Last Week

Introduction

- Lecturer
- Course Details
- A.M. Turing Award Winners for Algorithms
- What Is This Course About
- What Are Algorithms
- What Does It Mean to Analyze An Algorithm
- Comparing Time Complexity

Asymptotic Notations and Recurrences

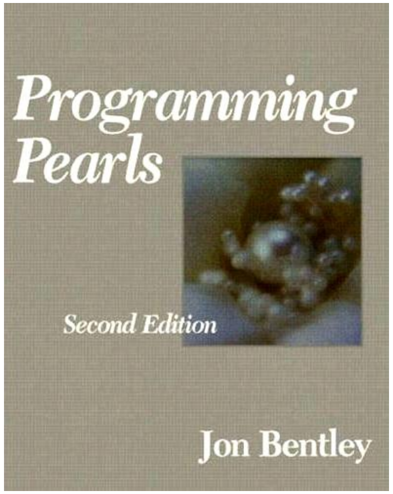
- Asymptotic Notations（渐近记号）
  - Big-Oh
  - Big-Omega
  - Big-Theta
  - Algorithm Design and Algorithm Turing
- Solving Recurrences
  - Recursion-tree Method（递归树法）
  - Substitution Method（代入法/替代法）
  - Master Method and Master Theorem (主方法)

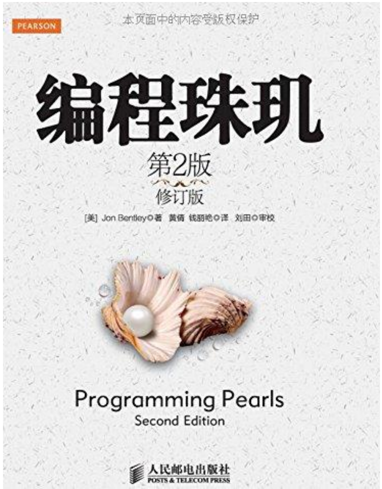
Some Thoughts on Algorithm Design

- Algorithm Design, as taught in this class, is mainly about designing algorithms that have big-Oh running times.
- As n gets larger and larger, O(nlogn) algorithms will run faster than $O(n^2)$ ones and O(n) algorithms will beat O(nlogn) ones.
- Good algorithm design & analysis allows you to identify the hard parts of your problem and deal with them effectively.
- Too often, programmers try to solve problems using brute force techniques and end up with slow complicated code!
- A few hours of abstract thought devoted to algorithm design often results in faster, simpler, and more general solutions.

Algorithm Tuning

- After algorithm design one can continue on to Algorithm tuning
  - concentrate on improving algorithms by cutting down on the constants in the big O() bounds.
  - needs a good understanding of both algorithm design principles and efficient use of data structures.
- In this course we will not go further into algorithm tuning
  - For a good introduction, see chapter 9 in Programming Pearls, 2nd ed by Jon Bentley





Outline

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Solving recurrences: Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
  - Each node represents the cost of a single subproblem.

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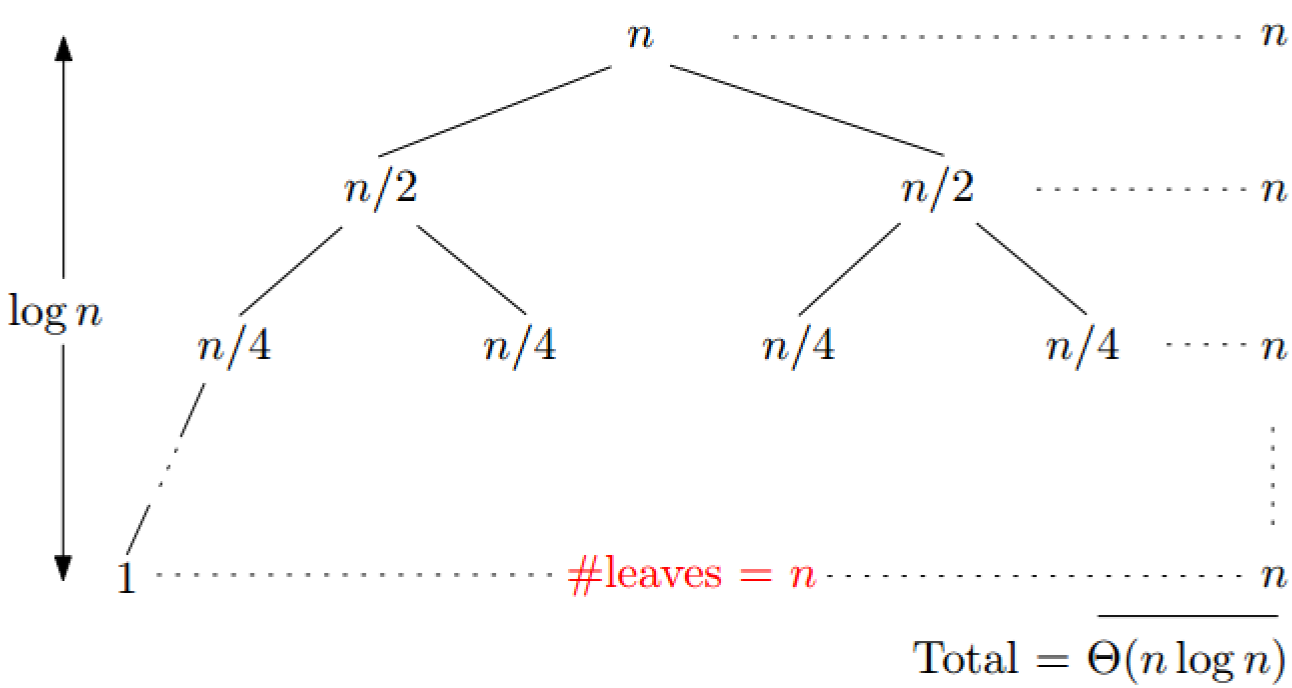

Recursion-tree method: Example

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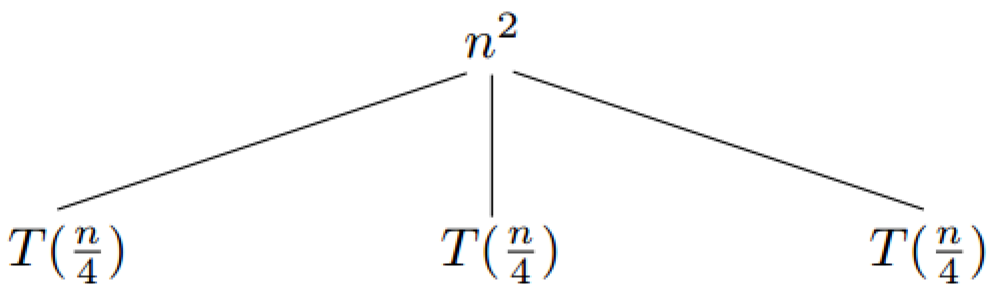

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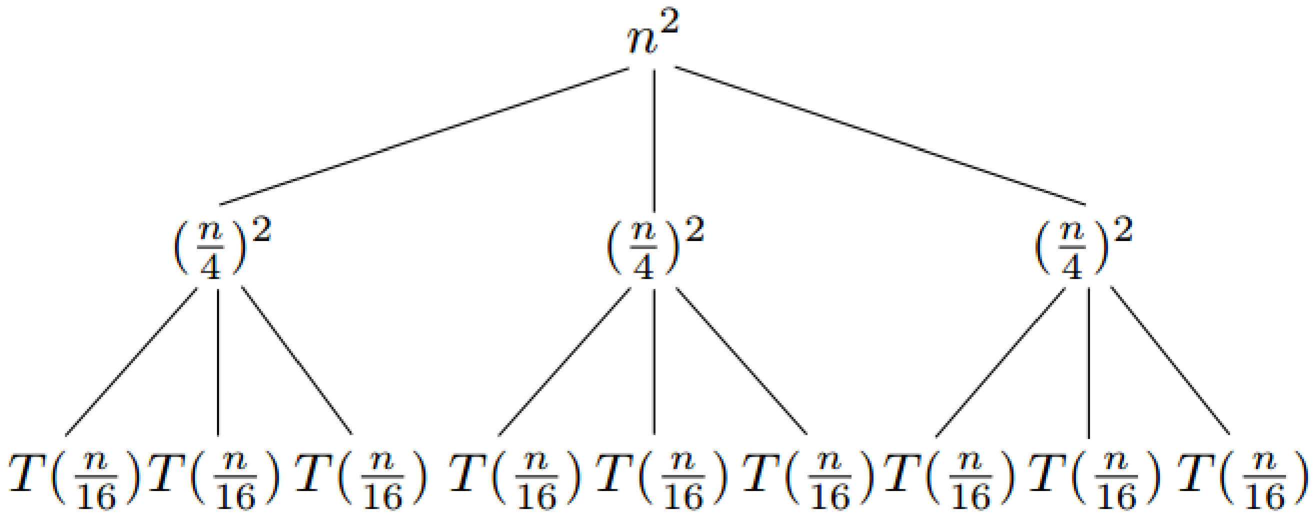

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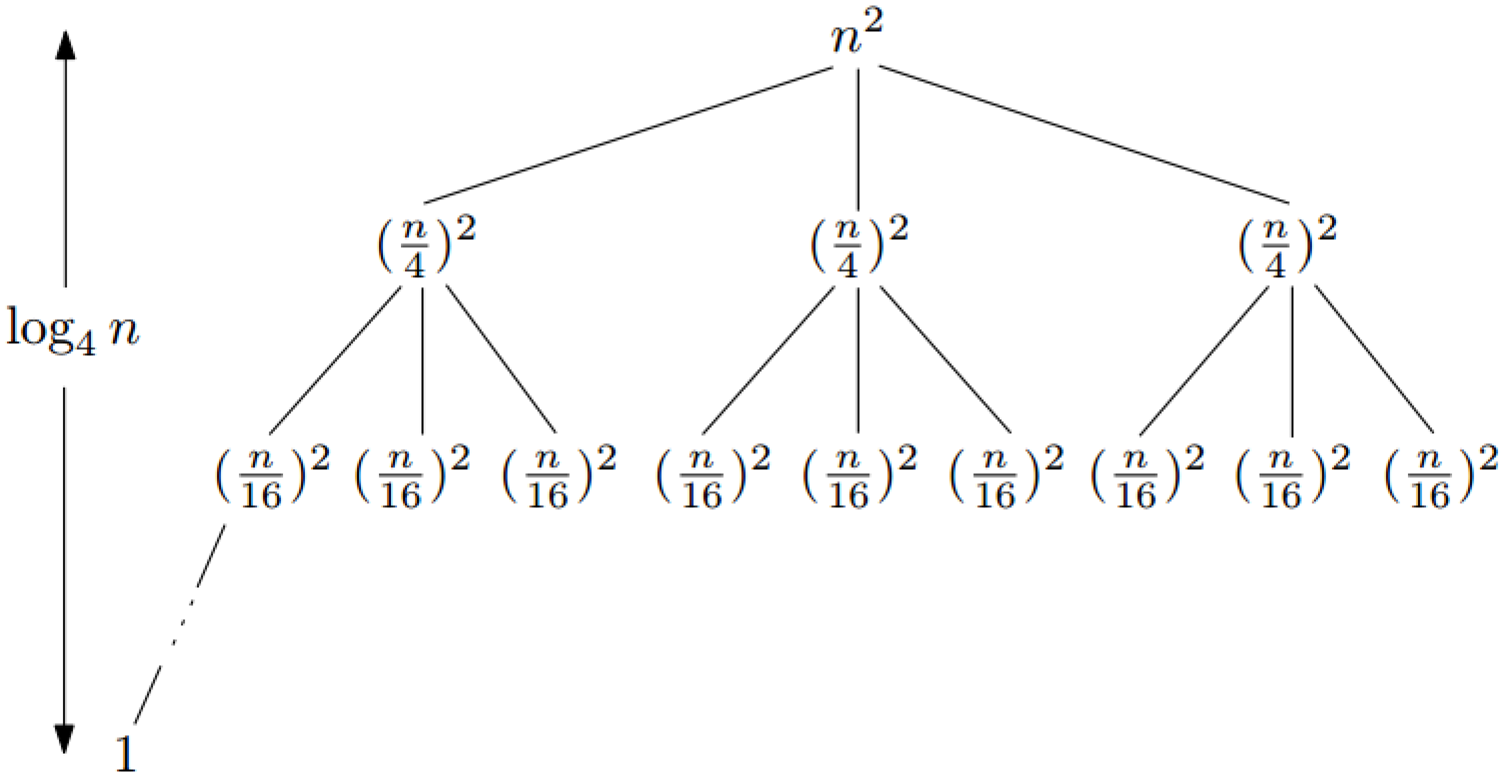

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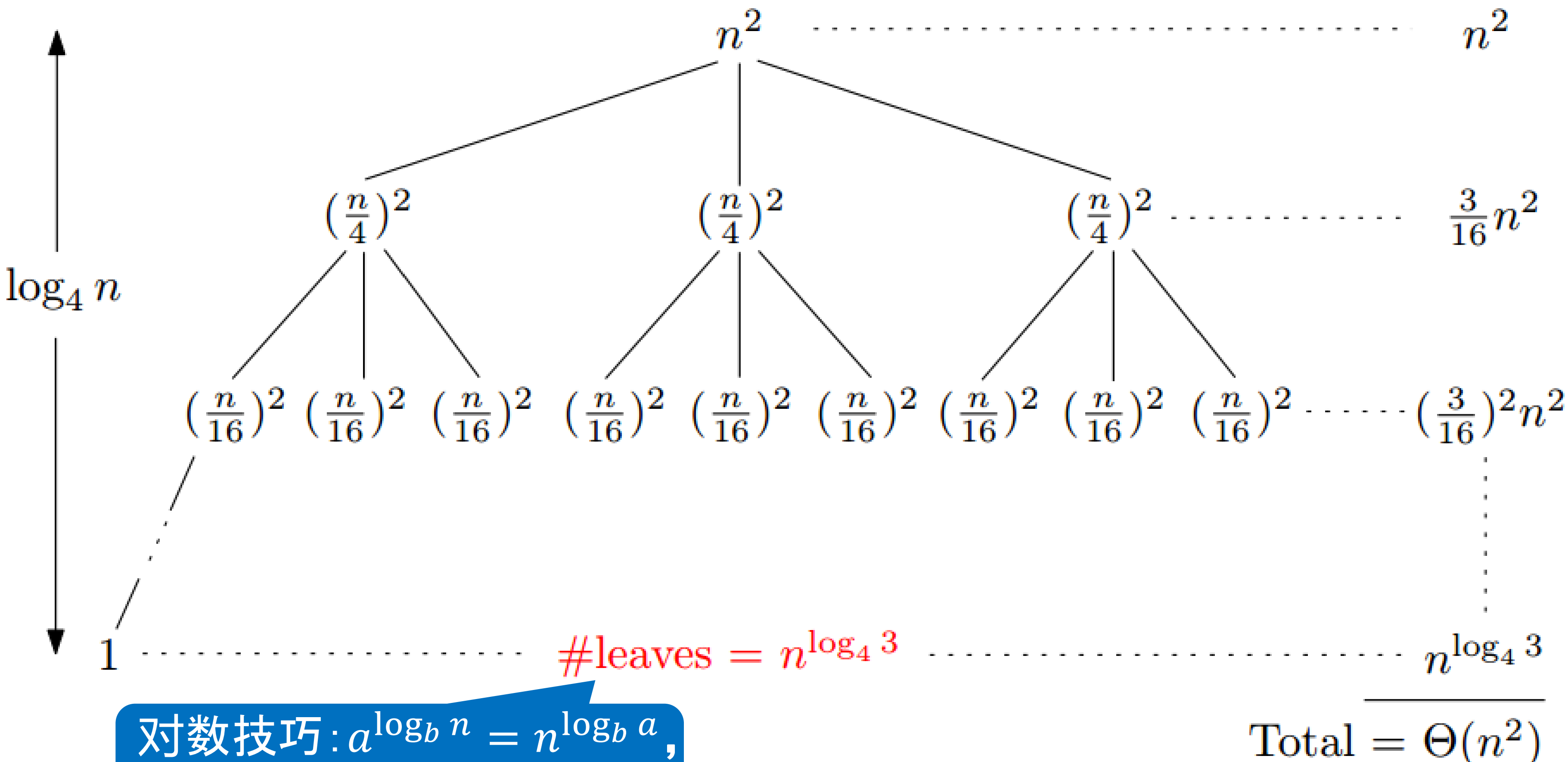

对数技巧：$a^{\log_b n} = n^{\log_b a}$,
$3^{\log_4 n} = n^{\log_4 3}$

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$$\begin{aligned} T(n) &\le n^2 + \frac{3}{16}n^2 + \left(\frac{3}{16}\right)^2 n^2 + \cdots \\ &= O(n^2). \end{aligned}$$

geometric series

几何级数（又称为等比级数）

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- Since $T(n) = 3T(n/4) + n^2$, it follows that $T(n) \geq n^2$
- So, $T(n) = \Omega(n^2)$.
- Thus, $T(n) = \Theta(n^2)$

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Substitution method: Example 1

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Prove $T(n) \leq cn^2$ by induction, where c is a large constant.

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Proof.

- Base (n=1) : obviously holds for any $c \ge 1$

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- Base (n=1) : obviously holds for any $c \geq 1$
- Induction:

$$\begin{aligned} T(n) &= 3T(n/4) + n^2 \\ &\leq 3c(n/4)^2 + n^2 \\ &= cn^2 - (13c/16 - 1)n^2 \\ &\leq cn^2, \end{aligned}$$

whenever $13c/16 - 1 \geq 0$, or $c \geq 16/13$.

□

Substitution method: Example 2

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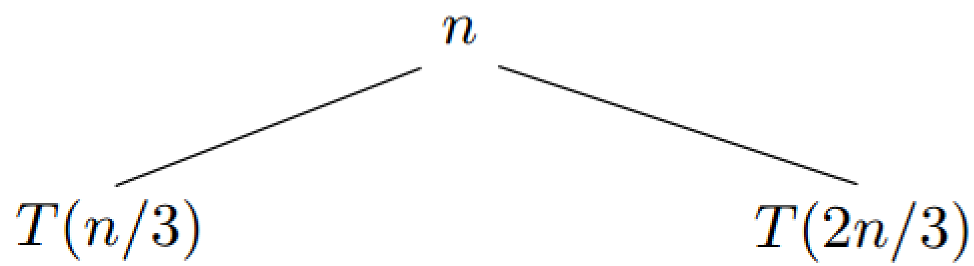

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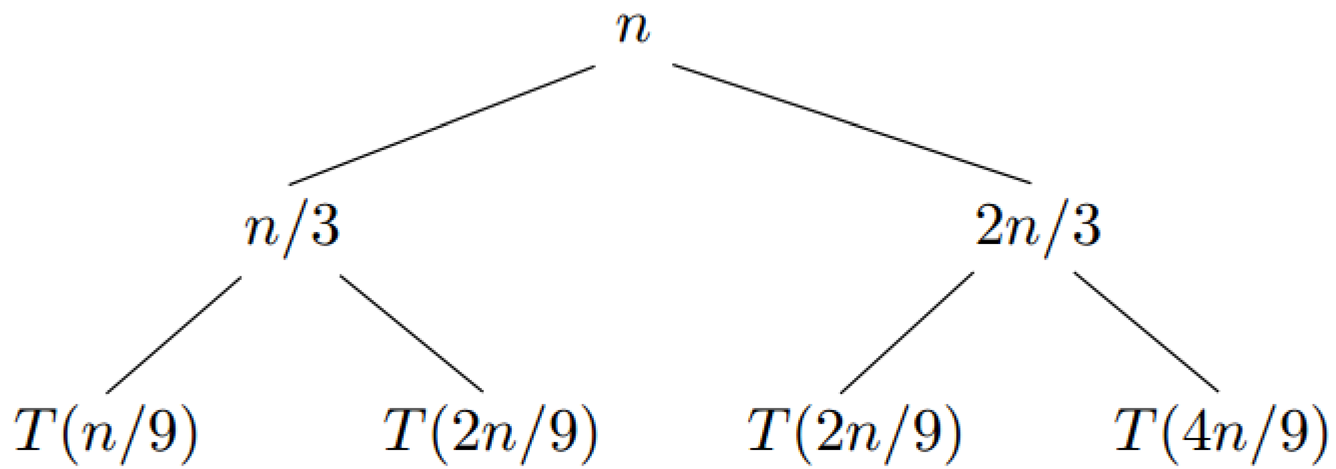

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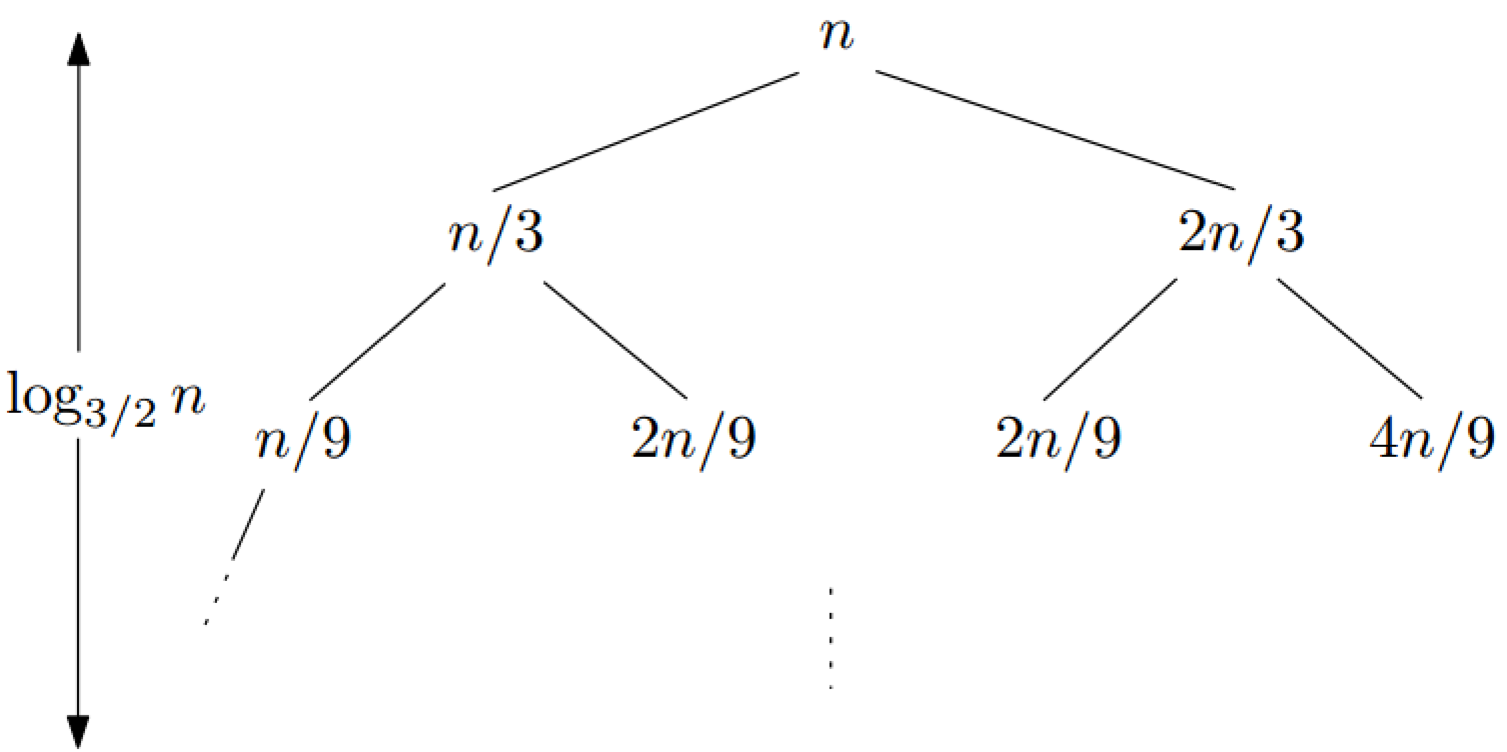

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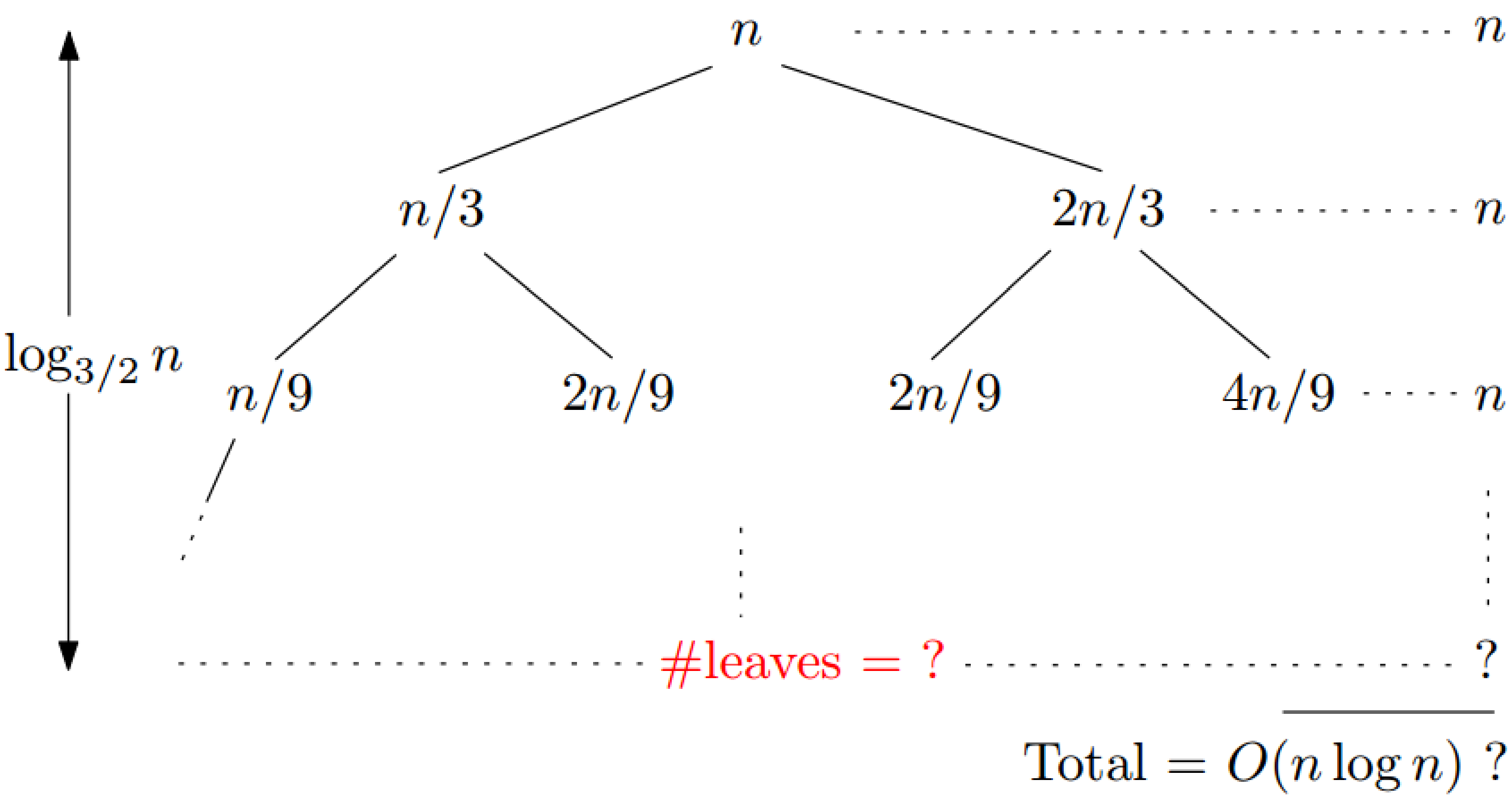

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as long as $c \geq 1/(\log 3 - 2/3)$.

$\square$

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Master Theorem

If $T(n) = aT\left(\left\lceil\frac{n}{b}\right\rceil\right) + O(n^d)$ for some constant a > 0, b > 1 and d ≥ 0, then

$$T(n) = \begin{cases} O\left(n^d\right), & if\ d > \log_b a \\ O\left(n^d \log n\right), & if\ d = \log_b a \\ O(n^{\log_b a}), & if\ d < \log_b a \end{cases}$$

Proof of the Master Theorem

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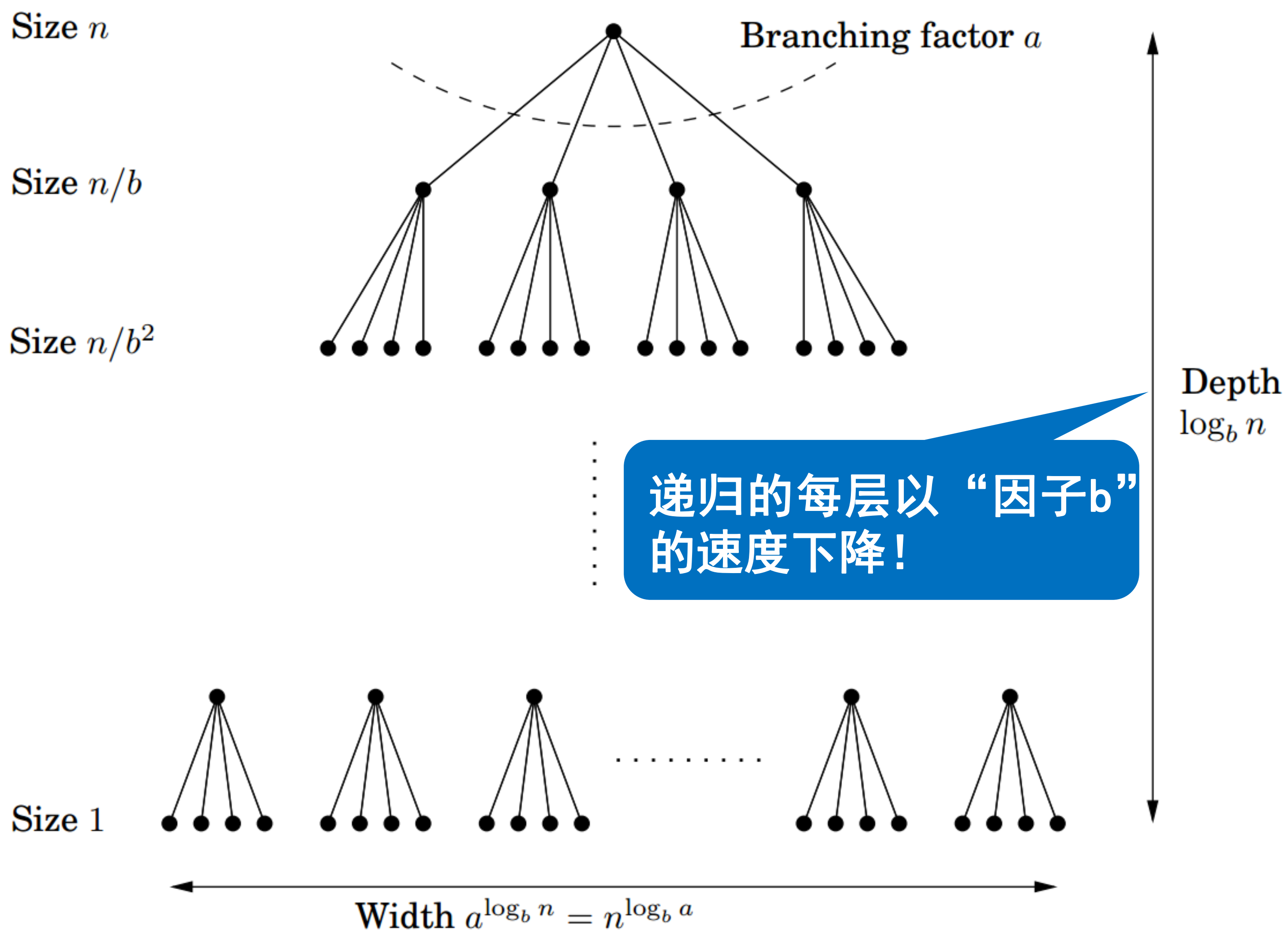
For the sake of convenience, we assume that n is a power of b. This will not influence the final bound in any important way—n is at most a multiplicative factor of b away from some power of b—and it will allow us to ignore the rounding effect in $\lceil\frac{n}{b}\rceil$.

Proof of the Master Theorem



$$a^k \times O\left(\frac{n}{b^k}\right)^d = O(n^d) \times \left(\frac{a}{b^d}\right)^k$$

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- The size of the subproblems decreases by a factor of b with each level of recursion, and therefore reaches the base case after $\log_b n$ levels. This is the height of the recursion tree.
- The k-th level of the tree is made up of a^k subproblems, each of size n/b^k
- The total work done at this level is

$$a^k \times O\left(\frac{n}{b^k}\right)^d = O\left(n^d\right) \times \left(\frac{a}{b^d}\right)^k$$

Proof of the Master theorem

It comes down to the following three cases.

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- **The ratio a/b^d is exactly 1 (a/b^d =1)**.
  In this case all O(log n) terms of the series are equal to $O(n^d)$.

Master theorem: Example

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- $T(n) = O\left(n^d \log n\right) = O(n^2 \log n)$

Part I: Divide and Conquer

Maximum Contiguous Subarray Problem and Counting Inversion Problem

Outline

- Introduction to Part I
- Maximum Contiguous Subarray Problem
  - Problem definition
  - A brute force algorithm
  - A data-reuse algorithm
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
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Introduction to Part I

- Divide-and-conquer (D&C) is an important algorithm design paradigm.

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  - **Combine**
    Combining the solutions of the subproblems into a global solution

Introduction to Part I

- In Part I, we will illustrate Divide-and-Conquer using several examples:
  - Maximum Contiguous Subarray (最大子数组)
  - Counting Inversions (逆序计数)
  - Integer Multiplication (整数乘法)
  - Polynomial Multiplication (多项式乘法)
  - QuickSort and Partition (快速排序与划分)
  - Deterministic and Randomized Selection (确定性与随机化选择)

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Maximum Contiguous Subarray (MCS) Problem

ACME Corp[1] – PROFIT HISTORY

Year	1	2	3	4	5	6	7	8	9
Profit M$	-3	2	1	-4	5	2	-1	3	-1

[1] A Company that Makes Everything

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如果所有数组元素都是非负数，整个数组和肯定是最大

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Answer: Year 5-8, 9 M$

[1] A Company that Makes Everything

Formal Definition

- Input: An array of reals $A[1\ldots n]$

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Definition (Maximum Contiguous Subarray Problem)

Find $i \leq j$ such that $V(i,j)$ is maximized.

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```
VMAX ← A[1];
for i ← 1 to n do
    for j ← i to n do
        // calculate V(i,j)
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        for x ← i to j do
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$O(n^3)$ arithmetic additions

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            VMAX ← V;
        end
    end
end
return VMAX
```

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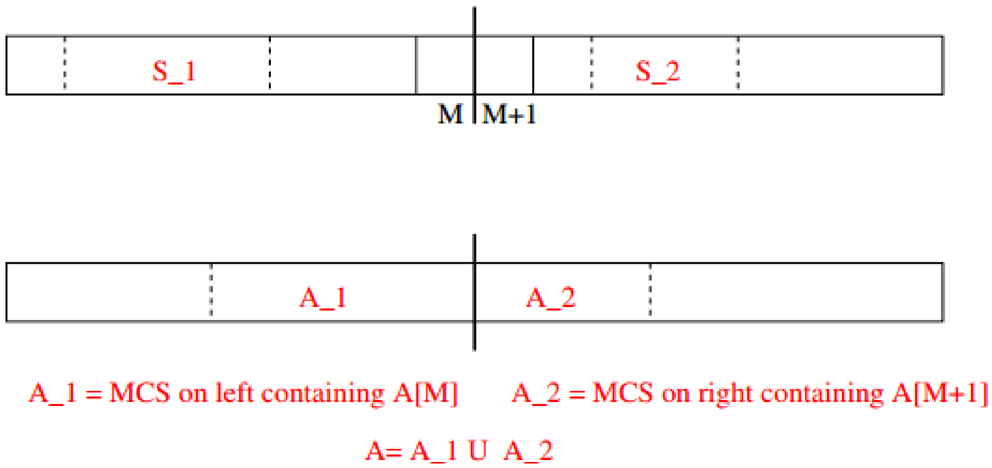
A Divide-and-Conquer Algorithm

Set $m = \lfloor (n+1)/2 \rfloor$



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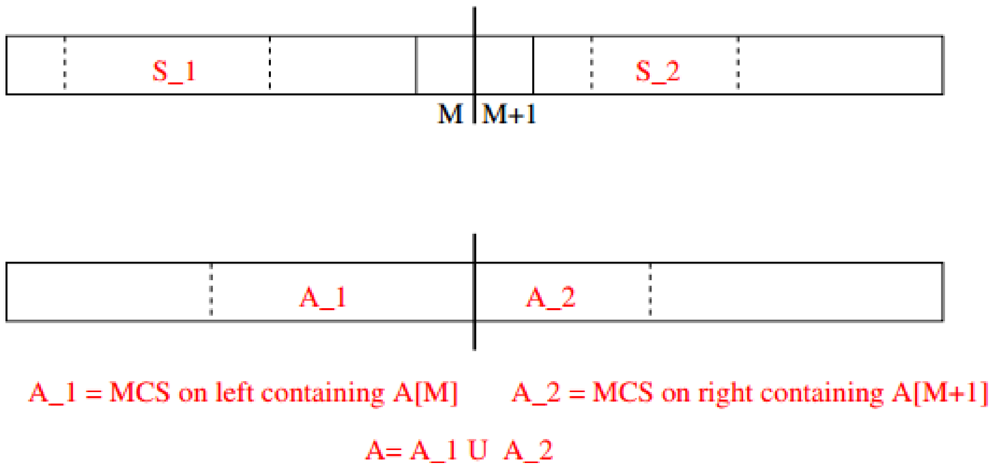

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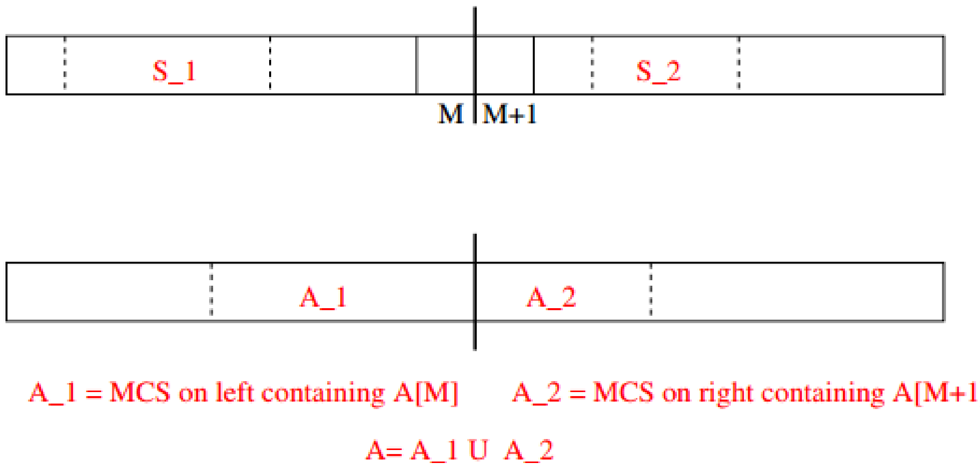

The MCS S must be one of

1. S_1: the MCS in $A[1 \ldots m]$

A Divide-and-Conquer Algorithm

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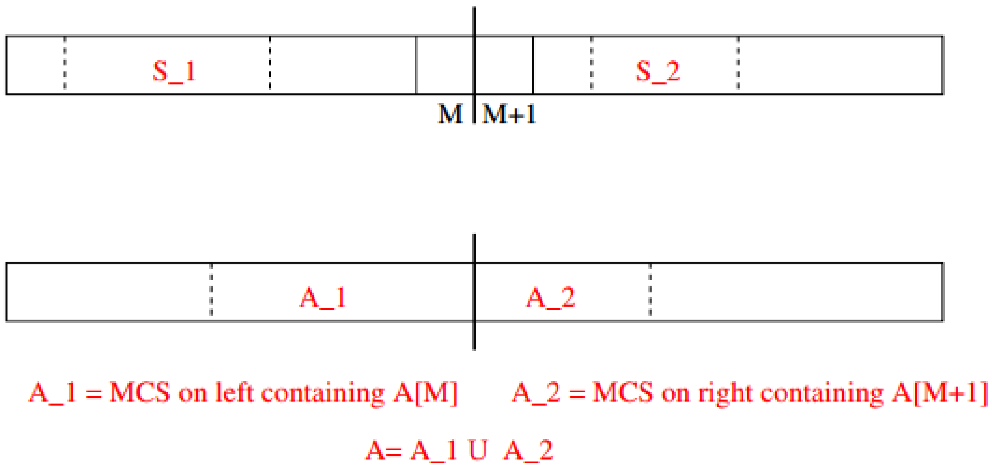

The MCS S must be one of

1. S_1: the MCS in $A[1 \ldots m]$
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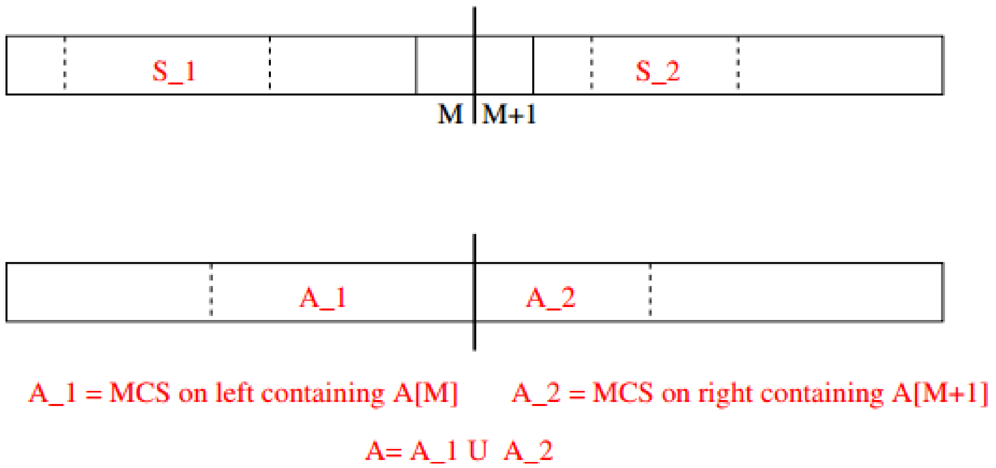

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1. S_1: the MCS in $A[1 \ldots m]$
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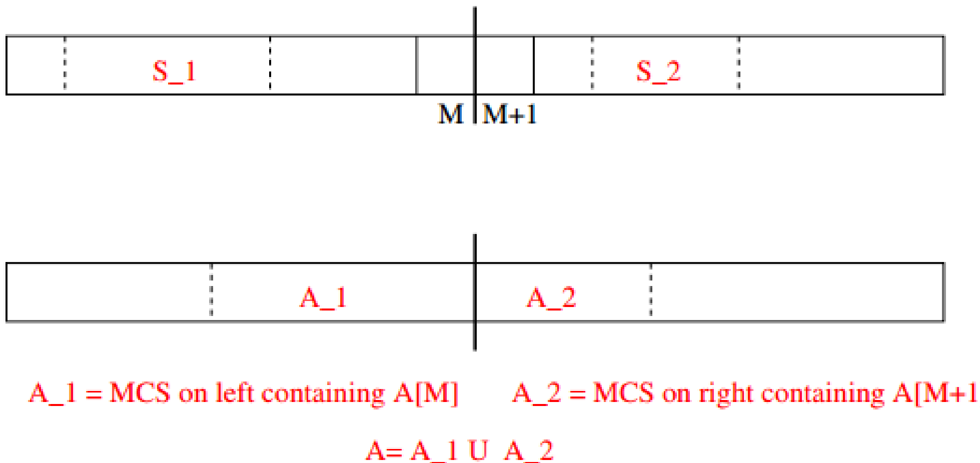

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So,

最终，在S_1，S_2和A（跨越中点的最大子数组）这三种情况中选取和最大者

$$S = \text{the best among } \{S_1, S_2, A\}$$

An Example of Divide-and-Conquer Algorithm

1	-5	4	2	-7	3	6	-1	2	-4	7	-10	2	6	1	-3

- $S_1 =$

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1	-5	4	2	-7	3	6	-1	2	-4	7	-10	2	6	1	-3

- $A_1 =$

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- $A_1 = [3, 6, -1]$ and $A_2 = [2, -4, 7]$
- $A = A_1 \cup A_2 =$

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- solution:

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Divide: MCS across The Cut

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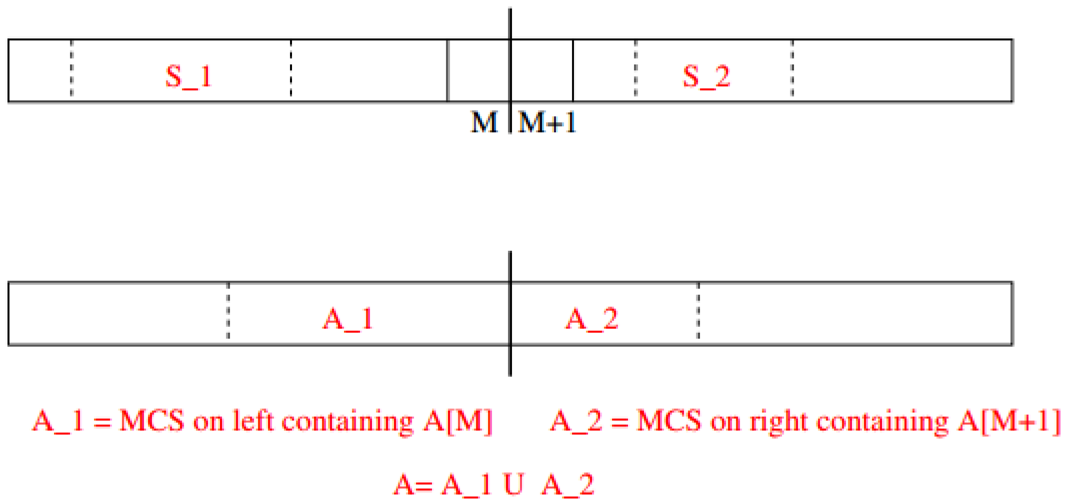

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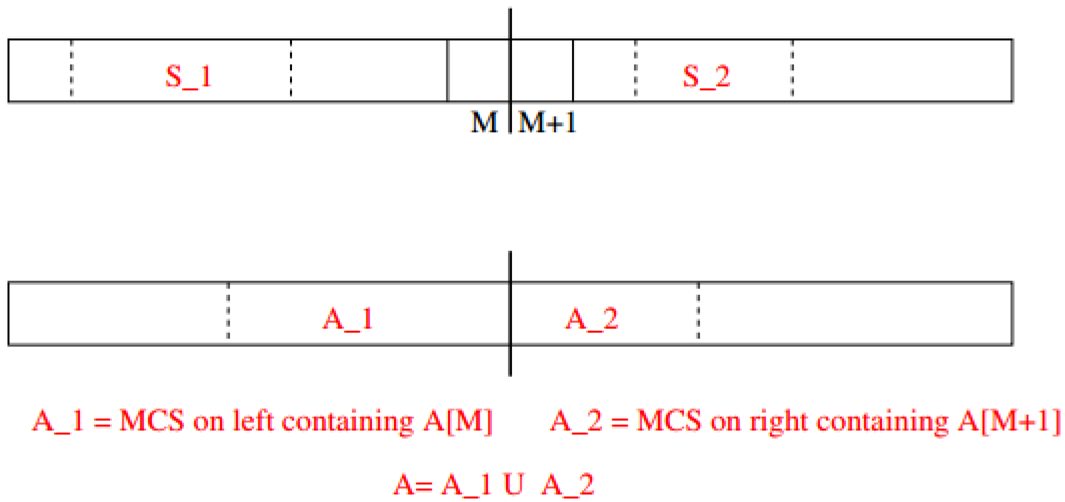

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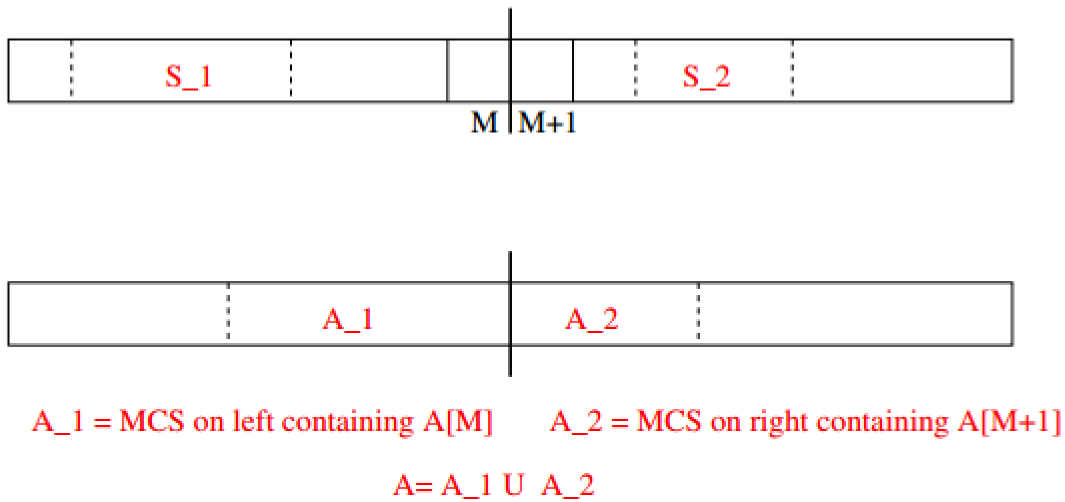

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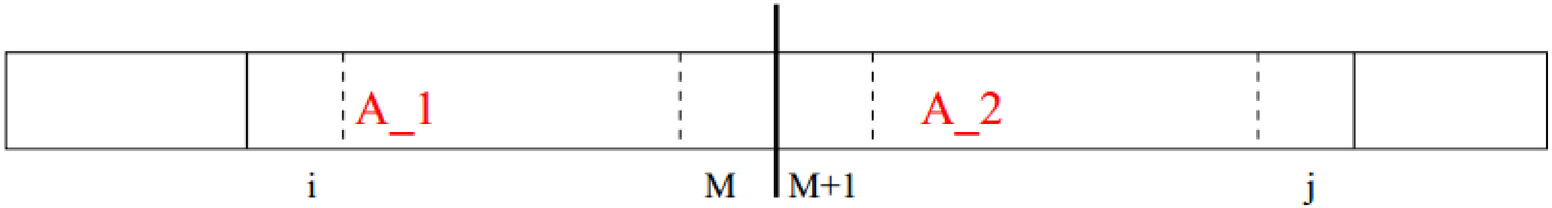

A_1 is in the form $A[i \ldots m]$, $V(i, m) = V(i+1, m) + A[i]$

MAX $\leftarrow A[m]$;
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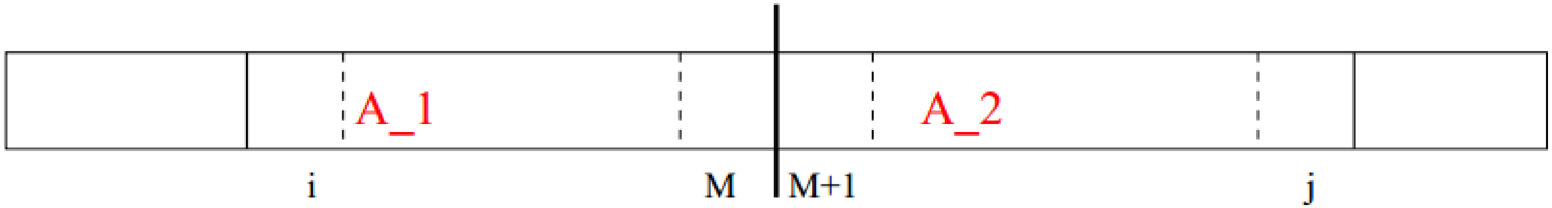

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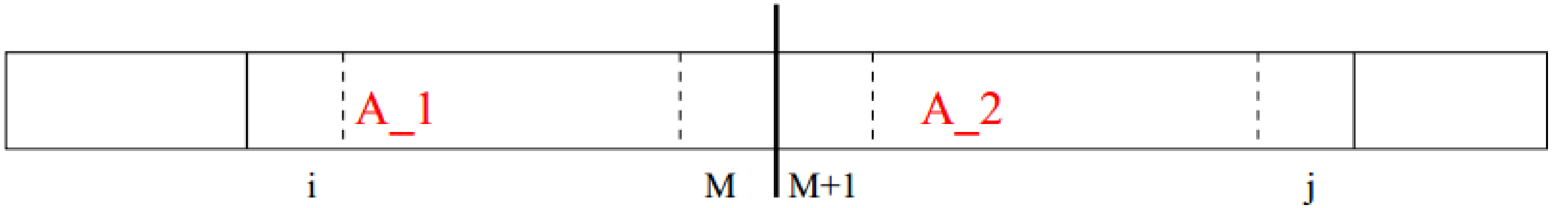

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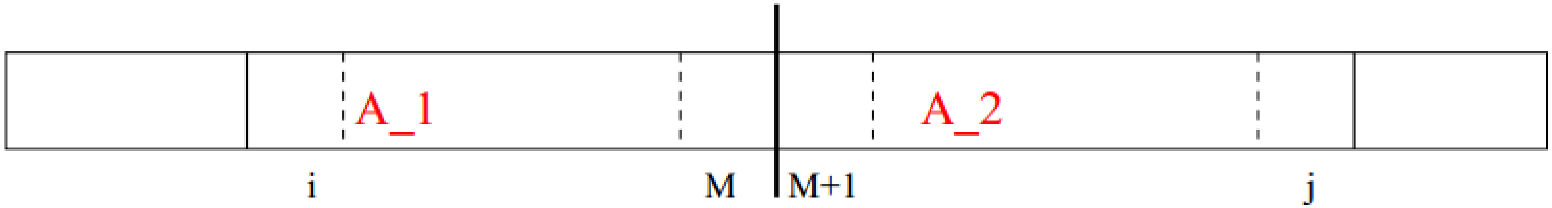

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        MAX $\leftarrow$ SUM;
    end
end
A_1=MAX;

Conquer: Finding "A" with A Linear Time

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Conquer: Finding "*A*" with A Linear Time

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  - linear to the input size

The Complete Divide-and-Conquer Algorithm

$MCS(A, s, t)$

Input: $A[s \ldots t]$ with $s \leq t$
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    Find $MCS(A, s, m)$;
    Find $MCS(A, m+1, t)$;
    Find MCS that contains both $A[m]$ and $A[m+1]$;

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    Find MCS that contains both $A[m]$ and $A[m+1]$;
    return maximum of the three sequences found
  end

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        Find $MCS(A, m+1, t)$;
        Find MCS that contains both $A[m]$ and $A[m+1]$;
        return maximum of the three sequences found
    end
end

The Complete Divide-and-Conquer Algorithm

$MCS(A, s, t)$

Input: $A[s \dots t]$ with $s \leq t$
Output: MCS of $A[s \dots t]$
begin
    if $s = t$ **then return** $A[s]$;
    else
        $m \leftarrow \lfloor \frac{s+t}{2} \rfloor$;
        Find $MCS(A, s, m)$;
        Find $MCS(A, m+1, t)$;
        Find MCS that contains both $A[m]$ and $A[m+1]$;
        return maximum of the three sequences found
    end
end

First Call: *MCS*(*A*, *1*, *n*)

A Full Illustration of the D&C Algorithm

6 -4 7 -4 0 1

A Full Illustration of the D&C Algorithm

6 -4 7 | -4 0 1

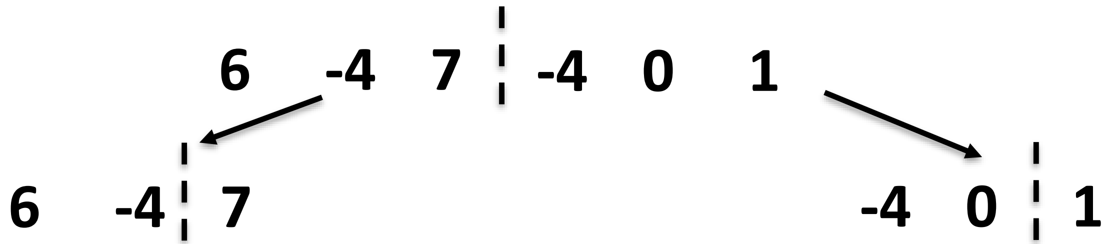
A Full Illustration of the D&C Algorithm

6 -4 7 | -4 0 1

6 -4 7

-4 0 1

A Full Illustration of the D&C Algorithm



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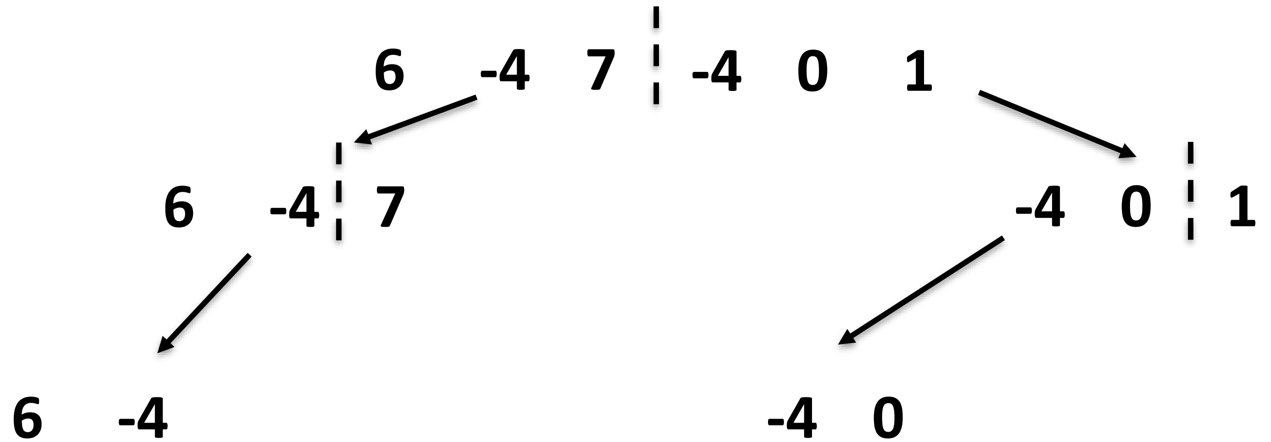

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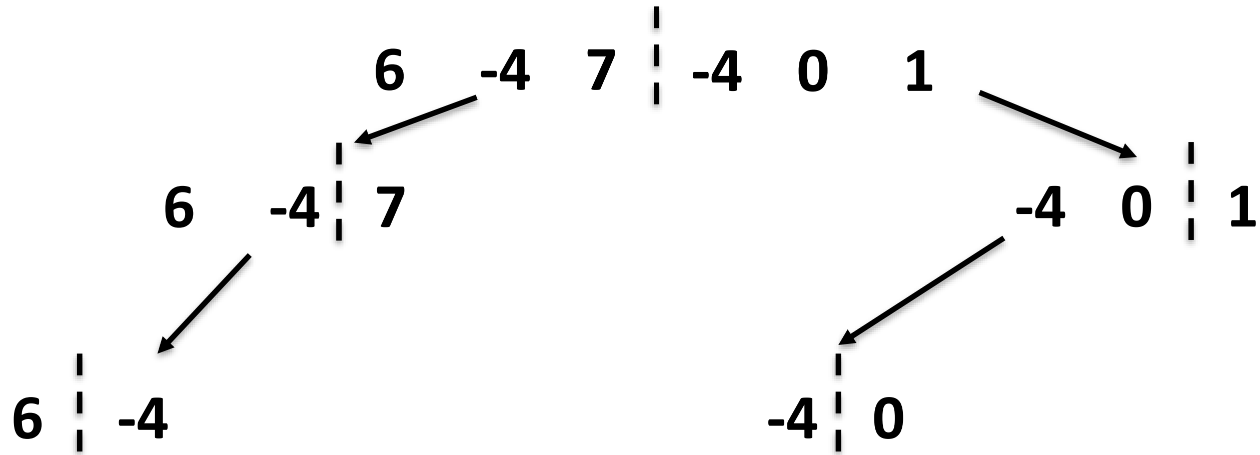


A Full Illustration of the D&C Algorithm

6 -4 7 | -4 0 1

6 -4 | 7

-4 0 | 1

6 | -4

-4 | 0

Divide

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

6 ¦ -4

-4 ¦ 0

Divide

Conquer

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7 **-4 0 ¦ 1**

6 ¦ -4 **-4 ¦ 0**

Divide

MCS={6} **MCS={-4}**

Conquer

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

6 ¦ -4

-4 ¦ 0

Divide

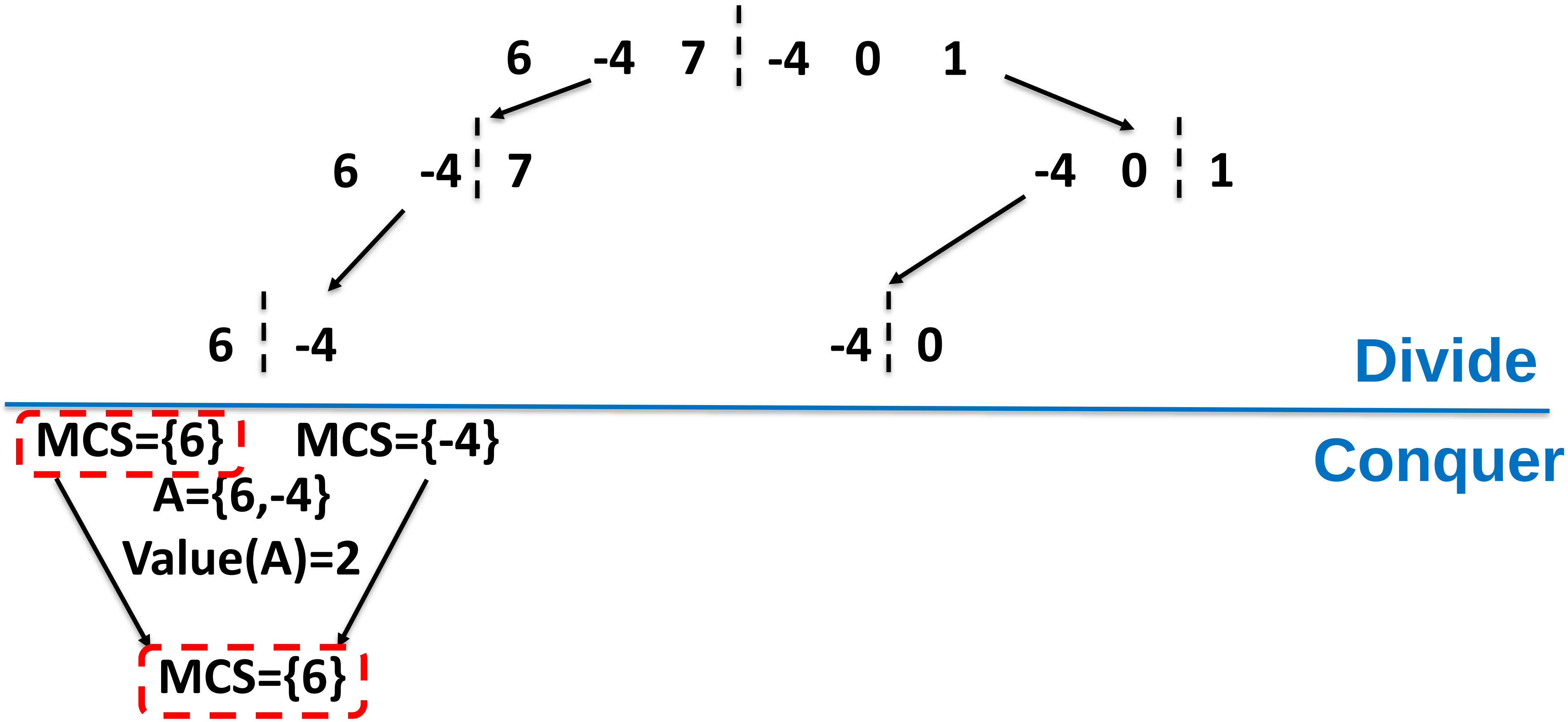
Conquer

MCS={6} **MCS={-4}**

A={6,-4}

Value(A)=2

A Full Illustration of the D&C Algorithm



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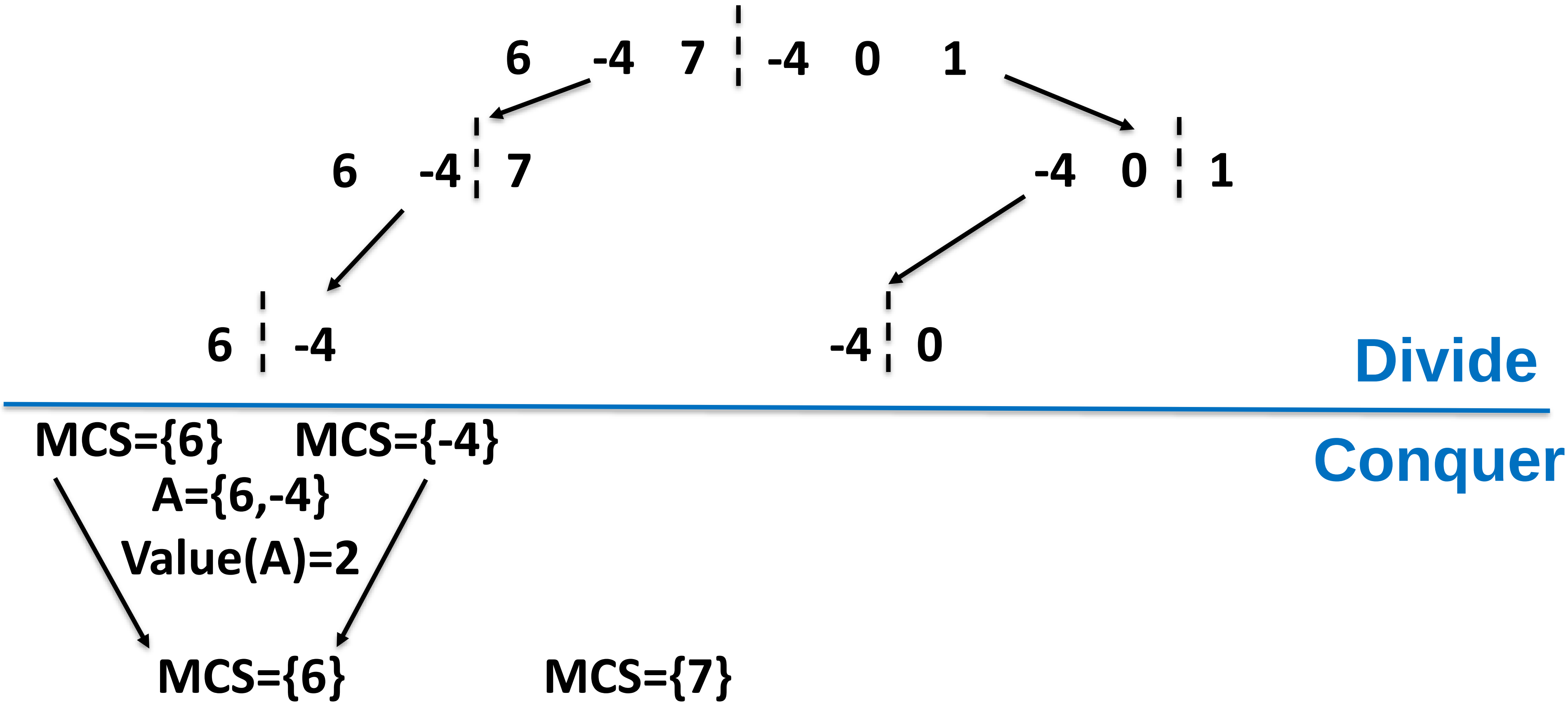


A Full Illustration of the D&C Algorithm

6 -4 7 | -4 0 1

6 -4 | 7

-4 0 | 1

6 | -4

-4 | 0

Divide

Conquer

MCS={6} **MCS={-4}**

A={6,-4}

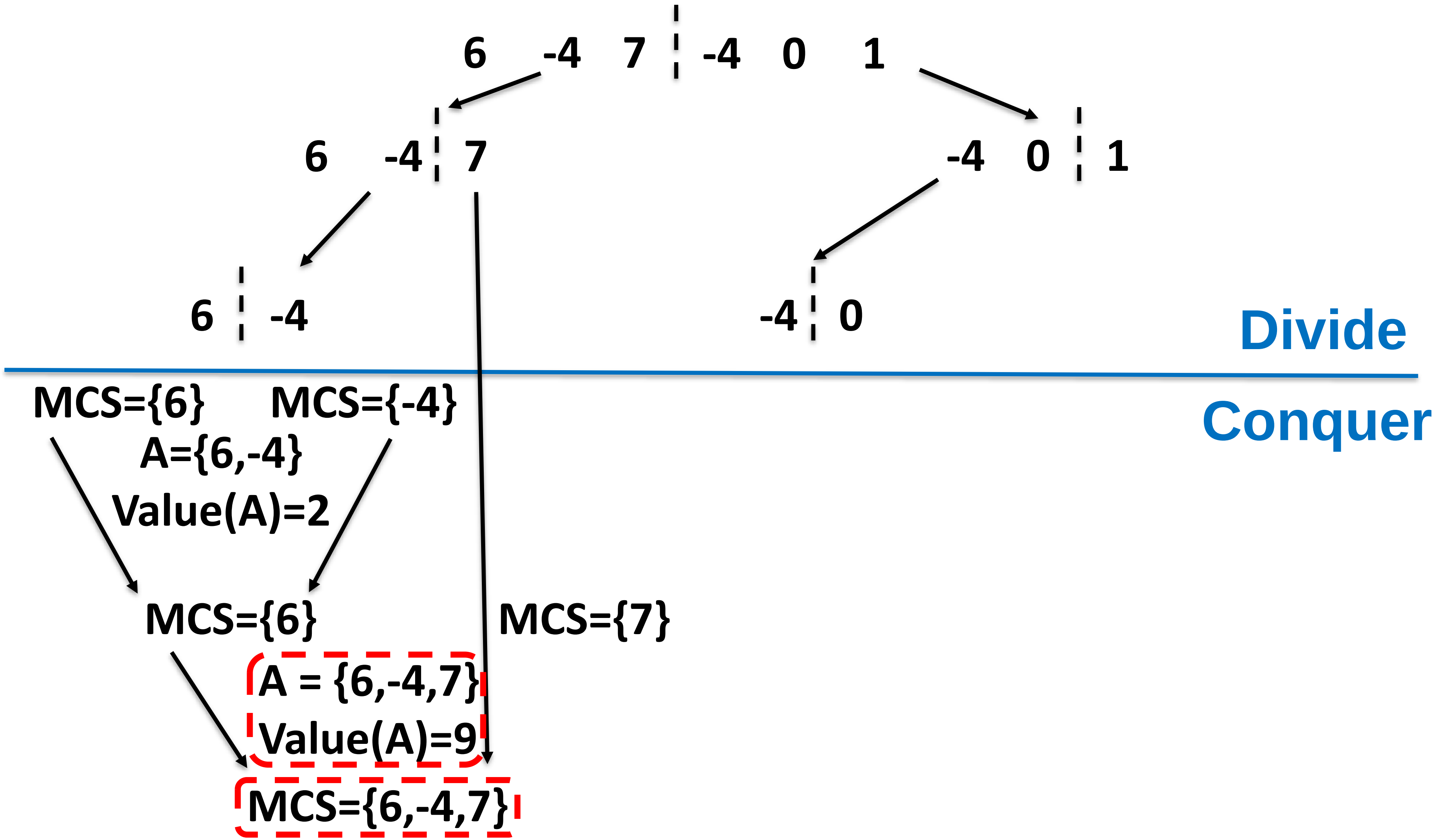
Value(A)=2

MCS={6} **MCS={7}**

A = {6,-4,7}

Value(A)=9

A Full Illustration of the D&C Algorithm



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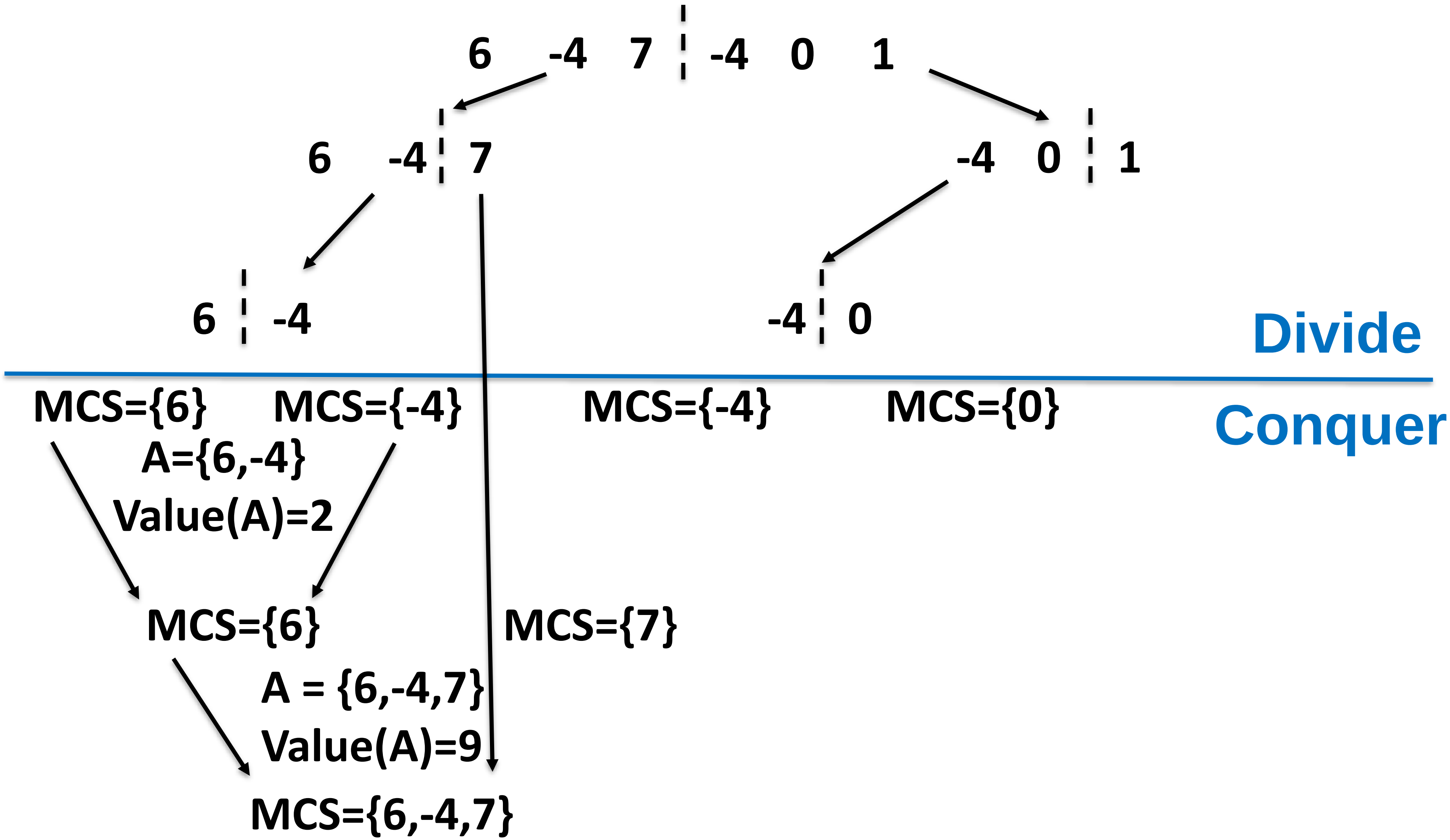

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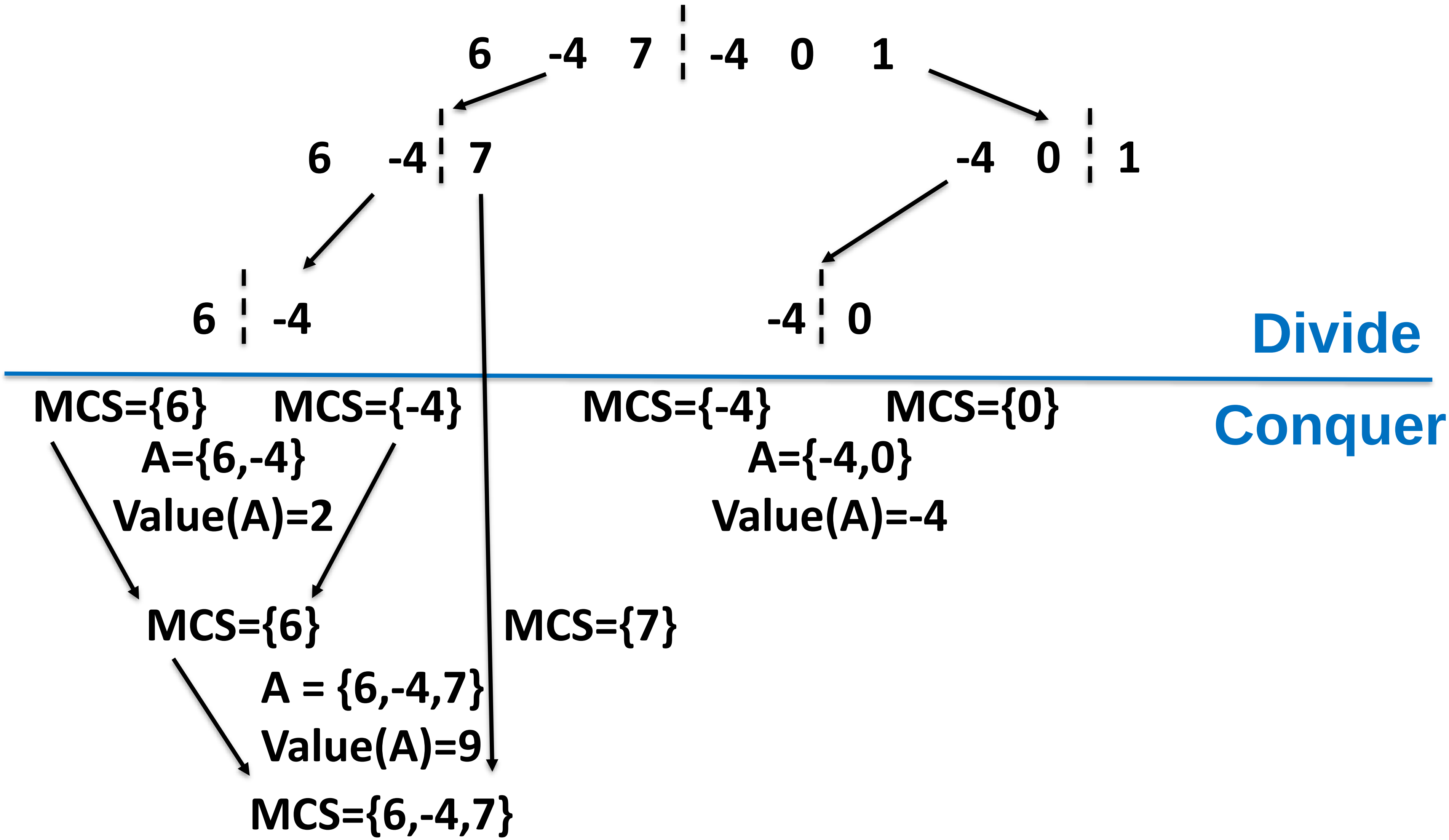

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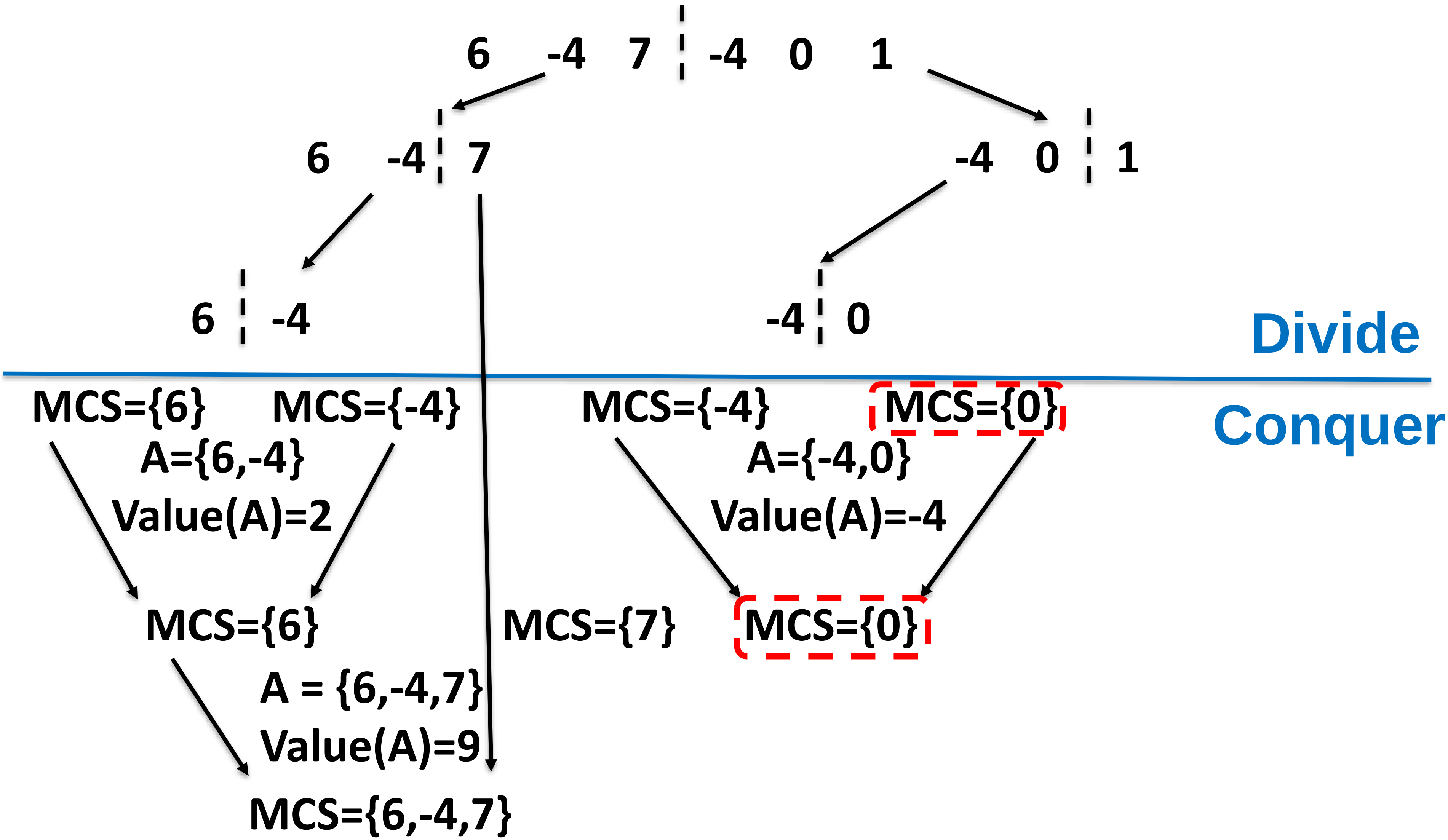

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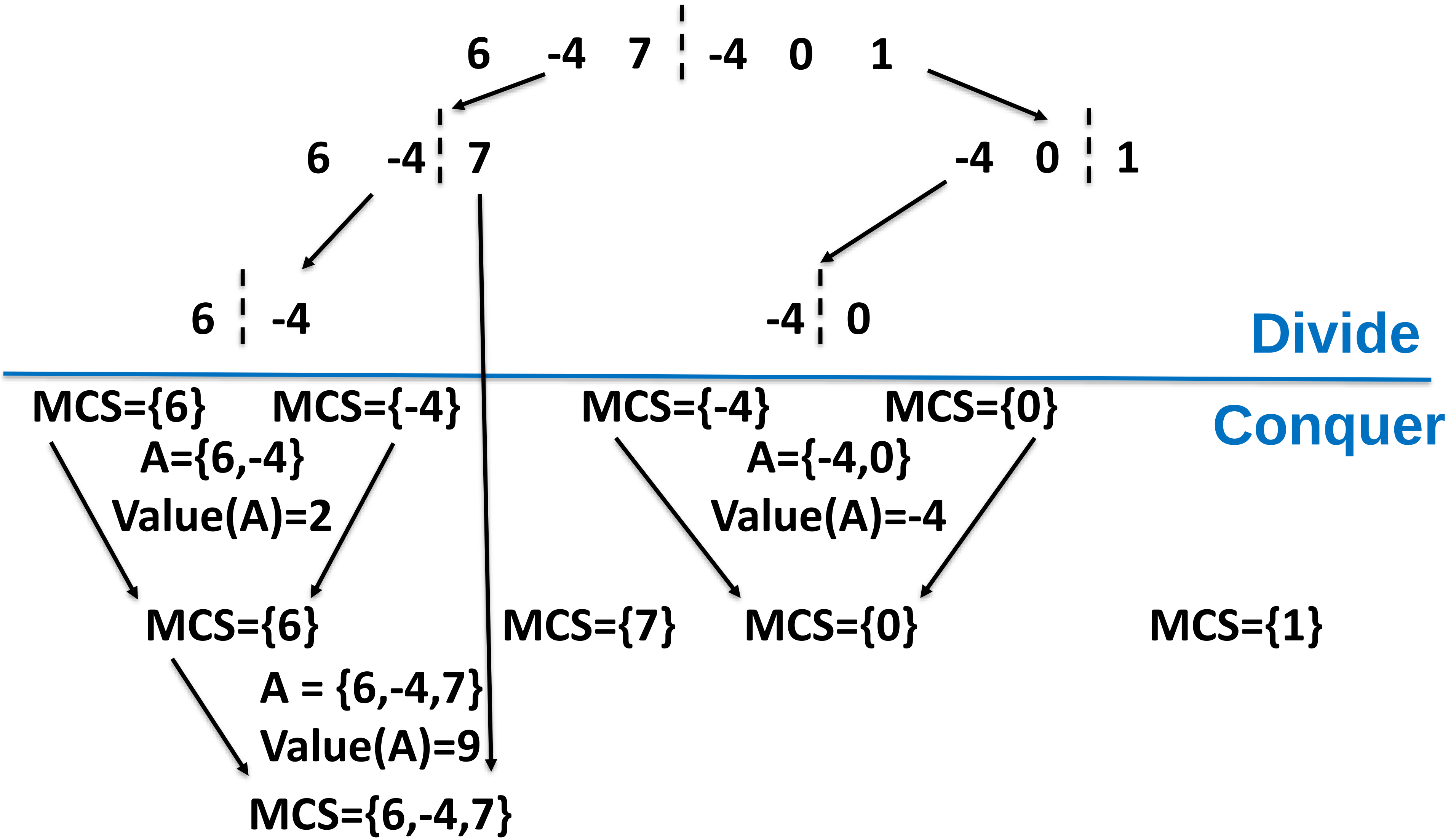

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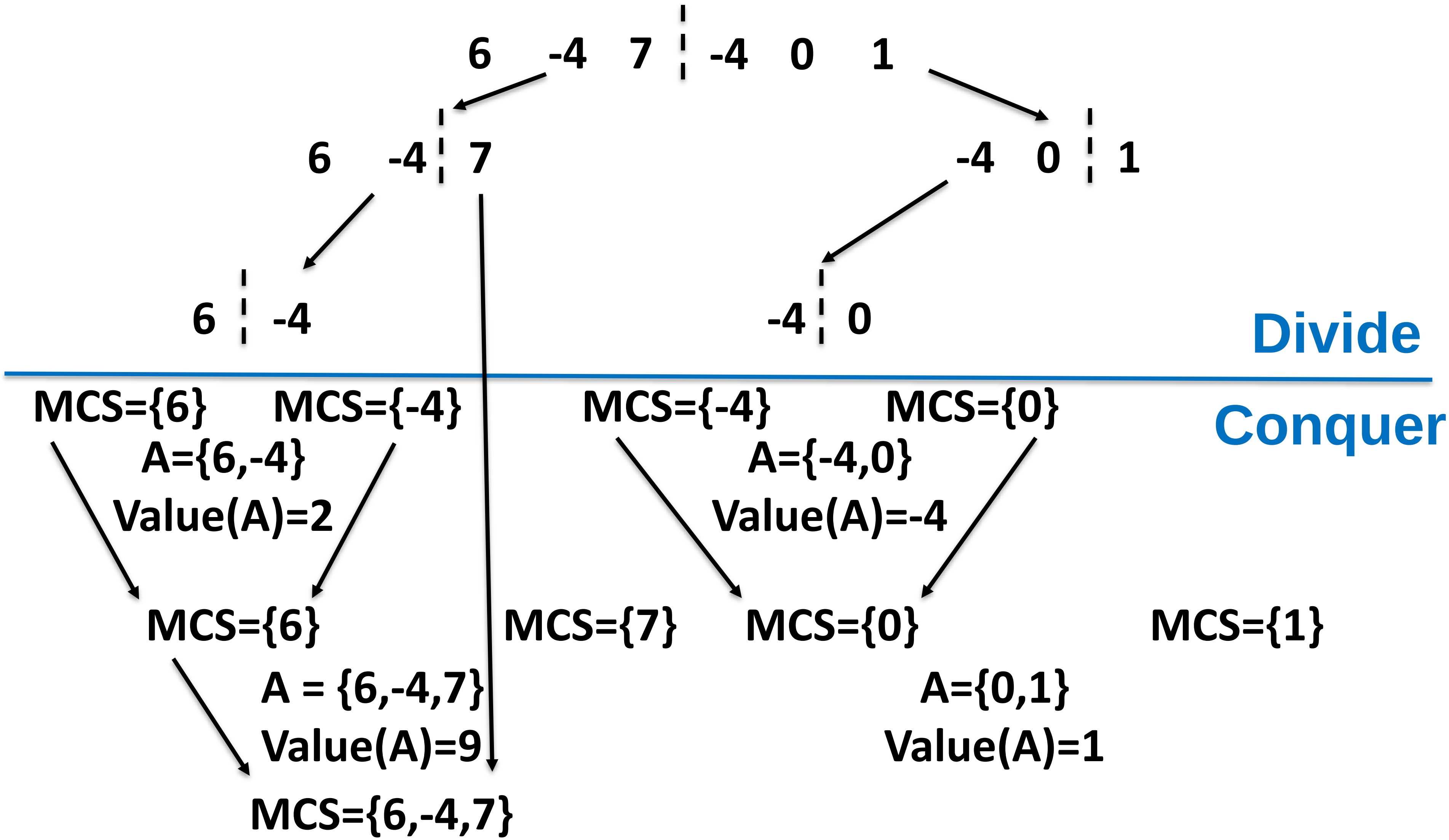

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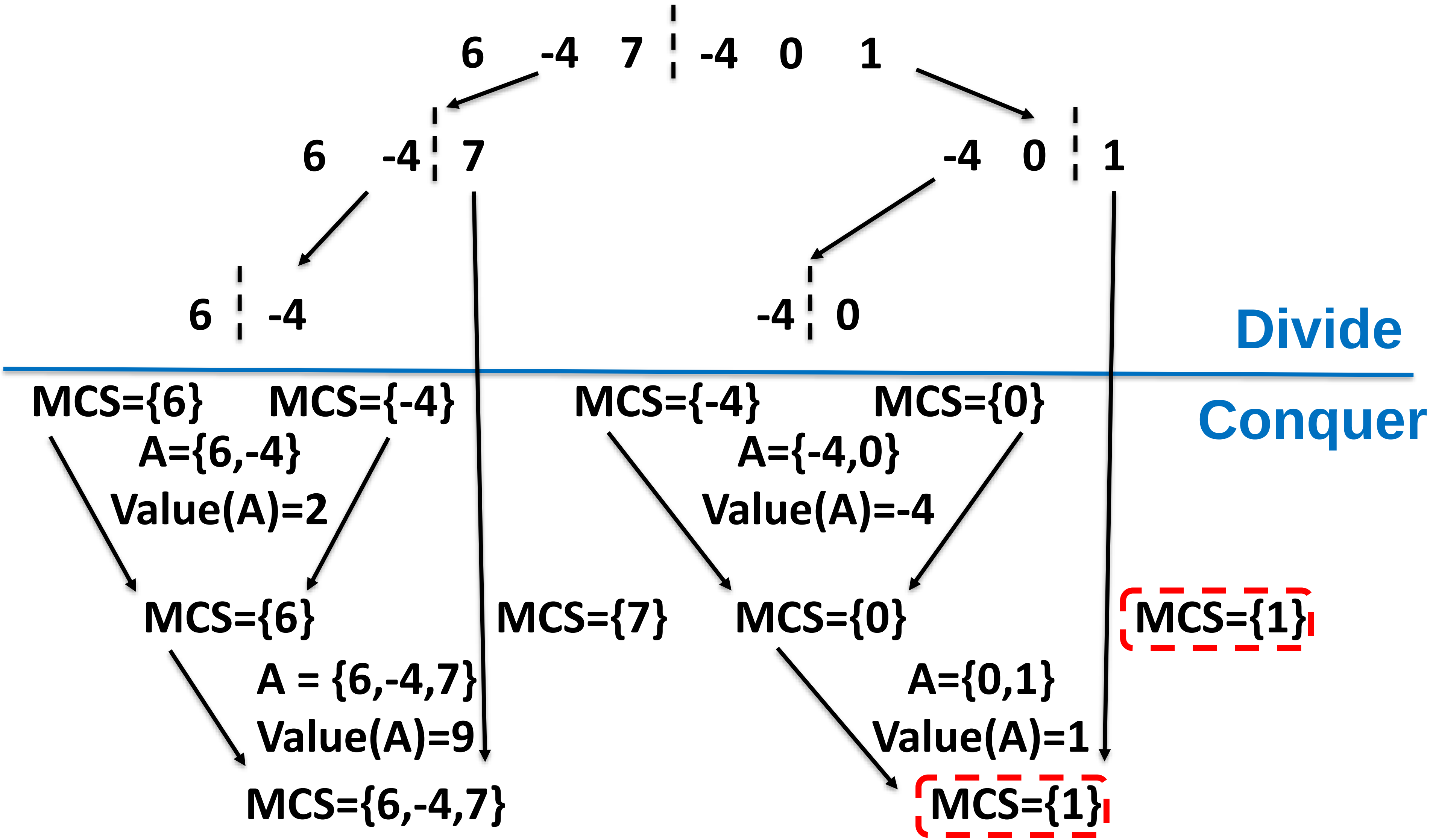

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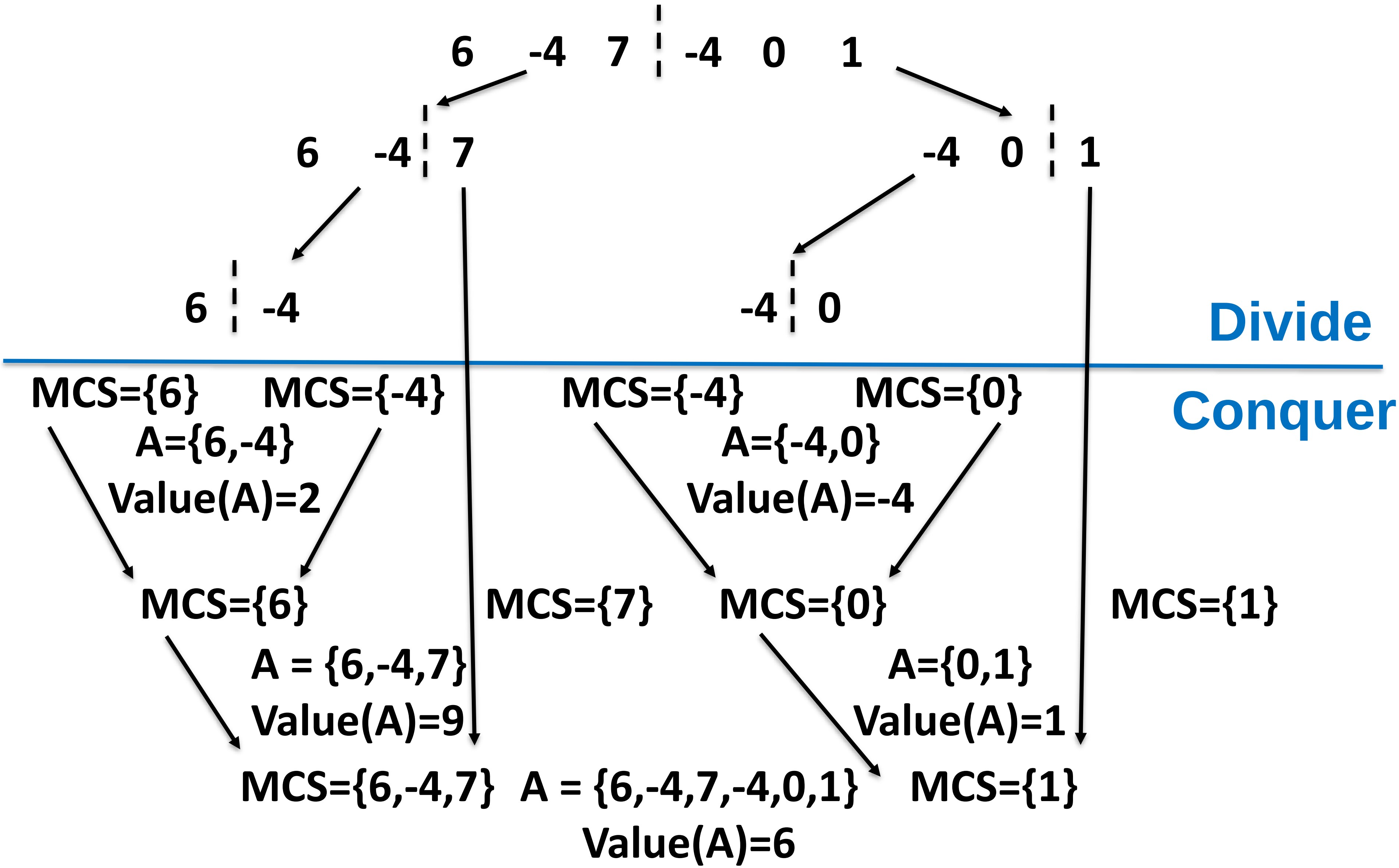

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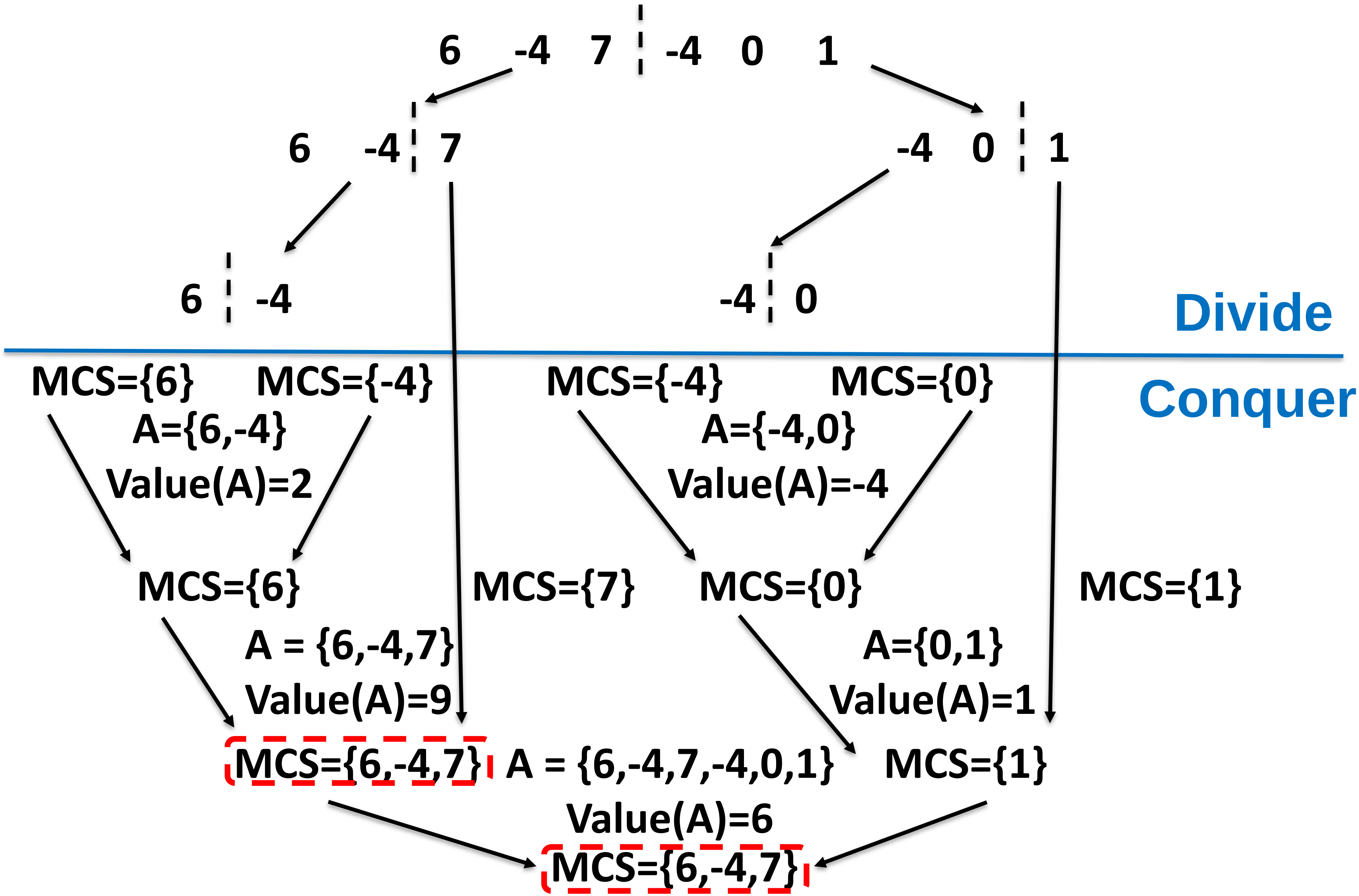

Outline

- Introduction to Part I
- Maximum Contiguous Subarray Problem
  - Problem definition
  - A brute force algorithm
  - A data-reuse algorithm
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Counting Inversions Problem
  - Problem definition
  - A brute force algorithm
  - A divide-and-conquer algorithm
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- $T(n)$: time needed to run $MCS(A, s, t)$

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    if s = t then return A[s] // O(1)
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    Find $MCS(A, s, m)$; // $T(\lceil \frac{n}{2} \rceil)$

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```

$$T(1) = O(1)$$

$$T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) \quad \text{for } n > 1$$

Analysis of the D&C Algorithm

To simplify the analysis, we assume that n is a power of 2

$$T(n) = \begin{cases} 2T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

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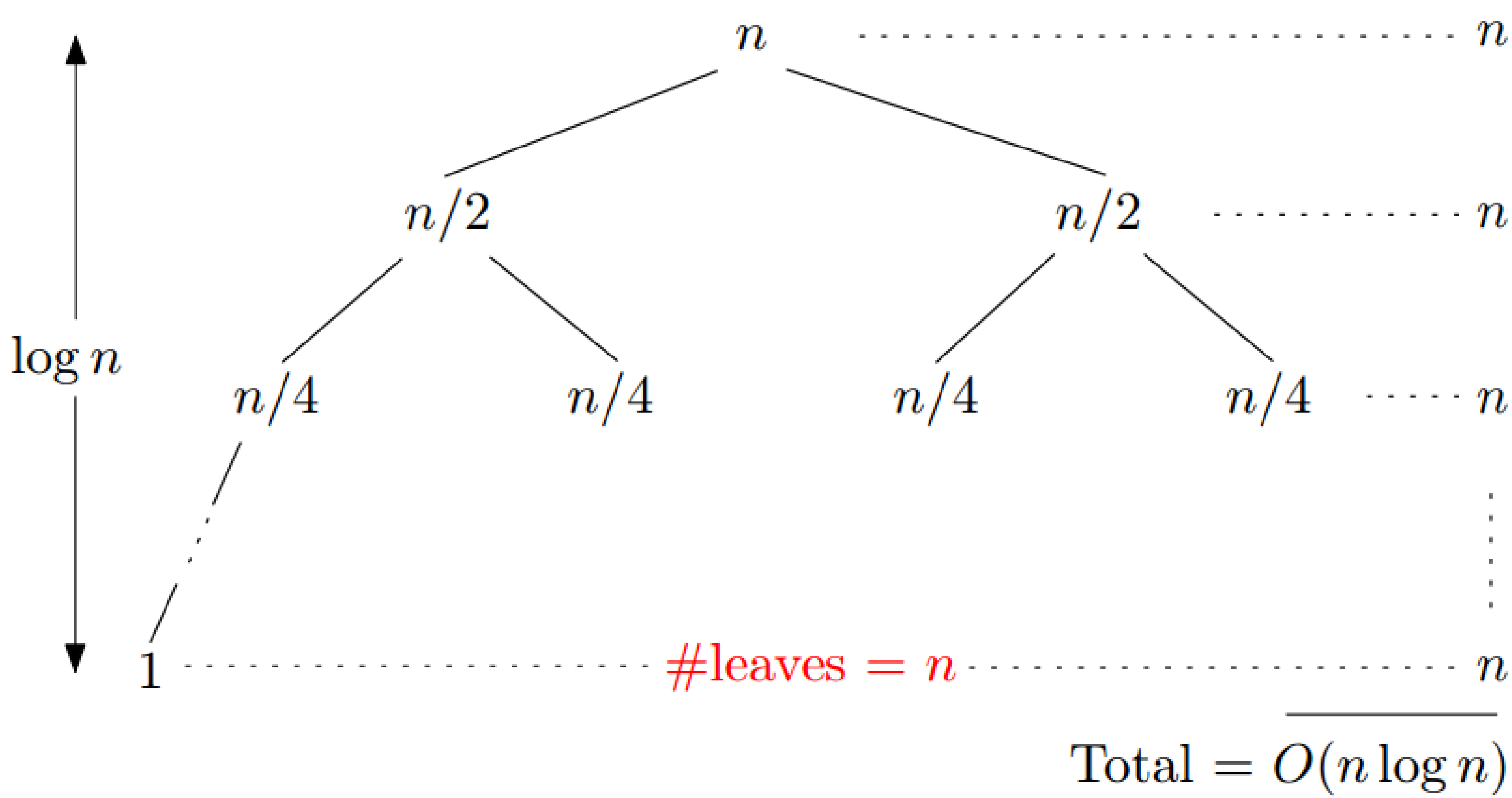

Summary

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Can you solve the problem in $O(n)$ time?

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