曹建铁　　2037517 7　　　201411　　可靠性与系统工程学院

3. (1) $\alpha(\mu_0)=P_{\mu_0}\{\bar{x}\ge c\}=P_{\mu_0}\left\{\frac{\bar{x}-\mu_0}{\sqrt{\frac{\sigma^2}{n}}}\ge\frac{c-\mu_0}{\sqrt{\frac{\sigma^2}{n}}}\right\}=1-\Phi\left(\frac{\sqrt{n}(c-\mu_0)}{\sigma}\right)$

(2) $\beta(\mu_1)=P_{\mu_1}\{\bar{x}<c\}=P_{\mu_1}\left\{\frac{\bar{x}-\mu_1}{\sqrt{\frac{\sigma^2}{n}}}<\frac{c-\mu_1}{\sqrt{\frac{\sigma^2}{n}}}\right\}=\Phi\left(\frac{\sqrt{n}(c-\mu_1)}{\sigma}\right)$　$\mu_1>\mu_0$

(3) 对固定n，当 $\alpha(\mu_0)\downarrow$，由(1) $c\uparrow$，又由(2) $\beta(\mu_1)\uparrow$

当 $\beta(\mu_1)\downarrow$，由(2) $c\downarrow$，由(1)得 $\alpha(\mu_0)\uparrow$

(4) 由 $\mu_0<c<\mu_1$，当n增大时，$\frac{\sqrt{n}(c-\mu_0)}{\sigma}$ 增大，而 $\frac{\sqrt{n}(c-\mu_1)}{\sigma}$ 减小

$\Rightarrow \lim\limits_{n\to+\infty}\frac{\sqrt{n}(c-\mu_0)}{\sigma}=+\infty$　　$\lim\limits_{n\to+\infty}\frac{\sqrt{n}(c-\mu_1)}{\sigma}=-\infty$

$\Rightarrow \lim\limits_{n\to+\infty}\alpha(\mu_0)=1-\lim\limits_{n\to+\infty}\Phi\left(\frac{\sqrt{n}(c-\mu_0)}{\sigma}\right)=1-\Phi(+\infty)=0$

$\lim\limits_{n\to+\infty}\beta(\mu_1)=\lim\limits_{n\to+\infty}\left(\frac{\sqrt{n}(c-\mu_1)}{\sigma}\right)=\Phi(-\infty)=0$

故当n增大时可使 $\alpha(\mu_0)$ 和 $\beta(\mu_1)$ 任意小

(5) 由 $\alpha(\mu_0)=1-\Phi\left(\frac{\sqrt{n}(c-\mu_0)}{\sigma}\right)\le\alpha$ 可得 $c\ge\mu_0+\frac{\sigma}{\sqrt{n}}z_{1-\alpha}$

存在一个显著性水平为α的检验的拒绝域 $W=\{(x_1,x_2\cdots x_n):\bar{x}\ge\mu_0+\frac{\sigma}{\sqrt{n}}z_{1-\alpha}\}$

$\beta(\mu_1)=\Phi\left(\frac{\sqrt{n}(\mu_0-\mu_1)}{\sigma}+z_{1-\alpha}\right)$

又由 $\beta(\mu_1)\le\beta$ 可得 $\Phi\left(\frac{\sqrt{n}(\mu_0-\mu_1)}{\sigma}+z_{1-\alpha}\right)\le\beta$

$\Rightarrow \frac{\sqrt{n}(\mu_0-\mu_1)}{\sigma}+z_{1-\alpha}\le z_\beta$

$\Rightarrow n\ge\frac{(z_{1-\alpha}-z_\beta)^2}{(\mu_1-\mu_0)^2}\sigma^2$

当 $\frac{(z_{1-\alpha}-z_\beta)^2}{(\mu_1-\mu_0)^2}\sigma^2$ 为整数时 可取 $n_{min}=\frac{(z_{1-\alpha}-z_\beta)^2}{(\mu_1-\mu_0)^2}\sigma^2$

否则取其整数部分+1

解. 似然比统计量

$$\lambda(x) = \frac{\sup\limits_{\mu \in \omega_0} \{p(x_1, \cdots, x_n; \mu)\}}{\sup\limits_{\mu \in \omega} \{p(x_1, \cdots, x_n; \mu)\}}$$

其中 $\omega = \{\mu: -\infty < \mu < +\infty\}$，$\omega_0 = \{\mu_0\}$

当 $\mu \in \omega$ 时 μ 的极大似然估计为 $\hat{\mu} = \bar{x}$

$$\lambda(x) = \exp\{\frac{1}{2} \times (\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}})^2\}$$

令 $z = \dfrac{\bar{x} - \mu_0}{\frac{\sigma_0}{\sqrt{n}}}$ $\Rightarrow \lambda(x) = \exp^{\frac{1}{2}z^2}$

又 $\because P_{\mu_0}\{\lambda(x) \geq c\} = P_{\mu_0}\{|z| \geq c_1\} = \alpha$

又 $\because$ 当 H_0 成立时 $z = \dfrac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma_0} \sim N(0,1)$

$\Rightarrow c_1 = z_{1-\frac{\alpha}{2}}$ 取 $z = \dfrac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma_0}$ 为检验统计量

在显著性水平 α 下，拒绝域为 $W = \{(x_1, x_2 \cdots x_n): |z| \geq z_{1-\frac{\alpha}{2}}\}$

37.

解 (1) $p(x_1,x_2\cdots,x_n;\lambda)=\prod_{i=1}^{n}\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}=e^{-n\lambda}\prod_{i=1}^{n}\frac{1}{x_i!}\exp\{(\ln\lambda)\sum_{i=1}^{n}x_i\}$

取 $c(\lambda)=\ln\lambda$，$T(x)=\sum_{i=1}^{n}x_i$.

检验函数为

$$\varphi(x)=\begin{cases}1, & T(x)>c\\ r, & T(x)=c\\ 0, & T(x)<c\end{cases}=\begin{cases}1, & \sum_{i=1}^{n}x_i>c\\ r, & \sum_{i=1}^{n}x_i=c\\ 0, & \sum_{i=1}^{n}x_i<c\end{cases}$$

⇒ 水平为 α 的 UMPT 的检验函数为

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i>c\\ \dfrac{\alpha-\sum_{k=c+1}^{+\infty}\frac{(n\lambda_0)^k}{k!}e^{-n\lambda_0}}{\frac{(n\lambda_0)^c}{c!}e^{-n\lambda_0}}, & \sum_{i=1}^{n}x_i=c\\ 0, & \sum_{i=1}^{n}x_i<c\end{cases}$$

(2) 令 $\theta=-\lambda$，$\theta_0=-\lambda_0$

则 假设检验问题可以化为

$H_0:\theta\leq\theta_0$ $\quad H_1:\theta>\theta_0$

⇒ $p(x_1,x_2\cdots x_n;\theta)=e^{n\theta}\prod_{i=1}^{n}\frac{1}{x_i!}\exp\{[\ln(-\theta)\sum_{i=1}^{n}x_i]\}$

检验函数

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i<c_2\\ r, & \sum_{i=1}^{n}x_i=c_2\\ 0, & \sum_{i=1}^{n}x_i>c_2\end{cases}$$

水平为 α 的 UMPT 检验函数:

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i<c_2\\ \dfrac{\alpha-\sum_{k=1}^{c_2-1}\frac{(n\lambda_0)^k}{k!}e^{-n\lambda_0}}{\frac{(n\lambda_0)^{c_2}}{c_2!}e^{-n\lambda_0}}, & \sum_{i=1}^{n}x_i=c_2\\ 0, & \sum_{i=1}^{n}x_i>c_2\end{cases}$$

(1) $p(x_1,x_2\cdots,x_n;\lambda)=\lambda^n I_{\{x_{(1)}>0\}}(x_1,x_2\cdots x_n)\exp\{-\lambda\sum_{i=1}^{n}x_i\}$

检验函数为

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i\le c_2\\ 0, & \sum_{i=1}^{n}x_i>c_2\end{cases}$$

由 $E_{\lambda_0}(\varphi(x))=P_{\lambda_0}\{\sum_{i=1}^{n}x_i\le c_2\}=\alpha$ $\Rightarrow c_2=T_\alpha(n,\lambda_0)$

$\Rightarrow$ 水平为 α 的 UMPT 的检验函数为

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i\le T_\alpha(n,\lambda_0)\\ 0, & \sum_{i=1}^{n}x_i>T_\alpha(n,\lambda_0)\end{cases}$$

(2) 令 $\theta=-\lambda$，$\theta_0=-\lambda_0$

假设检验问题可化为 $H_0: \theta\le\theta_0$；$H_1: \theta>\theta_0$

$p(x_1,x_2\cdots,x_n;\theta)=(-\theta)^n I_{\{x_{(1)}>0\}}(x_1,x_2\cdots x_n)\exp\{\theta\sum_{i=1}^{n}x_i\}$

检验函数为

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i\ge c\\ 0, & \sum_{i=1}^{n}x_i<c\end{cases}$$

由 $\sum_{i=1}^{n}(2\lambda_0x_i)\sim\chi^2(2n)$ $\Rightarrow E(\varphi(x))=\alpha$ $\Rightarrow P_{\lambda_0}\{\sum_{i=1}^{n}(2\lambda_0x_i)\ge2\lambda_0c\}=\alpha$

$\Rightarrow c=\frac{1}{2\lambda_0}\chi^2_{1-\alpha}(2n)$

水平为 α 的 UMPT 检验函数为

$$\varphi(x)=\begin{cases}1, & \sum_{i=1}^{n}x_i\ge\frac{1}{2\lambda_0}\chi^2_{1-\alpha}(2n)\\ 0, & \sum_{i=1}^{n}x_i<\frac{1}{2\lambda_0}\chi^2_{1-\alpha}(2n)\end{cases}$$