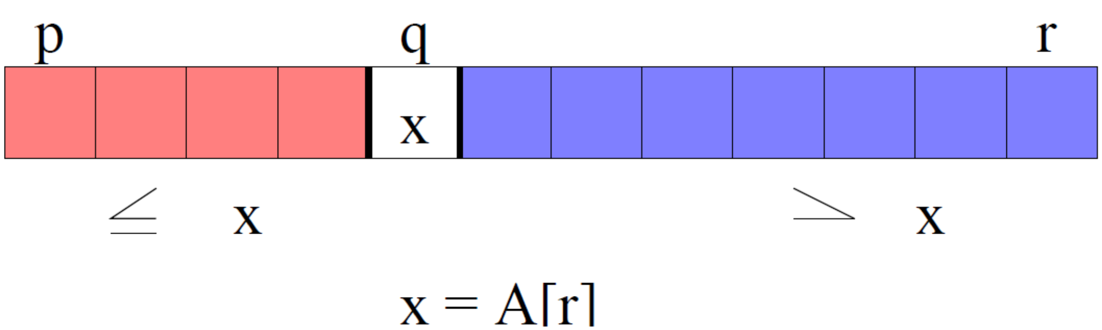

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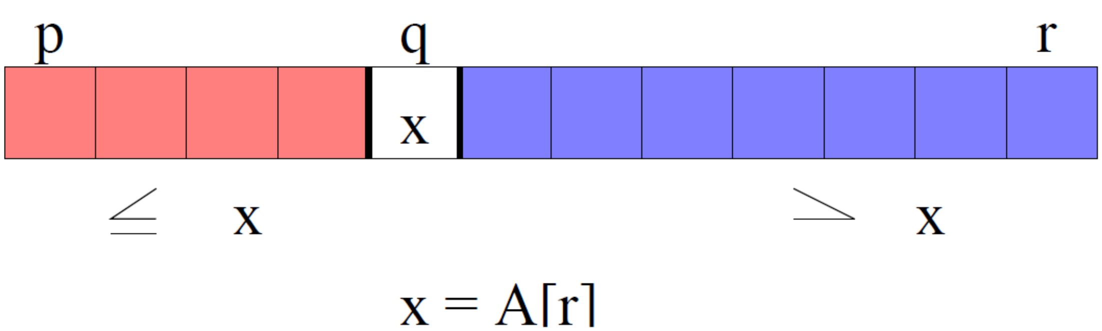

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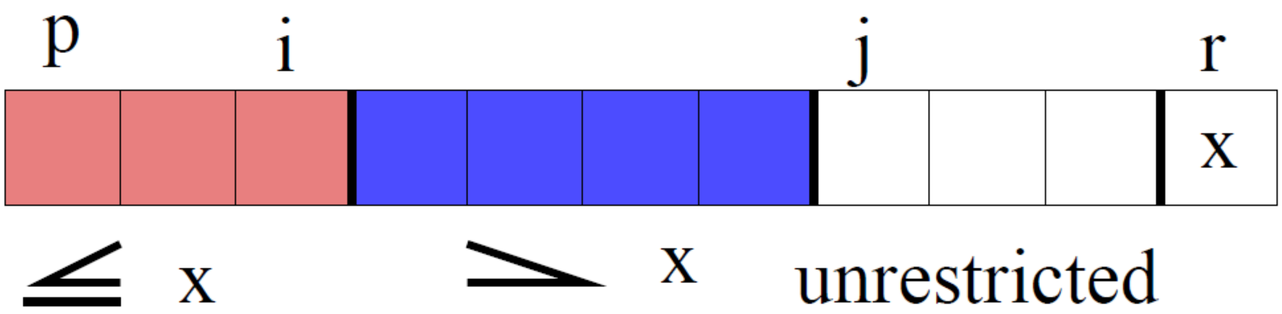
Partition

- The idea of Partition(A, p, r)
  - Use A[r] as the pivot, and grow partition from left to right



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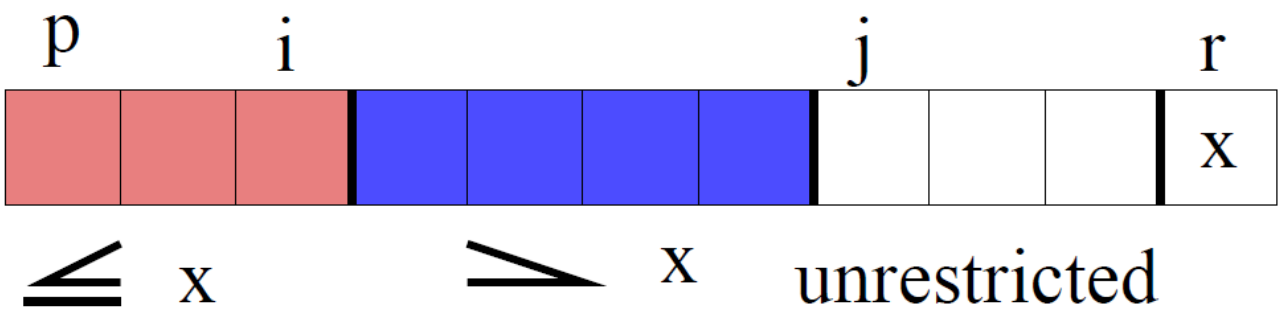

  - Initially (i, j) = (p–1, p)
  - Increase j by 1 each time to find a place for A[j]
    At the same time increase i when necessary
  - Stops when j = r

Partition

- One Iteration of the Procedure Partition
  - Increase j by 1 each time to find a place for A[j]

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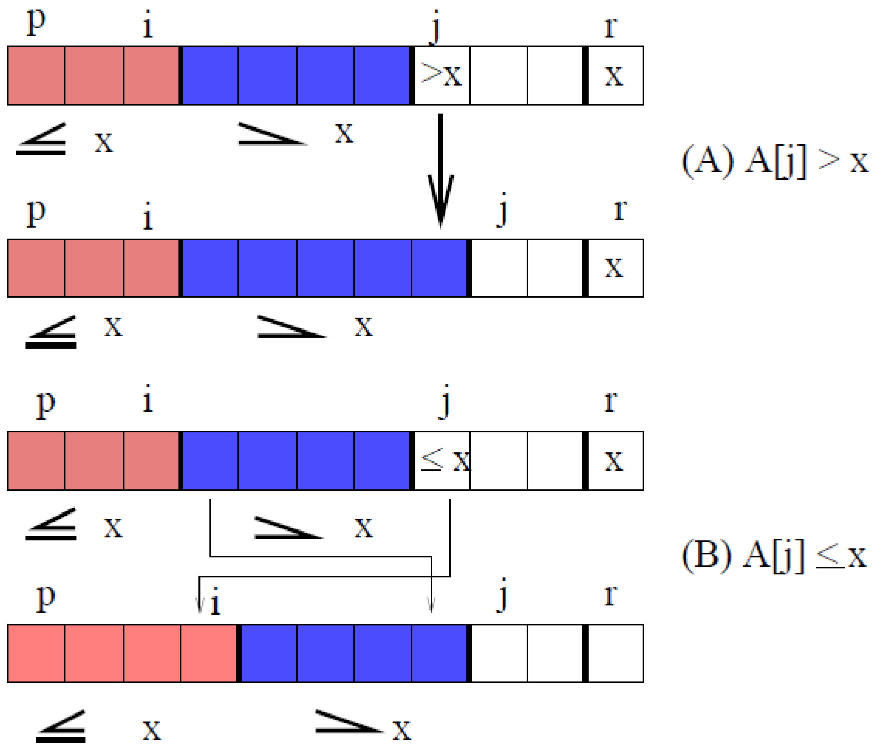

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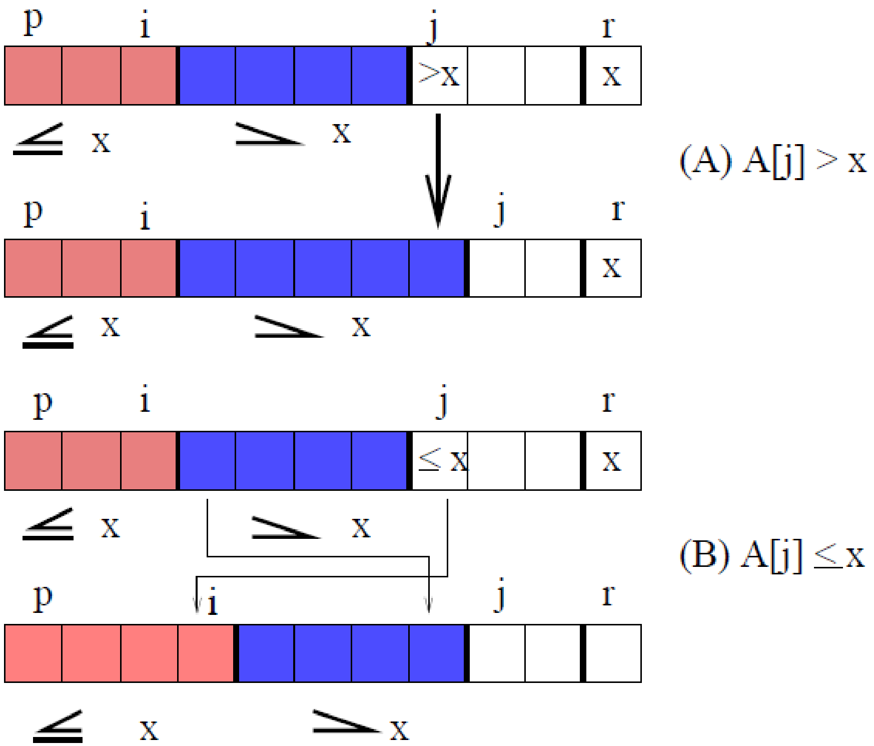

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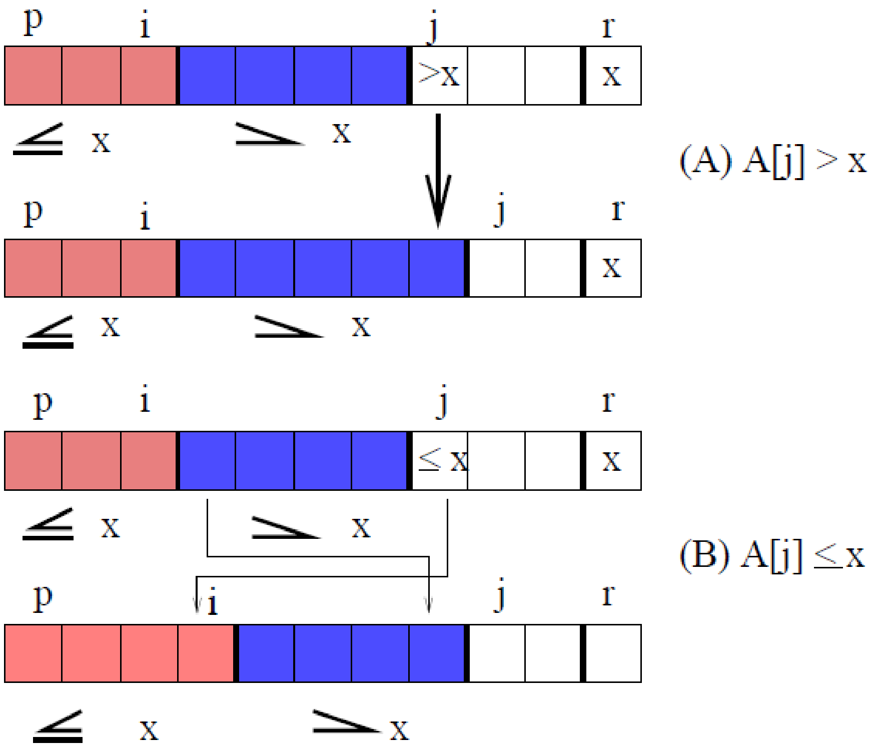

  - Case (A): Only increase j by 1
  - Case (B): i = i + 1; A[i] ↔ A[j]; j = j + 1.

Partition-Example

- The Operation of Partition(A, p, r)

i	p,j						r
	2	8	7	1	3	5	6	4

Partition-Example

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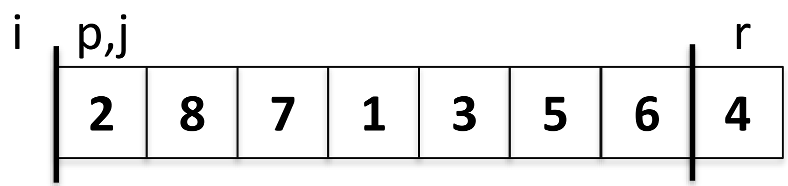

$$A[j] < A[r]$$

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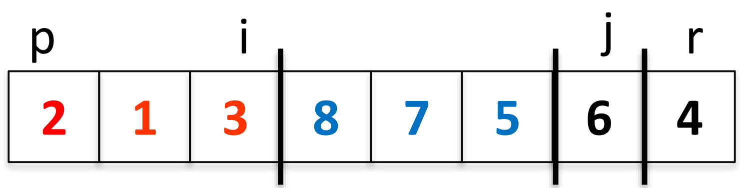

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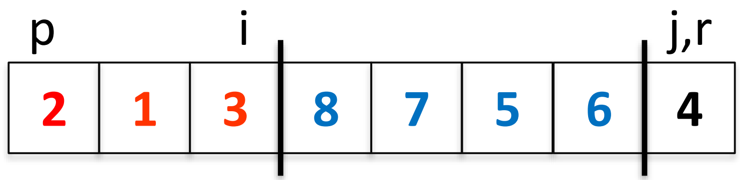

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$$A[i+1] \leftrightarrow A[r]$$

Partition - Pseudocode

Partition(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Index of the pivot after partition
$x \leftarrow A[r]$;//$A[r]$ is the pivot element

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- Running time is O(r – p)
  - linear in the length of the array A[p..r]

Quicksort

Quicksort(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Sorted array $\boldsymbol{A}$
if $p < r$ **then**
    $q \leftarrow$Partition(A, p, r);
    Quicksort(A,   );
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end
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A Divide-and-Conquer Framework

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- If we could always partition the array into halves, then we have the recurrence $T(n) \le 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.

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A Divide-and-Conquer Framework

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.
- However, if we always get unlucky with very unbalanced partitions, then $T(n) \leq T(n-1) + O(n)$, hence $T(n) = O(n^2)$.

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  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

Randomized-Partition(A, p, r)

- Idea
  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot *x* to partition the array A[p..r].

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Randomized-Partition(A, p, r)

- Idea
  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot x to partition the array A[p..r].
  - In the algorithm Randomized-Partition(A, p, r), we randomly choose an j , p ≤ j ≤ r , and use A[j] as pivot.
  - Idea is that if we choose randomly, then the chance that we get unlucky every time is extremely low.

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Partition
  - Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r.

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Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: A random index in $[p..j]$

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Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Quicksort
  - We make use of the Randomized-Partition idea to develop a new version of quicksort.

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Randomized-Quicksort(A,p,r)

```
Input: An array A waiting to be sorted,the range of index p,r
Output: Sorted array A
if p < r then
    q ← Randomized-Partition(A, p, r);
    Randomized-Quicksort(A,          );
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end
return A;
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Quicksort - Example

2	**8**	**7**	**1**	**3**	**5**	**6**	**4**

Quicksort - Example

Divide

2	8	7	1	3	5	6	4

Quicksort - Example

Divide

2	8	7	1	3	5	6	4

Quicksort - Example

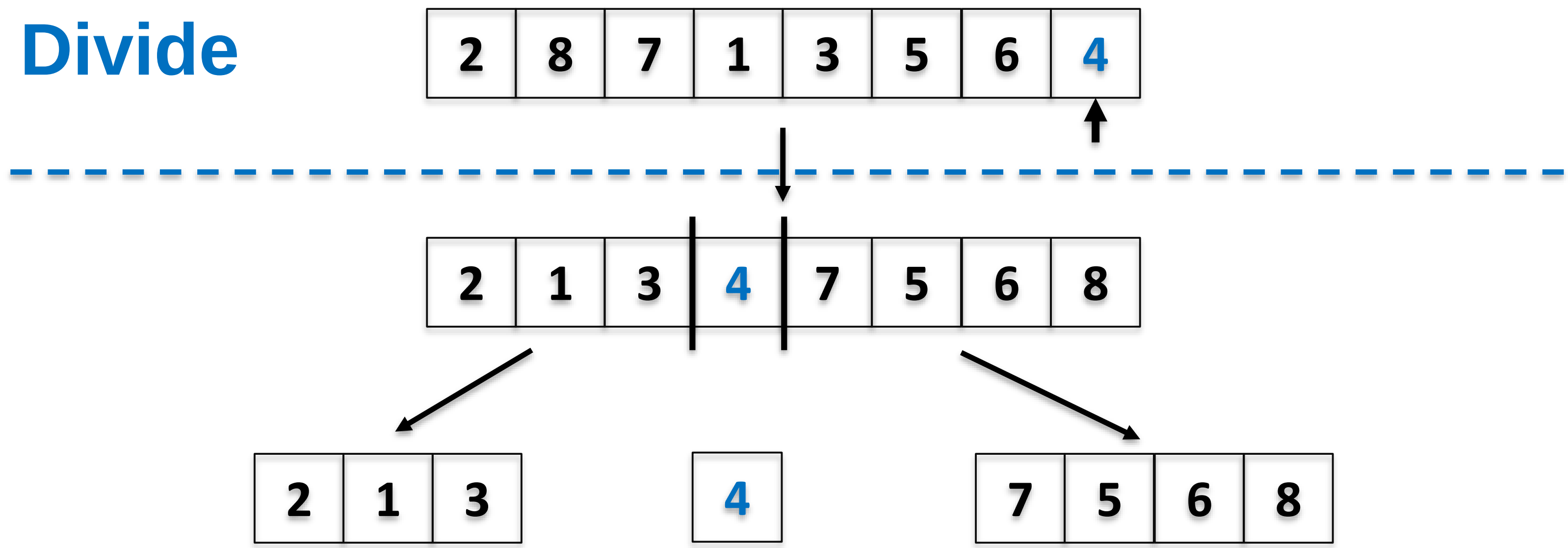
Divide

2	8	7	1	3	5	6	4

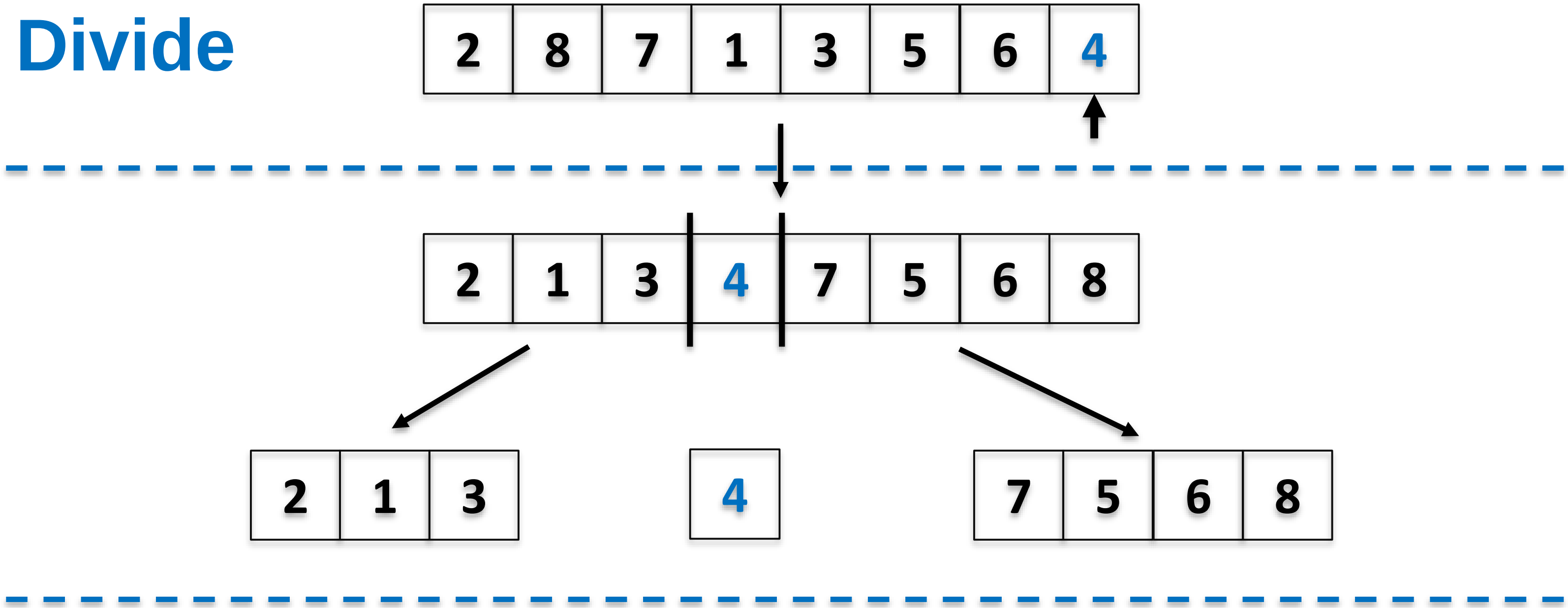
2	1	3	4	7	5	6	8

Quicksort - Example

Divide



Quicksort - Example



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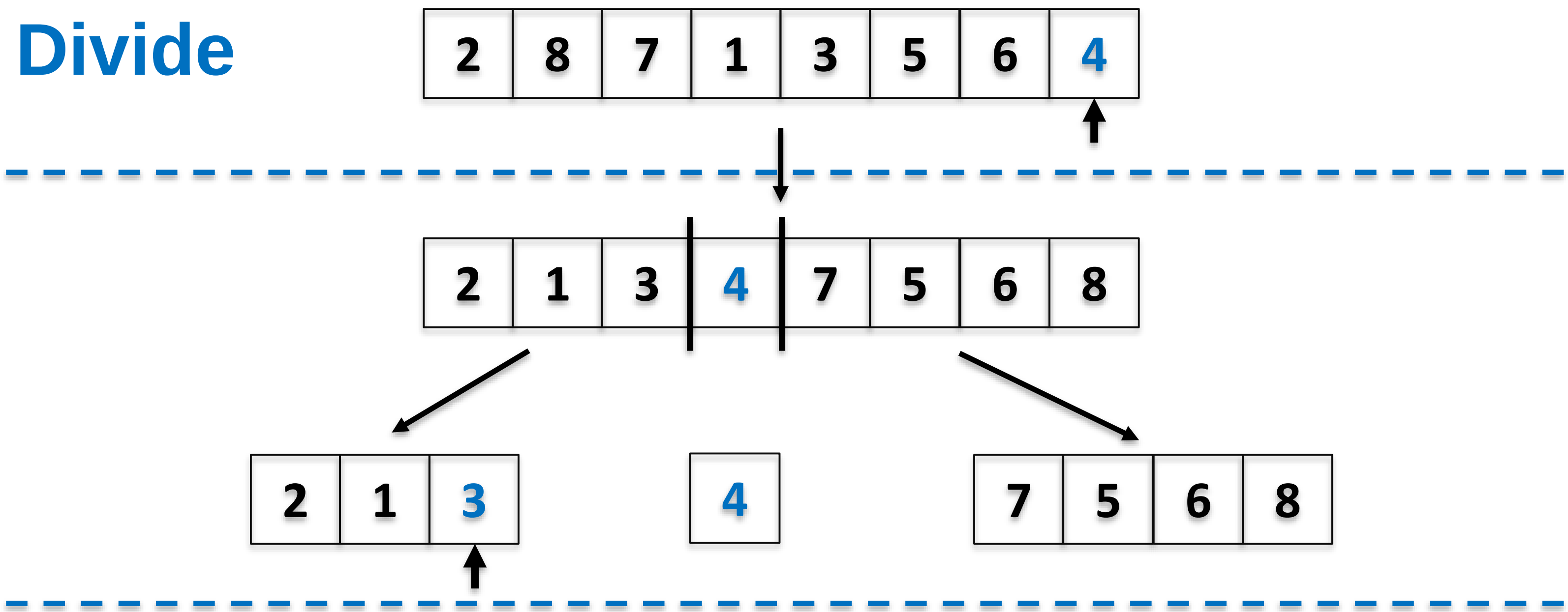

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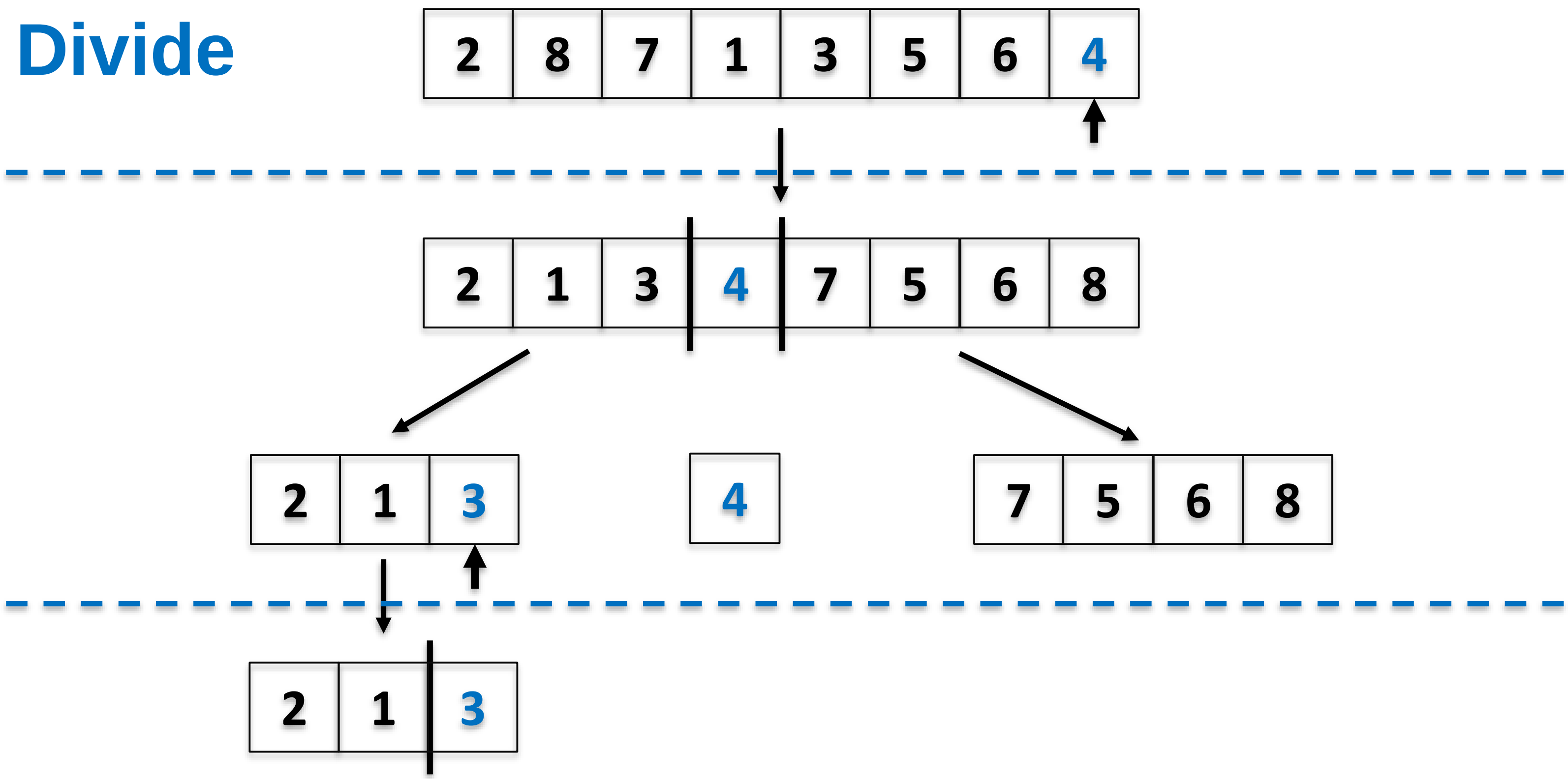

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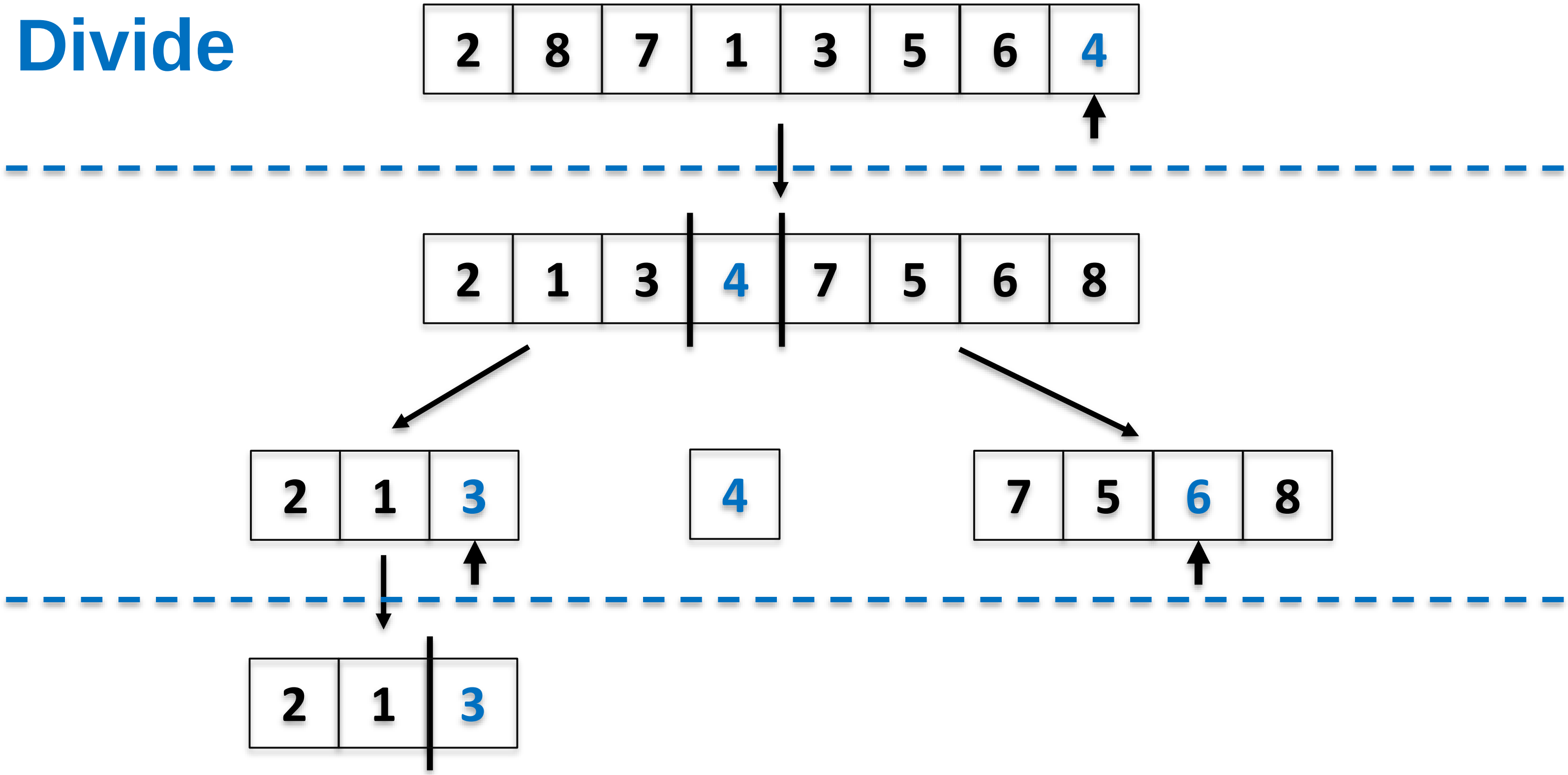

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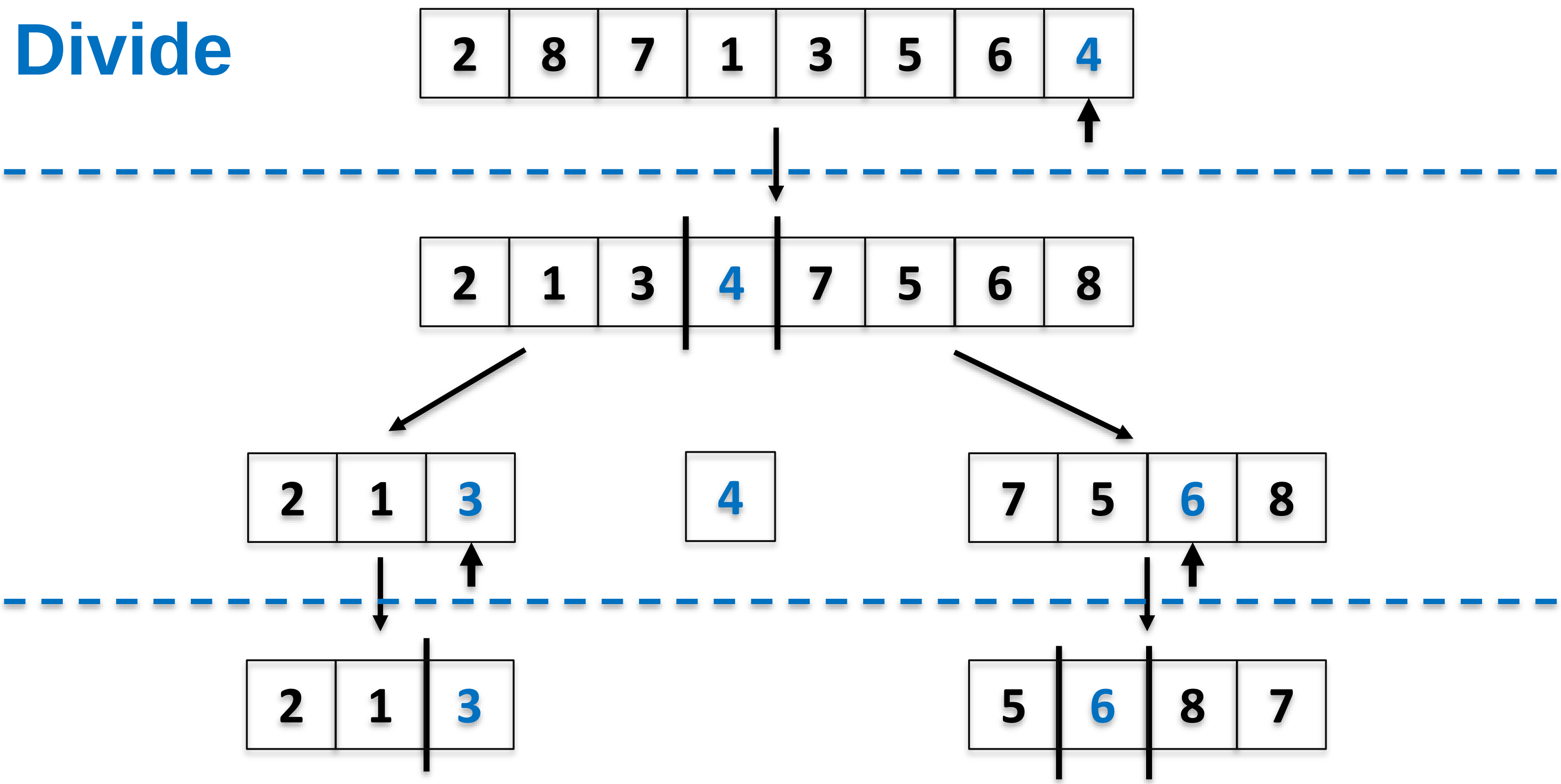

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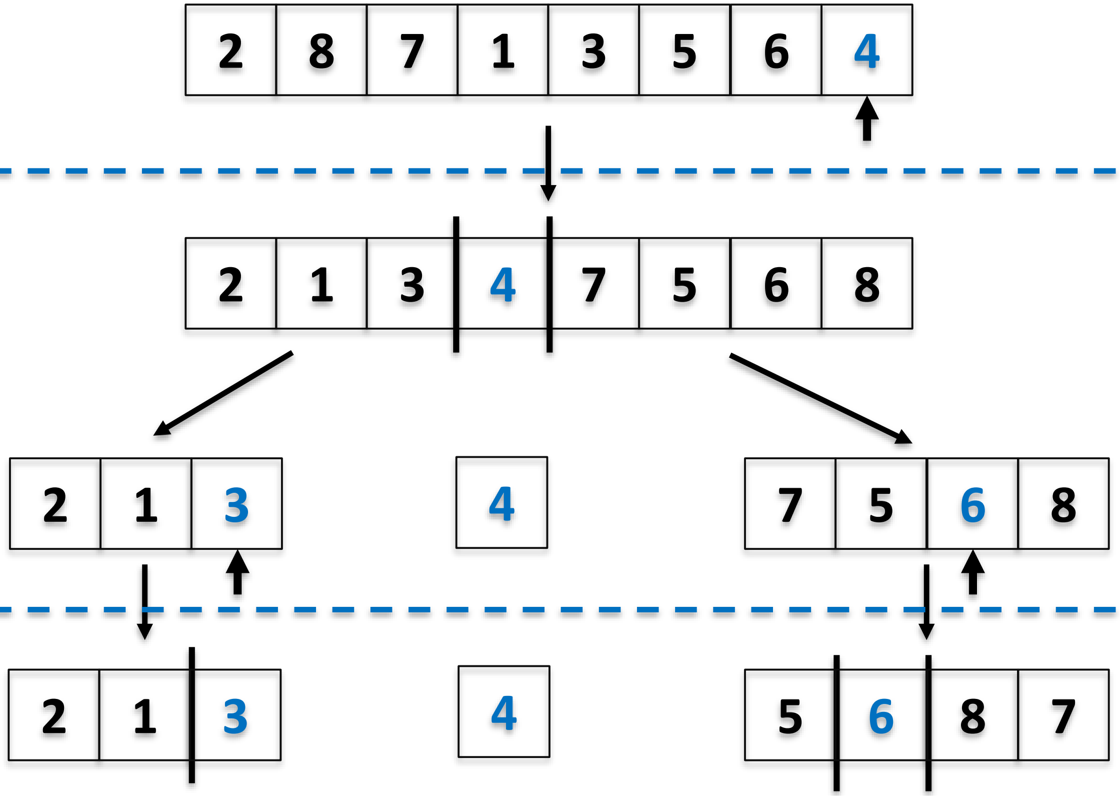

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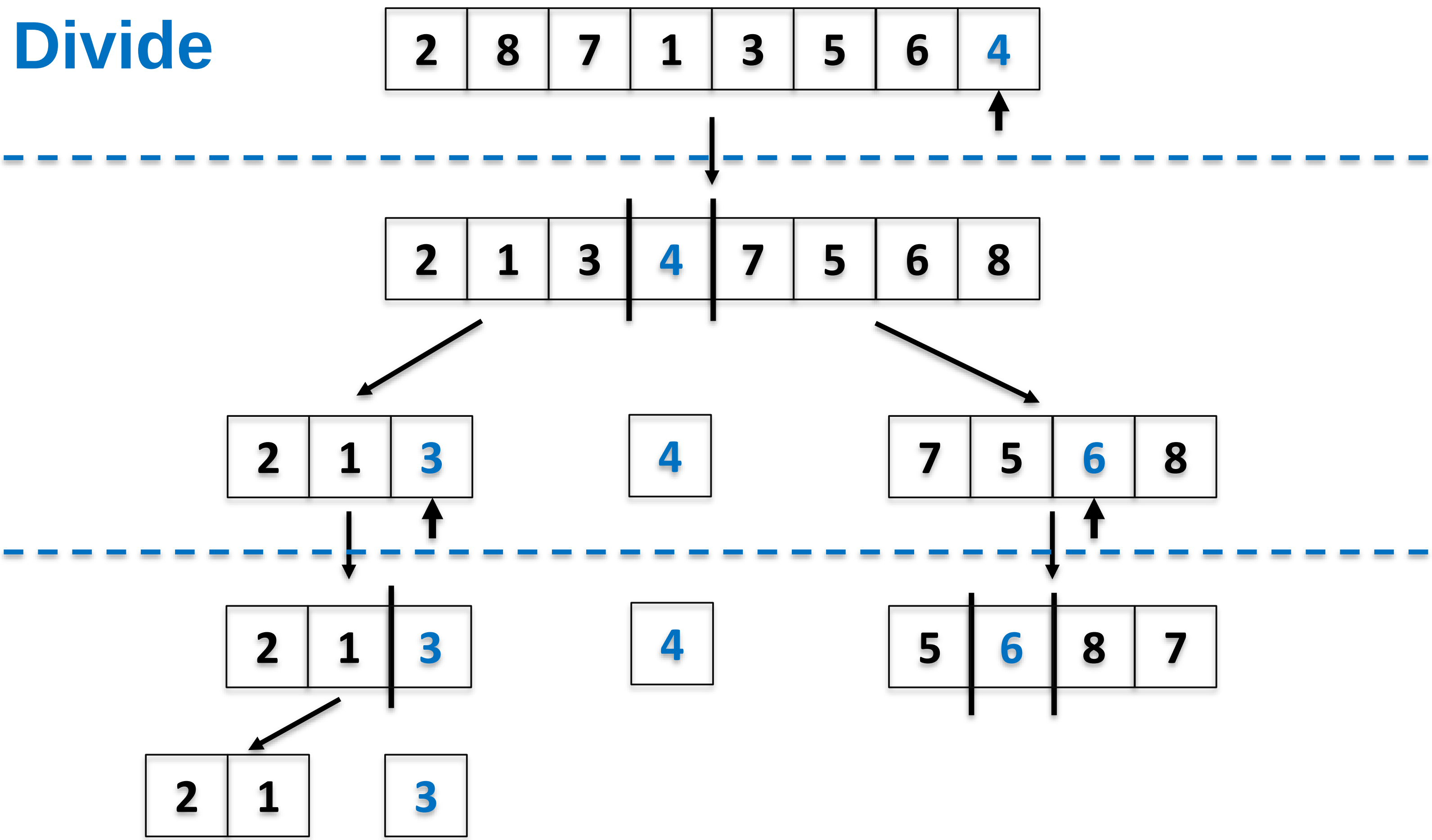

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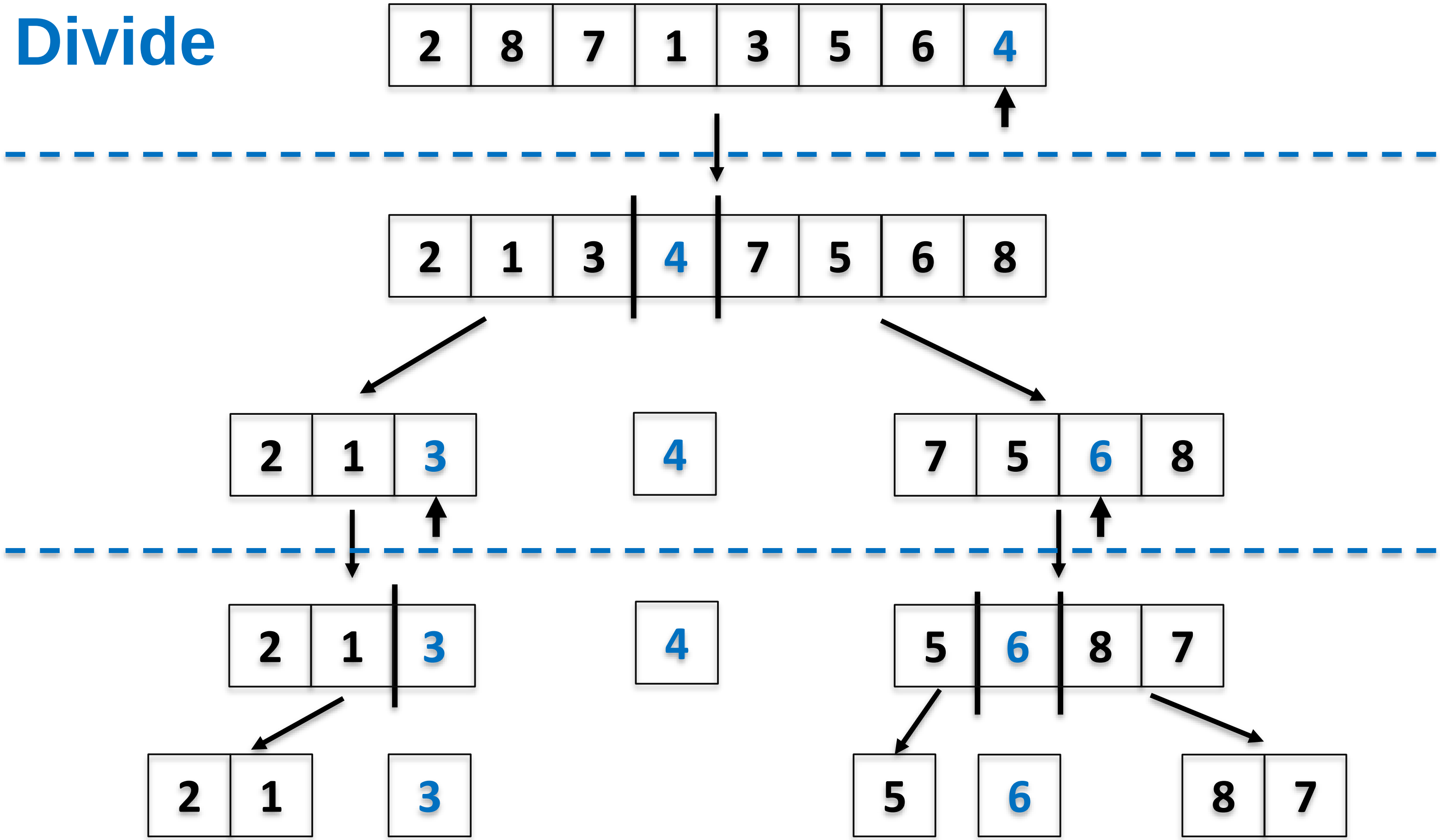

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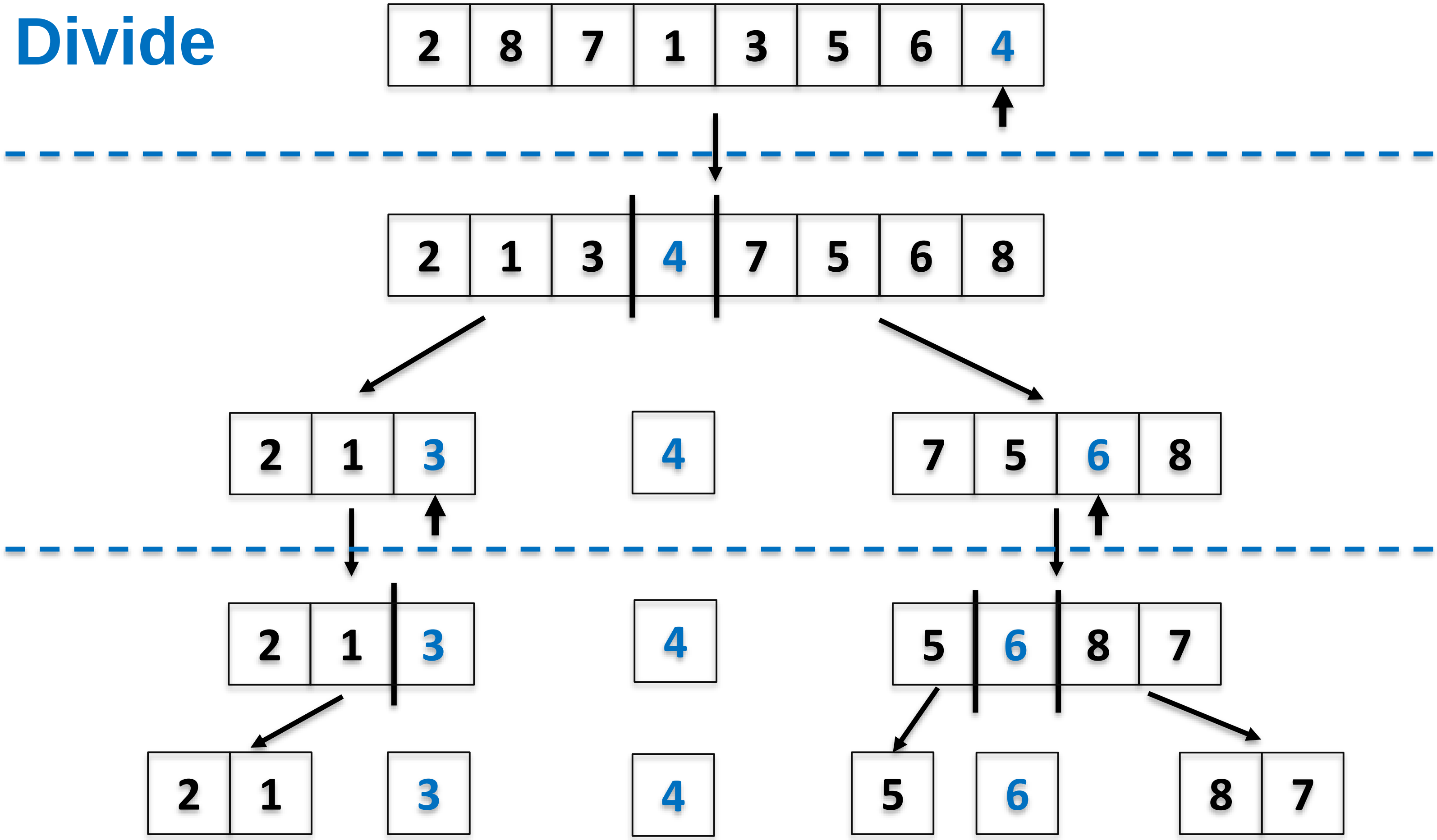

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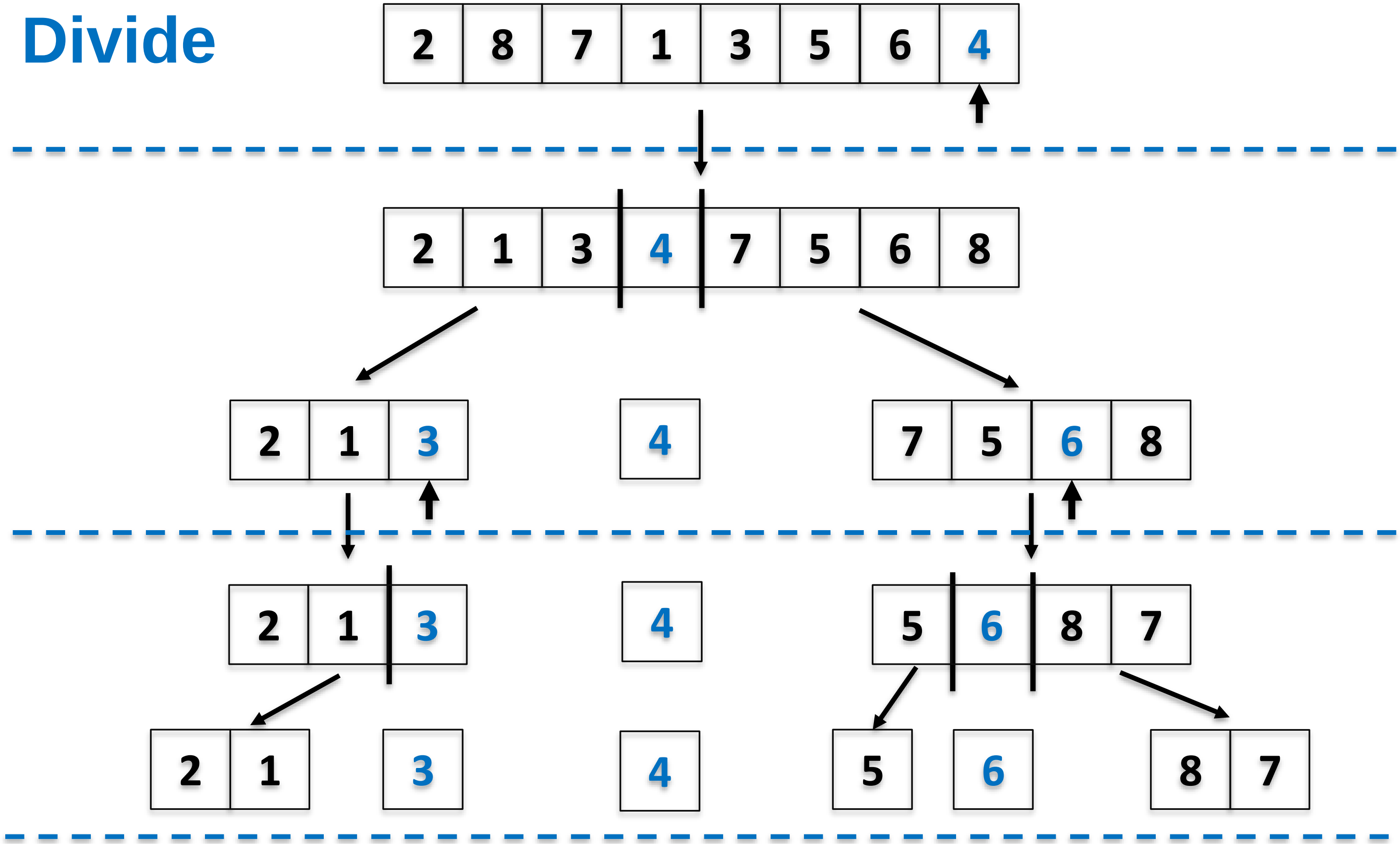

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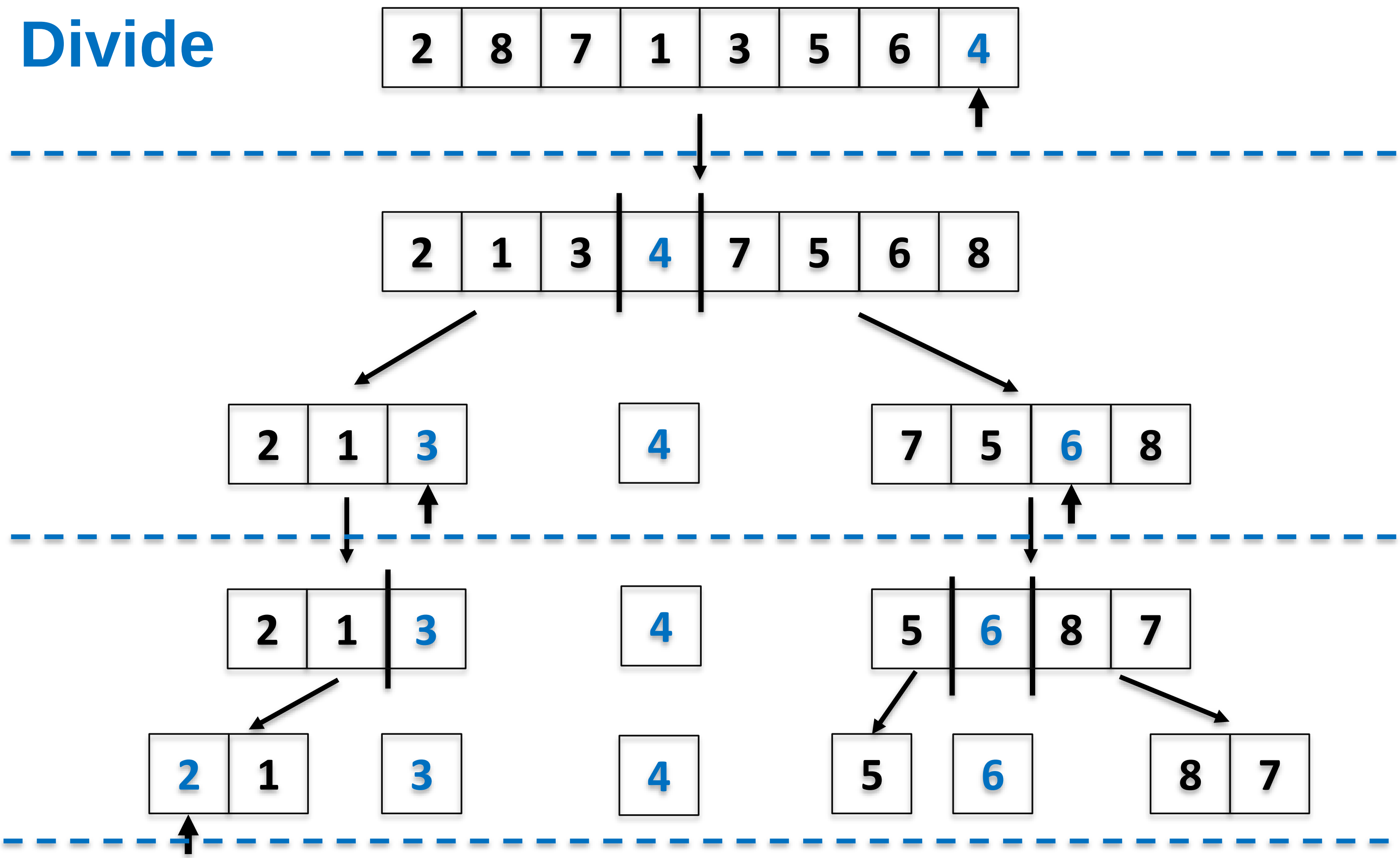

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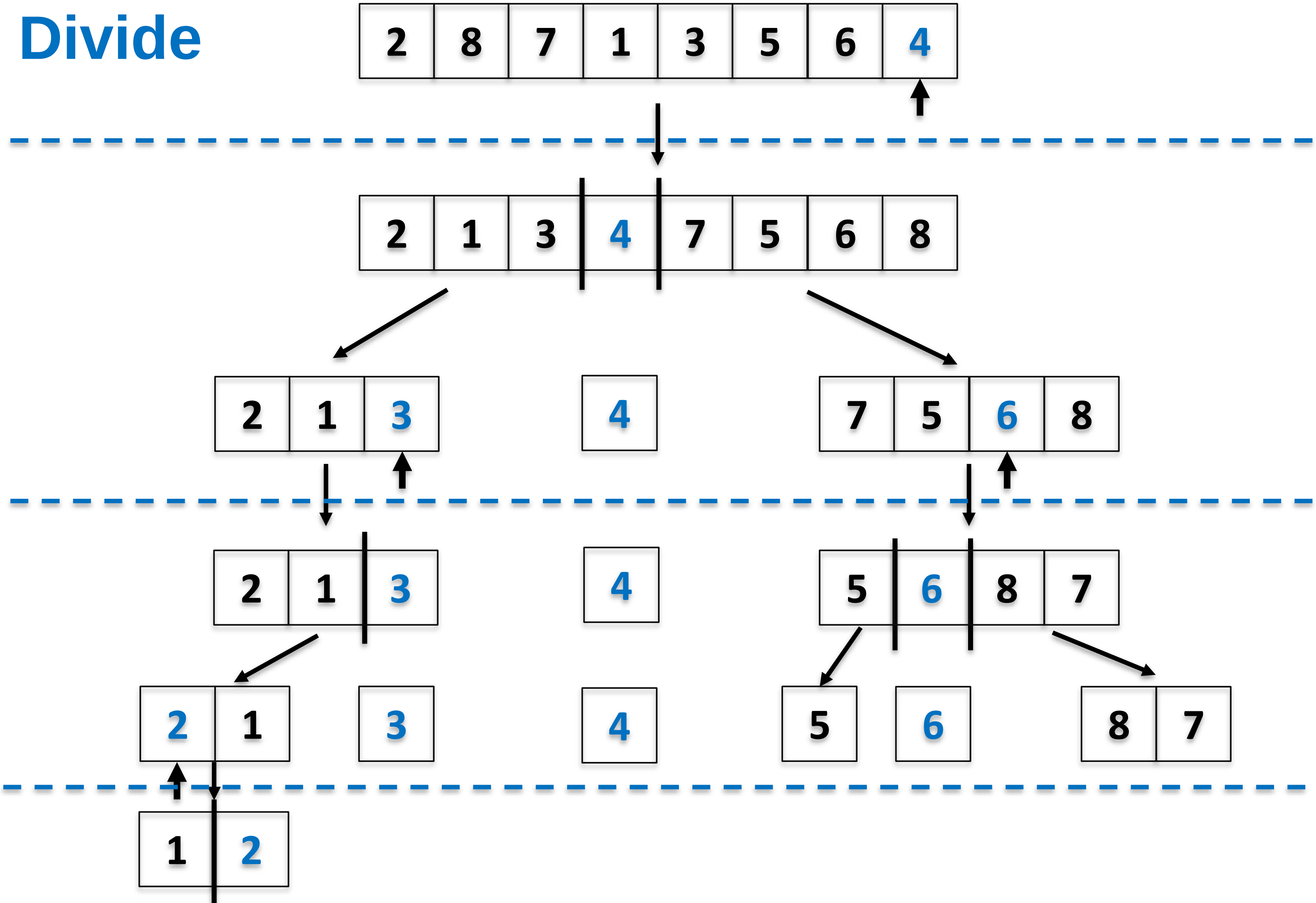

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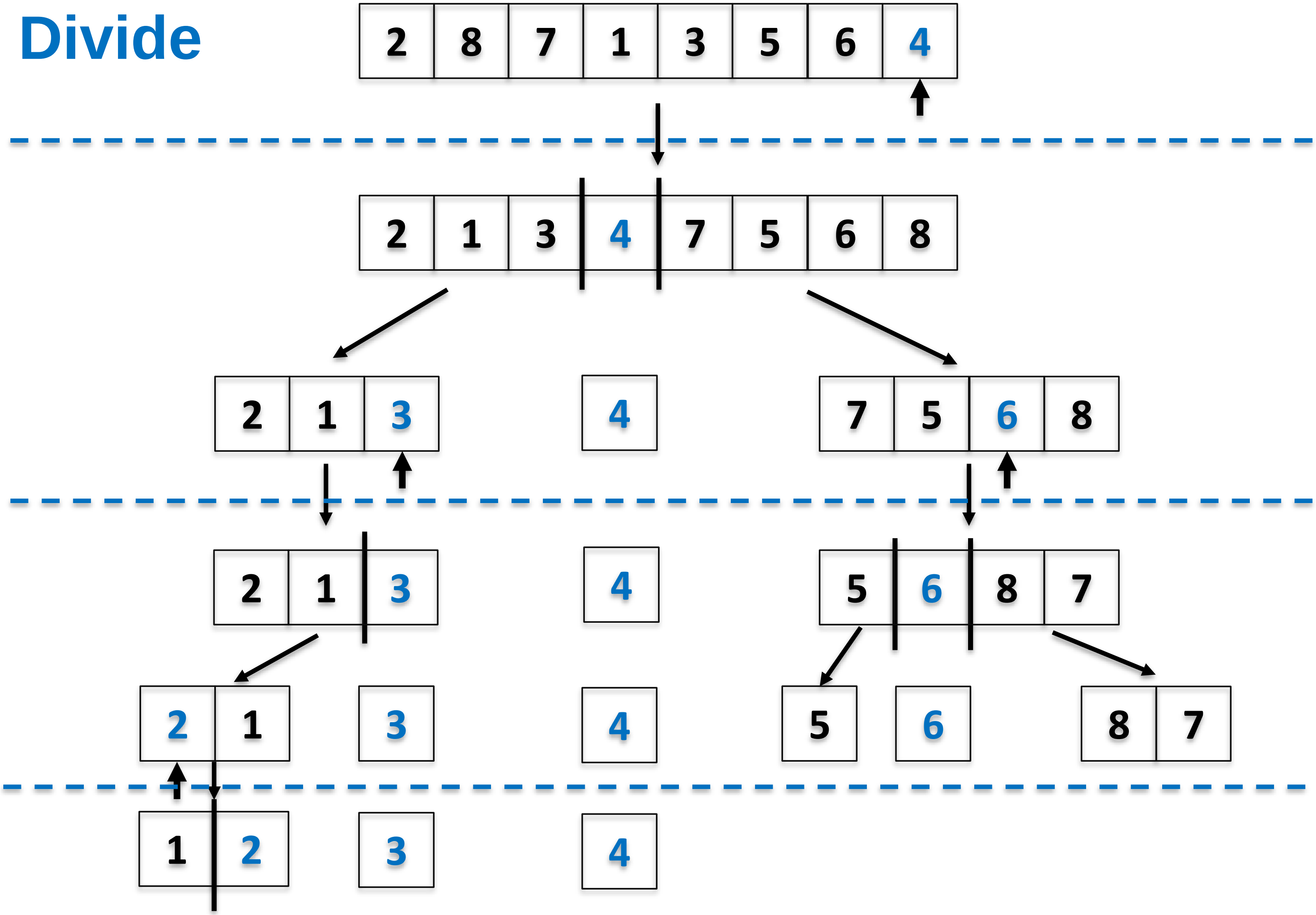

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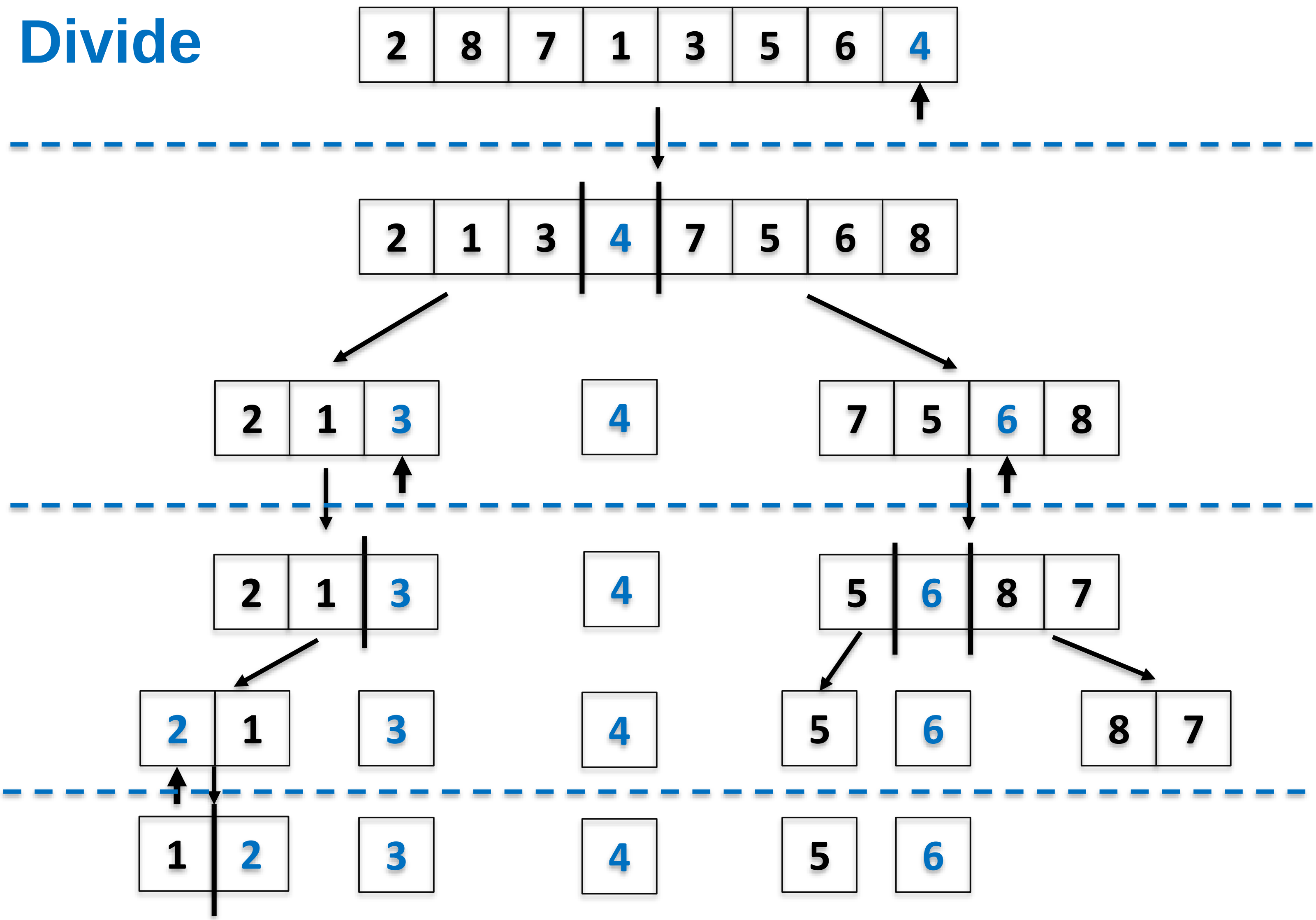

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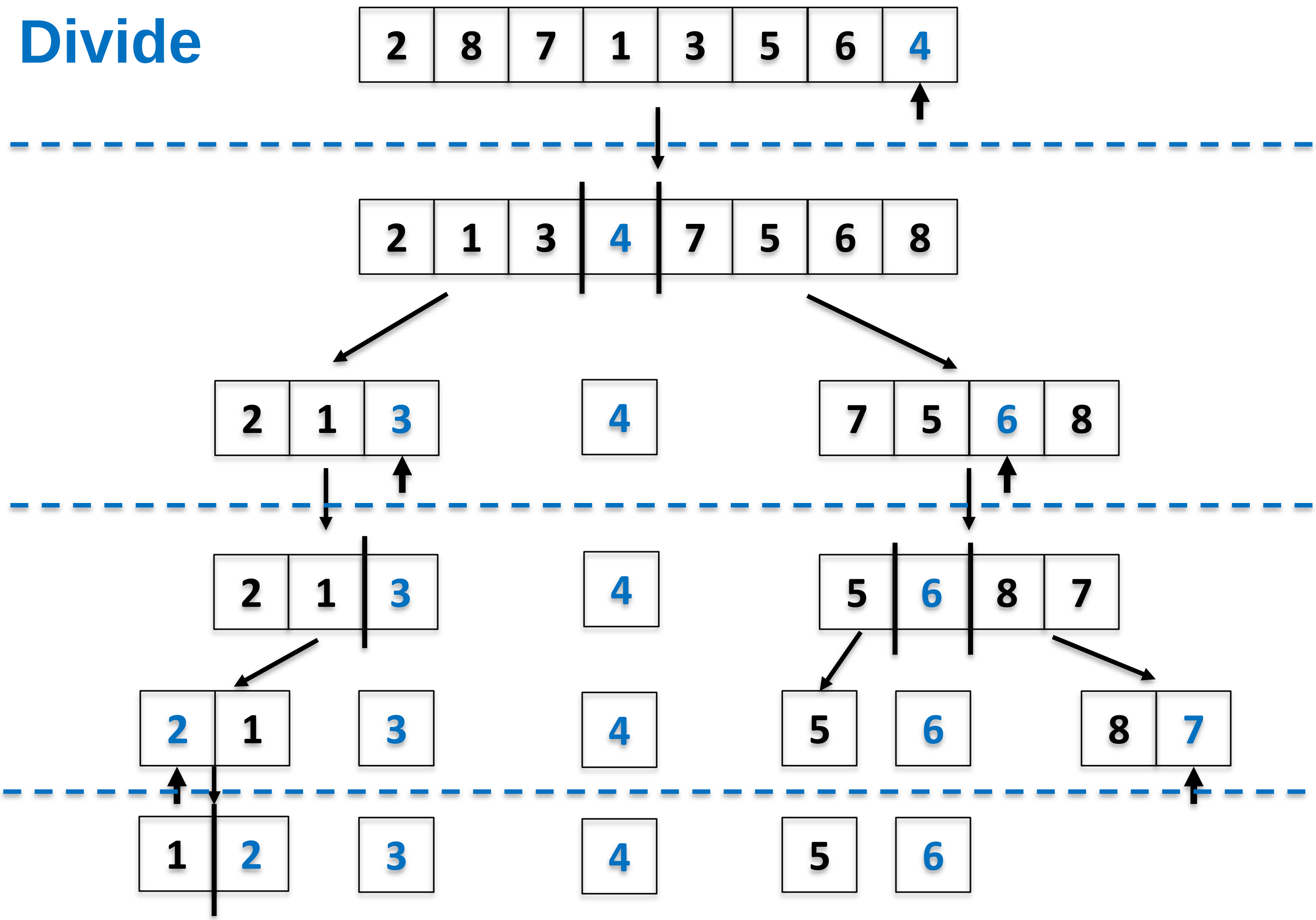

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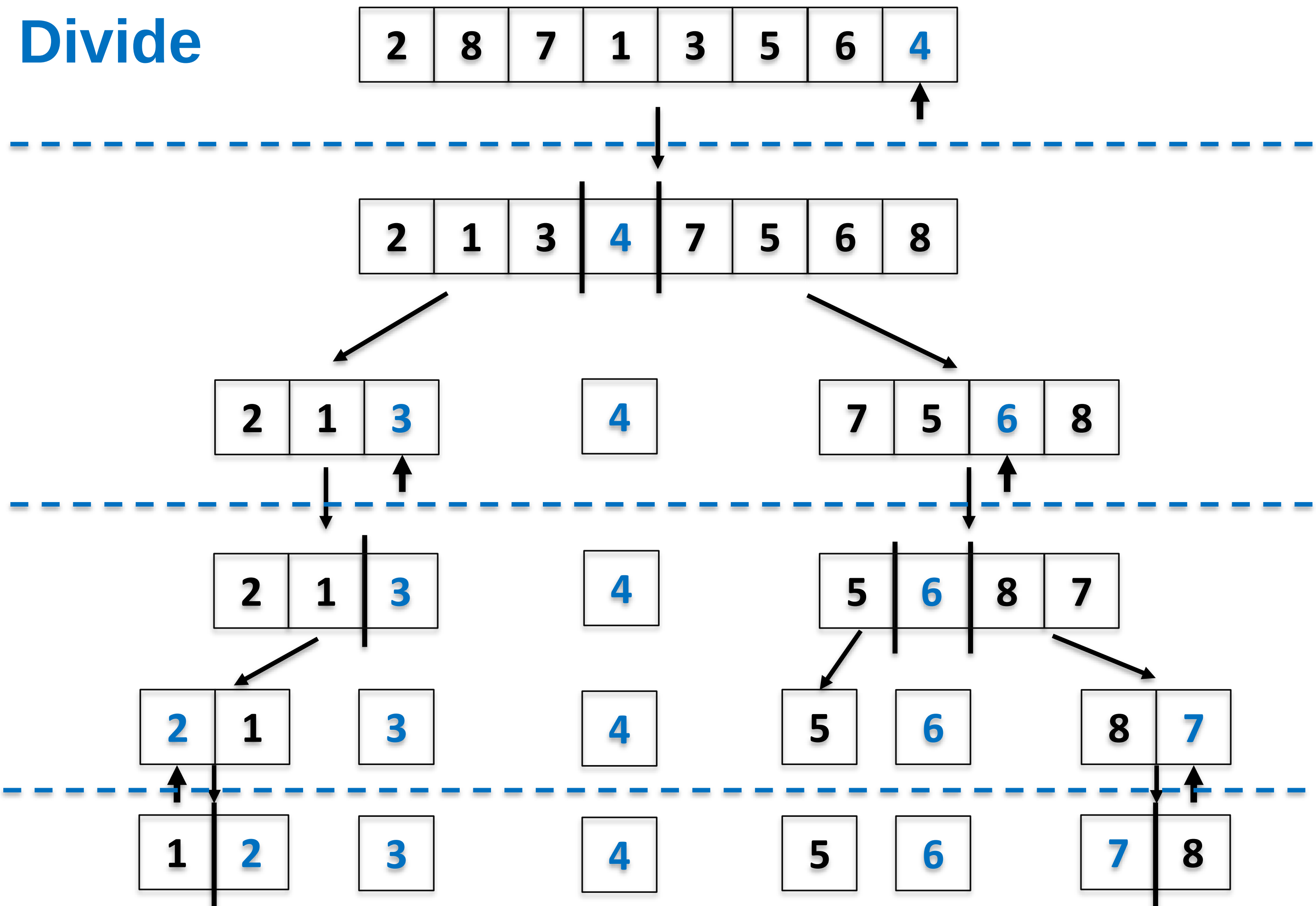

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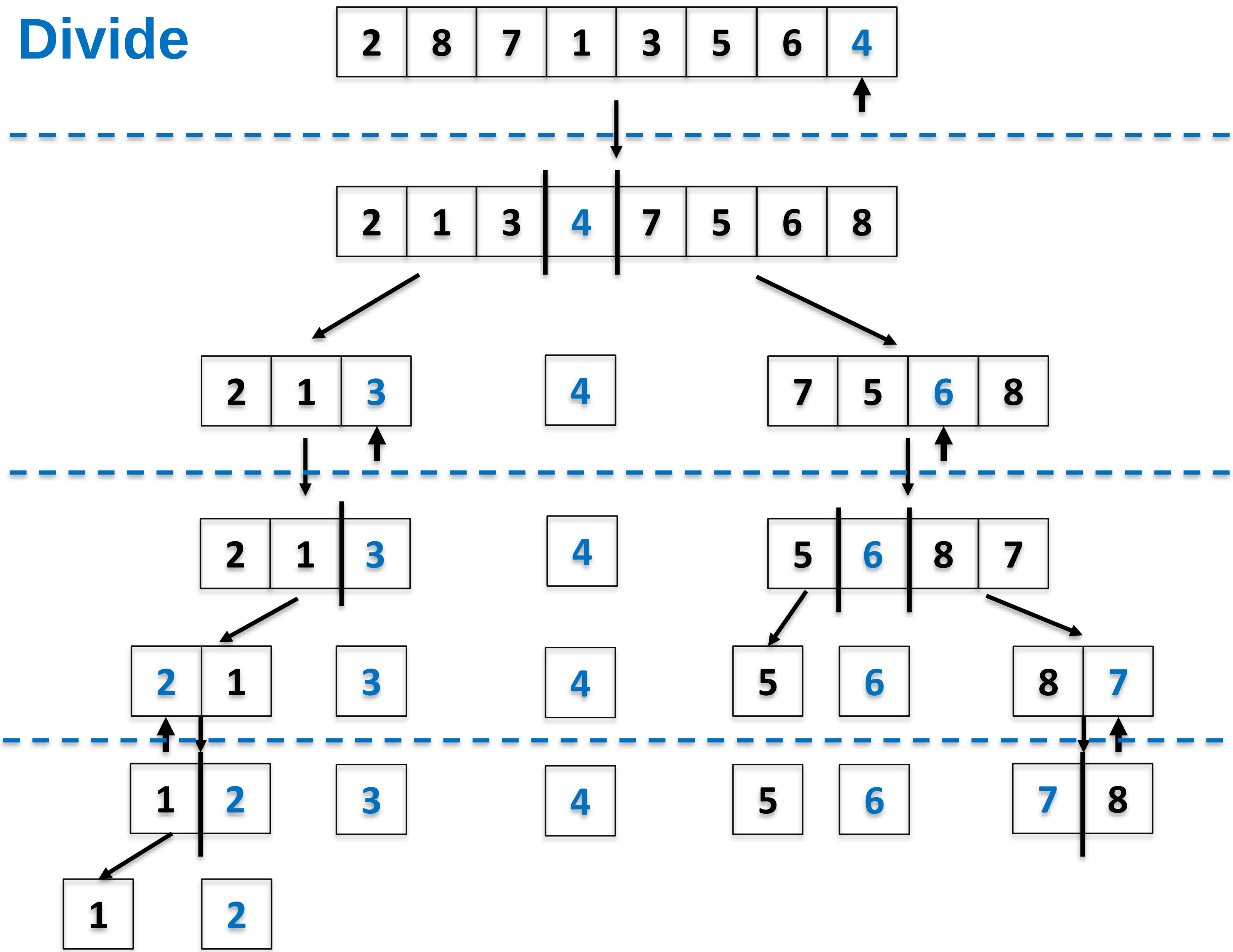

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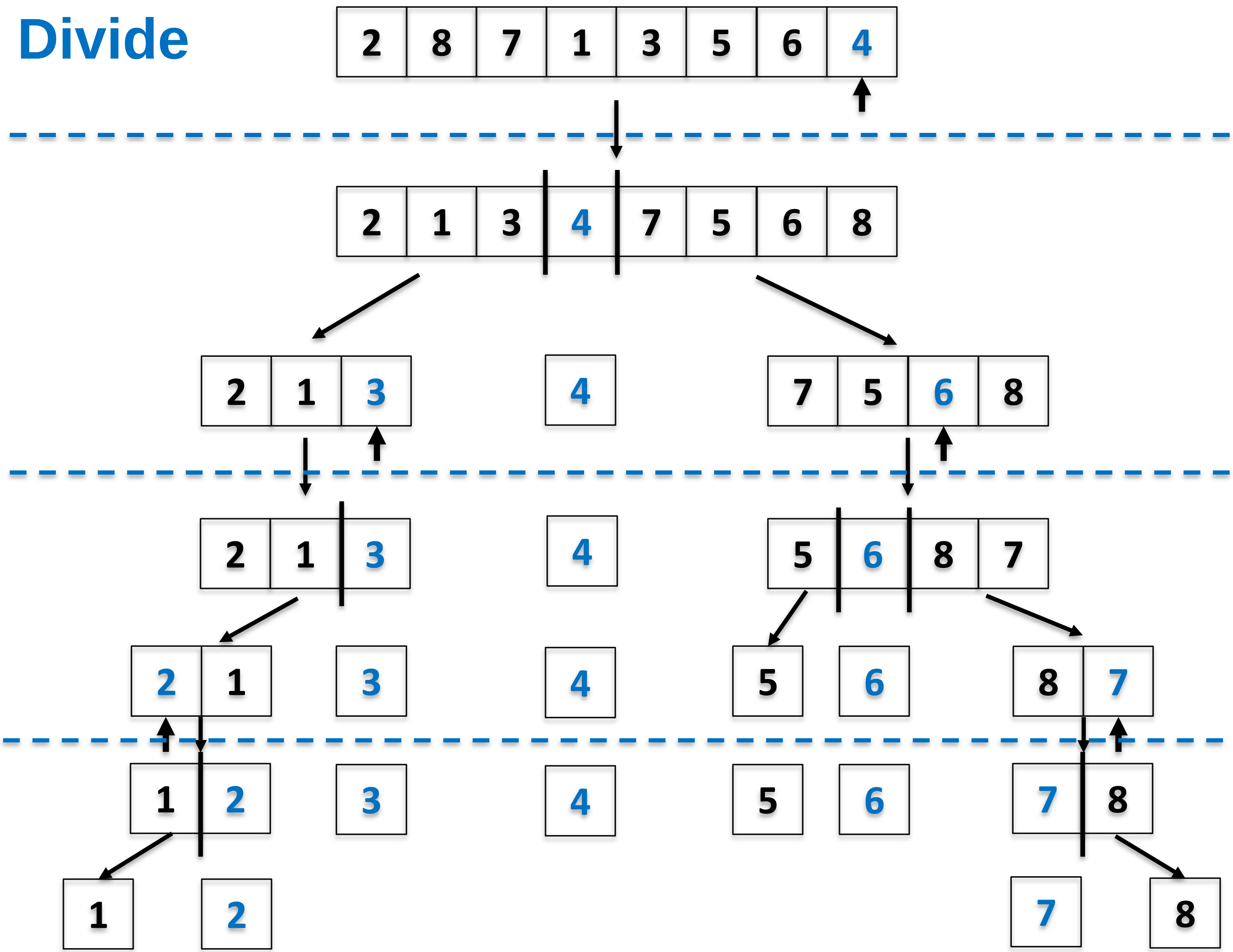

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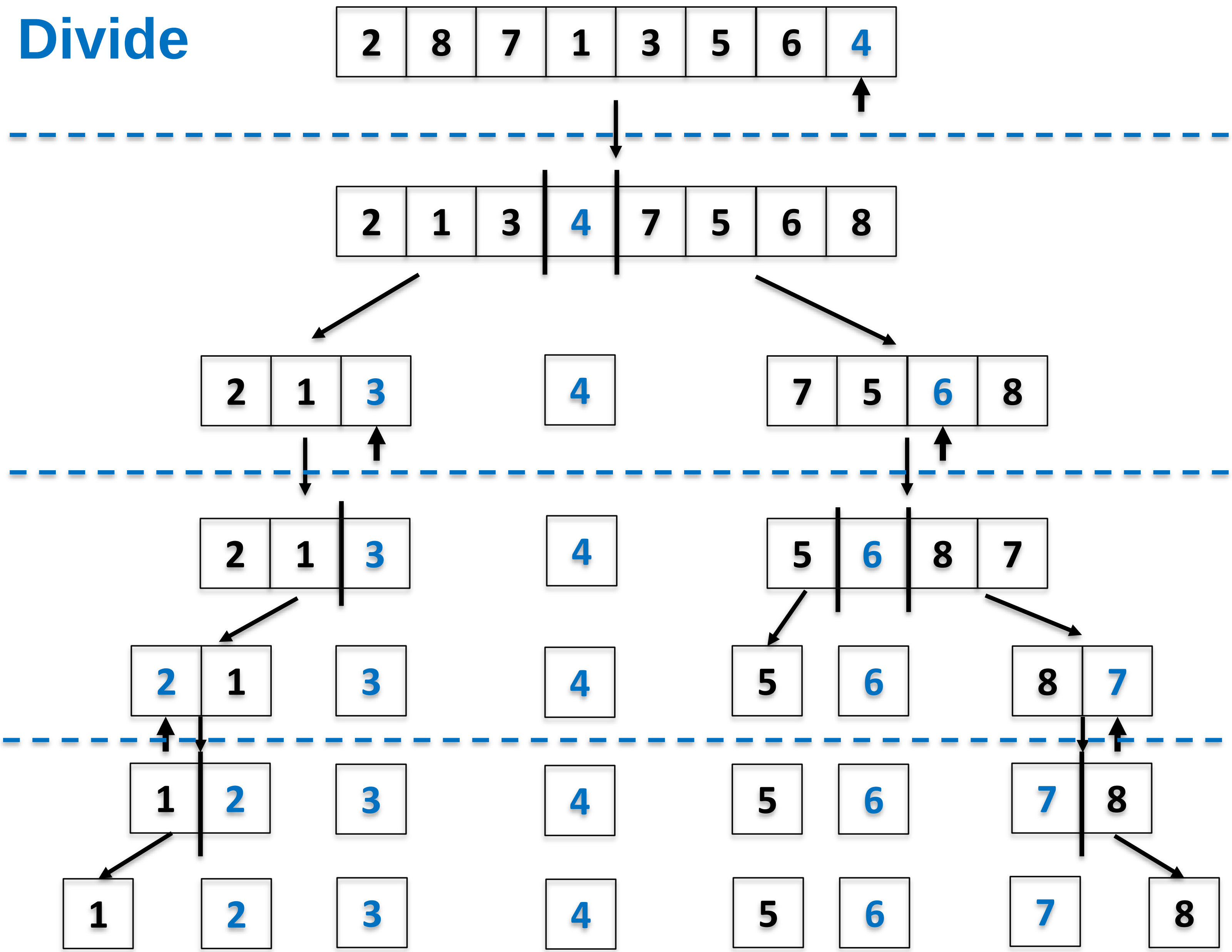


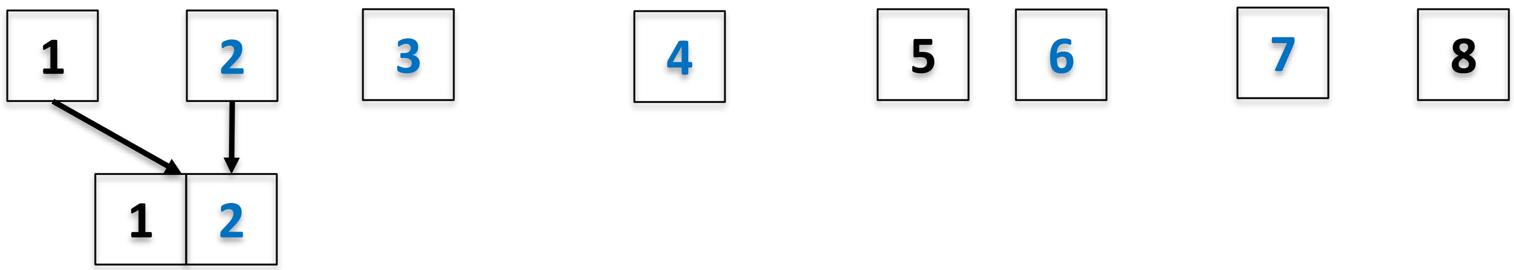
Quicksort - Example

Conquer

1	2	3	4	5	6	7	8

Quicksort - Example

Conquer



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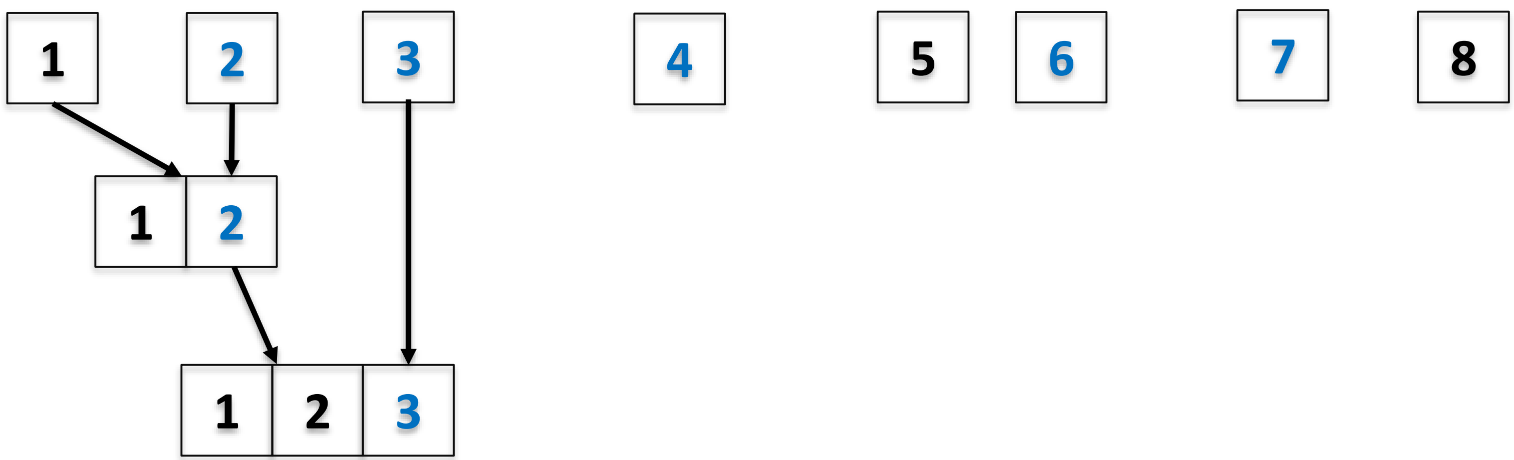

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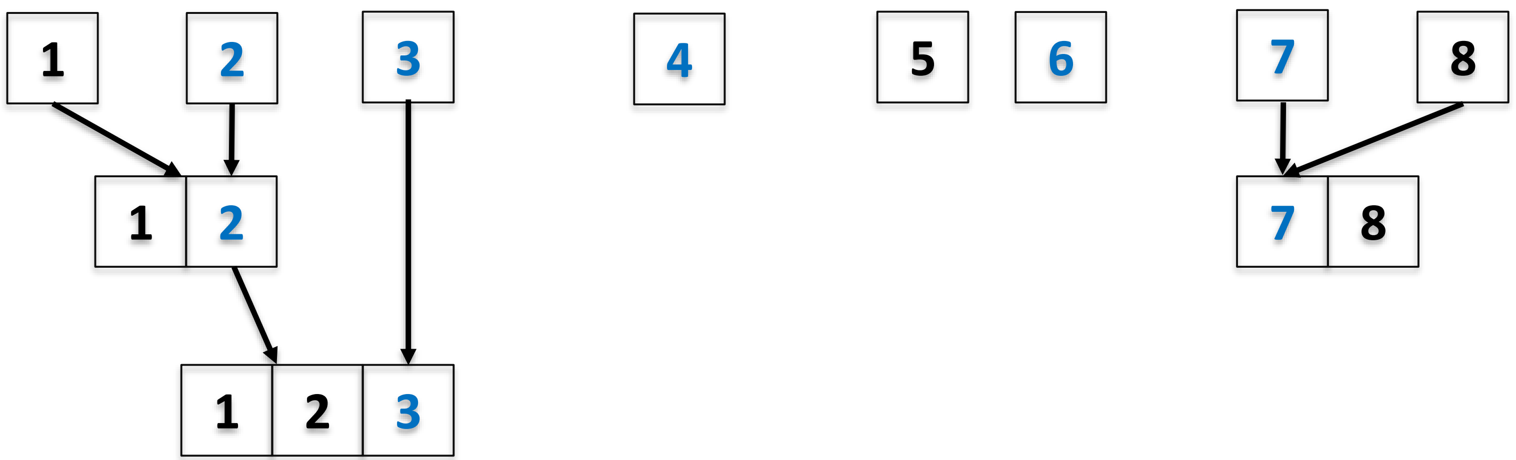

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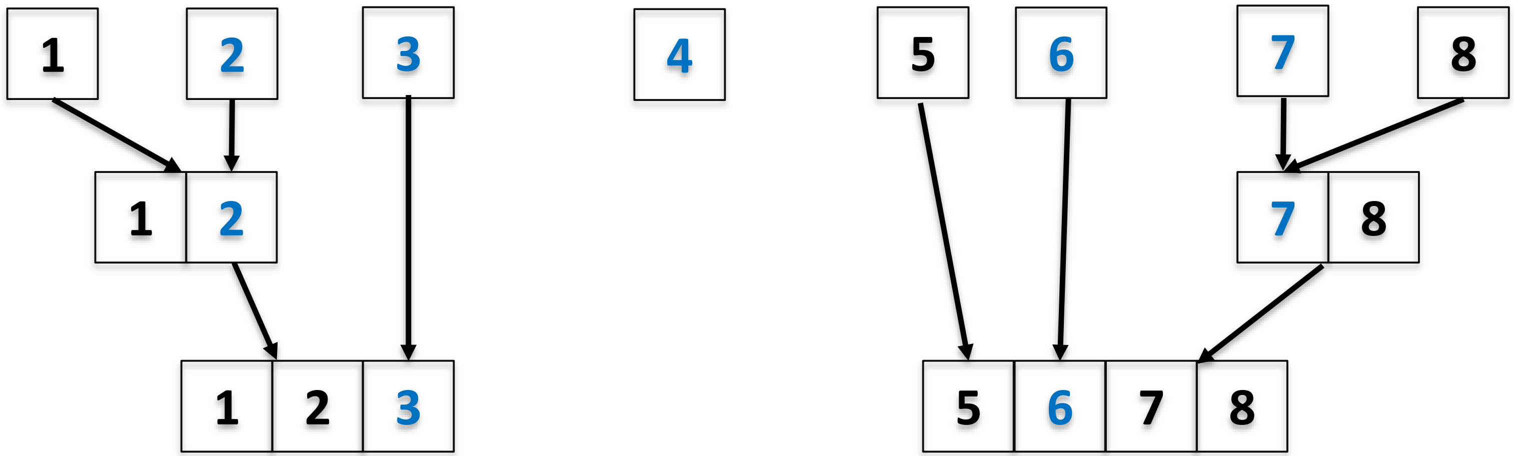

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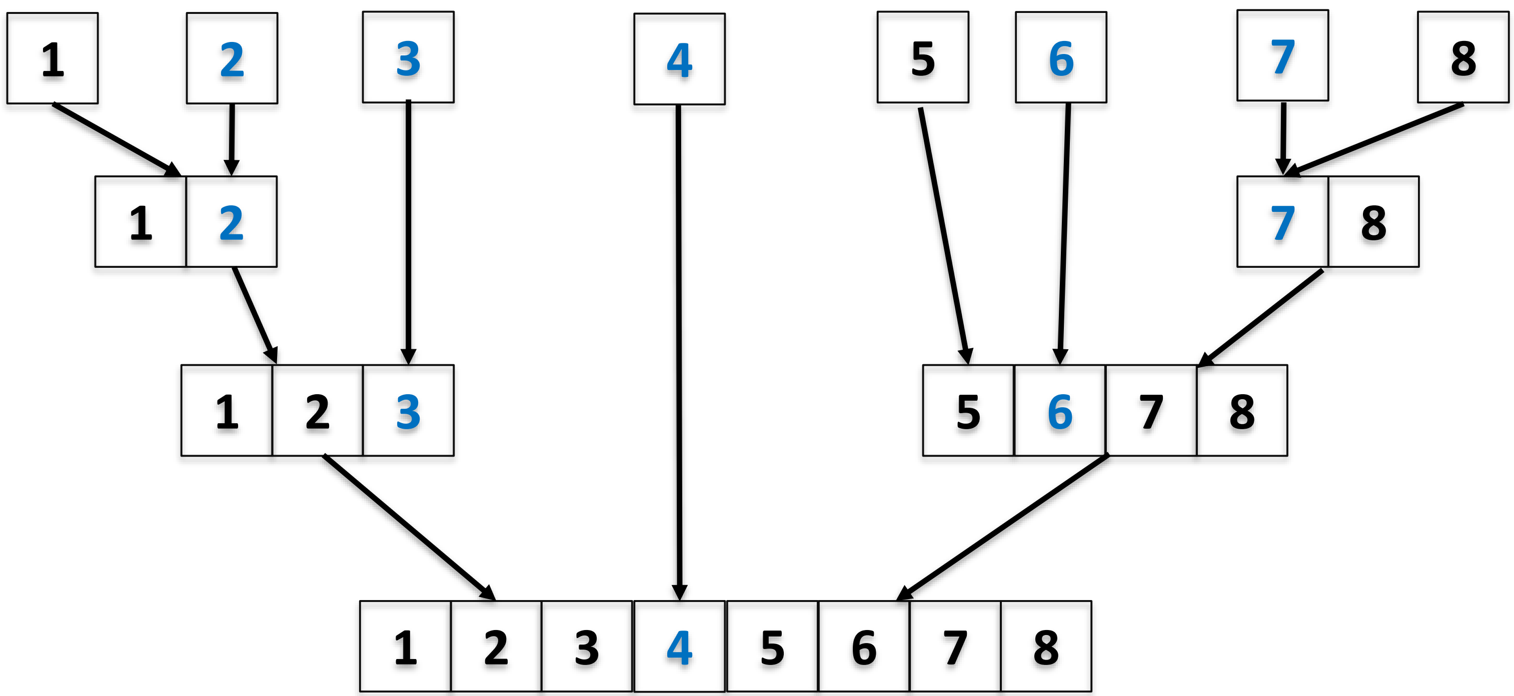

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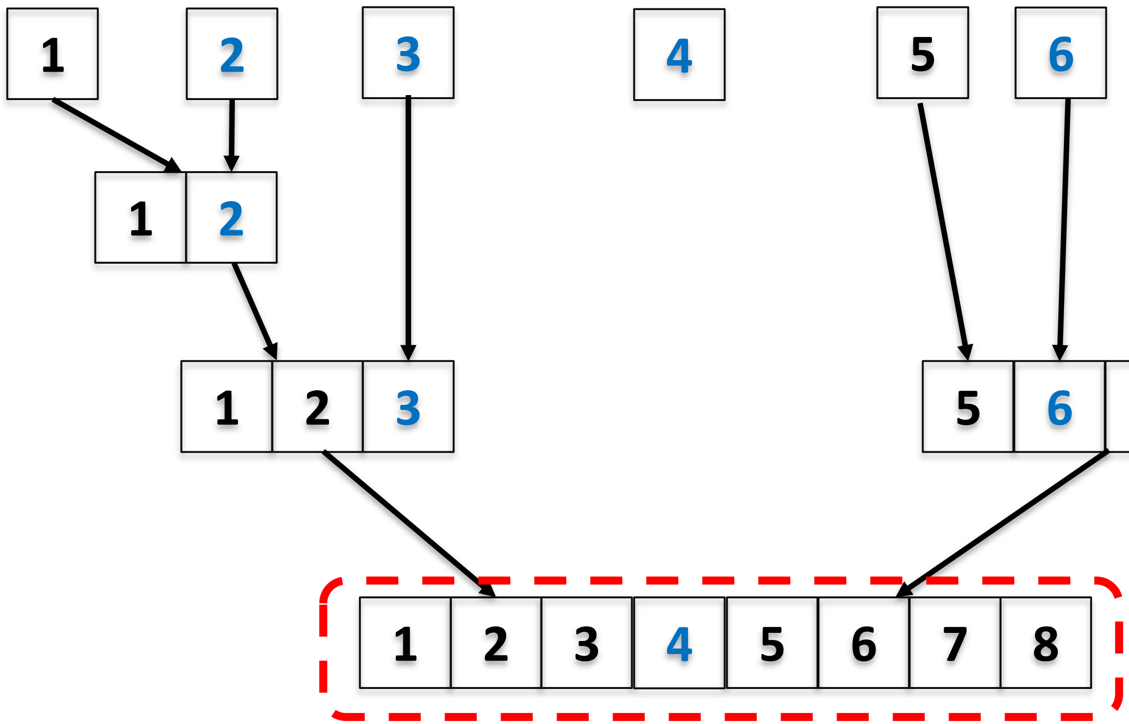

Outline

- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

Running Time of the Quicksort

- Measuring running time:

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- Two methods to analyze the expected running time of a divide-and-conquer randomized algorithm:
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  - New: Indicator variables.
    —— Simple and elegant, but needs practice to master.

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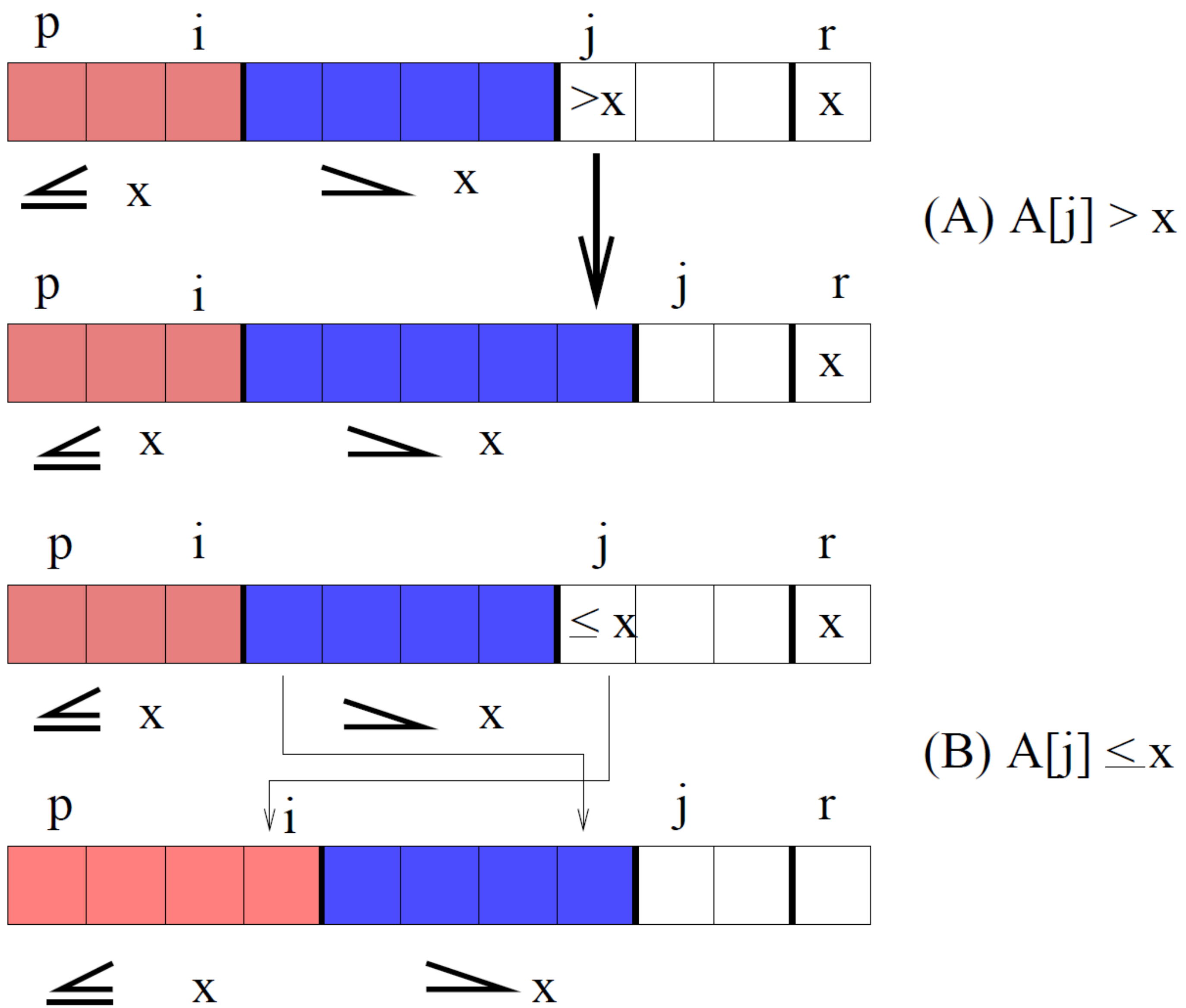

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- If the pivot is any element in Z_{ij} other than z_i or z_j
  - z_i and z_j are not compared with each other in all randomized-partition calls

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \ is \ compared \ with \ z_j\}$$

$$=$$

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$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$

$=$

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$$
E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} =
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Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

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$$

$$
= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n)
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Note: $\sum_{k=1}^{n} \frac{1}{k} \le \log(n)$

Hence, the expected number of comparisons is $O(n \log n)$, which is the expected running time of Randomized-Quicksort

Selection Problem

Outline

- Review to Divide-and-Conquer Paradigm

- Selection Problem
  - Problem Definition
  - First solution: Selection by sorting
  - A randomized divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

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Review to Divide-and-Conquer Paradigm

- Divide-and-conquer (D&C) is an important algorithm design paradigm.
  - **Divide**
    Dividing a given problem into two or more subproblems (ideally of approximately equal size)
  - **Conquer**
    Solving each subproblem (directly if small enough or recursively)
  - **Combine**
    Combining the solutions of the subproblems into a global solution

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Linear Time Selection

Definition (Selection Problem)

Given a sequence of numbers $\langle a_1, \ldots, a_n \rangle$, and an integer i, $1 \leq i \leq n$, find the ith smallest element. When $i = \lceil n/2 \rceil$, it is called the median problem.

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Example

Given $\langle 1, 8, 23, 10, 19, 33, 100 \rangle$, the 4th smallest element is 19.

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Example

Given $\langle 1, 8, 23, 10, 19, 33, 100 \rangle$, the 4th smallest element is 19.

Question

How do you solve this problem?

Outline

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First Solution: Selection by Sorting

- Sort the elements in ascending order with any algorithm of complexity O(n log n).

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Can we do better?

First Solution: Selection by Sorting

- Sort the elements in ascending order with any algorithm of complexity O(n log n).
- Return the *i*th element of the sorted array.

The complexity of this solution is O(n log n)

Question

Can we do better?

Answer: YES, but we need to recall Partition(A,p,r) used in Quicksort!

Outline

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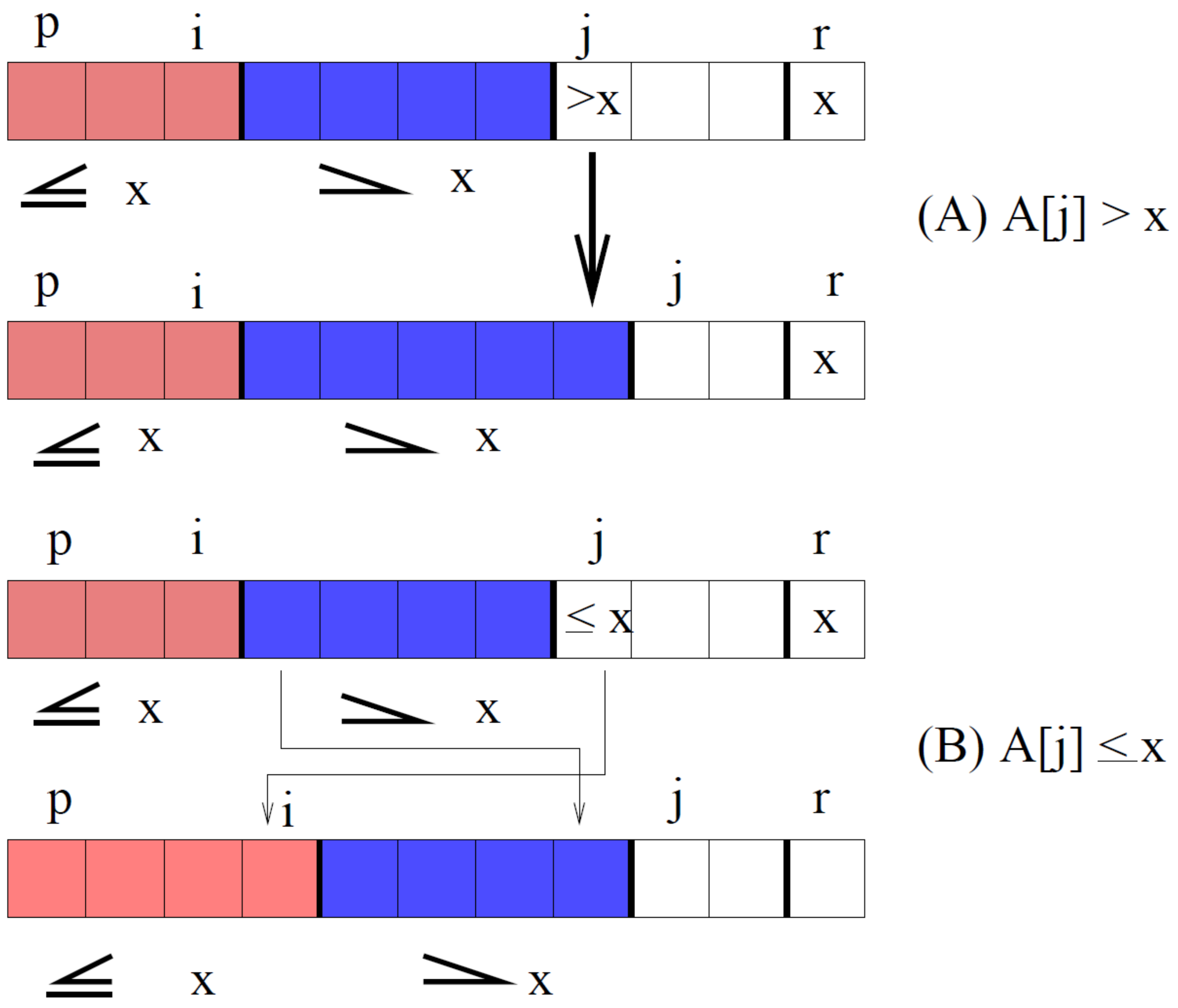
Review of Randomized-Partition (A,p,r)



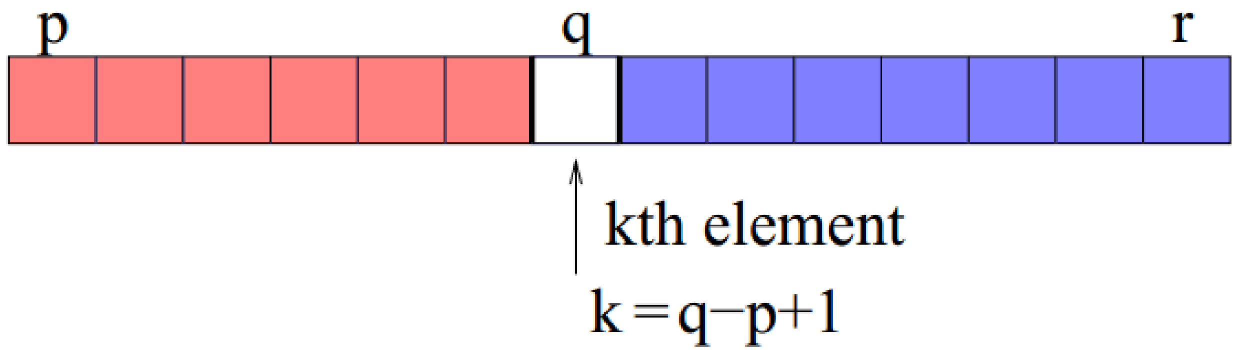
Randomized-Select(A,p,r,i), $1 \le i \le r-p+1$

Problem: Select the ith smallest element in A[p..r], where $1 \le i \le r-p+1$

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Problem: Select the *i*th smallest element in A[p..r], where 1 ≤ i ≤ r−p+1

Solution: Apply Randomized-Partition(A, p, r), getting



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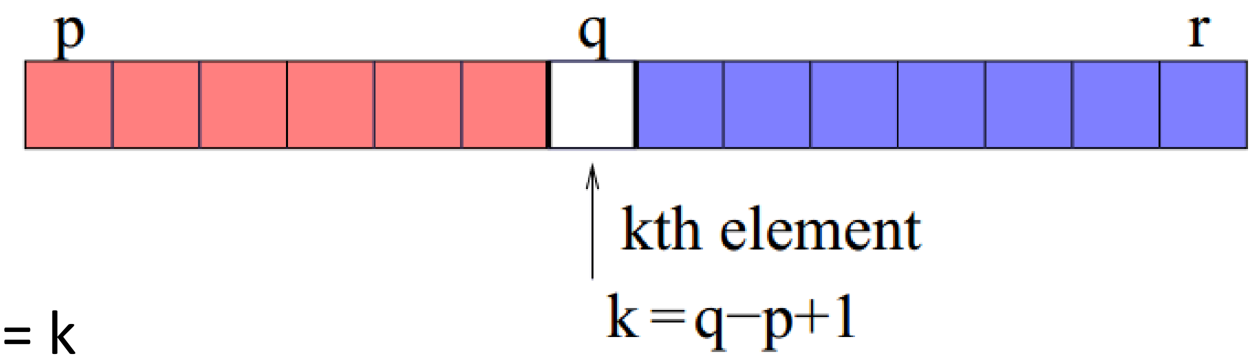

- i = k
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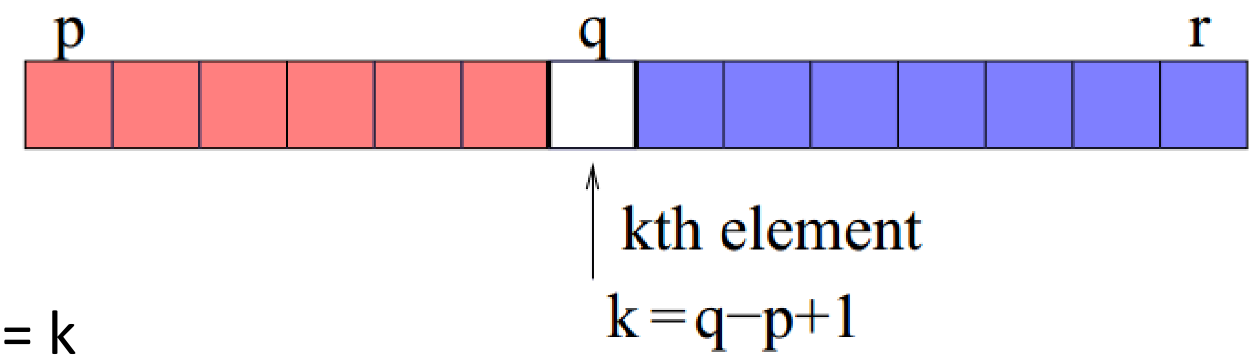

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- i < k
  - the *i*th smallest element in A[p..r] must be the *i*th smallest element in A[p..q−1]

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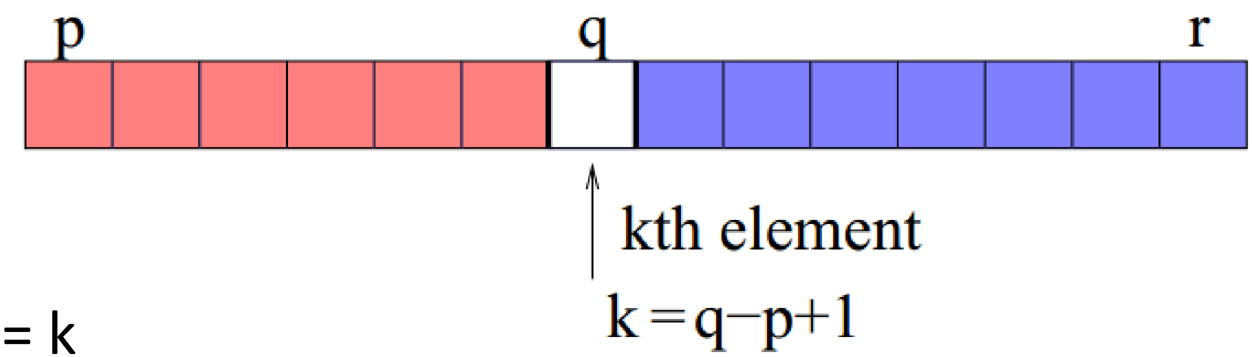

- i = k
  - pivot is the solution
- i < k
  - the *i*th smallest element in A[p..r] must be the *i*th smallest element in A[p..q−1]
- i > k
  - the *i*th smallest element in A[p..r] must be the (*i* − *k*)th smallest element in A[q+1..r]

If necessary, recursively call the same procedure to the subarray

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

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if p *is equal to* r **then**
  | **return** $A[p]$;
end

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

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if p *is equal to* r **then**

  return $A[p]$;

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$q \leftarrow$Randomized-Partition(A, p, r);

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

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Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

if p *is equal to* r **then**
  return $A[p]$;
end
$q \leftarrow$Randomized-Partition(A, p, r);
$k \leftarrow q - p + 1$;
if $i \leftarrow k$ **then**

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

```
Input: An array A,the range of index p,r,the ith smallest element that
       we want to select
Output: The ith smallest element A[i]
if p is equal to r then
 | return A[p];
end
q ←Randomized-Partition(A, p, r);
k ← q − p + 1;
if i ← k then
 | return A[q];//The pivot is the answer
end
```

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

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Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select
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if p *is equal to* r **then**
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$q \leftarrow$Randomized-Partition(A, p, r);
$k \leftarrow q - p + 1$;
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end
else if $i < k$ **then**
  | **return** Randomized-Select$(A, p, q - 1, i)$;
end

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Output: The ith smallest element A[i]
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else
 |  return Randomized-Select(A, q + 1, r, i − k);
end
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Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

if p *is equal to* r **then**
  | **return** $A[p]$;
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$q \leftarrow$Randomized-Partition(A, p, r);
$k \leftarrow q - p + 1$;
if $i \leftarrow k$ **then**
  | **return** $A[q]$;//The pivot is the answer
end
else if $i < k$ **then**
  | **return** Randomized-Select$(A, p, q - 1, i)$;
end
else
  | **return** Randomized-Select$(A, q + 1, r, i - k)$;
end

To find the ith smallest element in A[1..n], call Randomized-Select(A, 1, n, i)

Randomized Selection - Example

- Find the 8th smallest element of the following list of numbers:
  - 8 25 2 14 3 20 15 13 12 11 7 5 9 10 16 18 1 23 26 21

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i-k*)th smallest element in A[q+1..r]

p																			r
8	25	2	14	3	20	15	13	12	11	7	5	9	10	16	18	1	23	26	21

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$$i = 8, p = 1, r = 20$$

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p															q				r
8	2	14	3	15	13	12	11	7	5	9	10	16	18	1	20	25	23	26	21

$$i = 8, p = 1, r = 20$$
$$q = 16, \mathrm{k} = 16$$

Randomized Selection - Example

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p															q				r
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$$q = 16, \mathrm{k} = 16$$

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p														r					
8	2	14	3	15	13	12	11	7	5	9	10	16	18	1	20	25	23	26	21

$$i = 8, p = 1, r = 15$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
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  - i > k : the (*i*-*k*)th smallest element in A[q+1..r]

p														**r**					
8	**2**	**14**	**3**	**15**	**13**	**12**	**11**	**7**	**5**	**9**	**10**	**16**	**18**	**1**	20	25	23	26	21

$$i = 8, p = 1, r = 15$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
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p						q								r					
8	2	3	7	5	1	9	11	14	15	13	10	16	18	12	20	25	23	26	21

$$i = 8, p = 1, r = 15$$
$$q = 7, k = 7$$

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p						q								r					
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							p							r					
8	2	3	7	5	1	9	11	14	15	13	10	16	18	12	20	25	23	26	21

$$i = 1, p = 8, r = 15$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
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							p							r					
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$$i = 1, p = 8, r = 15$$

Randomized Selection - Example

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10 is the 8th smallest element of the array.

Outline

- Review to Divide-and-Conquer Paradigm

- Selection Problem
  - Problem Definition
  - First solution: Selection by sorting
  - A randomized divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

Running Time of Randomized-Select(A,1,n,i)

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Linearity of expectation

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$$
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&= \sum_{k=1}^{n} E[X_k \cdot T(\max(k-1, n-k))] + O(n) \\
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Independence between X_k and $T(max(k-1, n-k))$

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$$\max(k-1, n-k) = \begin{cases} k-1 \;\; if \; k > \lceil n/2 \rceil \\ n-k \; if \; k \leq \lceil n/2 \rceil \end{cases}$$

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- It is a complicated recurrence!
- Use Guess and Induction (Substitution Method)

Running Time of Randomized-Select(A,1,n,i)

- Guess:

$$T(n) \leq c\,n, \qquad for\ all\ n$$

  for some constant *c* to be figured out later.

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Running Time of Randomized-Select(A,1,n,i)

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\begin{aligned}
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&= c\left(\frac{3n}{4} + \frac{1}{2} - \frac{2}{n}\right) + an \\
&\le \frac{3cn}{4} + \frac{c}{2} + an \\
&= cn - \left(\frac{cn}{4} - \frac{c}{2} - an\right)
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If we choose $c \geq 12a$, then the induction step works for $n \geq 3$.

Induction basis:

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So if we choose $c = \max\{12a, T(1), T(2)/2\}$, then the entire proof works.

Running Time of Randomized-Select(A,1,n,i)

Worst Case:

$$T(n) = n - 1 + T(n-1), T(n) = O(n^2)$$

Running Time of Randomized-Select(A,1,n,i)

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$$T(n) = n - 1 + T(n-1), T(n) = O(n^2)$$

Expected running time much better than worst case!

Randomized Quicksort vs Randomized Selection

Question

Why does Randomized Selection take $O(n)$ time while Randomized Quicksort takes $O(n \log n)$ time?

Randomized Quicksort vs Randomized Selection

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Why does Randomized Selection take $O(n)$ time while Randomized Quicksort takes $O(n \log n)$ time?

Answer:

- Randomized Selection needs to work on only 1 of the two subproblems.

Randomized Quicksort vs Randomized Selection

Question

Why does Randomized Selection take $O(n)$ time while Randomized Quicksort takes $O(n \log n)$ time?

Answer:

- Randomized Selection needs to work on only 1 of the two subproblems.
- Randomize Quicksort needs to work on both of the two subproblems.

Summary of Divide-and-Conquer and Randomization

- Randomization is needed when it is not easy to divide evenly.
- Use expected case analysis for randomized algorithms.
- The running time is the expected running time for any given input, expectation is with respect to the random choices made by the algorithm internally.

End of Section