Design and Analysis of Algorithms

Part II: Dynamic Programming

Lecture 11: Maximum Contiguous Subarray Problem II

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动态规划篇概述

- 在算法课程第二部分动态规划主题中，我们将主要聚焦于如下经典问题：
  - 0-1 Knapsack (0-1背包问题)
  - Maximum Contiguous Subarray II (最大连续子数组 II)
  - Longest Common Subsequences (最长公共子序列)
  - Longest Common Substrings (最长公共子串)
  - Minimum Edit Distance (最小编辑距离)
  - Rod-Cutting (钢条切割)
  - Chain Matrix Multiplication (矩阵链乘法)

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最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	**-1**	**-3**	**3**	**5**	**-4**	**3**	**2**	**-2**	**3**	**6**

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组为数组***X***中连续的一段序列

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$X[3..7]$

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$X[3..7]$
  - 求和为：$3+5-4+3+2=9$

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$\boldsymbol{X[3..7]}$
  - 求和为：$\boldsymbol{3+5-4+3+2=9}$
- 子数组$\boldsymbol{X[1..10]}$

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$X[3..7]$
  - 求和为：$3+5-4+3+2=9$
- 子数组$X[1..10]$
  - 求和为：$-1-3+3+5-4+3+2-2+3+6=12$

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$\boldsymbol{X[3..7]}$
  - 求和为：$\boldsymbol{3+5-4+3+2=9}$
- 子数组$\boldsymbol{X[1..10]}$
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- 子数组$\boldsymbol{X[3..10]}$

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$X[3..7]$
  - 求和为：$3+5-4+3+2=9$
- 子数组$X[1..10]$
  - 求和为：$-1-3+3+5-4+3+2-2+3+6=12$
- 子数组$X[3..10]$
  - 求和为：$3+5-4+3+2-2+3+6=16$

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$\boldsymbol{X[3..7]}$
  - 求和为：$\boldsymbol{3+5-4+3+2=9}$
- 子数组$\boldsymbol{X[1..10]}$
  - 求和为：$\boldsymbol{-1-3+3+5-4+3+2-2+3+6=12}$
- 子数组$\boldsymbol{X[3..10]}$
  - 求和为：$\boldsymbol{3+5-4+3+2-2+3+6=16}$

问题：寻找数组X最大的非空子数组?

最大子数组问题

	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 子数组$X[3..7]$
  - 求和为：$3+5-4+3+2=9$
- 子数组$X[1..10]$
  - 求和为：$-1-3+3+5-4+3+2-2+3+6=12$
- 子数组$X[3..10]$
  - 求和为：$3+5-4+3+2-2+3+6=16$

问题：寻找数组X最大的非空子数组?
答案：$X[3..10]=16$

问题定义

- 形式化定义

最大子数组问题

Max Continuous Subarray, MCS

输入

- 给定一个数组$X[1..n]$，对于任意一对数组下标为l, r $(l \le r)$的非空子数组，其和记为

$$S(l,r) = \sum_{i=l}^{r} X[i]$$

输出

- 求出$S(l,r)$的最大值，记为S_{max}

蛮力枚举

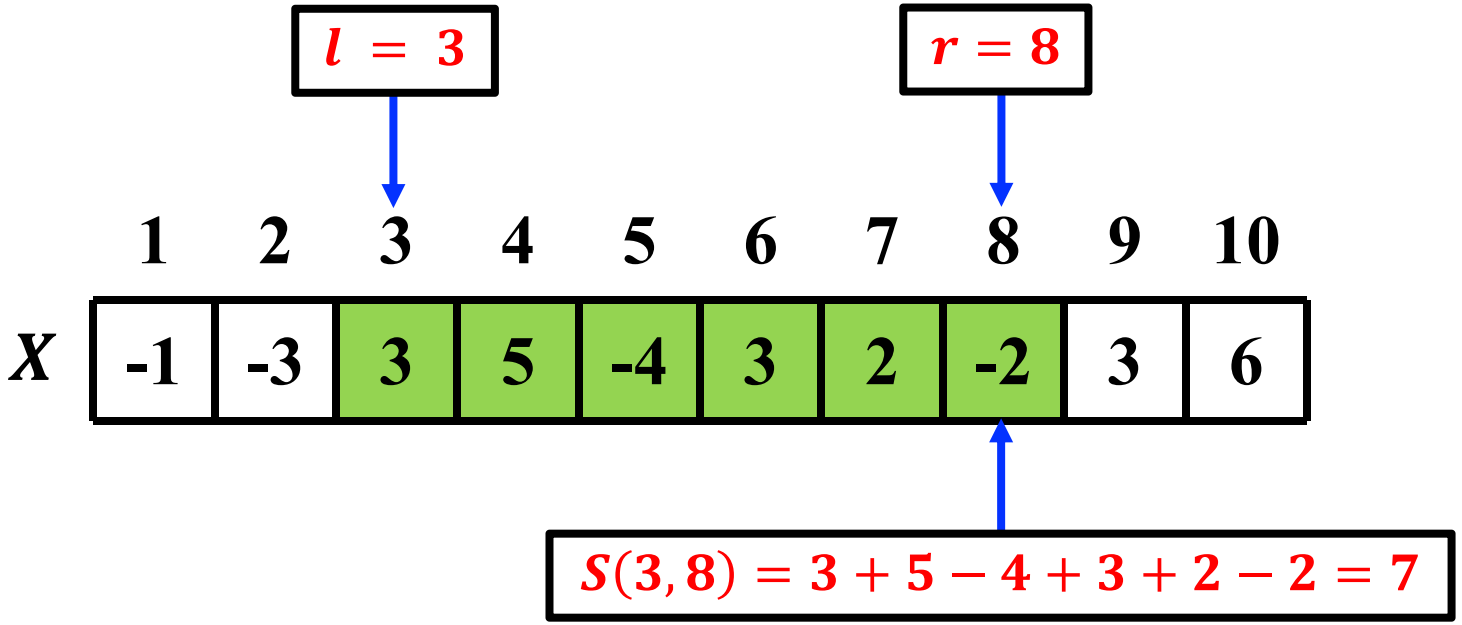
	1	2	3	4	5	6	7	8	9	10
X	-1	-3	3	5	-4	3	2	-2	3	6

- 数组$X[1..n]$，其所有的下标$l, r(l \le r)$组合分为以下两种情况
  - 当$l = r$时，一共$C_n^1 = n$种组合
  - 当$l < r$时，一共C_n^2种组合

- 枚举$n + C_n^2$种下标l, r组合，求出最大子数组之和

蛮力枚举

- $l = 3,\ r = 8$
- 计算$S(3, 8)$：



蛮力枚举

- 枚举$n + C_n^2$种可能的区间$[l, r](l \le r)$，求解最大子数组之和S_{max}

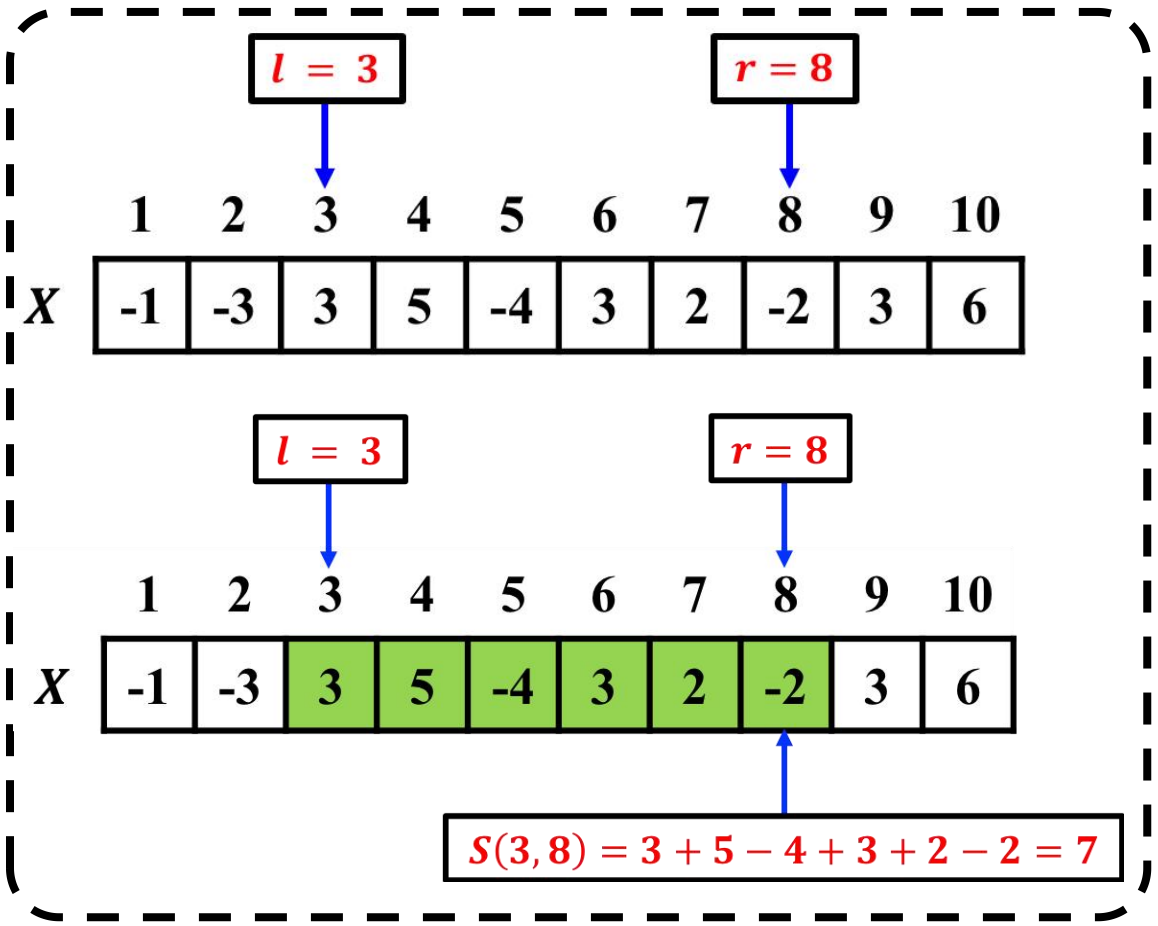
输入: 数组$X[1..n]$
输出: 最大子数组之和S_{max}

```
S_max ← −∞
for l ← 1 to n do
    for r ← l to n do
        S(l,r) ← 0
        for i ← l to r do
            S(l,r) ← S(l,r) + X[i]
        end
        S_max ← max{S_max, S(l,r)}
    end
end
return S_max
```

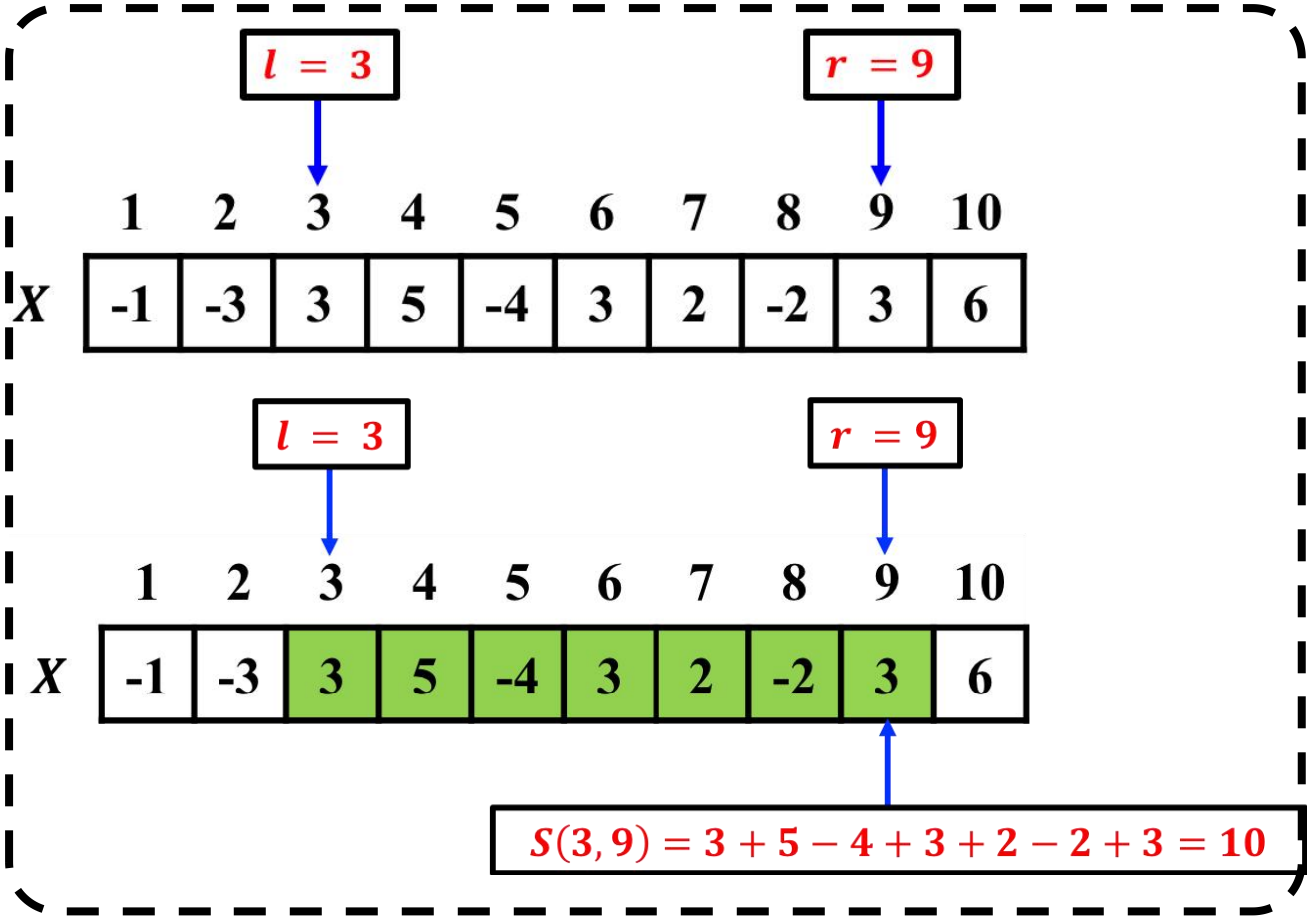
$O(n)$ $O(n^2)$ $O(n^3)$

时间复杂度：$O(n^3)$

从蛮力枚举到优化枚举

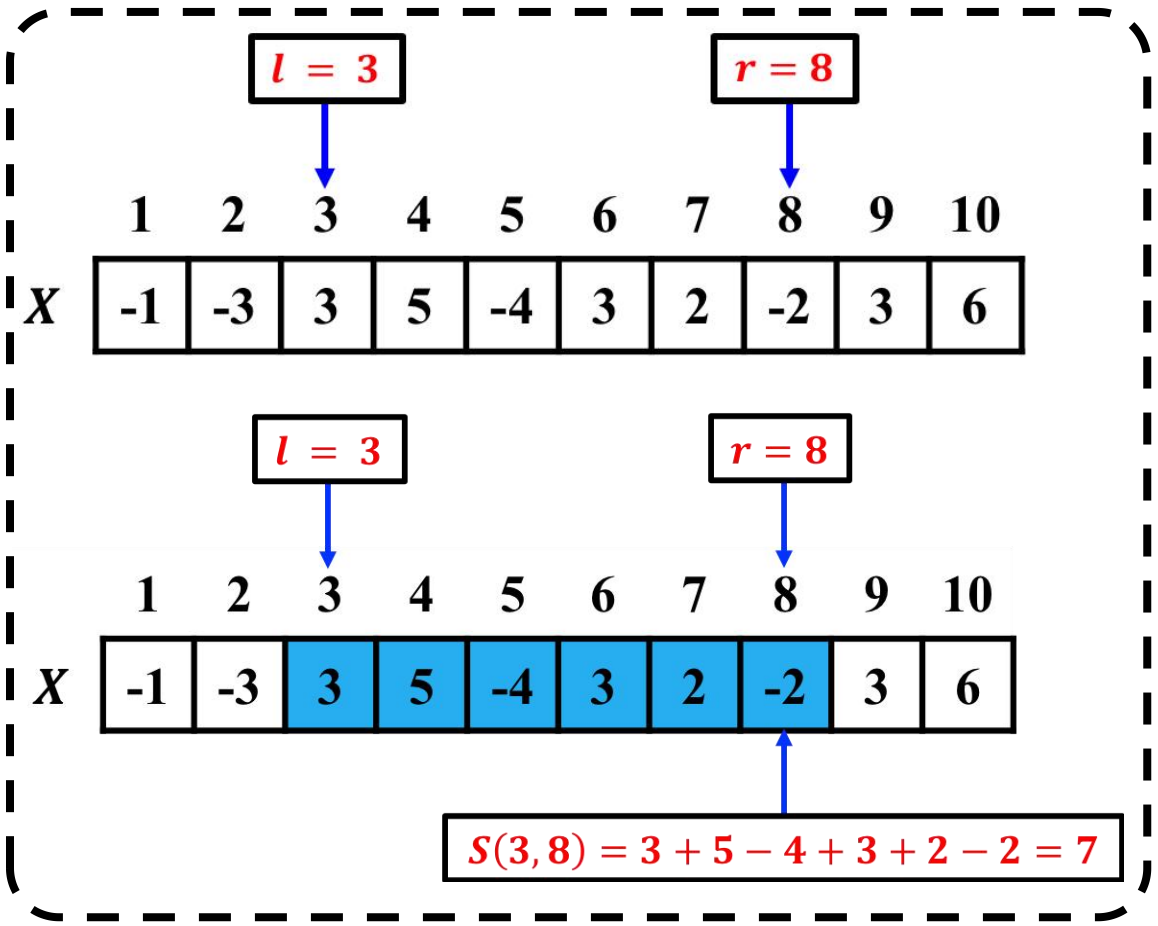


计算$S(3,8)$循环6次

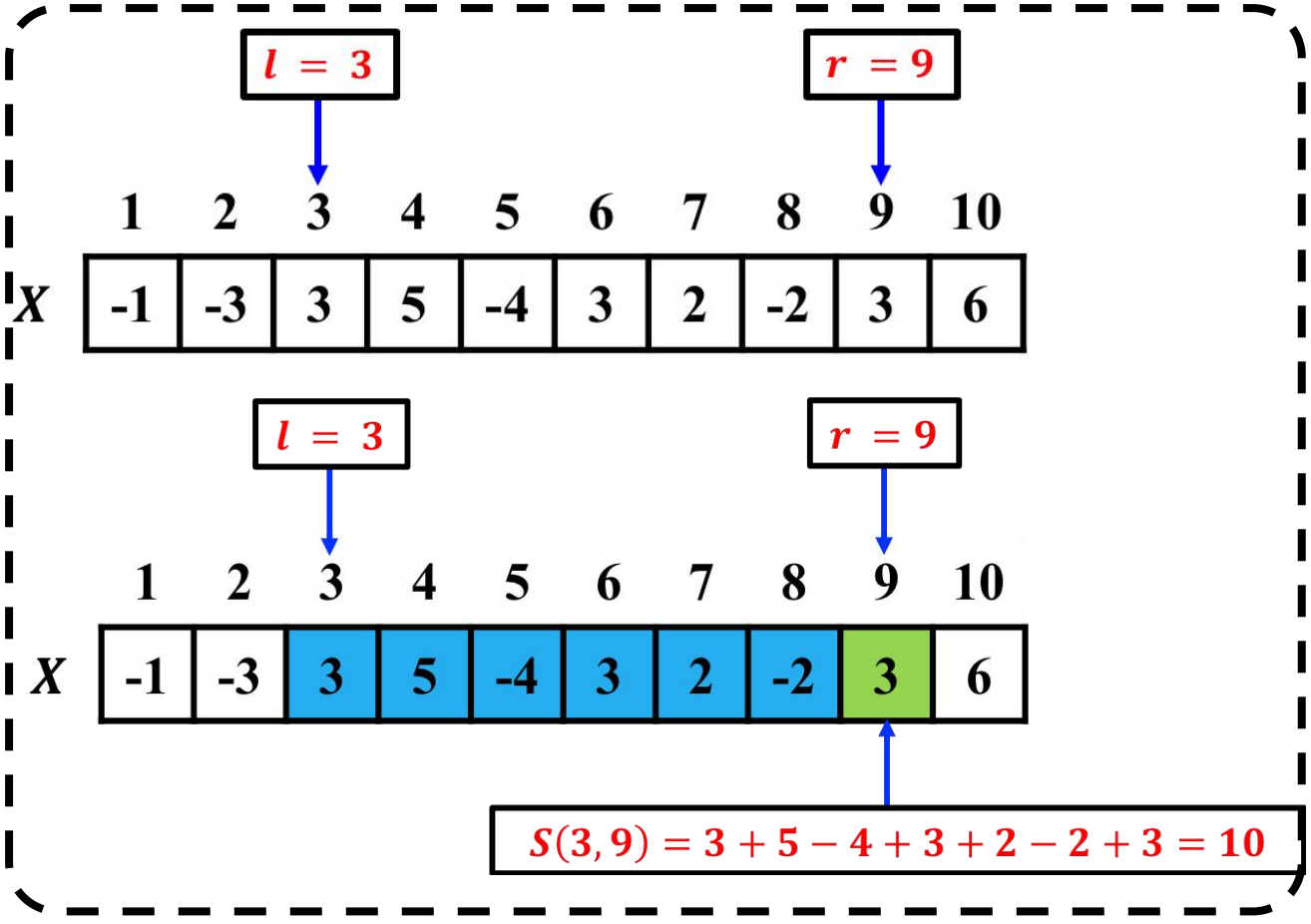


计算$S(3,9)$循环7次

从蛮力枚举到优化枚举



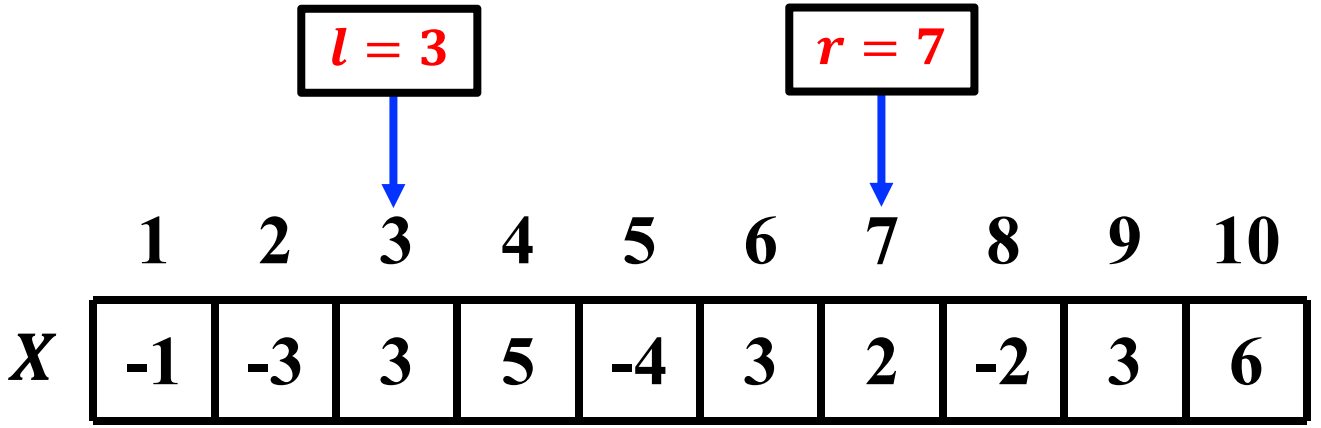
计算$S(3,8)$循环6次



计算$S(3,9)$循环7次

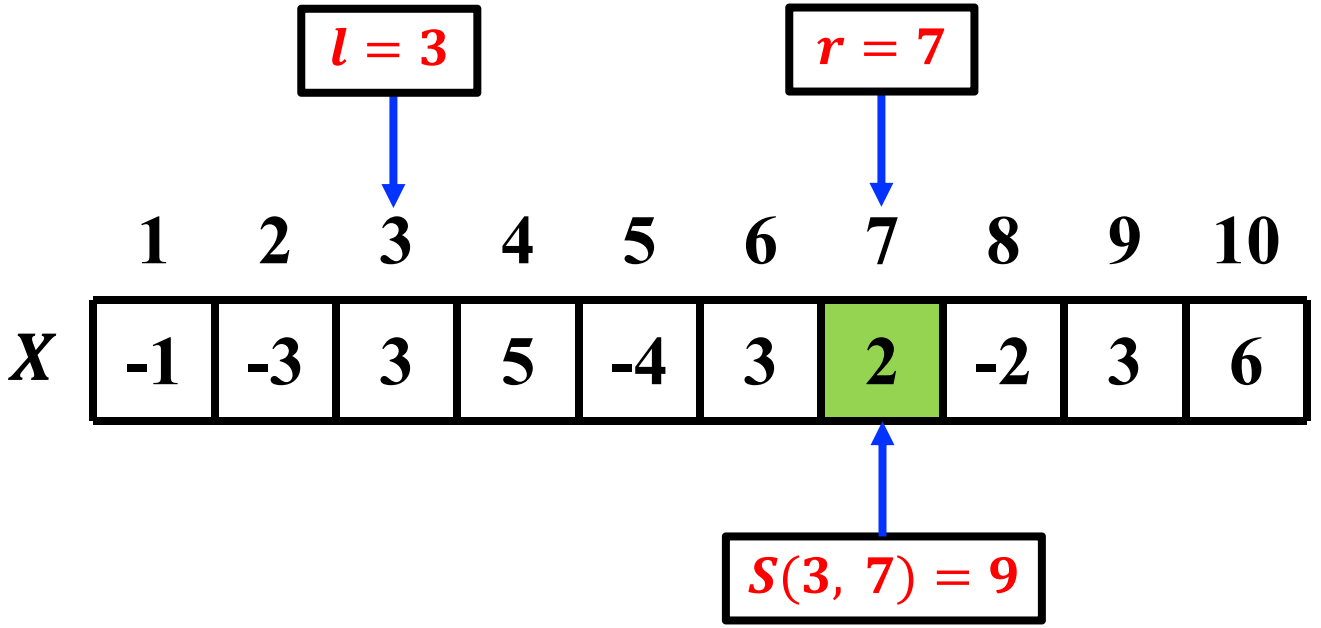
优化枚举

- $l = 3,\ r = 7$



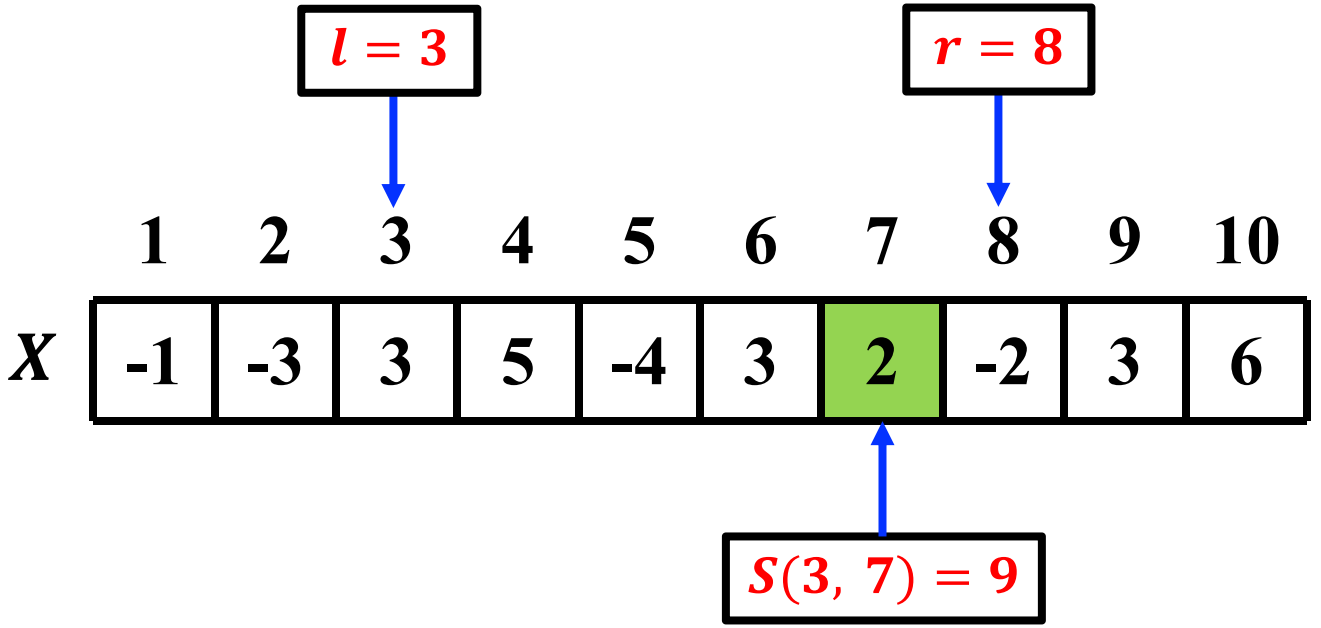
优化枚举

- $l = 3,\ r = 7,\ S(3,\ 7) = 9$



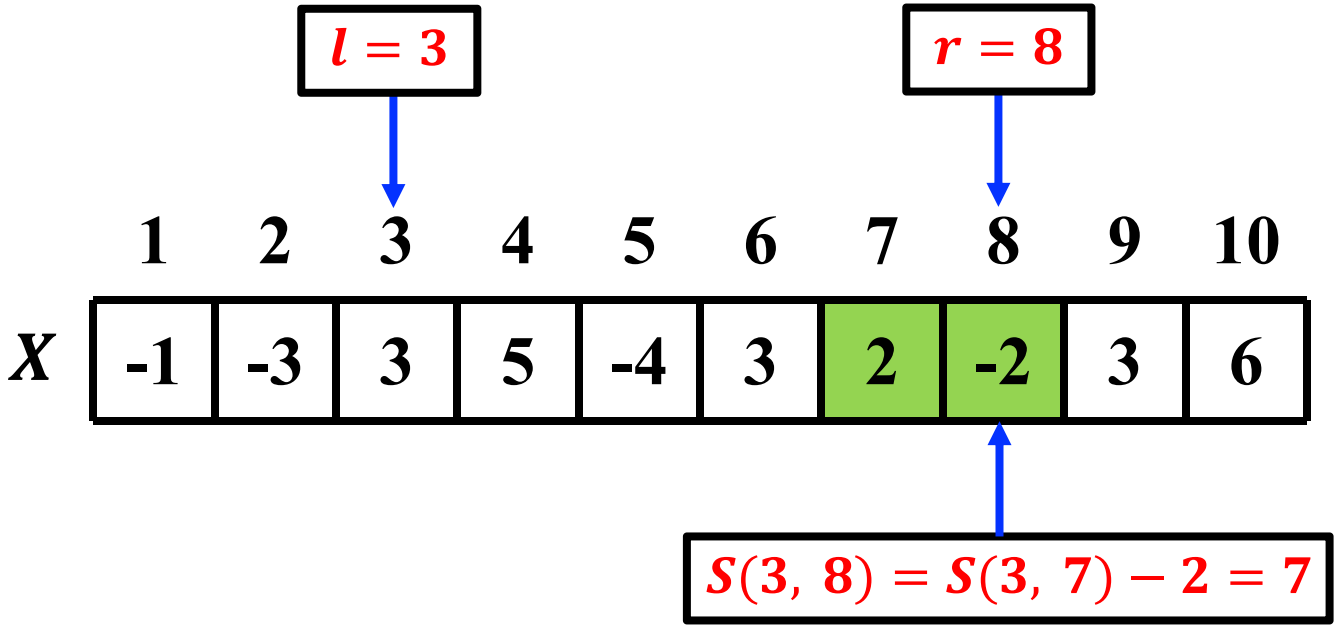
优化枚举

- $l = 3,\ r = 8,\ S(3,\ 8) = \ ?$



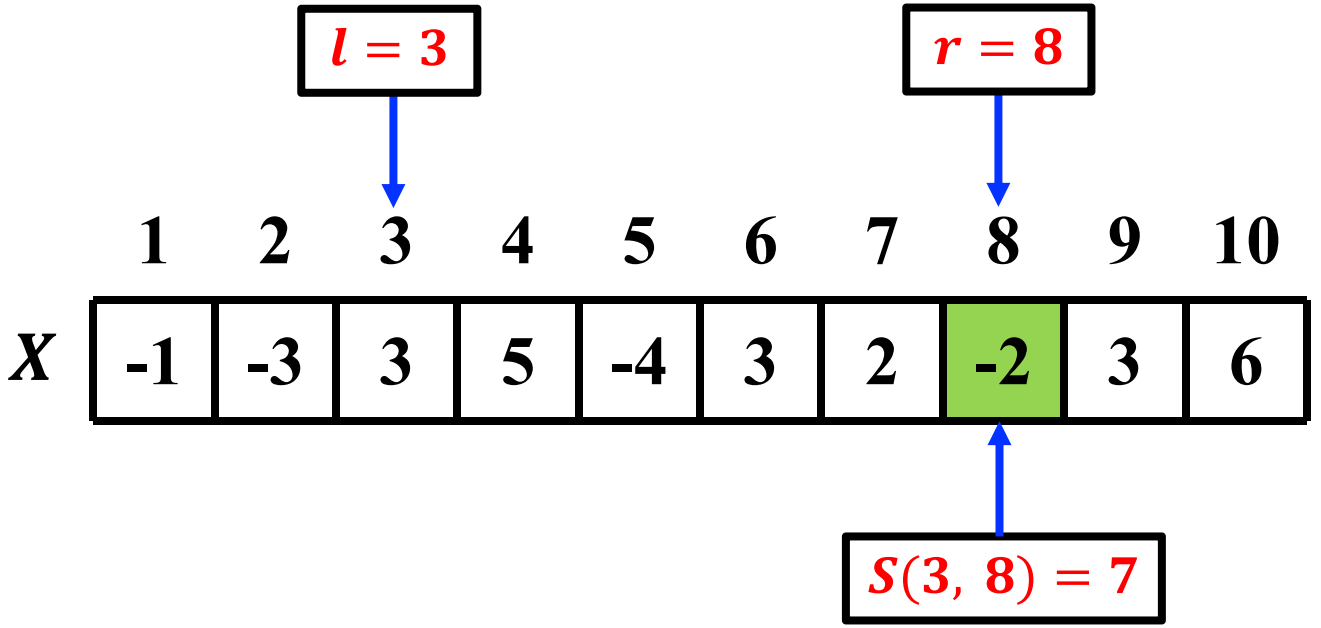
优化枚举

- $l = 3,\ r = 8,\ S(3,\ 8) = 7$



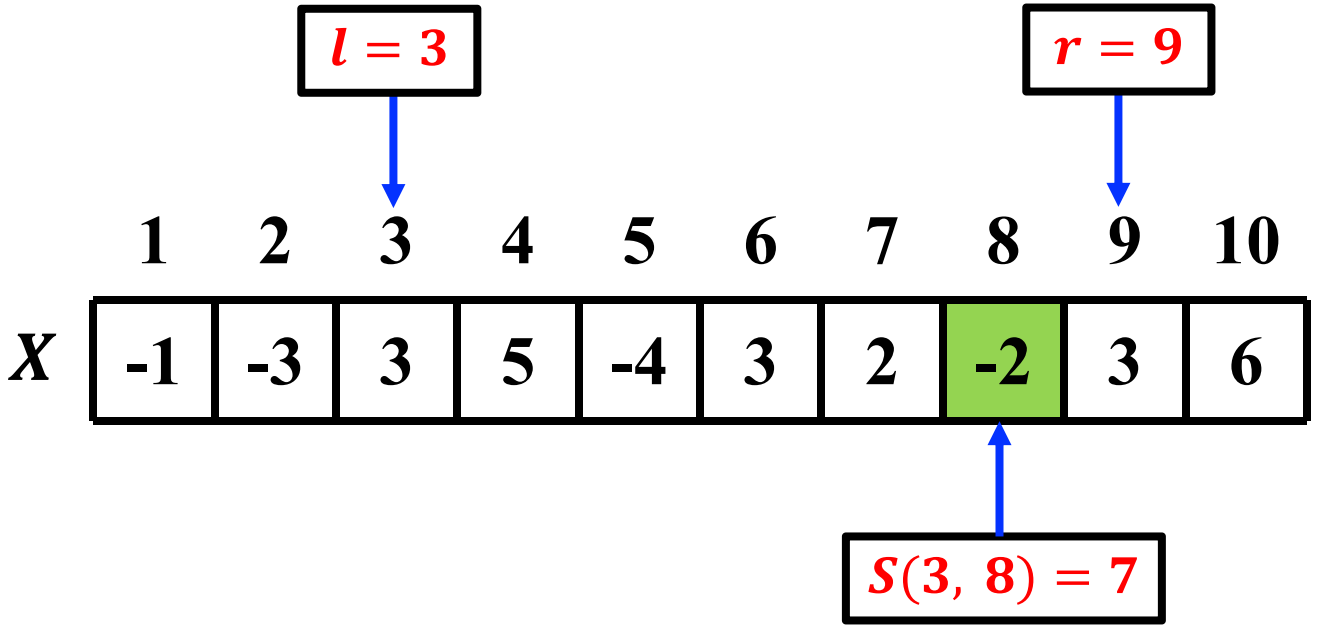
优化枚举

- $l = 3,\ r = 8,\ S(3, 8) = 7$



优化枚举

- $l = 3,\ r = 9,\ S(3,\ 9) = \ ?$



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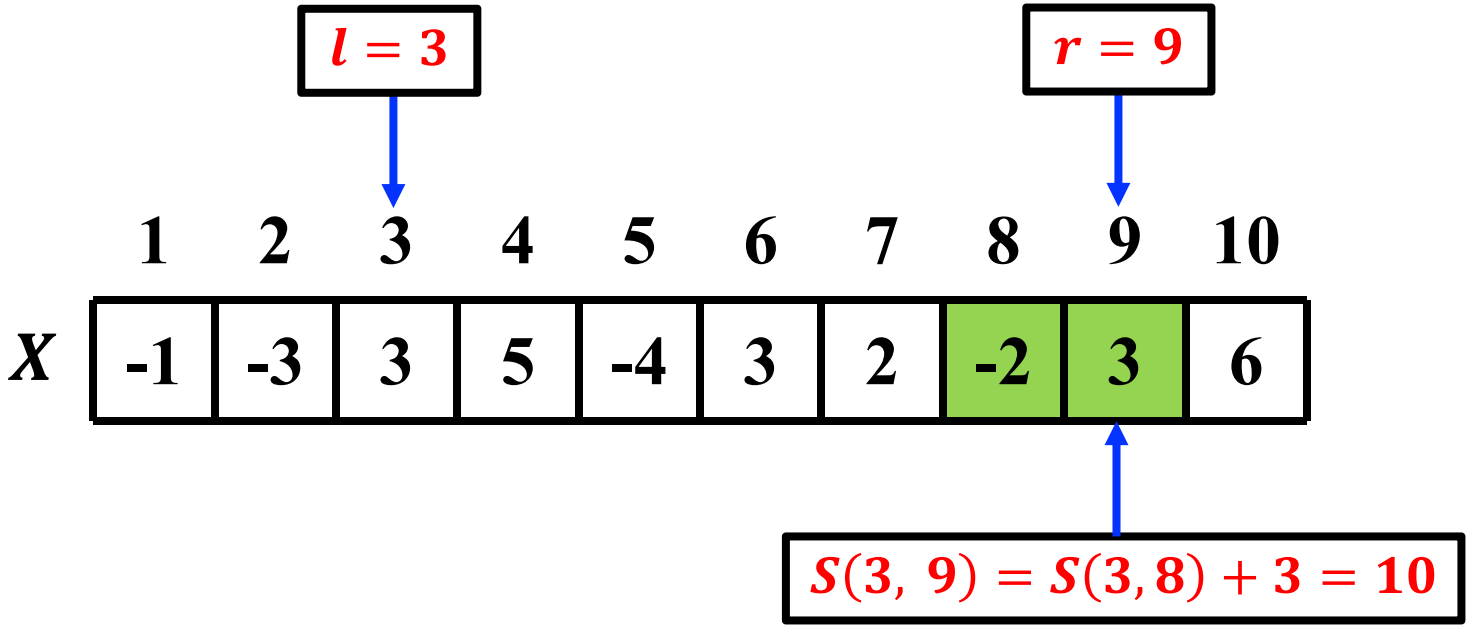
- $l = 3,\ r = 9,\ S(3,\ 9) = \ ?$



优化枚举

- 核心思想：$S(l, r) = \sum_{i=l}^{r} X[i] = S(l, r-1) + X[r]$

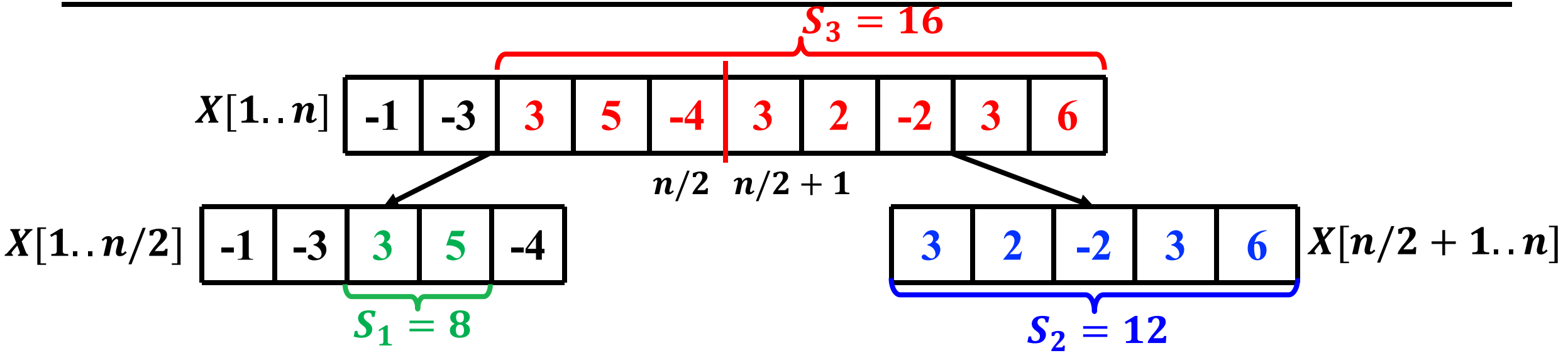
输入：数组$X[1..n]$
输出：最大子数组之和S_{max}

```
S_max ← −∞
for l ← 1 to n do
    S ← 0
    for r ← l to n do
        S ← S + X[r]
        S_max ← max{S_max, S}
    end
end
return S_max
```

$O(n)$ $O(n^2)$

时间复杂度：$O(n^2)$

分而治之



- 将数组$X[1..n]$分为$X[1..n/2]$和$X[n/2+1..n]$
- 递归求解子问题
  - S_1：数组$X[1..n/2]$ 的最大子数组
  - S_2：数组$X[n/2+1..n]$ 的最大子数组
- 合并子问题，得到S_{max}
  - S_3：跨中点的最大子数组
  - 数组X的最大子数组之和$S_{max} = max\{S_1, S_2, S_3\}$

分解原问题

解决子问题

合并问题解

分而治之

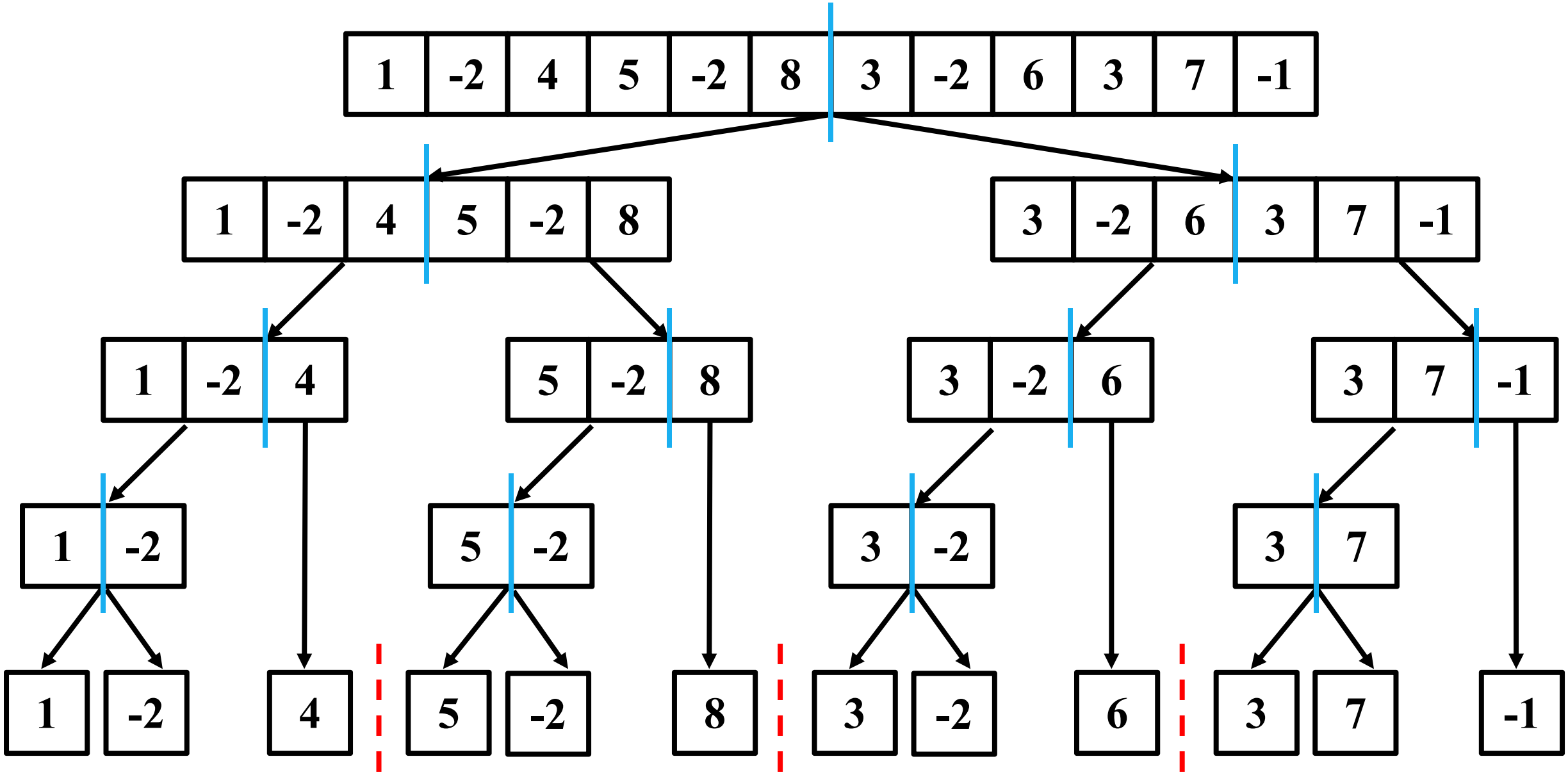
- 时间复杂度分析

输入: 数组X, 数组下标$low, high$
输出: 最大子数组之和S_{max}

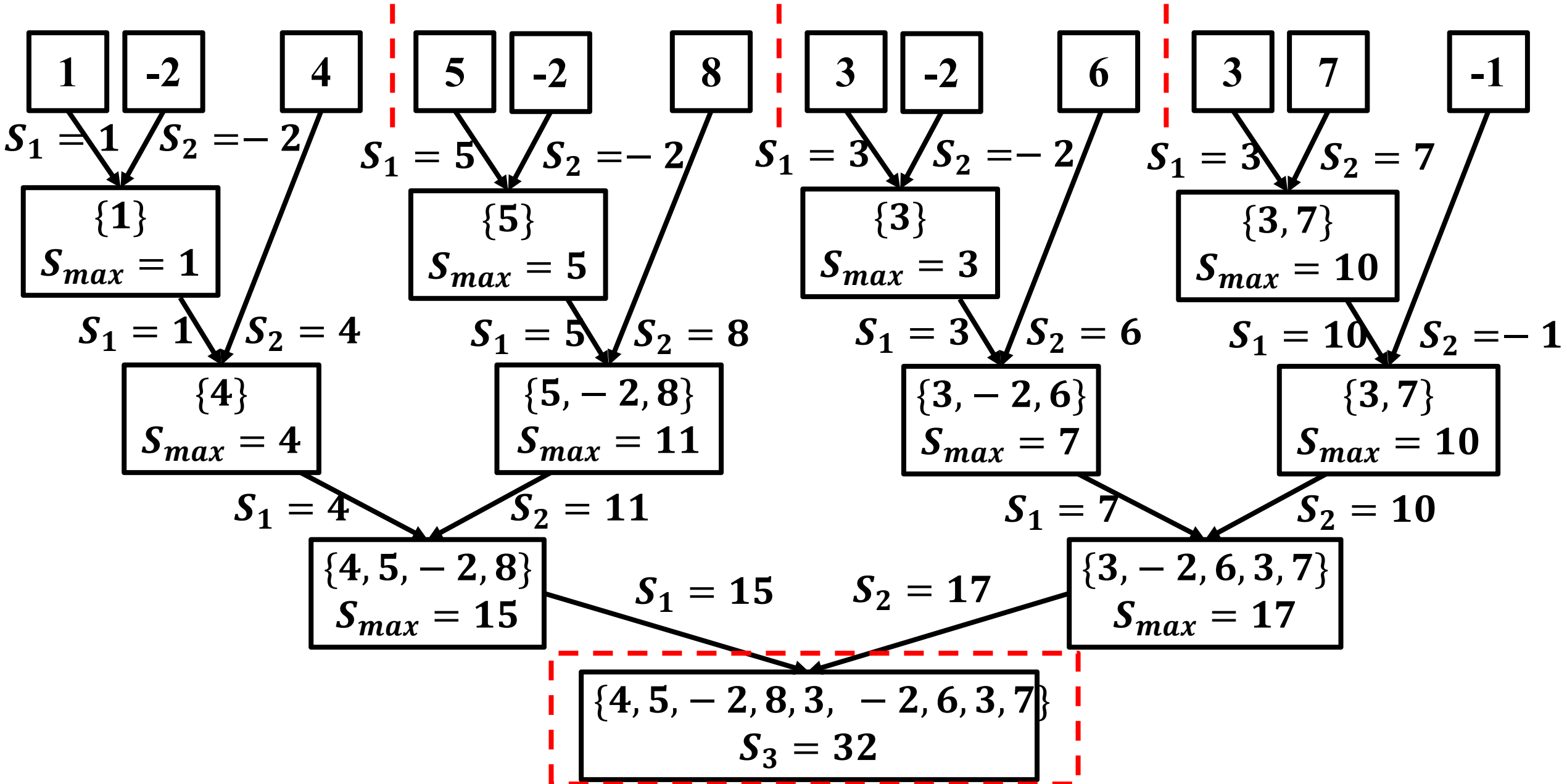
```
if low = high then
    return X[low]
end
else
    mid ← ⌊(low+high)/2⌋
    S1 ← MaxSubArray(X,low,mid)
    S2 ← MaxSubArray(X,mid+1,high)
    S3 ← CrossingSubArray(X,low,mid,high)
    Smax ← max{S1, S2, S3}
    return Smax
end
```

时间复杂度：$O(n \log n)$

分而治之


1 -2 4 5 -2 8 3 -2 6 3 7 -1
1 -2 4 5 -2 8
3 -2 6 3 7 -1
1 -2 4
5 -2 8
3 -2 6
3 7 -1
1 -2
5 -2
3 -2
3 7
1
-2
4
5
-2
8
3
-2
6
3
7
-1

算法实例


1
-2
4
5
-2
8
3
-2
6
3
7
-1
$S_1 = 1$ $S_2 = -2$
$S_1 = 5$ $S_2 = -2$
$S_1 = 3$ $S_2 = -2$
$S_1 = 3$ $S_2 = 7$
{1}
$S_{max} = 1$
{5}
$S_{max} = 5$
{3}
$S_{max} = 3$
{3, 7}
$S_{max} = 10$
$S_1 = 1$ $S_2 = 4$
$S_1 = 5$ $S_2 = 8$
$S_1 = 3$ $S_2 = 6$
$S_1 = 10$ $S_2 = -1$
{4}
$S_{max} = 4$
{5, −2, 8}
$S_{max} = 11$
{3, −2, 6}
$S_{max} = 7$
{3, 7}
$S_{max} = 10$
$S_1 = 4$ $S_2 = 11$
$S_1 = 7$ $S_2 = 10$
{4, 5, −2, 8}
$S_{max} = 15$
{3, −2, 6, 3, 7}
$S_{max} = 17$
$S_1 = 15$ $S_2 = 17$
{4, 5, −2, 8, 3, −2, 6, 3, 7}
$S_3 = 32$

算法比较

算法名称	时间复杂度
蛮力枚举	$O(n^3)$
优化枚举	$O(n^2)$
分而治之	$O(n\log n)$
?	$O(n)$

问题：是否可以设计一个时间复杂度为$O(n)$的算法？

算法比较

算法名称	时间复杂度
蛮力枚举	$O(n^3)$
优化枚举	$O(n^2)$
分而治之	$O(n\log n)$
动态规划	$O(n)$

算法比较

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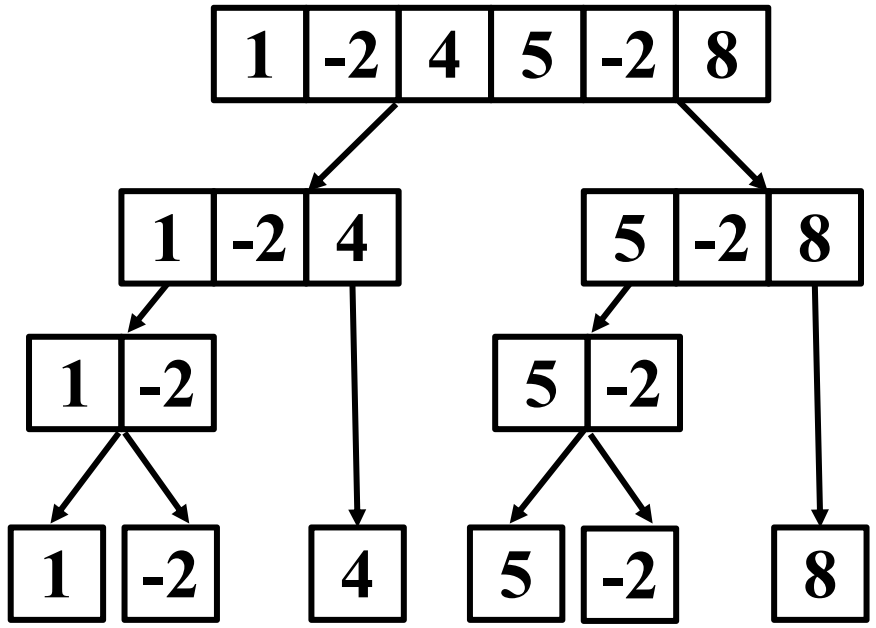
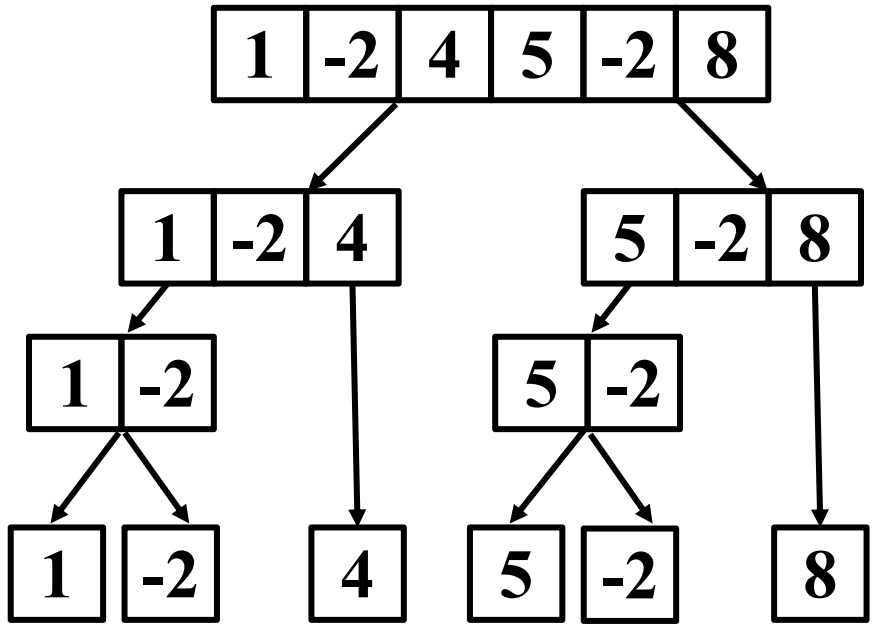


子问题相互独立

算法比较

基于枚举

算法名称	时间复杂度
蛮力枚举	$O(n^3)$
优化枚举	$O(n^2)$
分而治之	$O(n\log n)$
动态规划	$O(n)$



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算法比较

基于枚举

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蛮力枚举	$O(n^3)$
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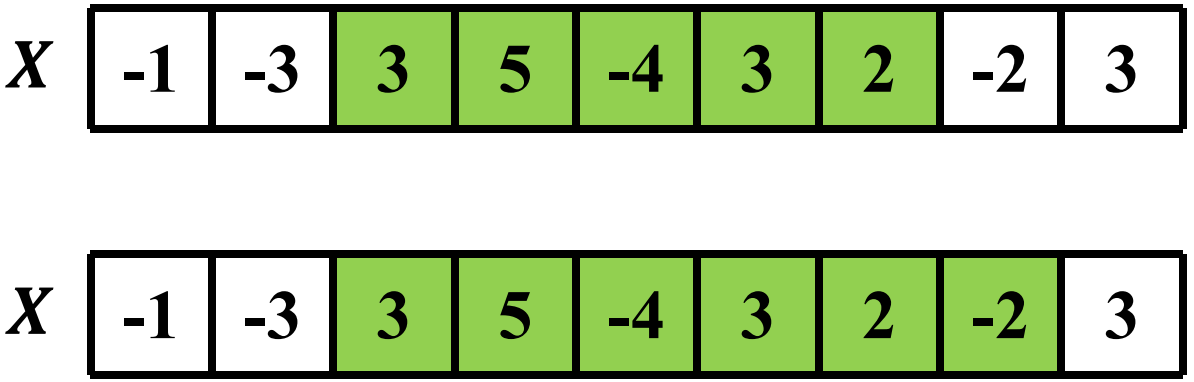
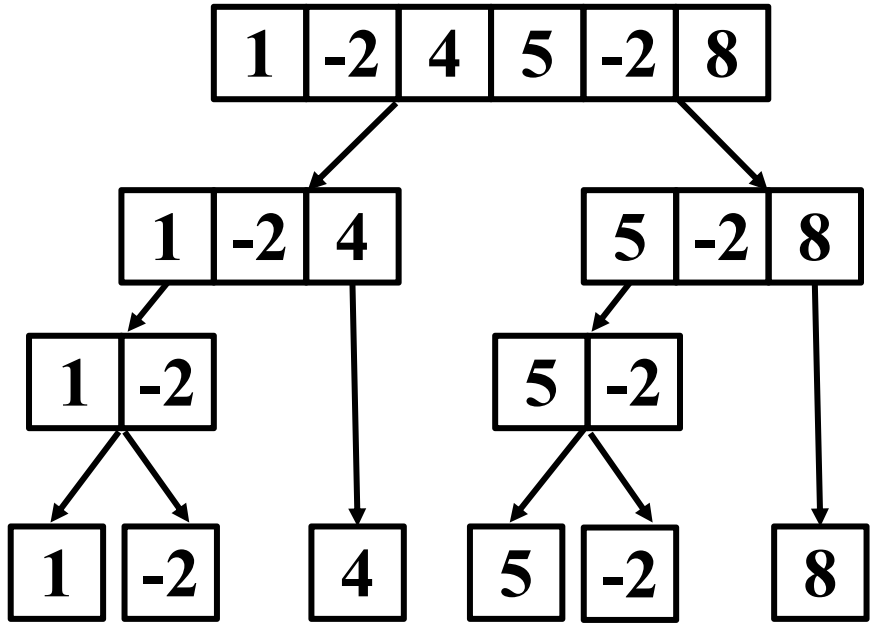
分而治之	$O(n\log n)$
动态规划	$O(n)$

- 枚举区间



存在重叠子问题



子问题相互独立

枚举过程分析

- 两层枚举
  - 第**1**层：枚举位置***i***作为区间开头

枚举过程分析

- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

当前最大值$=4, S(i,j)=4$

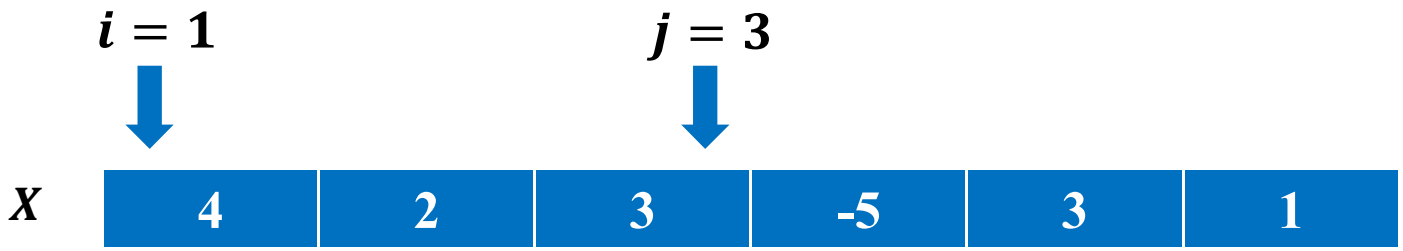
枚举过程分析

- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

当前最大值$= 6, S(i, j) = 6$

枚举过程分析

- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾



当前最大值$= 9, S(i, j) = 9$

枚举过程分析

- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

当前最大值$= 9, S(i, j) = 4$

枚举过程分析

- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

当前最大值$= 9, S(i, j) = 7$

枚举过程分析

- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

当前最大值$= 9, S(i, j) = 8$

枚举过程分析

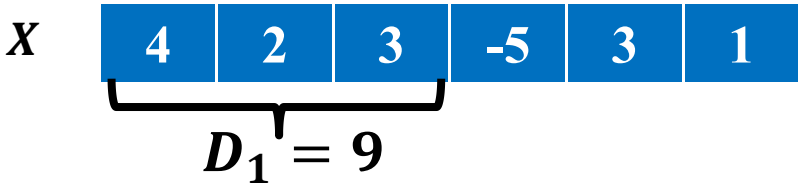
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

当前最大值= 9

D_i：以$X[i]$开头的最大子数组

枚举过程分析

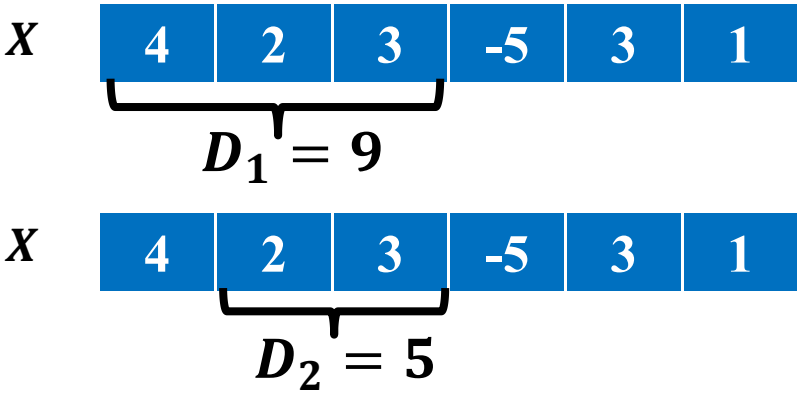
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾



当前最大值$= 9$

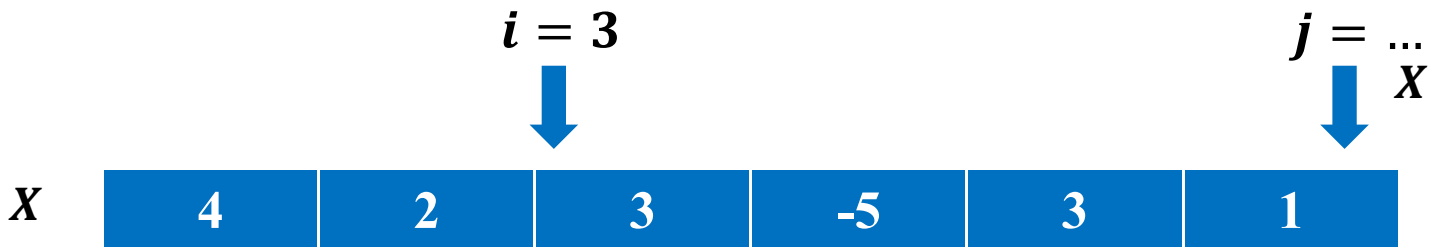
枚举过程分析

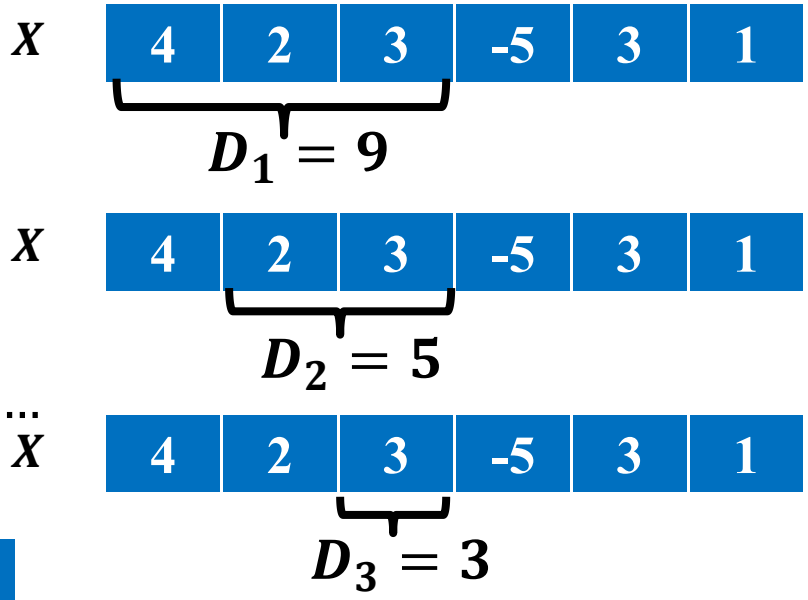
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾



枚举过程分析

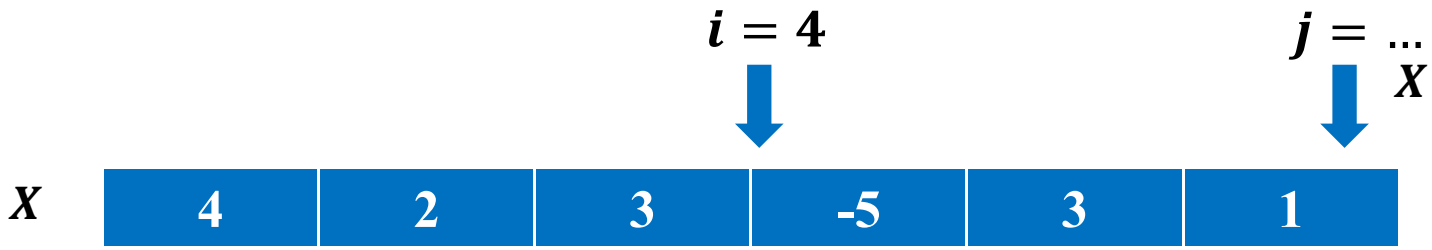
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

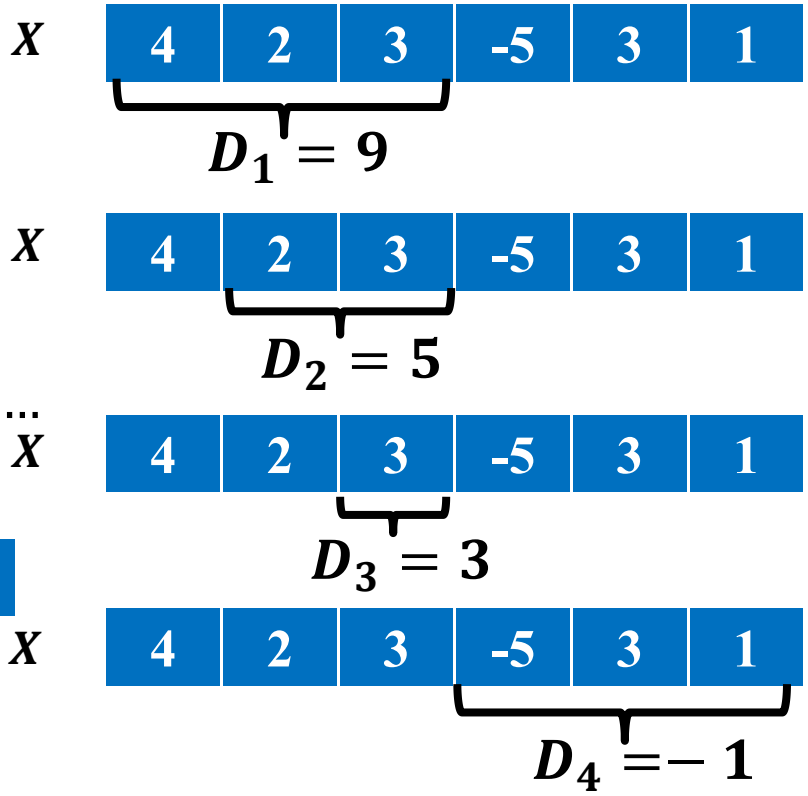




枚举过程分析

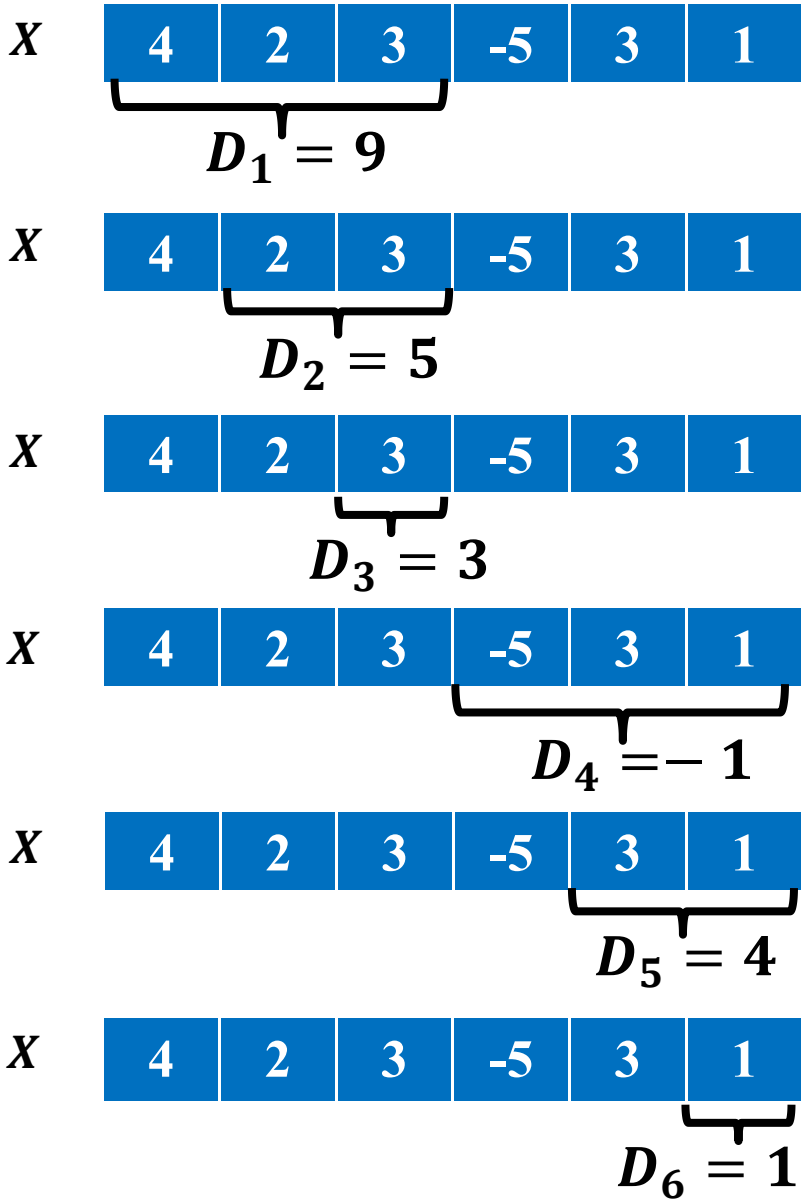
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾





枚举过程分析

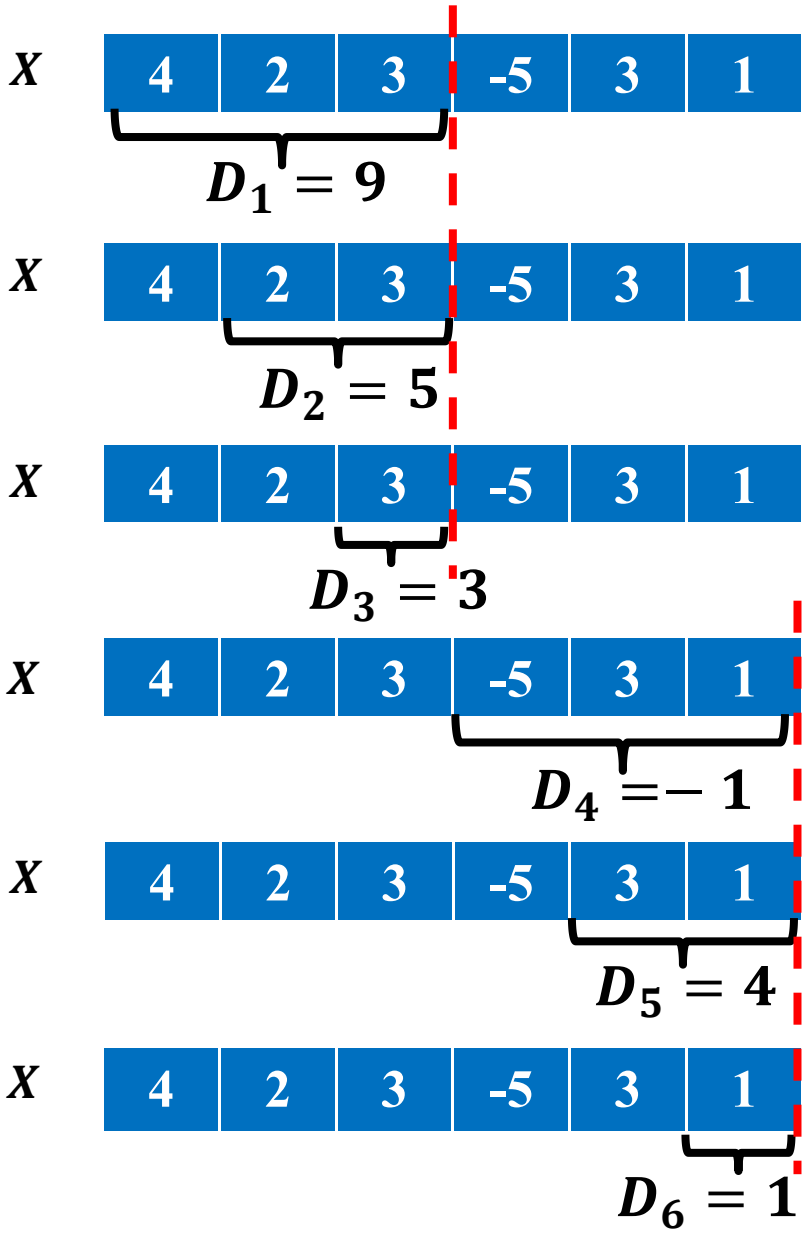
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾



枚举过程分析

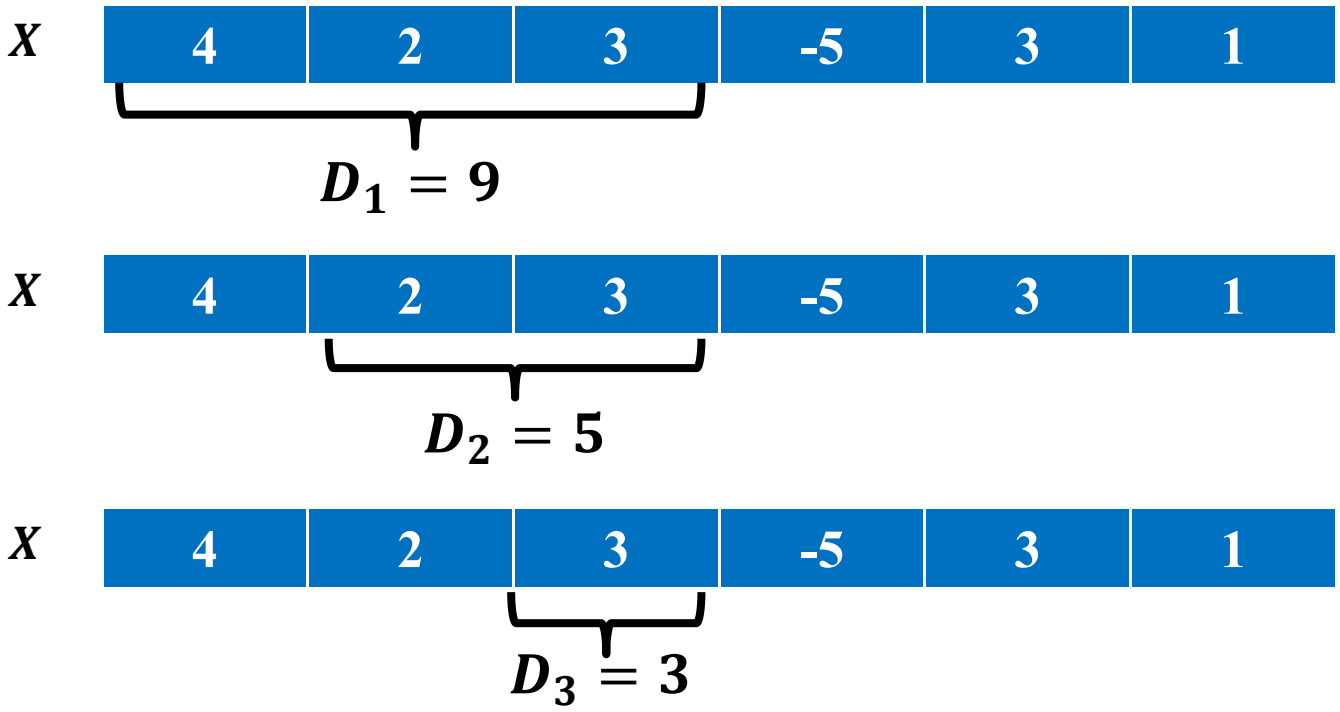
- 两层枚举
  - 第1层：枚举位置i作为区间开头
  - 第2层：枚举位置j作为区间结尾

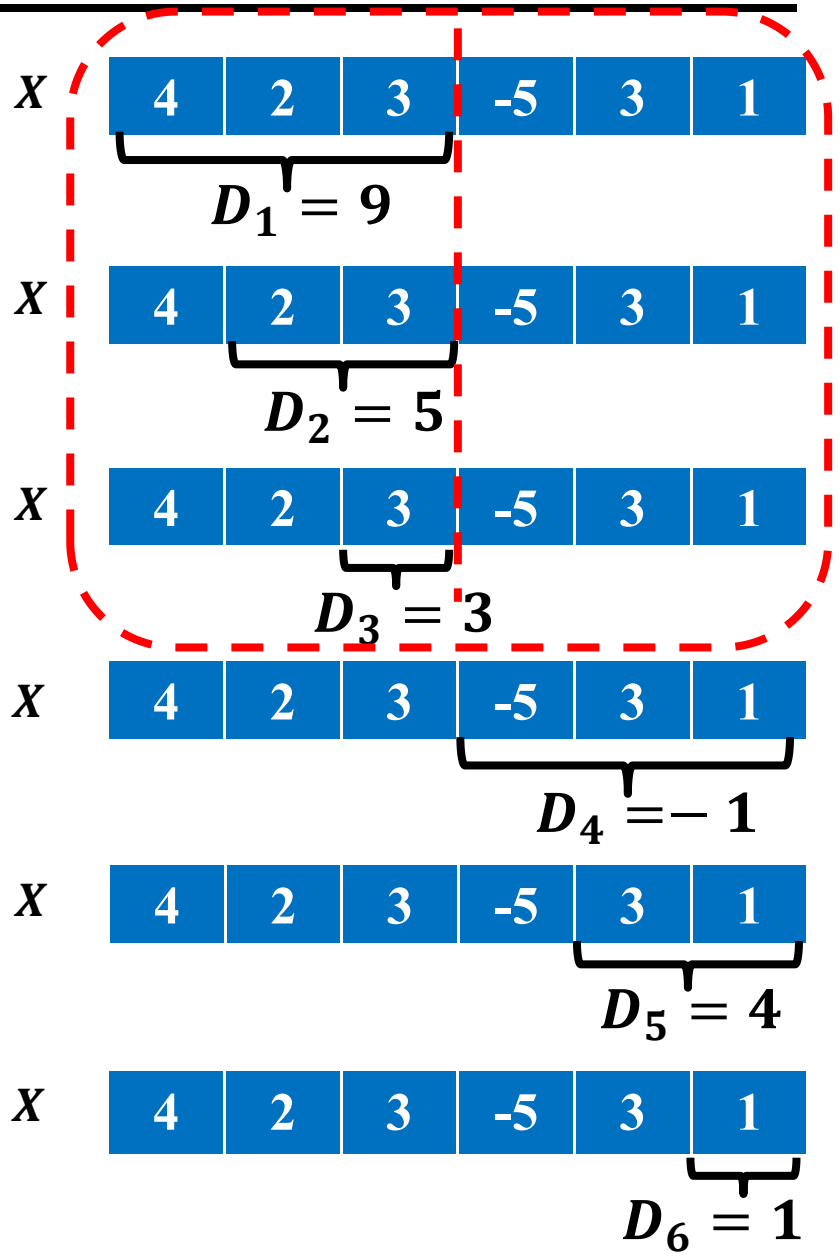
- 观察D_i
  - 结尾相同：D_1, D_2, D_3
  - 结尾不同：D_3, D_4



规律观察

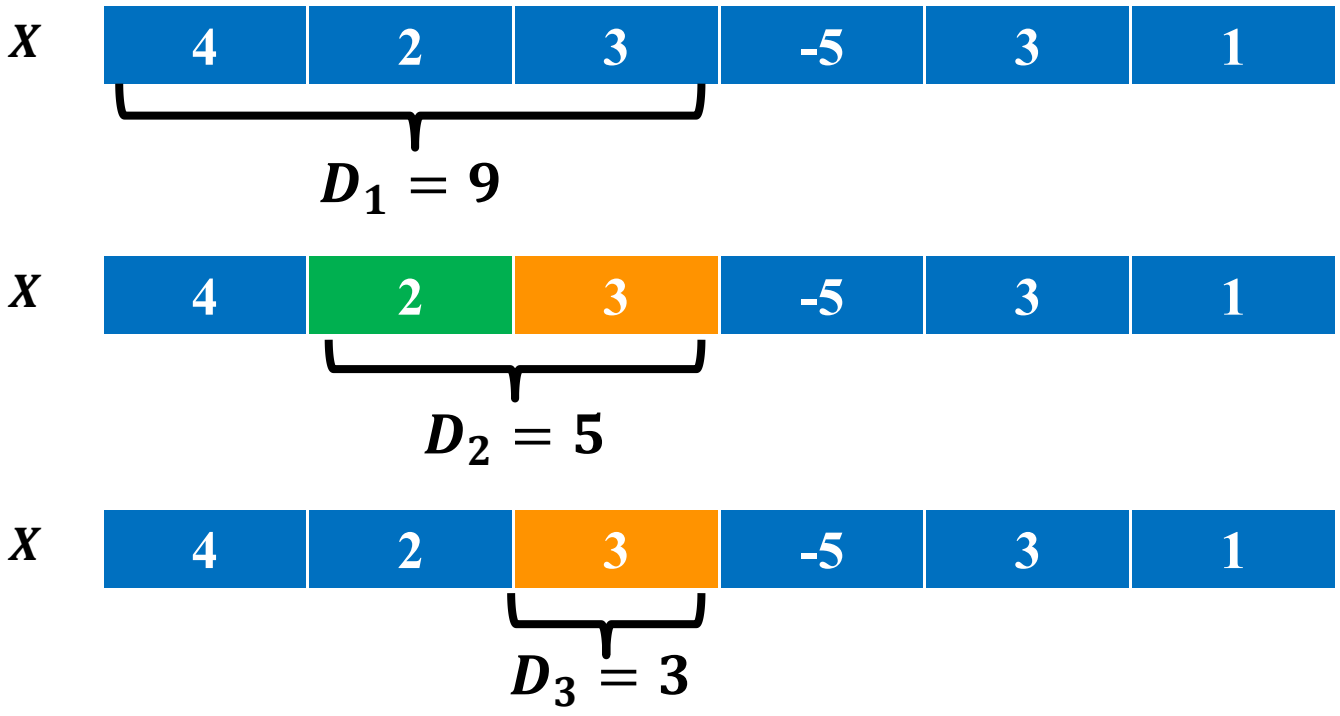
- 结尾位置相同
  - D_1, D_2, D_3

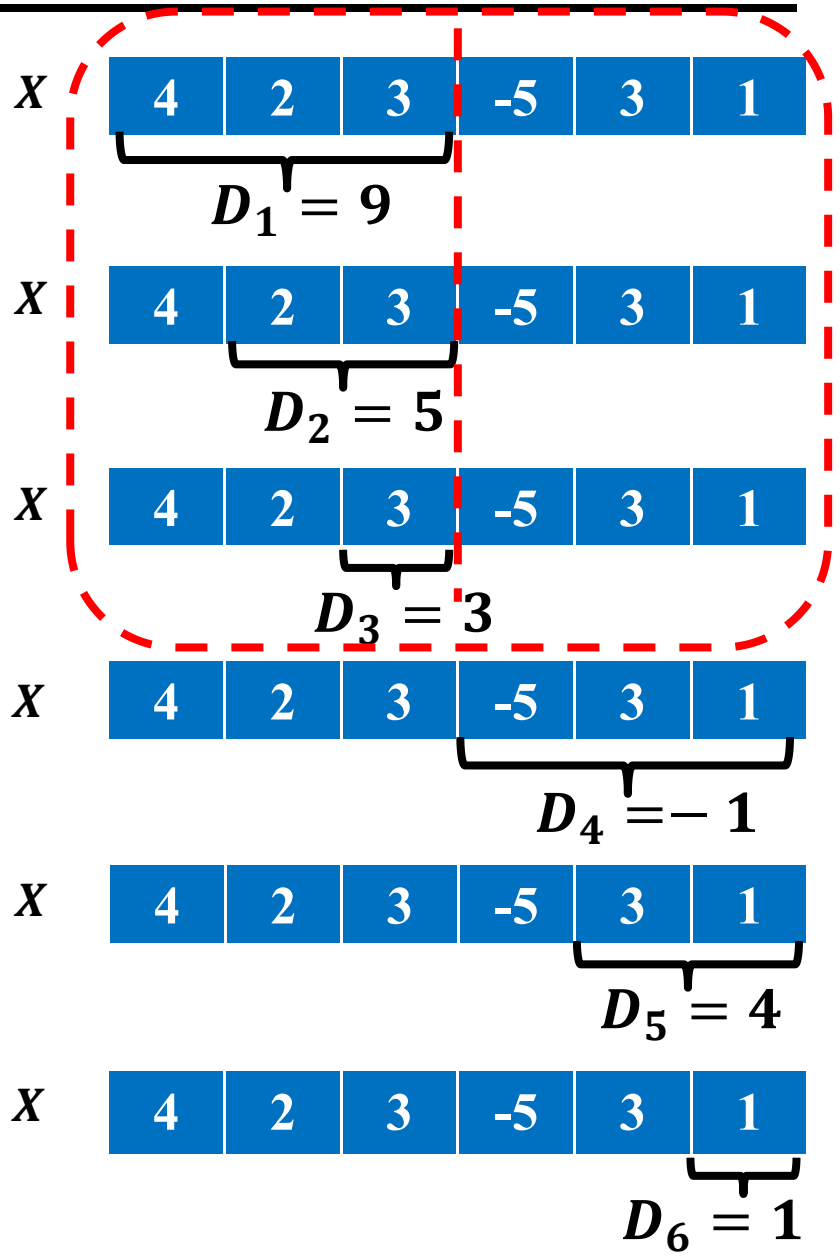




规律观察

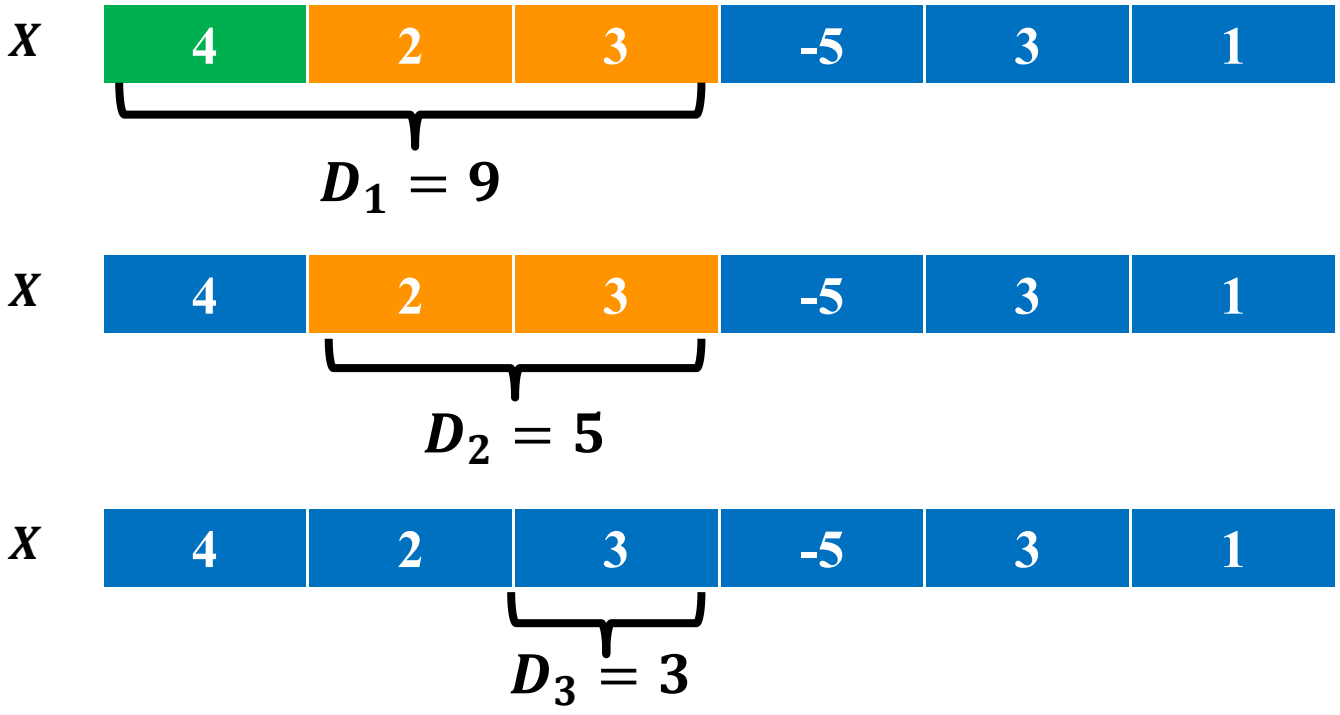
- 结尾位置相同
  - D_1, D_2, D_3
    - $D_2 = X[2] + D_3$

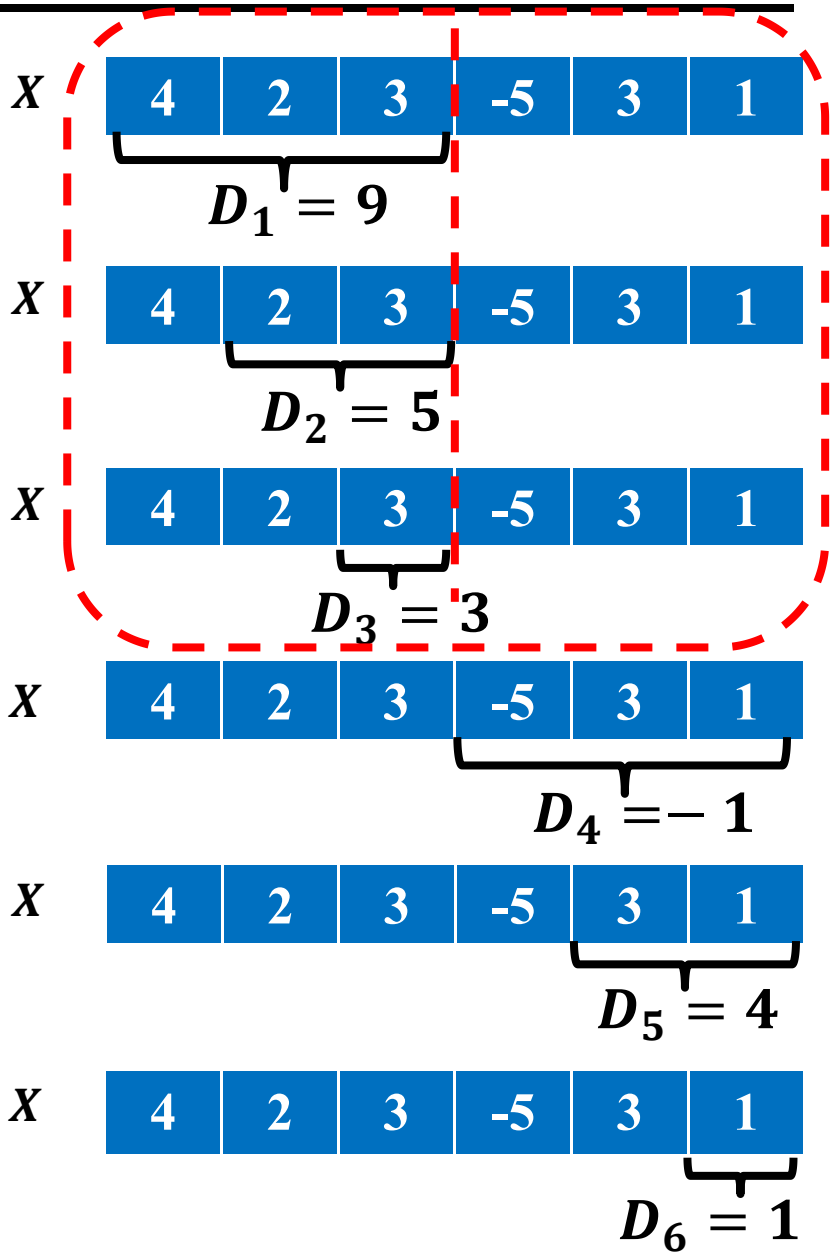




规律观察

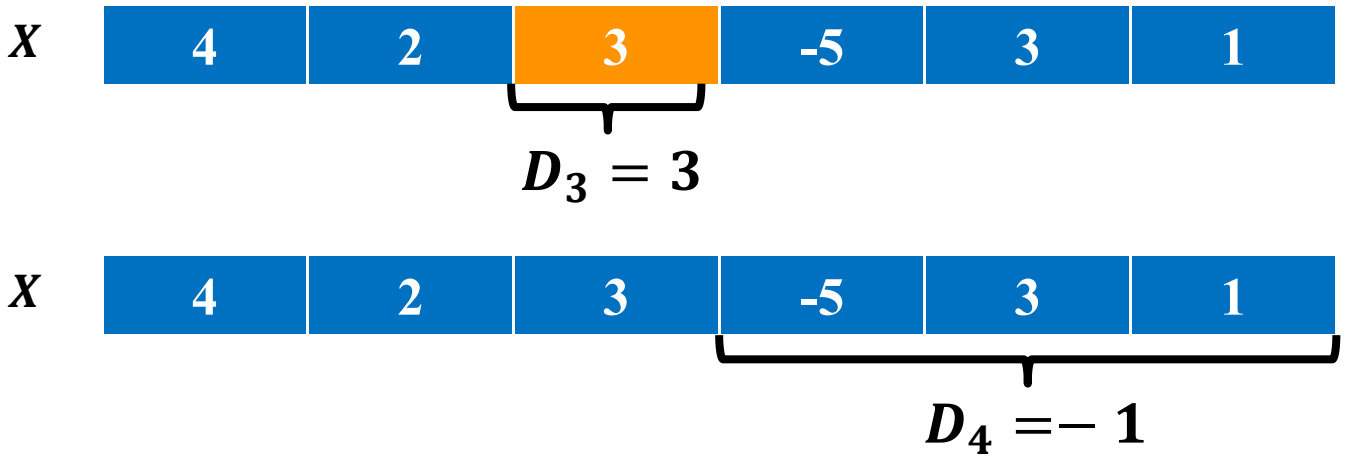
- 结尾位置相同
  - D_1, D_2, D_3
    - $D_2 = X[2] + D_3$
    - $D_1 = X[1] + D_2$

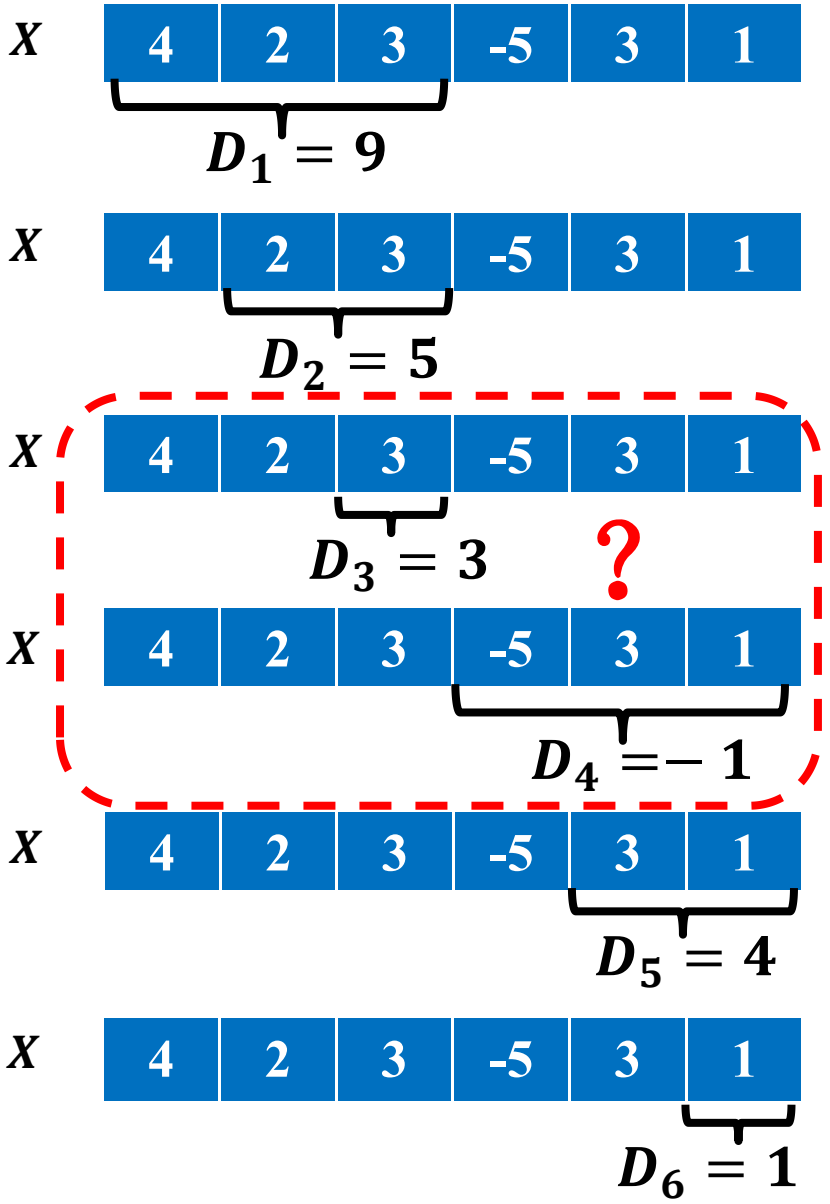




规律观察

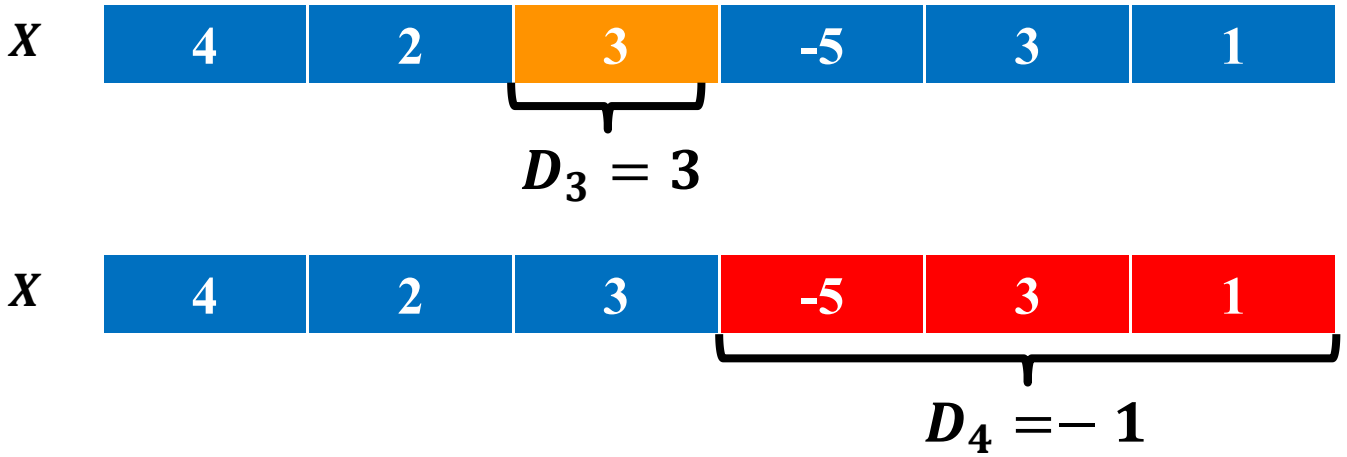
- 结尾位置不同
  - D_3, D_4
    - $D_3 = X[3]$

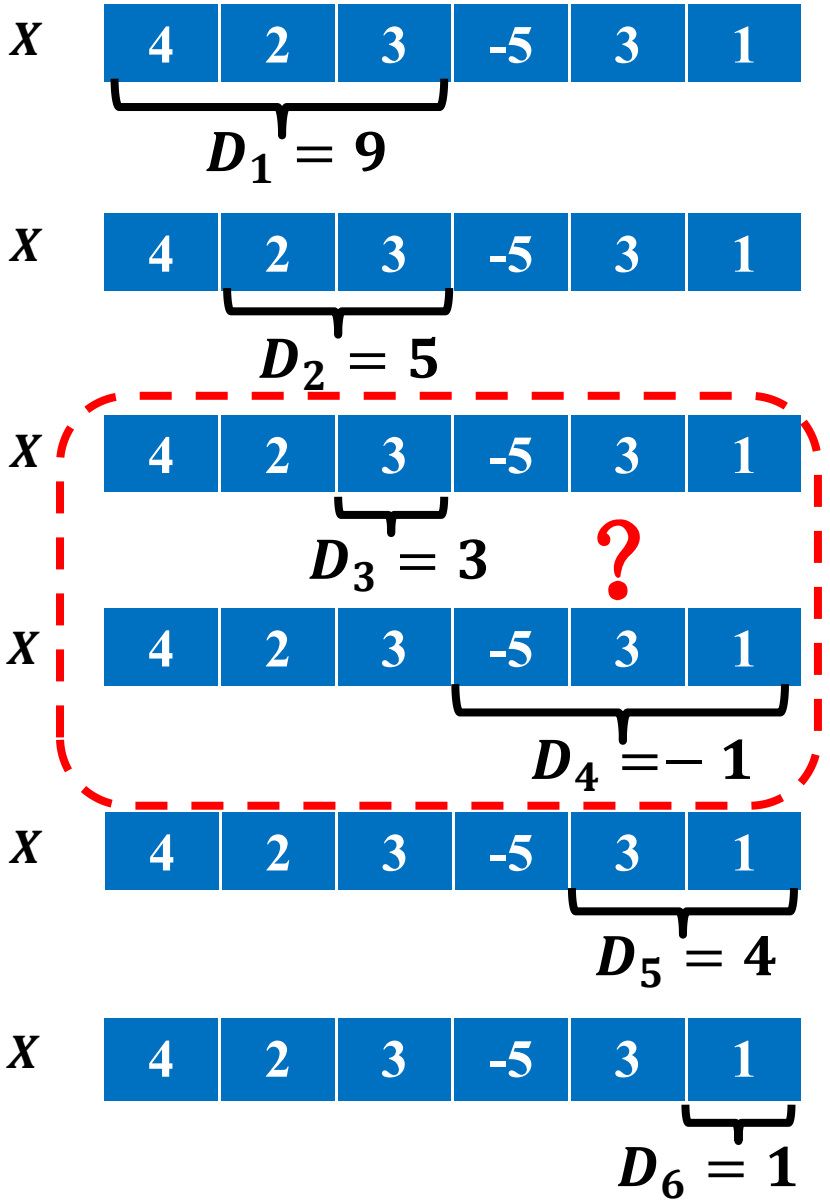




规律观察

- 结尾位置不同
  - D_3, D_4
    - $D_3 = X[3]$
    - $D_4 < 0$

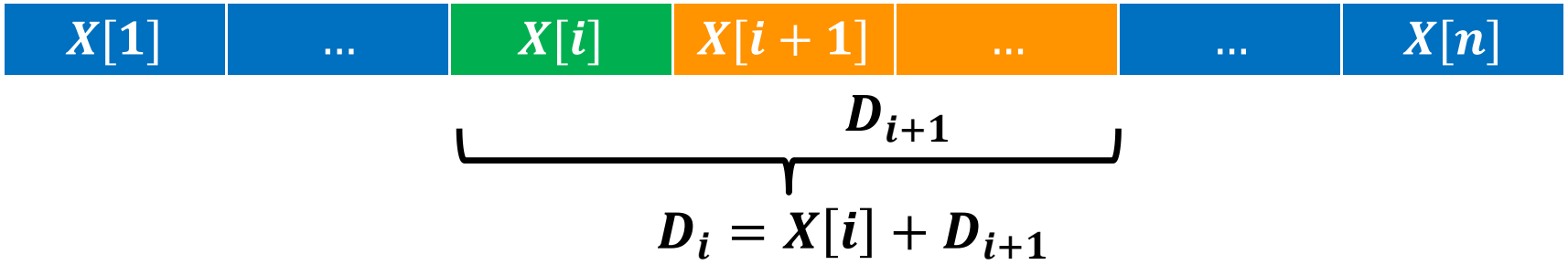




规律描述

- $\boldsymbol{D_i}$：以$\boldsymbol{X[i]}$开头的最大子数组和

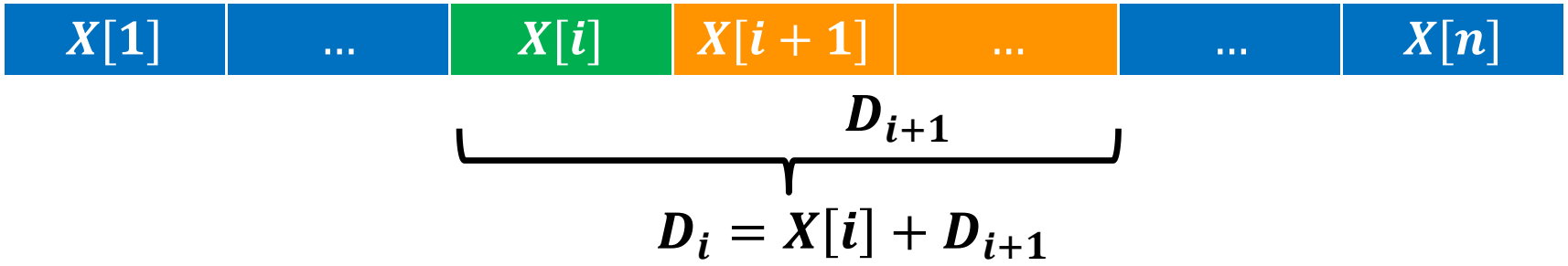
- 情况**1**：$\boldsymbol{D_{i+1} > 0}$时



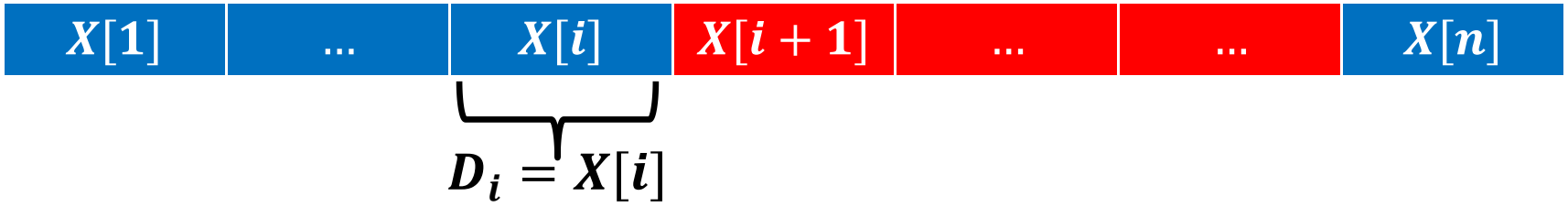
规律描述

- D_i：以$X[i]$开头的最大子数组和

- 情况1：$D_{i+1} > 0$时

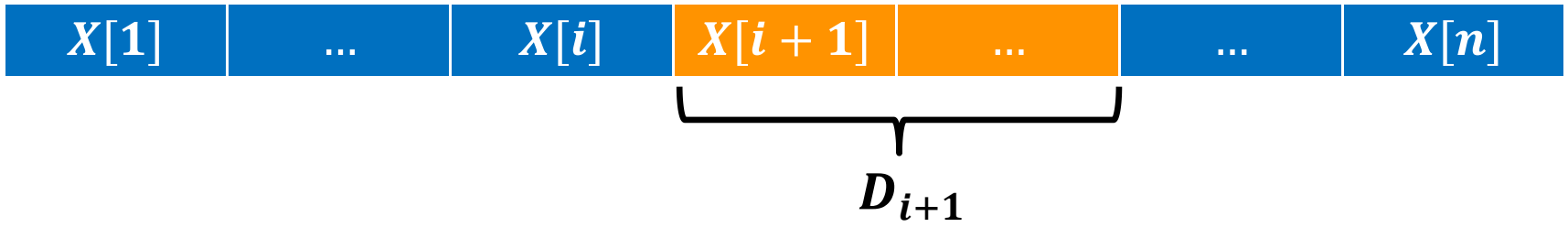


- 情况2：$D_{i+1} \leq 0$时



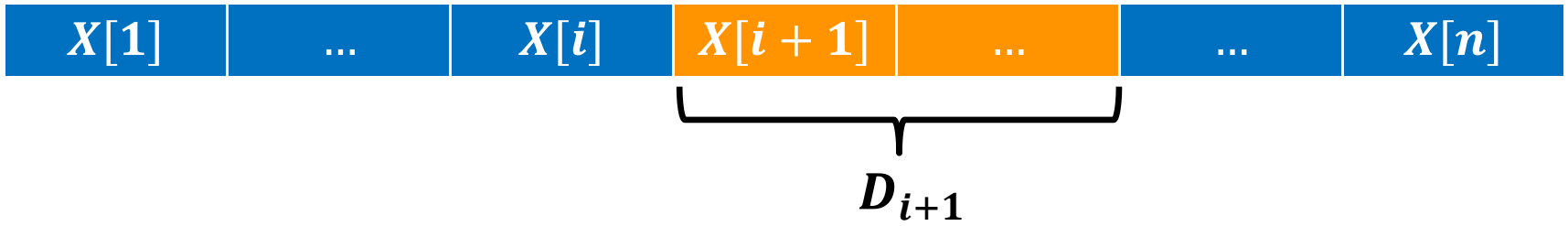
规律证明：情况1

- $\boldsymbol{D_{i+1}}$：以$\boldsymbol{X[i+1]}$开头的最大子数组和（$\boldsymbol{D_{i+1} > 0}$）

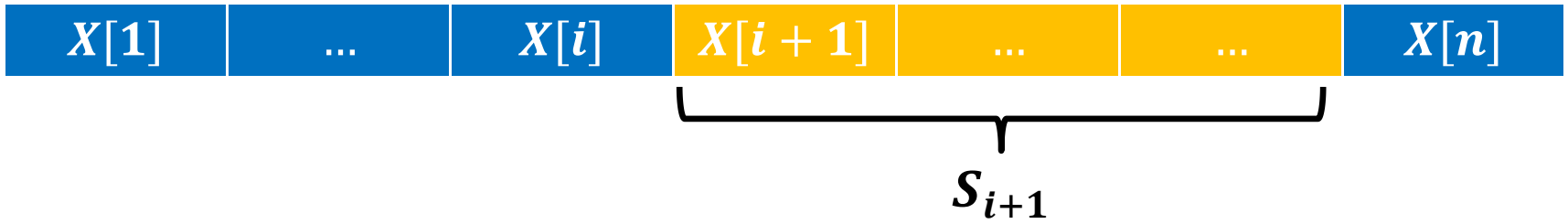


规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）

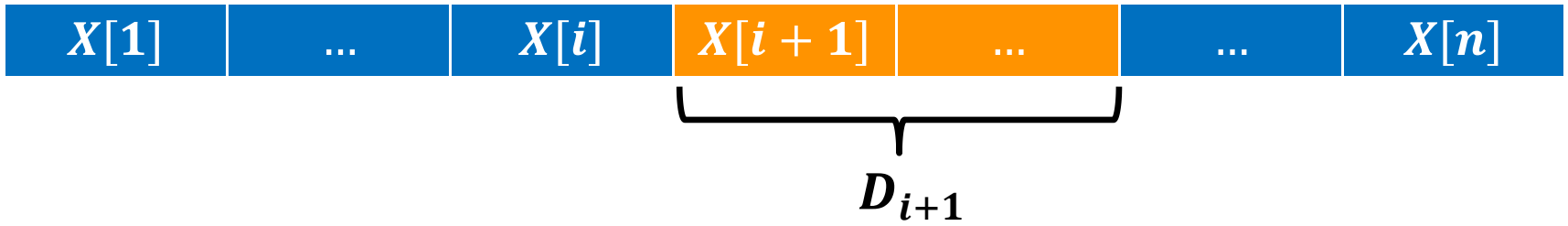


- S_{i+1}：以$X[i+1]$开头的任一子数组和

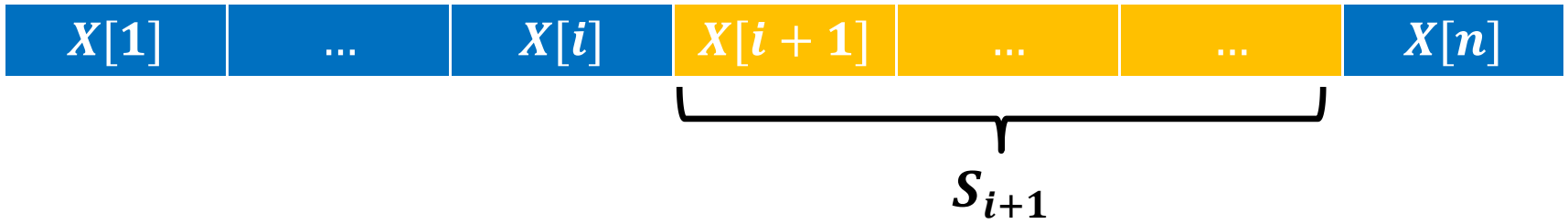


规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）

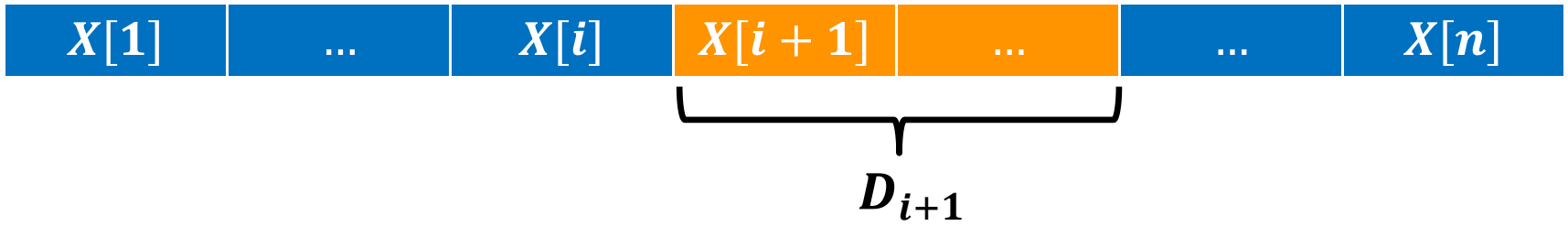


- S_{i+1}：以$X[i+1]$开头的任一子数组和

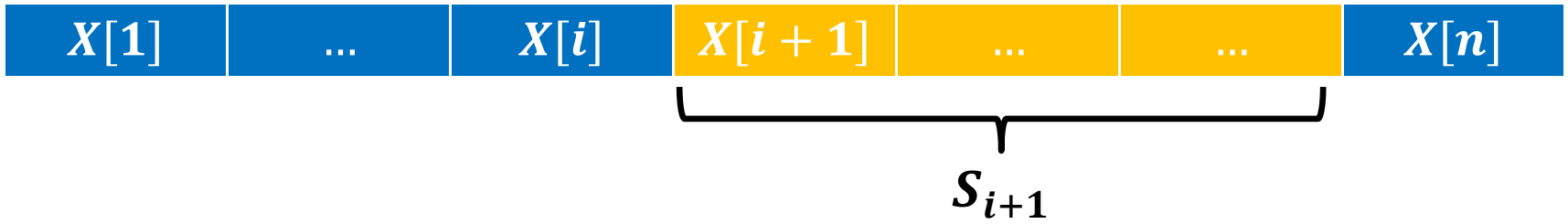


规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）

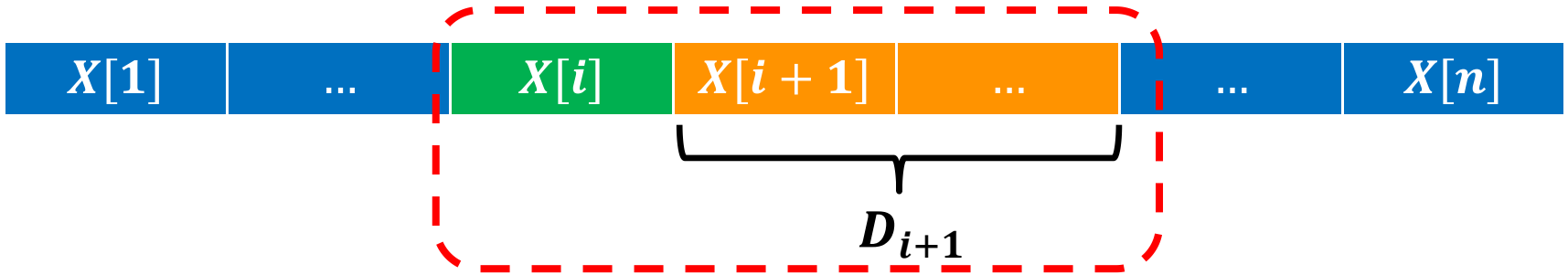


- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）

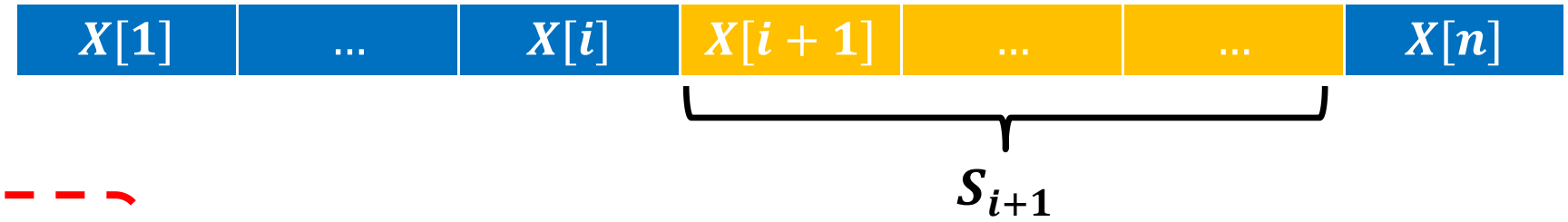


规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）



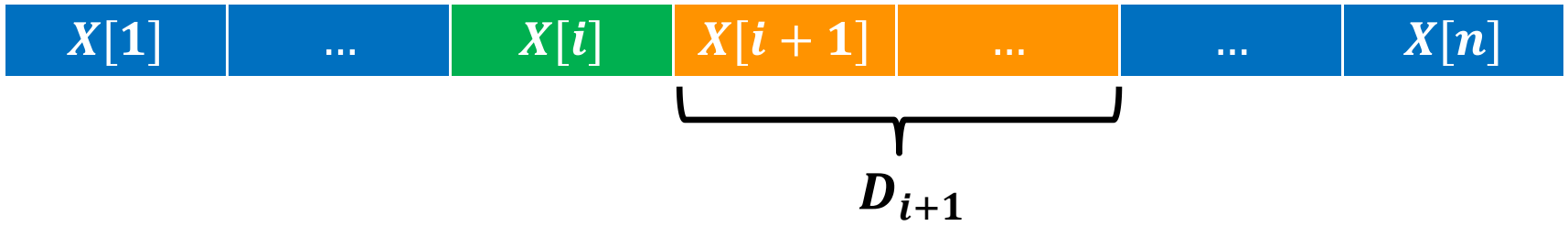
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



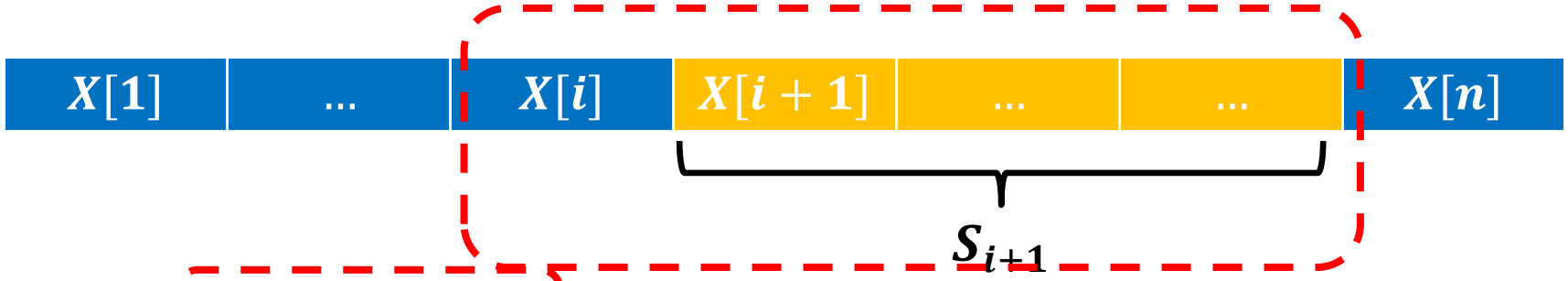
- $X[i] + D_{i+1}$

规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）



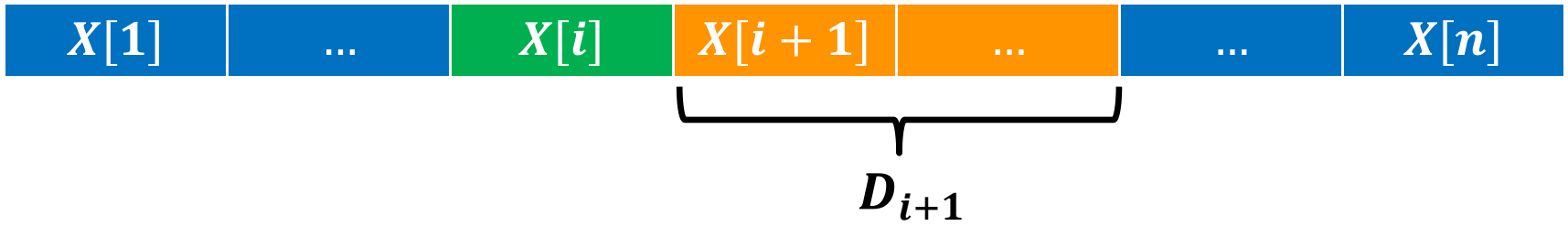
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



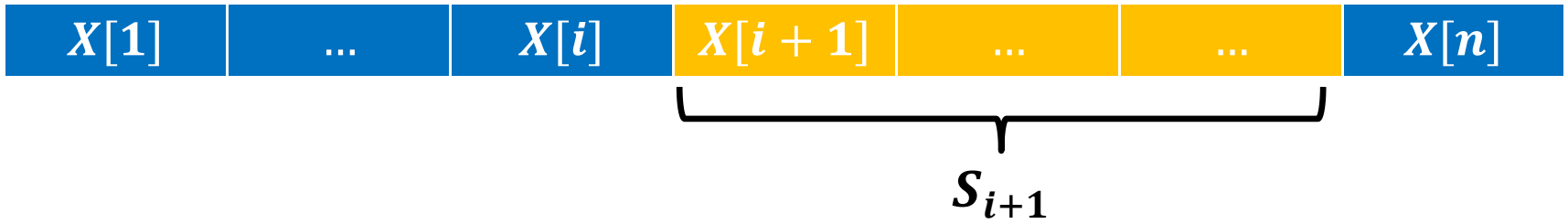
- $X[i] + D_{i+1}$ $X[i] + S_{i+1}$

规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）



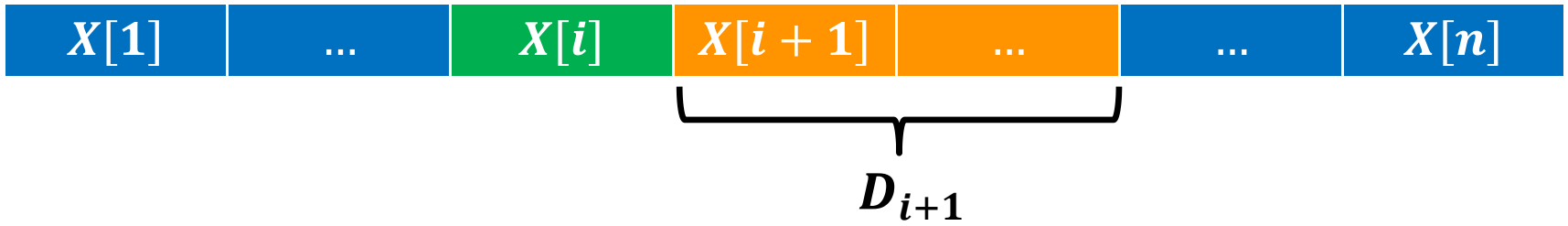
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



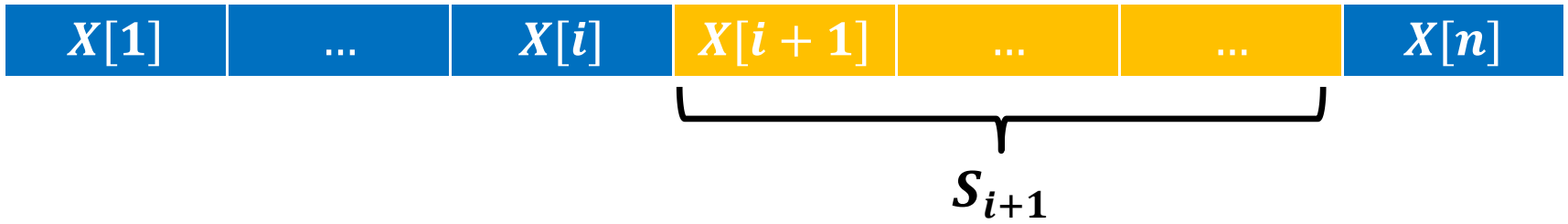
- $X[i] + D_{i+1} \geq X[i] + S_{i+1}$

规律证明：情况1

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} > 0$）



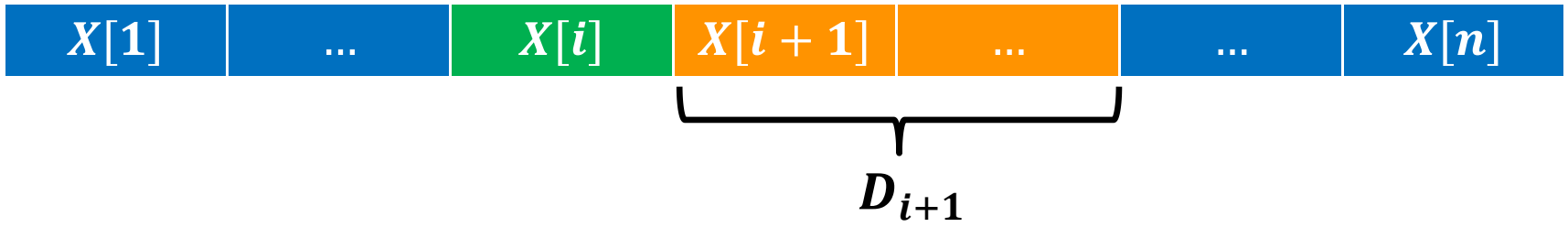
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



- $X[i] + D_{i+1} \geq X[i] + S_{i+1}$
- $D_i = X[i] + D_{i+1}$

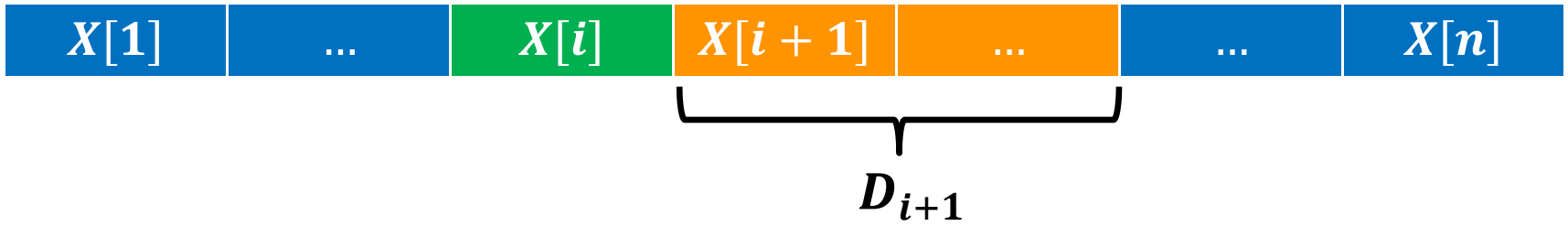
规律证明：情况2

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} \leq 0$）

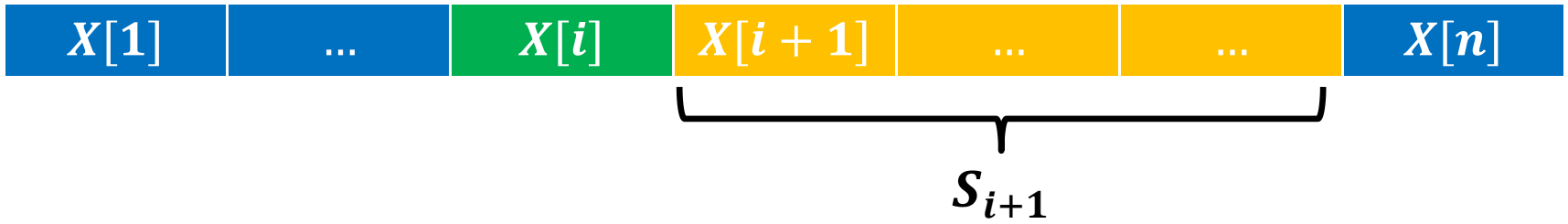


规律证明：情况2

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} \leq 0$）

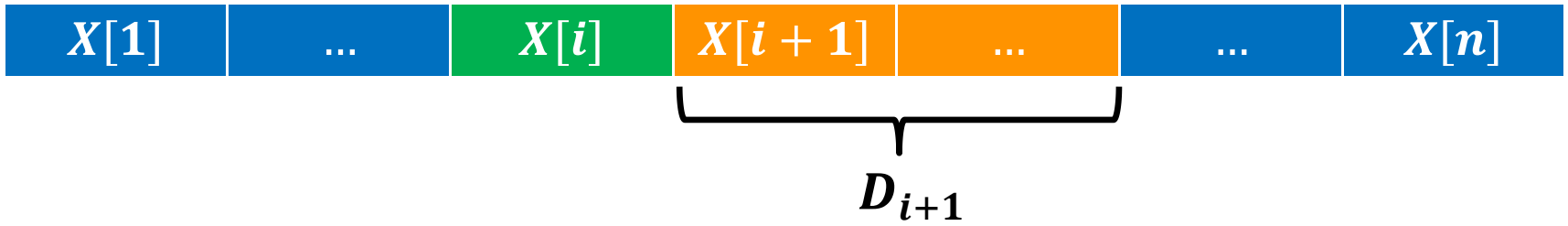


- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）

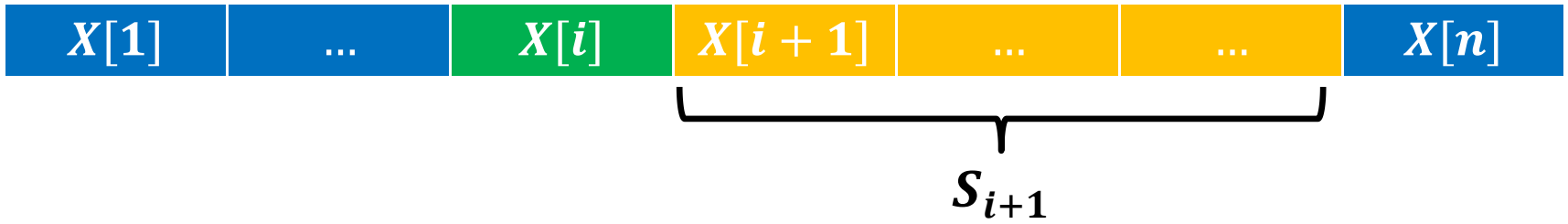


规律证明：情况2

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} \leq 0$）



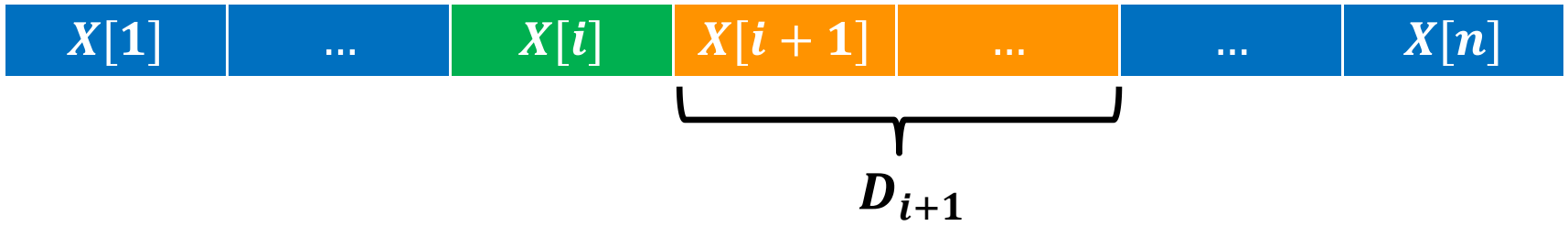
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



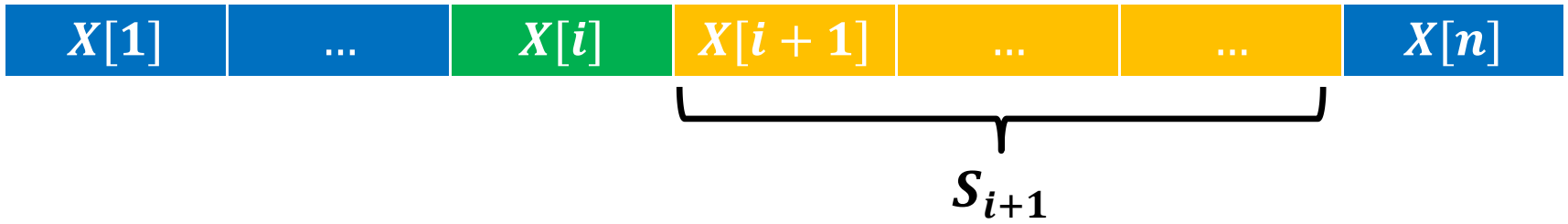
- $X[i] + S_{i+1} \leq X[i] + D_{i+1}$

规律证明：情况2

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} \leq 0$）



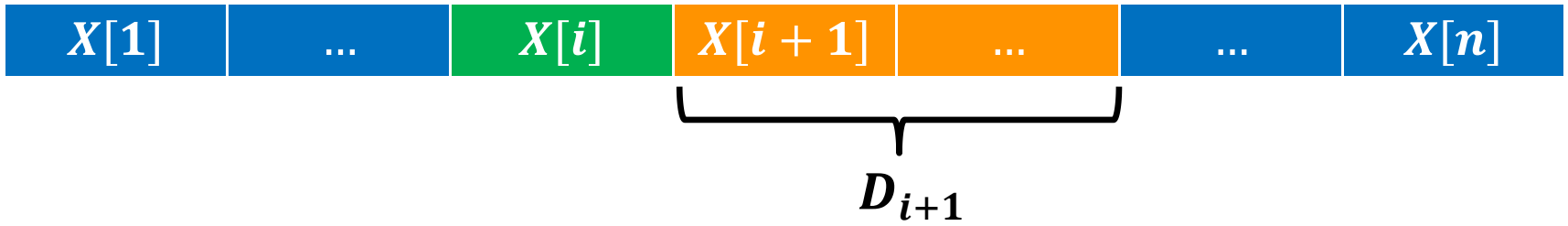
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



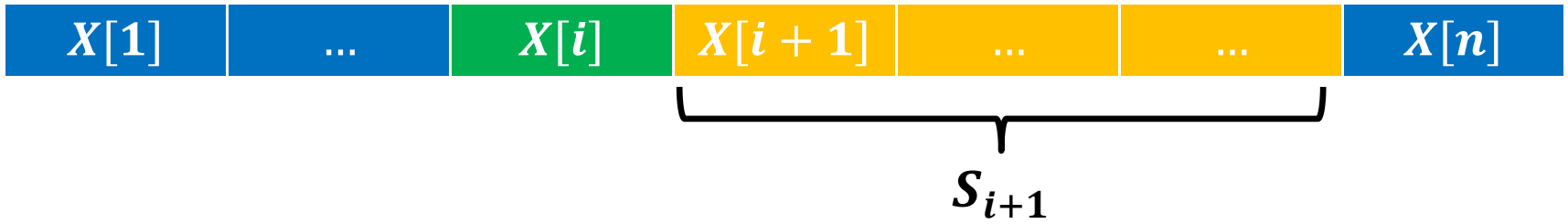
- $X[i] + S_{i+1} \leq X[i] + D_{i+1} \leq X[i]$

规律证明：情况2

- D_{i+1}：以$X[i+1]$开头的最大子数组和（$D_{i+1} \leq 0$）



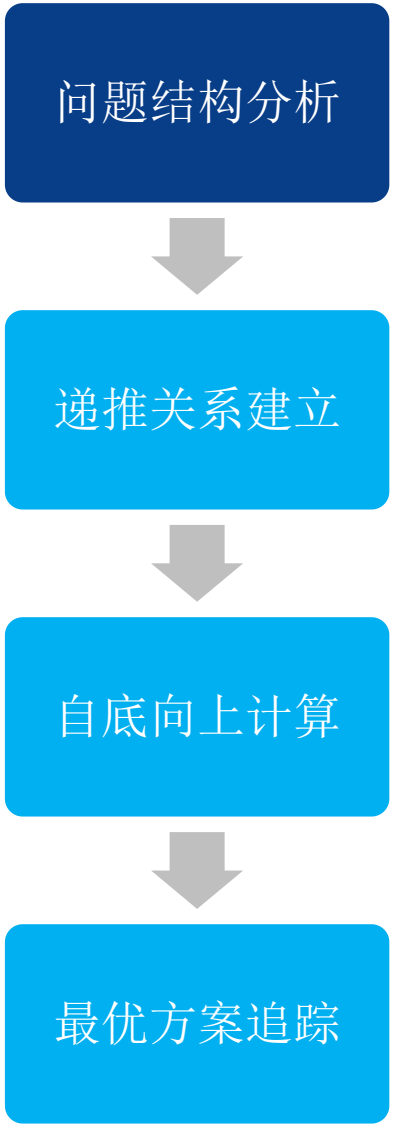
- S_{i+1}：以$X[i+1]$开头的任一子数组和（$S_{i+1} \leq D_{i+1}$）



- $X[i] + S_{i+1} \leq X[i] + D_{i+1} \leq X[i]$
- $D_i = X[i]$

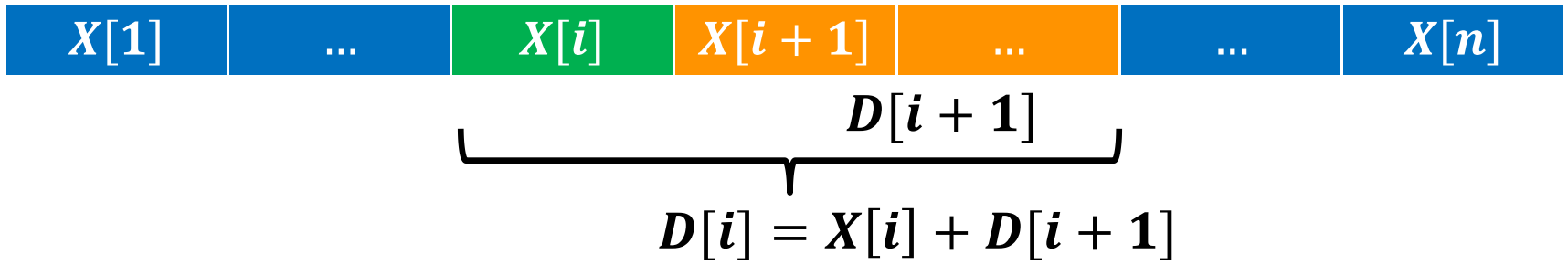
问题结构分析

- 给出问题表示
  - $\boldsymbol{D[i]}$：以$\boldsymbol{X[i]}$开头的最大子数组和
- 明确原始问题
  - $\boldsymbol{S_{max} = \max\limits_{1 \le i \le n} \{D[i]\}}$

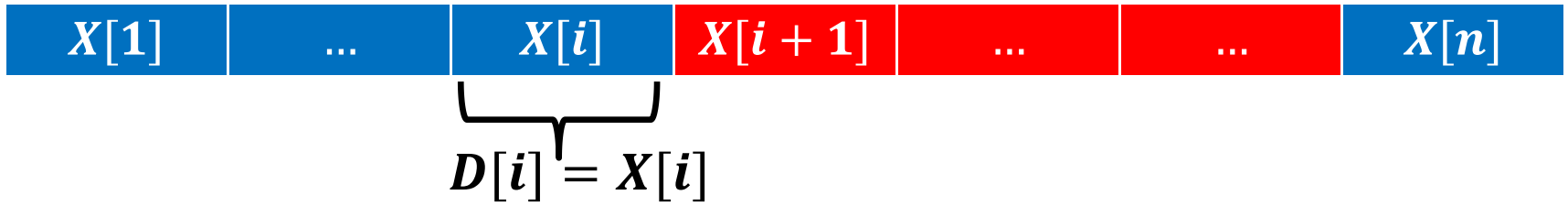


递推关系建立：分析最优（子）结构

- $D[i]$：以$X[i]$开头的最大子数组和
- 情况1：$D[i+1] > 0$



- 情况2：$D[i+1] \leq 0$



问题结构分析

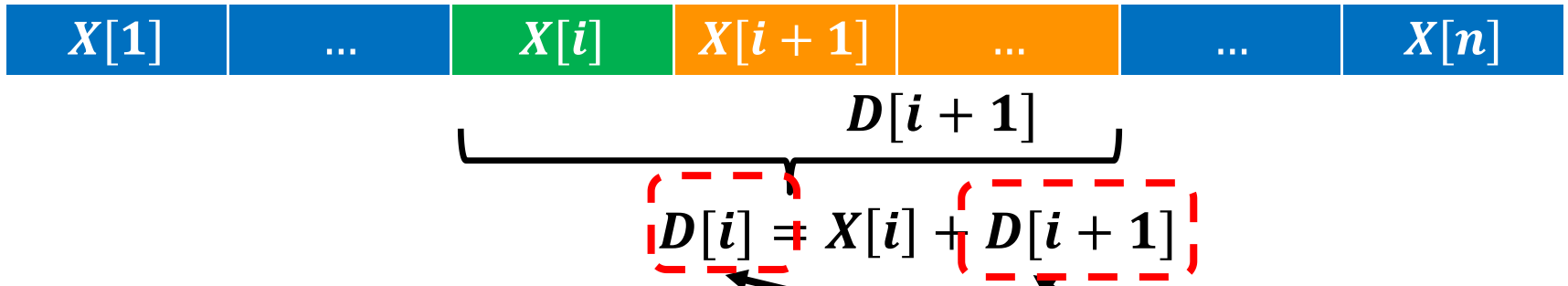
递推关系建立

自底向上计算

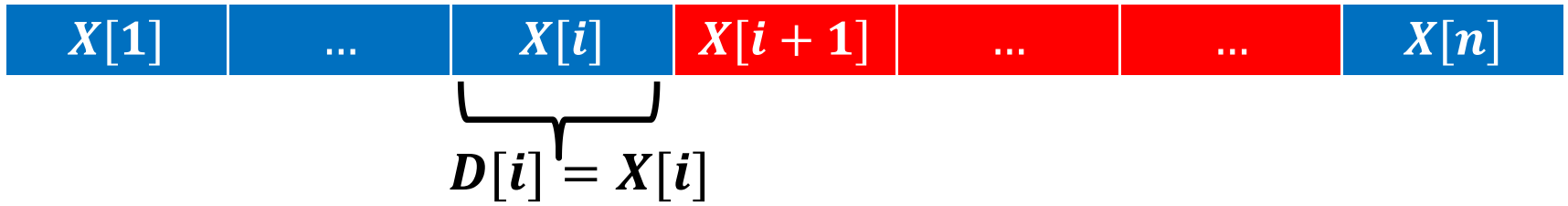
最优方案追踪

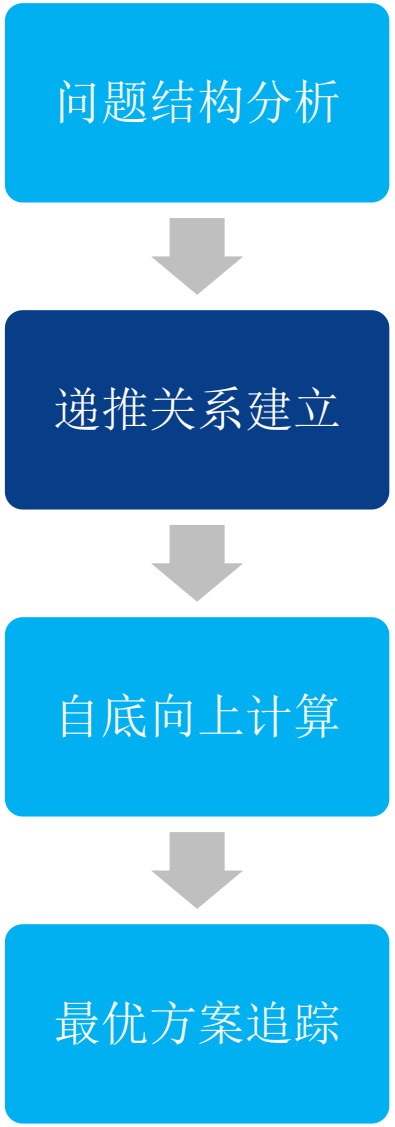
递推关系建立：分析最优（子）结构

- $D[i]$：以$X[i]$开头的最大子数组和
- 情况1：$D[i+1] > 0$

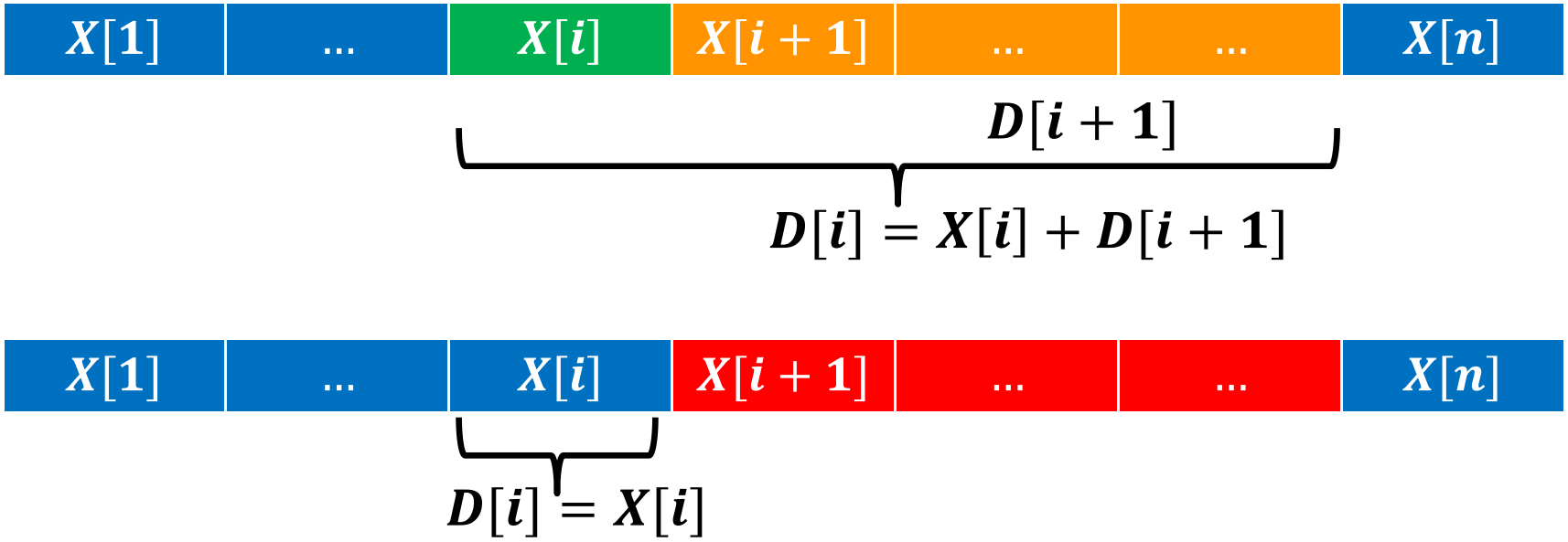


- 情况2：$D[i+1] \leq 0$





递推关系建立：构造递推公式



- $D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$

问题结构分析

递推关系建立

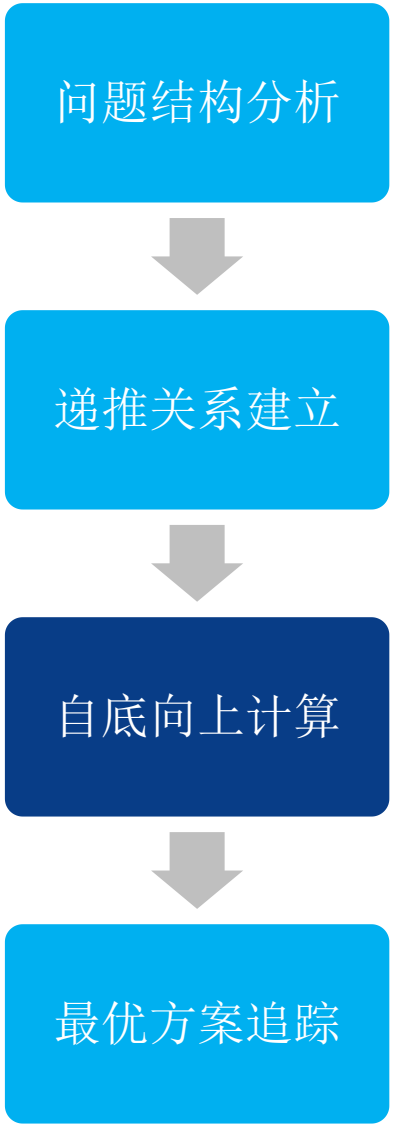
自底向上计算

最优方案追踪

自底向上计算：确定计算顺序

已知

	1	2	…	i	i+1	…	$n-1$	n
X	$X[1]$	$X[2]$		$X[i]$				$X[n]$



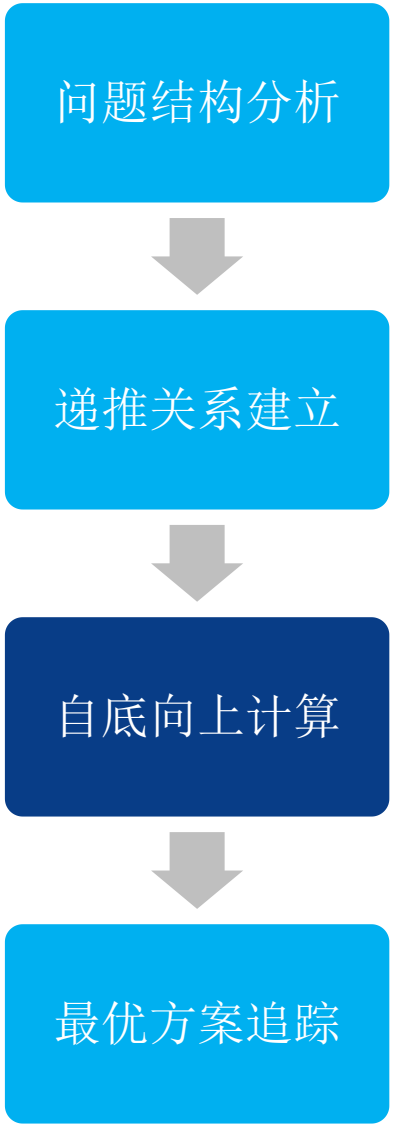
自底向上计算：确定计算顺序

- 初始化
  - $D[n] = X[n]$

已知

	1	2	…	i	i+1	…	$n-1$	n
X	$X[1]$	$X[2]$		$X[i]$				$X[n]$

	1	2	…	i	i+1	…	$n-1$	n
D								$X[n]$



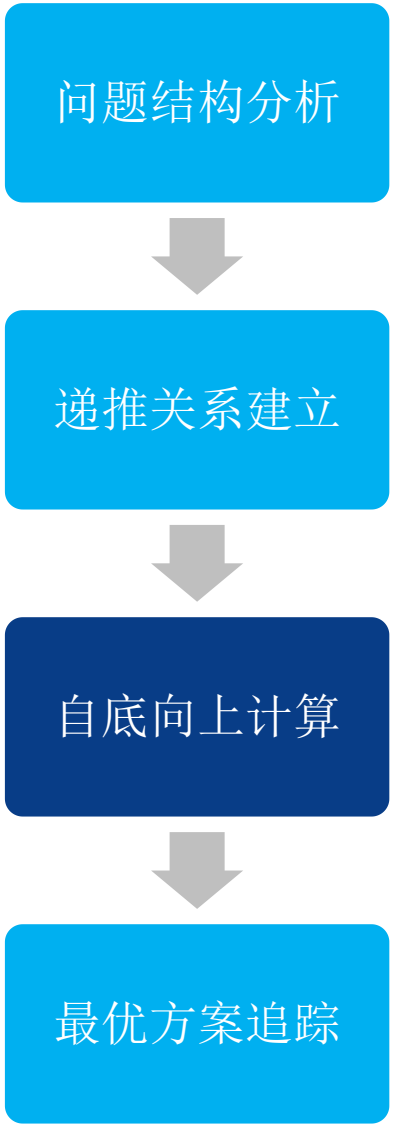
自底向上计算：确定计算顺序

- 初始化
  - $D[n] = X[n]$
- 递推公式
  - $$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

已知

	1	2	…	i	i+1	…	$n-1$	n
X	$X[1]$	$X[2]$		$X[i]$				$X[n]$

	1	2	…	i	i+1	…	$n-1$	n
D								$X[n]$



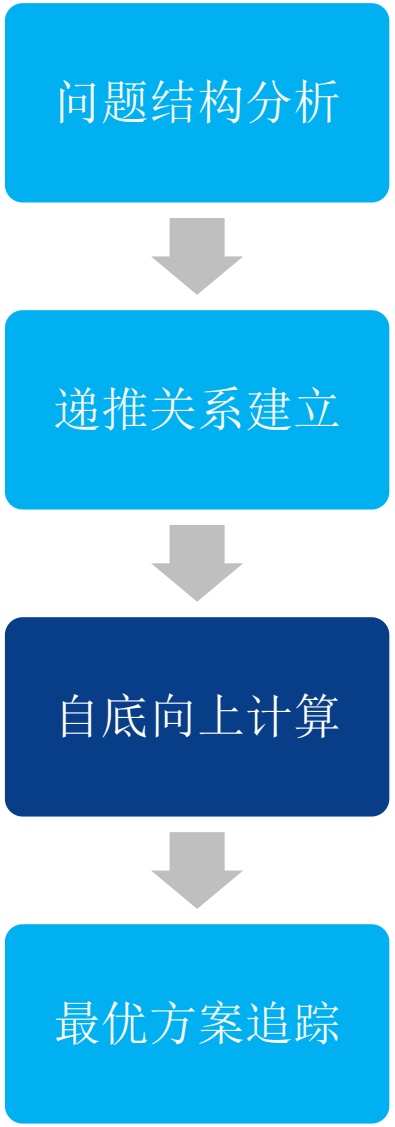
自底向上计算：确定计算顺序

- 初始化
  - $D[n] = X[n]$
- 递推公式
  - $$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

已知

	1	2	…	i	i+1	…	$n-1$	n
X	$X[1]$	$X[2]$		$X[i]$				$X[n]$

	1	2	…	i	i+1	…	$n-1$	n
D					$D[i+1]$			$X[n]$



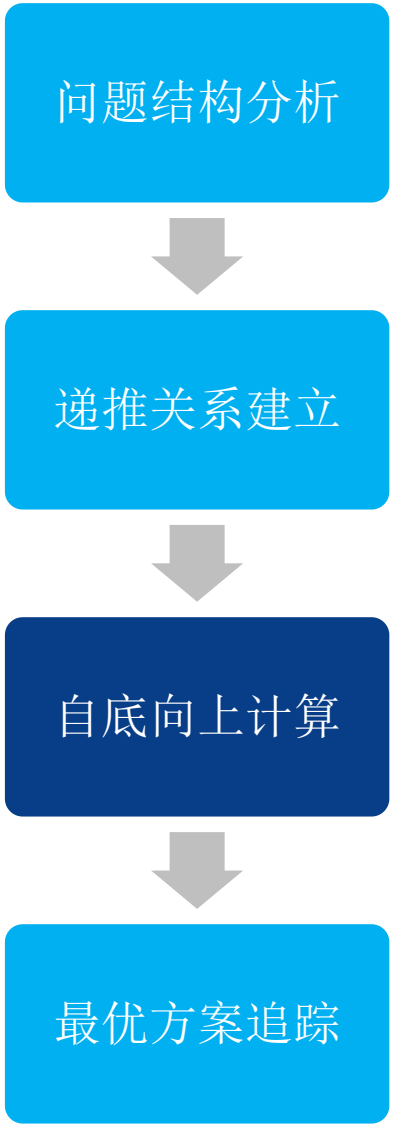
自底向上计算：确定计算顺序

- 初始化
  - $D[n] = X[n]$
- 递推公式
  - $$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

已知

	1	2	…	i	i+1	…	$n-1$	n
X	$X[1]$	$X[2]$		$X[i]$				$X[n]$

	1	2	…	i	i+1	…	$n-1$	n
D				$D[i]$	$D[i+1]$			$X[n]$



自底向上计算：依次求解问题

- 初始化
  - $D[n] = X[n]$
- 递推公式
  - $$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

已知

	1	2	…	i	i+1	…	$n-1$	n
X	$X[1]$	$X[2]$		$X[i]$				$X[n]$

	1	2	…	i	i+1	…	$n-1$	n
D	←							$X[n]$

自底向上计算

问题结构分析

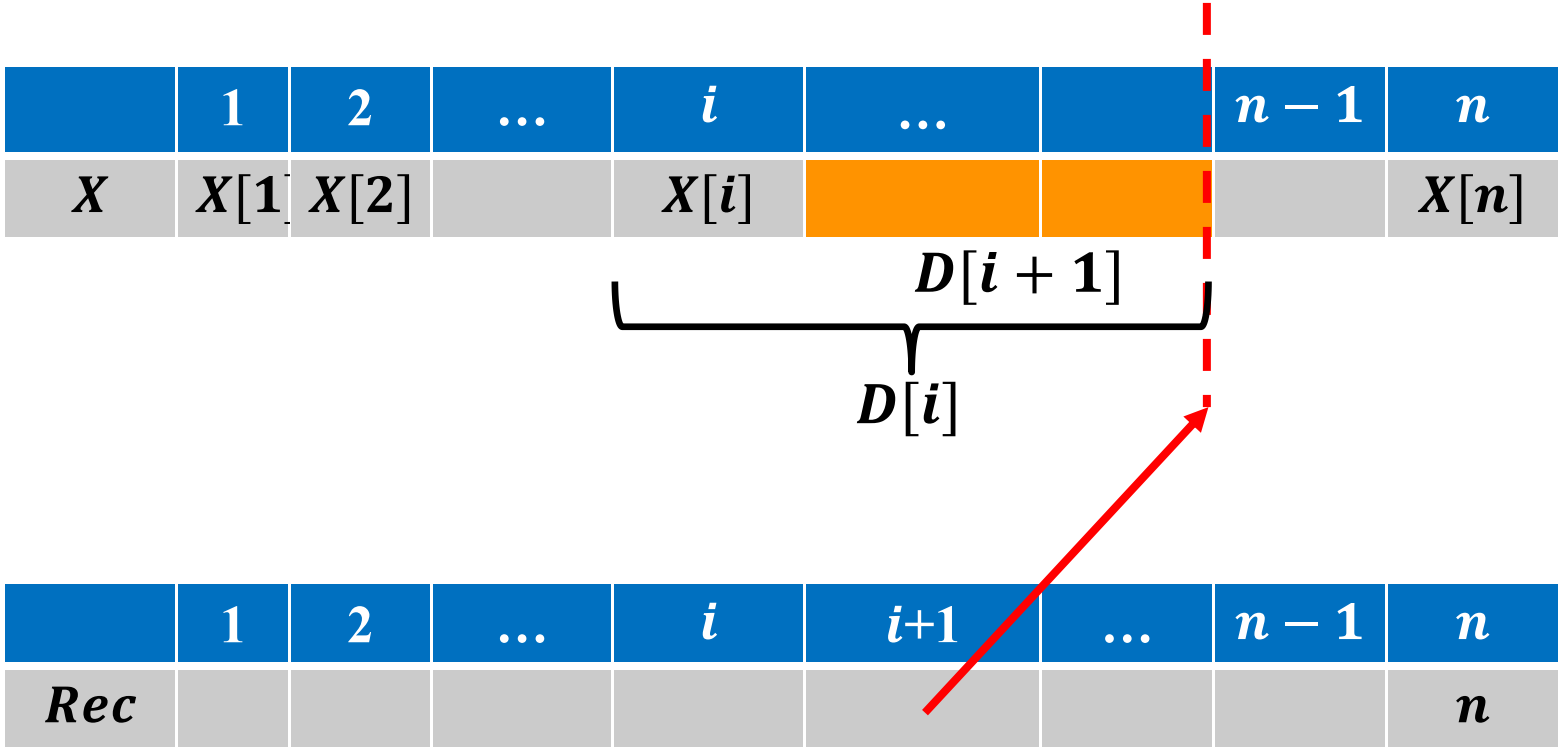
递推关系建立

自底向上计算

最优方案追踪

最优方案追踪：记录决策过程

- 构造追踪数组$Rec[1..n]$
  - 情况1：结尾相同
    - $Rec[i] = Rec[i+1]$



问题结构分析

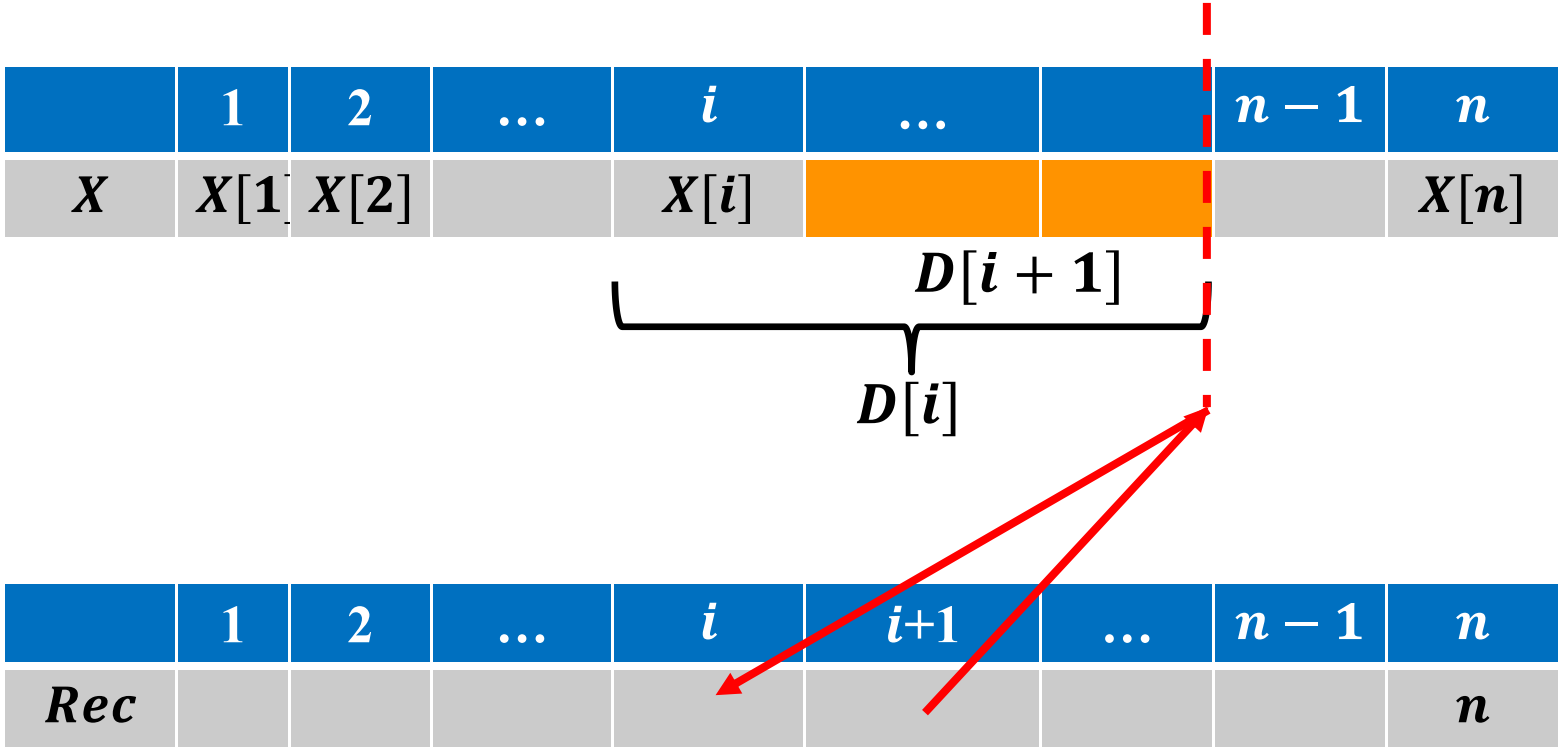
递推关系建立

自底向上计算

最优方案追踪

最优方案追踪：记录决策过程

- 构造追踪数组$Rec[1..n]$
  - 情况1：结尾相同
    - $Rec[i] = Rec[i+1]$



问题结构分析

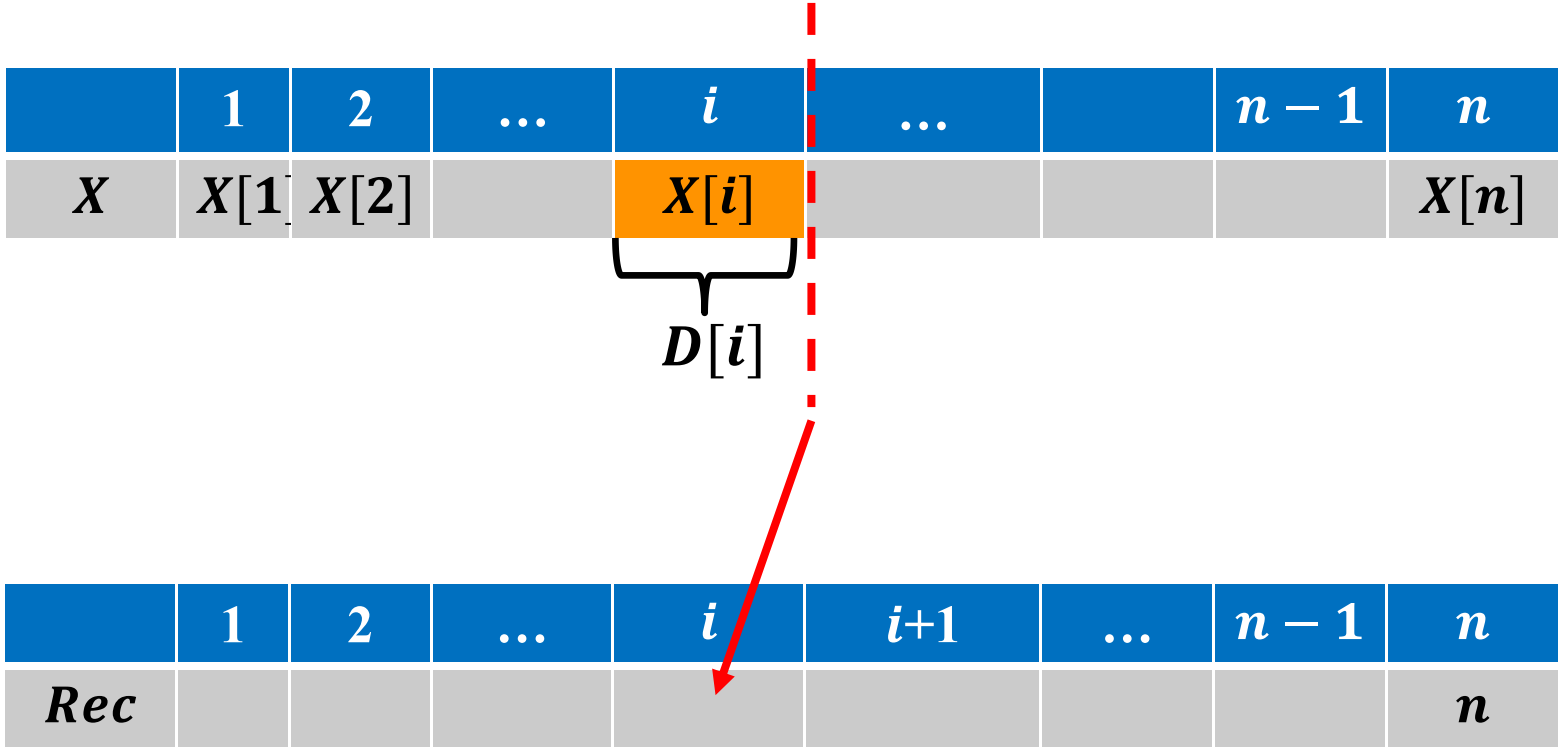
递推关系建立

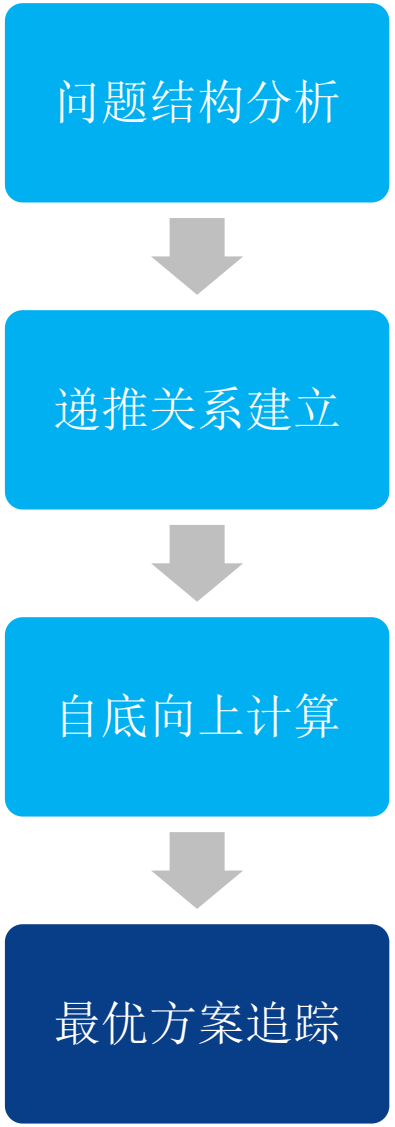
自底向上计算

最优方案追踪

最优方案追踪：记录决策过程

- 构造追踪数组$Rec[1..n]$
  - 情况2：结尾不同
    - $Rec[i]=i$





最优方案追踪：输出最优方案

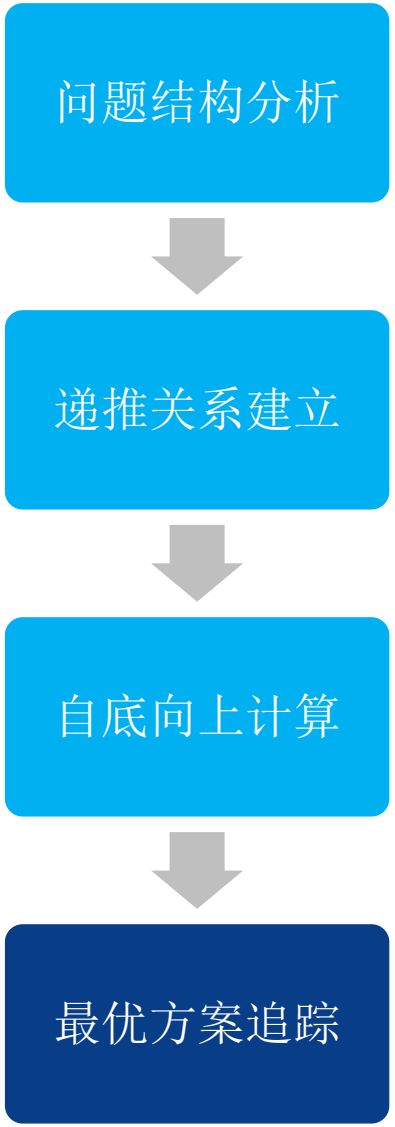
- 从子问题中查找最优解

	1	2	…	…	i		$n-1$	n
X	$X[1]$	$X[2]$			$X[i]$			$X[n]$

已求得

	1	2	…		i		$n-1$	n
D	$D[1]$	$D[2]$			$D[i]$			$D[n]$

	1	2	…		i	…	$n-1$	n
Rec								n



最优方案追踪：输出最优方案

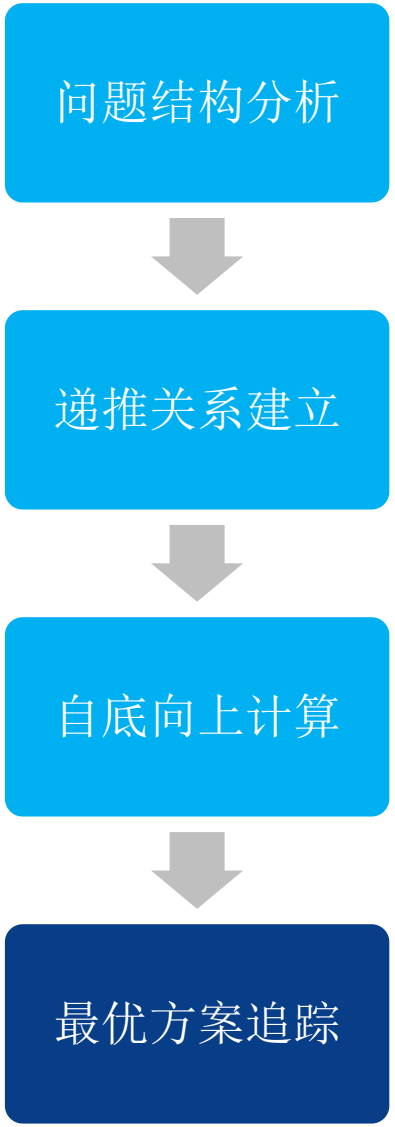
- 从子问题中查找最优解

	1	2	…	…	i		$n-1$	n
X	$X[1]$	$X[2]$			$X[i]$			$X[n]$

	1	2	…		i		$n-1$	n
D	$D[1]$	$D[2]$			$D[i]$			$D[n]$

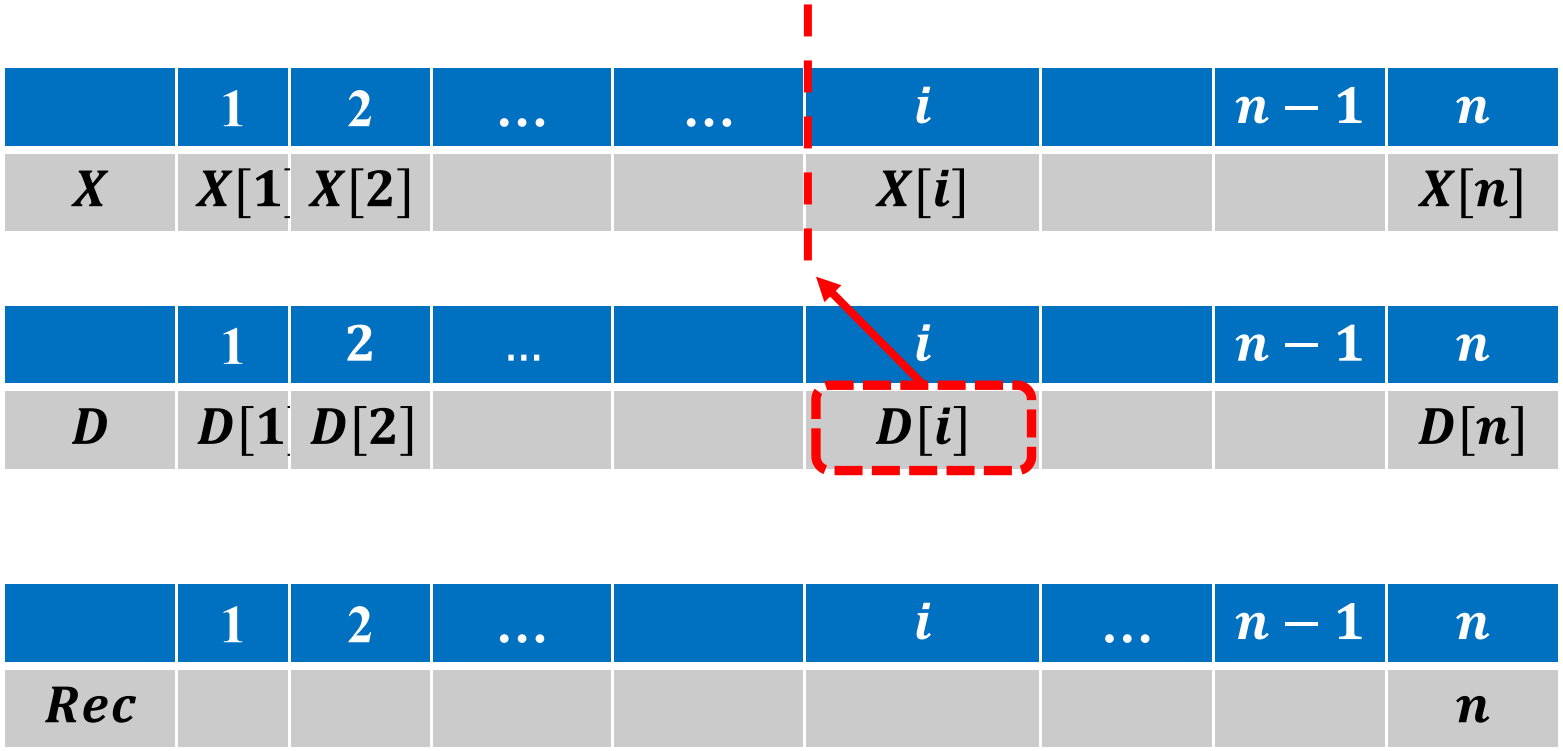
最优解

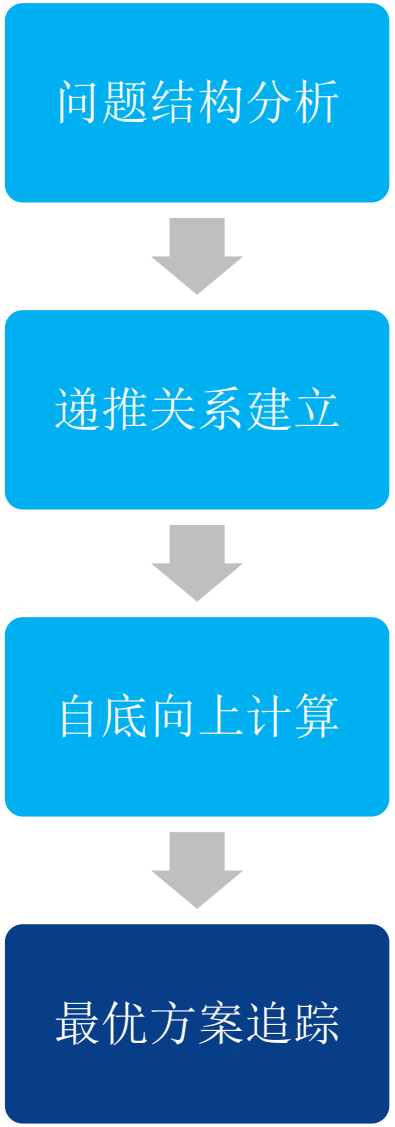
	1	2	…		i	…	$n-1$	n
Rec								n



最优方案追踪：输出最优方案

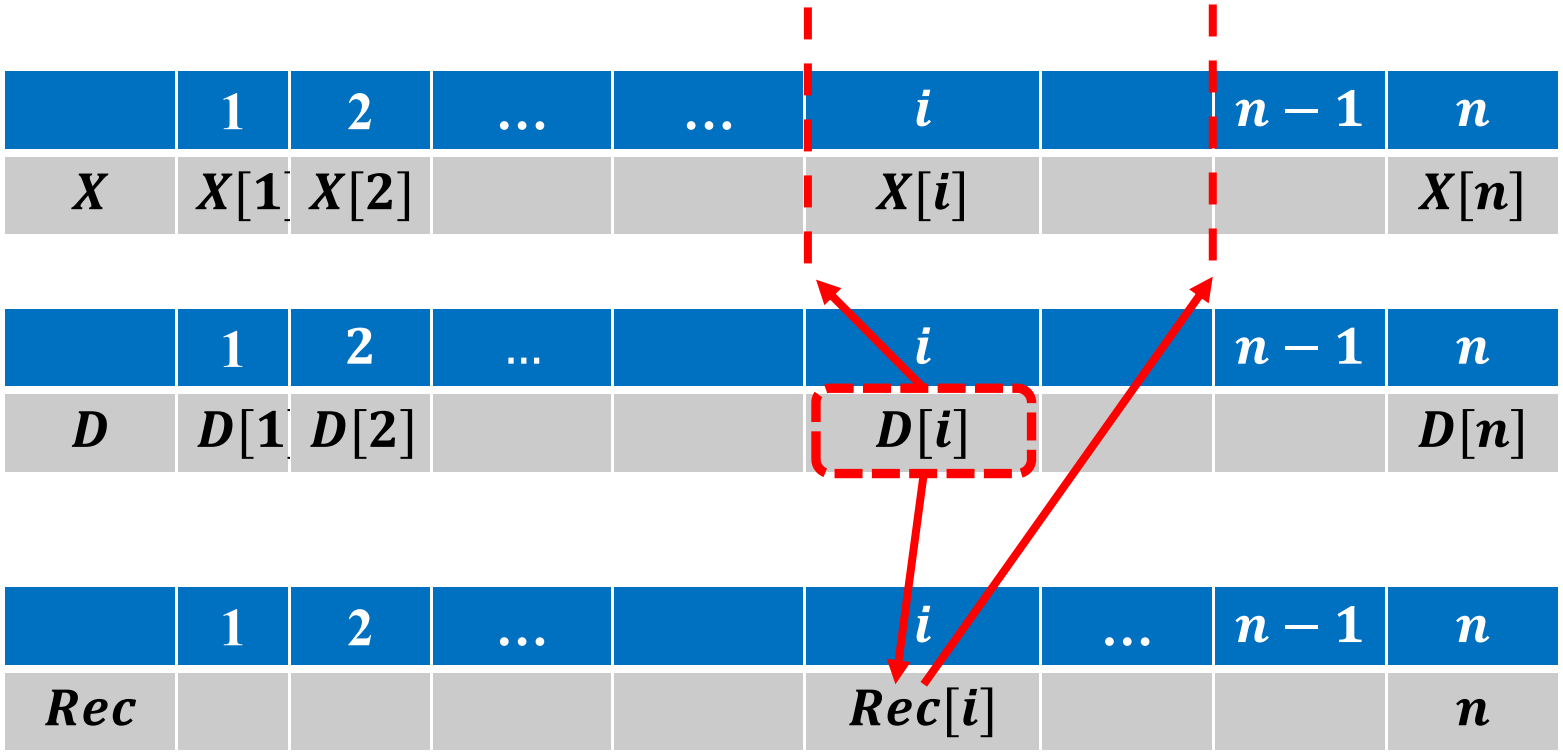
- 从子问题中查找最优解
- 最大子数组开头位置：i

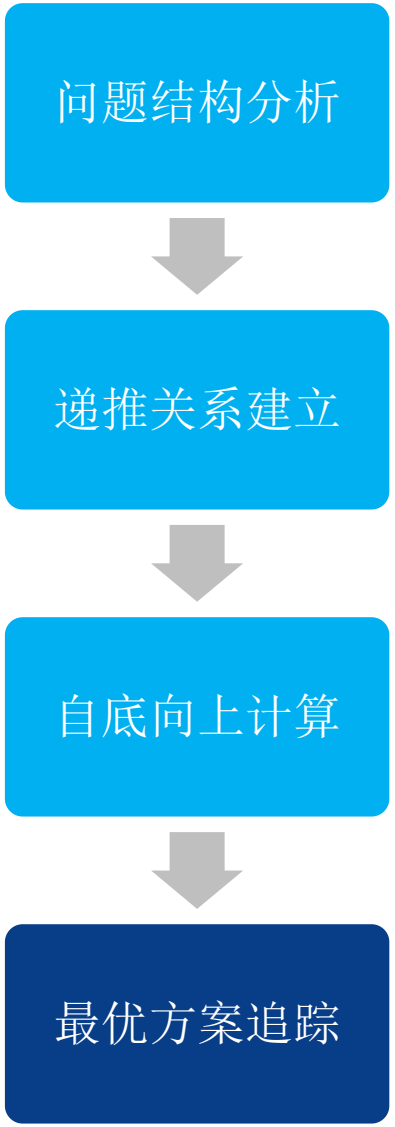




最优方案追踪：输出最优方案

- 从子问题中查找最优解
- 最大子数组开头位置：i
- 最大子数组结尾位置：$Rec[i]$





算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12

算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12
												-1

初始化

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12
												12

算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12
											7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \leq 0 \end{cases}$$

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12
												12

算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12
											7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12
											11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D										10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec										11	11	12

算法实例

X	i	1	2	3	4	5	6	7	8	9	10	11	12
		1	-2	4	5	-2	8	3	-2	6	3	7	-1

D	i	1	2	3	4	5	6	7	8	9	10	11	12
										16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

Rec	i	1	2	3	4	5	6	7	8	9	10	11	12
										11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D								14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec								11	11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D							17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec							11	11	11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D						25	17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \leq 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec						11	11	11	11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D					23	25	17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \leq 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec					11	11	11	11	11	11	11	12

算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12
				28	23	25	17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12
				11	11	11	11	11	11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D			32	28	23	25	17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \leq 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec			11	11	11	11	11	11	11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D		30	32	28	23	25	17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \le 0 \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec		11	11	11	11	11	11	11	11	11	11	12

算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12
	31	30	32	28	23	25	17	14	16	10	7	-1

$$D[i] = \begin{cases} X[i] + D[i+1], & if\ D[i+1] > 0 \\ X[i], & if\ D[i+1] \leq 0 \end{cases}$$

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12
	11	11	11	11	11	11	11	11	11	11	11	12

算法实例

X

i	1	2	3	4	5	6	7	8	9	10	11	12
	1	-2	4	5	-2	8	3	-2	6	3	7	-1

D

i	1	2	3	4	5	6	7	8	9	10	11	12
	31	30	32	28	23	25	17	14	16	10	7	-1

最优解

Rec

i	1	2	3	4	5	6	7	8	9	10	11	12
	11	11	11	11	11	11	11	11	11	11	11	12

算法实例

i	1	2	3	4	5	6	7	8	9	10	11	12
X	1	-2	4	5	-2	8	3	-2	6	3	7	-1

i	1	2	3	4	5	6	7	8	9	10	11	12
D	31	30	32	28	23	25	17	14	16	10	7	-1

最优解

i	1	2	3	4	5	6	7	8	9	10	11	12
Rec	11	11	11	11	11	11	11	11	11	11	11	12

终止位置

算法实例

X	i	1	2	3	4	5	6	7	8	9	10	11	12
		1	-2	4	5	-2	8	3	-2	6	3	7	-1

D	i	1	2	3	4	5	6	7	8	9	10	11	12
		31	30	32	28	23	25	17	14	16	10	7	-1

最优解

Rec	i	1	2	3	4	5	6	7	8	9	10	11	12
		11	11	11	11	11	11	11	11	11	11	11	12

终止位置

$$S = \{4,\ 5,\ -2,\ 8,\ 3,\ -2,\ 6,\ 3,\ 7\}$$

动态规划：伪代码

- **Max-Continuous-Subarray-DP(X, n)**

输入: 数组X, 数组长度n
输出: 最大子数组和S_{max},子数组起止位置l, r
新建一维数组$D[1..n]$和$Rec[1..n]$
//初始化
$D[n] \leftarrow X[n]$
$Rec[n] \leftarrow n$
//动态规划
for $i \leftarrow n-1$ *to* 1 **do**
  if $D[i+1] > 0$ **then**
    $D[i] \leftarrow X[i] + D[i+1]$
    $Rec[i] \leftarrow Rec[i+1]$
  end
  else
    $D[i] \leftarrow X[i]$
    $Rec[i] \leftarrow i$
  end
end

初始化

动态规划：伪代码

- **Max-Continuous-Subarray-DP(X, n)**

输入: 数组X, 数组长度n
输出: 最大子数组和S_{max},子数组起止位置l, r
新建一维数组$D[1..n]$和$Rec[1..n]$
//初始化
$D[n] \leftarrow X[n]$
$Rec[n] \leftarrow n$
//动态规划
for $i \leftarrow n-1$ *to* 1 **do**
  if $D[i+1] > 0$ **then**
    $D[i] \leftarrow X[i] + D[i+1]$
    $Rec[i] \leftarrow Rec[i+1]$
  end
  else
    $D[i] \leftarrow X[i]$
    $Rec[i] \leftarrow i$
  end
end

自底向上计算

动态规划：伪代码

- **Max-Continuous-Subarray-DP(X, n)**

输入: 数组X, 数组长度n
输出: 最大子数组和S_{max},子数组起止位置l, r
新建一维数组$D[1..n]$和$Rec[1..n]$
//初始化
$D[n] \leftarrow X[n]$
$Rec[n] \leftarrow n$
//动态规划
for $i \leftarrow n-1$ *to* 1 **do**
  if $D[i+1] > 0$ **then**
    $D[i] \leftarrow X[i] + D[i+1]$
    $Rec[i] \leftarrow Rec[i+1]$
  end
  else
    $D[i] \leftarrow X[i]$
    $Rec[i] \leftarrow i$
  end
end

情况1：$\boldsymbol{D[i+1] > 0}$

动态规划：伪代码

- **Max-Continuous-Subarray-DP(X, n)**

输入: 数组X, 数组长度n
输出: 最大子数组和S_{max},子数组起止位置l, r
新建一维数组$D[1..n]$和$Rec[1..n]$
//初始化
$D[n] \leftarrow X[n]$
$Rec[n] \leftarrow n$
//动态规划
for $i \leftarrow n-1$ *to* 1 **do**
  if $D[i+1] > 0$ **then**
    $D[i] \leftarrow X[i] + D[i+1]$
    $Rec[i] \leftarrow Rec[i+1]$
  end
  else
    $D[i] \leftarrow X[i]$
    $Rec[i] \leftarrow i$
  end
end

记录子问题结果与决策

动态规划：伪代码

- **Max-Continuous-Subarray-DP(X, n)**

输入: 数组X, 数组长度n
输出: 最大子数组和S_{max},子数组起止位置l, r
新建一维数组$D[1..n]$和$Rec[1..n]$
//初始化
$D[n] \leftarrow X[n]$
$Rec[n] \leftarrow n$
//动态规划
for $i \leftarrow n-1$ *to* 1 **do**
  if $D[i+1] > 0$ **then**
    $D[i] \leftarrow X[i] + D[i+1]$
    $Rec[i] \leftarrow Rec[i+1]$
  end
  else
    $D[i] \leftarrow X[i]$
    $Rec[i] \leftarrow i$
  end
end

情况2： $D[i+1] \leq 0$

动态规划：伪代码

- **Max-Continuous-Subarray-DP(X, n)**

```
//查找解
S_max ← D[1]
for i ← 2 to n do
    if S_max < D[i] then
        S_max ← D[i]
        l ← i
        r ← Rec[i]
    end
end
return S_max, l, r
```

时间复杂度分析

输入: 数组X, 数组长度n
输出: 最大子数组和S_{max},子数组起止位置l, r
新建一维数组$D[1..n]$和$Rec[1..n]$

```
//初始化
D[n] ← X[n]
Rec[n] ← n
//动态规划
for i ← n − 1 to 1 do
    if D[i + 1] > 0 then
        D[i] ← X[i] + D[i + 1]
        Rec[i] ← Rec[i + 1]
    end
    else
        D[i] ← X[i]
        Rec[i] ← i
    end
end
```

$O(n)$

时间复杂度：$O(n)$

谢谢