北京航空航天大學
BEIHANG UNIVERSITY

数值分析

主讲教师：贺慧霞

北京航空航天大学数学科学学院

第三章 矩阵特征值与特征向量的算法

----Jacobi方法

- 只适用于实对称方阵
- 可以求出所有特征值和特征向量

一、理论依据：

（1）如 A 为实对称矩阵，则一定存在正交矩阵 Q，使之相似于一个对角矩阵，而该对角矩阵的对角元正是A的特征值。

$$Q^{\mathrm{T}} = Q^{-1}, \quad QQ^{\mathrm{T}} = I, \quad QAQ^{\mathrm{T}} = \Lambda$$

（2）一个矩阵左乘一个正交矩阵或右乘一个正交矩阵，其F范数(Frobenius)不变。

$$\|A\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = trace(A^{\mathrm{T}}A) = trace(A^{\mathrm{T}}Q^{\mathrm{T}}QA) = \|QA\|_F^2$$

(3)在正交相似变换下,矩阵元素的平方和不变.

定理 设A是n阶实对称矩阵，则必有n阶正交矩阵Q使

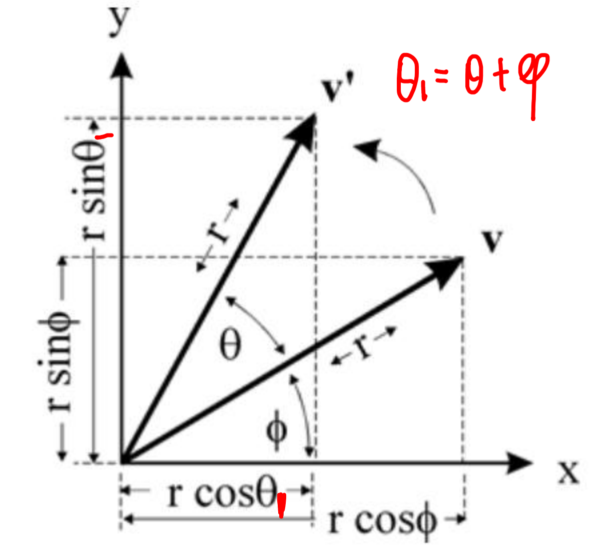
$$Q^T AQ = Q^{-1} AQ = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

其中Q的列是A的n个相互正交的单位特征向量，$\lambda_1,\lambda_2,\cdots,\lambda_n$是$A$的全部实特征值.

二、Jacobi方法的基本原理

对一个实对称矩阵A一定存在一个正交矩阵 R ($R^{-1}=R^T$)使得 $R^TAR=D$，其中 $D=\mathrm{diag}[d_1, d_2, \ldots, d_n]$，并且D的对角元素即为A 的特征值，对应的R的列向量即为相应的特征向量 。

基本思路： 通过一系列的**旋转变换（正交变换）**把A中非对角线上的非零元变为零.通过变换使得非对角元素的平方和减小.反复以上过程,使变换后的矩阵的非对角元素的平方和趋近于零,从而使该矩阵近似为对角矩阵,得到全部特征值和特征向量.



$\theta_1 = \theta + \varphi$

$\vec{v} = (x, y) = (r\cos\varphi, r\sin\varphi)$

$\vec{v'} = (x', y') = (r\cos\theta_1, r\sin\theta_1)$

$= (r\cos\theta\cos\varphi - r\sin\theta\sin\varphi,\ r\sin\theta\cos\varphi + r\cos\theta\sin\varphi)$

$= (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underline{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

矩阵 $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ 通常叫做**旋转变换矩阵**.

对应的变换称做**旋转变换**.

$$U(1,2)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\qquad A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$U(1,2)A=\begin{pmatrix}\mathrm{c}\cdot a_{11}-s\cdot a_{21} & \mathrm{c}\cdot a_{12}-s\cdot a_{22} & \mathrm{c}\cdot a_{31}-s\cdot a_{23}\\ s\cdot a_{11}+c\cdot a_{21} & s\cdot a_{12}+c\cdot a_{22} & s\cdot a_{13}+c\cdot a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$AU(1,2)=\begin{pmatrix}\mathrm{c}\cdot a_{11}+s\cdot a_{12} & \mathrm{c}\cdot a_{12}-s\cdot a_{11} & a_{13}\\ s\cdot a_{22}+c\cdot a_{21} & -s\cdot a_{21}+c\cdot a_{22} & a_{23}\\ \mathrm{c}\cdot a_{31}+s\cdot a_{32} & -s\cdot a_{31}+c\cdot a_{32} & a_{33}\end{pmatrix}$$

定义下面的 n 阶正交矩阵：

$u_{pp}=u_{qq}=\cos\varphi$　$u_{ii}=1, i\neq p,q$

$u_{pq}=-\sin\varphi$　$u_{qp}=\sin\varphi$

$u_{ij}=0, i\neq j, i,j\neq p,q$

$$
U(p,q,\varphi)=\begin{array}{c}\left[\begin{array}{ccccccc}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & \cos\varphi & & -\sin\varphi & & \\
 & & & \ddots & & & \\
 & & & 1 & & & \\
 & & & & \ddots & & \\
 & & \sin\varphi & & \cos\varphi & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{array}\right]\end{array}
\begin{array}{c} \\ \\ (p) \\ \\ \\ \\ (q) \\ \\ \\ \end{array}
$$

平面旋转矩阵

称为R^n中平面$\{x_p, x_q\}$的旋转变换，也称为吉文斯变换.

平面旋转矩阵的性质：

1. U_{pq}与单位矩阵I只在$(p,p),(p,q),(q,q),(q,p)$四个位置元素不同;

2. U_{pq}为正交矩阵$(U_{pq}^T = U_{pq}^{-1})$;

$U_{pq}^T A U_{pq} \longrightarrow$ 改变A的第 $\left.\begin{matrix}\text{i, j行}\\ \text{i, j列}\end{matrix}\right\}$ 元素

3. $U_{pq}A$只需计算第i行第j行元素,即对矩阵$A=(a_{ij})$,

$$U_{pq}A=(a'_{ij})=(a_{ij}),\quad \begin{pmatrix} a'_{pl} \\ a'_{ql} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{pl} \\ a_{ql} \end{pmatrix}$$

4. AU_{pq}只需计算第i列第j列元素,即对矩阵$A=(a_{ij})$,

$$(a'_{pl},a'_{ql})=(a_{pl},a_{ql})\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},l=1,2,\cdots,n.$$

利用平面旋转变换，可将向量x中的指定元素变为零.

- **变换过程：在保证相似条件下，使主对角线外元素趋于零！**

- 记 n 阶方阵 $A=[a_{ij}]$，对 A 做下面的变换：

$$\boldsymbol{A}_1=\boldsymbol{U}_{pq}{}^T\boldsymbol{A}\boldsymbol{U}_{pq}, \tag{3.12}$$

A_1 仍然是实对称阵，因为，$U_{pq}{}^T=U_{pq}{}^{-1}$，知 A_1 与 A 的特征值相同.

- 下面，以 4 阶矩阵为例，来计算 (3.12)

取 $p=2, q=3$，$A_1=U_{pq}^T A U_{pq}$.　　$A=\left(a_{ij}\right)$,

$$\boldsymbol{U}_{pq}=\boldsymbol{U}_{23}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U_{23}=\begin{pmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}I&&0\\&(CS)&\\0&&I\end{pmatrix},\quad U_{23}^T=\begin{pmatrix}1&0&0&0\\0&\cos\theta&\sin\theta&0\\0&-\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}I&&0\\&(CS)^T&\\0&&I\end{pmatrix},$$

$$A=(a_{ij})=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix},$$

记$A_1=(b_{ij})=\begin{pmatrix}b_{11}&b_{12}&b_{13}&b_{14}\\b_{21}&b_{22}&b_{23}&b_{24}\\b_{31}&b_{32}&b_{33}&b_{34}\\b_{41}&b_{42}&b_{43}&b_{44}\end{pmatrix}=U_{23}^T A U_{23}$

$$A_1=\begin{pmatrix}I&&0\\&(CS)^T&\\0&&I\end{pmatrix}\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}\begin{pmatrix}I&&0\\&(CS)&\\0&&I\end{pmatrix}$$

$$=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\(CS)^T A_{21}&(CS)^T A_{22}&(CS)^T A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}\begin{pmatrix}I&&0\\&(CS)&\\0&&I\end{pmatrix}=\begin{pmatrix}a_{11}&A_{12}(CS)&a_{14}\\(CS)^T A_{21}&\boxed{(CS)^T A_{22}(CS)}&(CS)^T A_{23}\\a_{41}&A_{32}(CS)&a_{44}\end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{14} \\ b_{41} & b_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{pmatrix}, \quad \{i, j\} \cap \{p, q\} = \varnothing, \quad (1)$$

$$\begin{pmatrix} b_{21} & b_{24} \\ b_{31} & b_{34} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{pmatrix}, \quad i = p, q, j \neq p, q, \quad (2)$$

$$\begin{pmatrix} b_{12} & b_{13} \\ b_{42} & b_{43} \end{pmatrix} = \begin{pmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad i \neq p, q, j = p, q. \quad (3)$$

$$\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad i = p, j = q, \quad (4)$$

$$\begin{pmatrix} a_{11} & A_{12}(CS) & a_{14} \\ (CS)^T A_{21} & \boxed{(CS)^T A_{22}(CS)} & (CS)^T A_{23} \\ a_{41} & A_{32}(CS) & a_{44} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

$$b_{11} = a_{11}, \; b_{14} = a_{14}, \\ b_{41} = a_{41}, \; b_{44} = a_{44}.$$

$$\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta a_{22} + \sin\theta a_{32} & \cos\theta a_{23} + \sin\theta a_{33} \\ -\sin\theta a_{22} + \cos\theta a_{32} & -\sin\theta a_{23} + \cos\theta a_{33} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$b_{22} = \cos^2\theta a_{22} + 2\sin\theta\cos\theta a_{32} + \sin^2\theta a_{33}$$

$$b_{33} = \sin^2\theta a_{22} - 2\sin\theta\cos\theta a_{32} + \cos^2\theta a_{33}$$

$$b_{23} = -\sin\theta\cos\theta a_{22} - \sin^2\theta a_{32} + \cos^2\theta a_{23} + \sin\theta\cos\theta a_{33}$$

$$b_{32} = -\sin\theta\cos\theta a_{22} - \sin^2\theta a_{23} + \cos^2\theta a_{32} + \sin\theta\cos\theta a_{33}$$

$$a_{ji} = a_{ij}$$

$$= b_{23} = \frac{1}{2}(a_{33} - a_{22})\sin 2\theta + a_{23}\cos 2\theta.$$

$$A_1 = U_{pq}^T A U_{pq} = (b_{ij})$$

$$\begin{cases} b_{pp} = \cos^2\theta a_{pp} + 2\sin\theta\cos\theta a_{pq} + \sin^2\theta a_{qq} \\ b_{qq} = \sin^2\theta a_{pp} - 2\sin\theta\cos\theta a_{pq} + \cos^2\theta a_{qq} \\ \boxed{b_{pq} = b_{qp} = \frac{1}{2}(a_{qq} - a_{pp})\sin 2\theta + a_{pq}\cos 2\theta.} \end{cases}$$

$$\begin{pmatrix} b_{pi} \\ b_{qi} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{pi} \\ a_{qi} \end{pmatrix}, \quad i \neq p, q,$$

$$\begin{pmatrix} b_{jp} \\ b_{jq} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{jp} \\ a_{jq} \end{pmatrix}, \quad j \neq p, q,$$

$$b_{ij} = b_{ji} = a_{ij}, i \neq p, q, j \neq p, q.$$

选取θ满足

$$\tan 2\theta = \frac{-2a_{pq}}{a_{qq} - a_{pp}},$$

则可得到

$$b_{pq} = b_{qp} = 0.$$

- 由此见到，矩阵 A_1 的第 p 行、列与第 q 行、列中的元素发生了变化，其它行、列中的元素不变。

$$\boxed{b_{pq} = b_{qp} = \frac{1}{2}\left(a_{qq} - a_{pp}\right)\sin 2\varphi + a_{pq}\cos 2\varphi}$$

$$\tan 2\varphi = \frac{-2a_{pq}}{a_{qq} - a_{pp}}$$

$$\begin{aligned} b_{pq} &= \frac{1}{2}\left(a_{qq} - a_{pp}\right)\sin 2\varphi + a_{pq}\cos 2\varphi \\ &= \sin 2\varphi\left[\frac{1}{2}\left(a_{qq} - a_{pp}\right) + a_{pq}\frac{a_{pp} - a_{qq}}{2a_{pq}}\right] \\ &= \sin 2\varphi\left[\frac{1}{2}\left(a_{qq} - a_{pp}\right) + \frac{a_{pp} - a_{qq}}{2}\right] = 0 \end{aligned}$$

为保证单值，限定 $|\varphi| \le \pi/4$

$$U_{pq}^T A U_{pq} = A^{(1)}$$ $A^{(1)}$ 的元素为：

$$(**)\begin{cases}
a_{pp}^{(1)} = a_{pp}\cos^2\phi + a_{qq}\sin^2\phi + 2a_{pq}\cos\phi\sin\phi \\
a_{qq}^{(1)} = a_{pp}\sin^2\phi + a_{qq}\cos^2\phi - 2a_{pq}\cos\phi\sin\phi \\
a_{pi}^{(1)} = a_{ip}^{(1)} = a_{pi}\cos\phi + a_{qi}\sin\phi, i \neq p,q, i = 1,2,\cdots,n \\
a_{qi}^{(1)} = a_{iq}^{(1)} = -a_{pi}\sin\phi + a_{qi}\cos\phi, i \neq p,q, i = 1,2,\cdots,n \\
a_{pq}^{(1)} = a_{qp}^{(1)} = \dfrac{1}{2}(a_{qq} - a_{pp})\sin 2\phi + a_{pq}\cos 2\phi \\
a_{ij}^{(1)} = a_{ji}^{(1)} = a_{ij}, i,j \neq p,q
\end{cases}$$

选取 ϕ 满足 $\tan 2\phi = \dfrac{2a_{pq}}{a_{pp} - a_{qq}}$ 我们就有 $a_{pq}^{(1)} = a_{qp}^{(1)} = 0$

Jacobi法的算法

1.给定矩阵A，收敛条件ε

2.找按模最大的元素a_{pq}

3.计算ϕ,$\sin\phi$和$\cos\phi$,其中ϕ满足$\tan 2\phi = \dfrac{2a_{pq}}{a_{pp} - a_{qq}}$

4. 用(**)式子计算 $A^{(1)} = U_{pq}{}^{T} A U_{pq}$中的元素$a_{ij}{}^{(1)}$,

5. 如果 $\sum_{i<j}^{n} |a_{ij}^{(1)}| < \varepsilon$,则停止，否则返回第 2 步.

$\boxed{\text{停止计算时,得特征值}\quad \lambda_i \approx a_{ii}^{(1)}.}$

特征向量的求解

设经过N次迭代，得到对角阵D,则做了下面的变换

$$A_1 = U_{p_1q_1}^T A U_{p_1q_1} \qquad A_2 = U_{p_2q_2}^T A_1 U_{p_2q_2} = U_{p_2q_2}^T U_{p_1q_1}^T A U_{p_1q_1} U_{p_2q_2}$$

$$\cdots\cdots$$

$$A_N = U_{p_Nq_N}^T \cdots U_{p_2q_2}^T U_{p_1q_1}^T A U_{p_1q_1} U_{p_2q_2} \cdots U_{p_Nq_N}$$

$$\cdots\cdots$$

记$U = U_{p_1q_1} U_{p_2q_2} \cdots U_{p_Nq_N}$,则$U$为正交矩阵,且$U^T A U = D$.

U的列向量为A的特征向量.

Jacobi算法的收敛性

定理：设A是实对称矩阵，由Jacobi方法第k次迭代得到的矩阵记为$A^{(k)}$，记

$$\eta_k = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (a_{i,j}^{(k)})^2 \qquad \text{则有} \lim_{k \to \infty} \eta_k = 0.$$

旋转矩阵 U_{pq} 的计算方法

$$\tan 2\varphi = \frac{2a_{pq}}{a_{pp}-a_{qq}},$$

(1) 当 $a_{pp}=a_{qq}$ 时, $\tan 2\varphi=\infty$， $\varphi=\frac{\pi}{4}\operatorname{sgn}(a_{pq})$.

(2) 当 $a_{pp}\neq a_{qq}$ 时, $\tan 2\varphi=\frac{2a_{pq}}{a_{pp}-a_{qq}}=\frac{2\tan\varphi}{1-\tan^2\varphi}=\frac{1}{c}$,

限定 $|\varphi|\le\frac{\pi}{4}$,

$|\tan\varphi|\le 1$,

$$\tan^2\varphi+2c\tan\varphi-1=0,$$

$$\tan\varphi=\frac{-2c\pm\sqrt{4c^2+4}}{2}=-c\pm\sqrt{c^2+1}=\frac{1}{c\pm\sqrt{c^2+1}},$$

故可取 $\tan\varphi=\frac{\operatorname{sgn}(c)}{|c|+\sqrt{c^2+1}}=t$,

$$\cos\varphi=\frac{1}{\sqrt{1+\tan^2\varphi}}=\frac{1}{\sqrt{1+t^2}},\ \sin\varphi=t\cdot\cos\varphi=\frac{t}{\sqrt{1+t^2}},$$

例 用Jacobi方法计算矩阵 $A=\begin{bmatrix}3.5 & -6 & 5\\ -6 & 8.5 & \boxed{-9}\\ 5 & -9 & 8.5\end{bmatrix}$ 的全部特征值和特征向量，误差 $\varepsilon=10^{-3}$.

$p=2,\ q=3$　U_{23}　a_{23}

解 第一步：选非对角线元素中的主元素

$$a_{23}=-9\ (p=2,q=3),\quad a_{22}=a_{33},\quad \tan 2\varphi=\infty$$

$$\therefore \varphi=\frac{\pi}{4}\operatorname{sgn}(a_{23})=-\frac{\pi}{4}\quad \cos\varphi=\frac{1}{\sqrt{2}},\quad \sin\varphi=-\frac{1}{\sqrt{2}}$$

$$\mathbf{U}_1=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$a_{12}^{(1)}$　$p=1,\ q=2$　U_{12}

$$A_1=U_{23}^TAU_{23}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\varphi\ \tfrac{1}{\sqrt{2}} & \sin\varphi\ \tfrac{1}{\sqrt{2}}\\ 0 & -\sin\varphi & \cos\varphi\end{bmatrix}\cdot\begin{bmatrix}3.5 & -6 & 5\\ -6 & 8.5 & -9\\ 5 & -9 & 8.5\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{bmatrix}=\begin{bmatrix}3.5 & \boxed{-7.7782} & -0.7071\\ -7.7782 & 17.5 & 0\\ -0.7071 & 0 & -0.5\end{bmatrix}$$

第二步:在 A_1 中选非对角线元素中的主元素

$$t=\frac{\operatorname{sgn}(c)}{|c|+\sqrt{c^2+1}}$$

$$a_{12}^{(1)}=-7.7782\,(p=1,q=2),\ \tan 2\varphi=\frac{2a_{12}^{(1)}}{a_{11}^{(1)}-a_{22}^{(1)}}=\frac{1}{c}$$

$$a_{11}^{(1)}=3.5,\ a_{11}^{(2)}=17.5$$

$$c=\frac{3.5-17.5}{-2\times 7.7782}=0.89995\approx 0.9,\quad t=\frac{1}{0.9+\sqrt{0.9^2+1}}=0.4454$$

$$\cos\varphi=\frac{1}{\sqrt{1+t^2}}=0.9135,\quad \sin\varphi=\frac{t}{\sqrt{1+t^2}}=0.4069$$

$$\mathbf{U}_2=\begin{pmatrix}0.9135 & -0.4069 & 0\\ 0.4069 & 0.9135 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A_2=U_{12}^TA_1U_{12}=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}3.5 & -7.7782 & -0.7071\\ -7.7782 & 17.5 & 0\\ -0.7071 & 0 & -0.\end{bmatrix}\cdot\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0.0358 & 0 & \boxed{-0.6459}\\ 0 & 20.9653 & 0.2877\\ -0.6459 & 0.2877 & -0.5\end{bmatrix}$$

$a_{13}^{(2)}$ $p=1$ $q=3$

第三步：在 A_2 中选非对角线元素中的主元素

$$\mathbf{U}_3=\begin{pmatrix}0.8316 & 0 & 0.5554\\ 0 & 1 & 0\\ -0.5554 & 0 & 0.8316\end{pmatrix}$$

$$a_{13}^{(2)}=-0.6459\ (p=1,q=3),\ \tan 2\varphi=\frac{2a_{13}^{(2)}}{a_{11}^{(2)}-a_{33}^{(2)}}=\frac{1}{c}$$

$$c=\frac{0.0358+0.5}{-2\times 0.6459}=-0.4148,\quad t=\frac{-1}{0.4148+\sqrt{0.4148^2+1}}=-0.6678$$

$$t=\frac{\mathrm{sgn}(c)}{|c|+\sqrt{c^2+1}}$$

$$\cos\varphi=\frac{1}{\sqrt{1+t^2}}=0.8316,\quad \sin\varphi=\frac{t}{\sqrt{1+t^2}}=-0.5554$$

$$a_{13}^{(2)}=0.0358,\ a_{13}^{(2)}=-0.5$$

$$A_3=U_{13}^T A_2 U_{13}=\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}\cdot\begin{bmatrix}0.0358 & 0 & -0.6459\\ 0 & 20.9653 & 0.2877\\ -0.6459 & 0.2877 & -0.5\end{bmatrix}\cdot\begin{bmatrix}\cos\varphi & 0 & -\sin\varphi\\ 0 & 1 & 0\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix}$$

$$=\begin{bmatrix}0.4672 & -0.1598 & 0\\ -0.1598 & 20.9653 & \boxed{0.2392}\\ 0 & 0.2392 & -0.9312\end{bmatrix}$$

$a_{23}^{(3)}$ $p=2,\ q=3$

第四步：在 A_3 中选非对角线元素中的主元素

$$a_{23}^{(3)}=0.2392\ (p=2,q=3),\ \tan 2\varphi=\frac{2a_{23}^{(3)}}{a_{22}^{(3)}-a_{33}^{(3)}}=\frac{1}{c}$$

$$\mathbf{U}_4=\begin{pmatrix}1 & 0 & 0\\ 0 & 0.9999 & 0.0107\\ 0 & -0.0107 & 0.9999\end{pmatrix}$$

$$c=\frac{20.9653+0.9312}{2\times 0.2392}=45.7703,\ t=\frac{1}{45.7703+\sqrt{45.7703^2+1}}=0.0109$$

$$\cos\varphi=\frac{1}{\sqrt{1+t^2}}=0.9999,\quad \sin\varphi=\frac{t}{\sqrt{1+t^2}}=0.0107$$

$$A_4=U_{23}^T A_3 U_{23}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\varphi & \sin\varphi\\ 0 & -\sin\varphi & \cos\varphi\end{bmatrix}\cdot\begin{bmatrix}0.4672 & -0.1598 & 0\\ -0.1598 & 20.9653 & 0.2392\\ 0 & 0.2392 & -0.9312\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{bmatrix}$$

$$=\begin{bmatrix}0.4672 & \boxed{-0.1598} & -0.0017\\ -0.1598 & -0.9340 & 0\\ -0.0017 & 0 & 20.9669\end{bmatrix}$$

$a_{12}^{(4)}$ $p=1$ $q=2$

第五步：在 A_4 中选非对角线元素中的主元素

$$U_5=\begin{pmatrix} 0.9937 & 0.1119 & 0 \\ -0.1119 & 0.9937 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_{12}^{(4)}=-0.1598\ (p=1,q=2),\ \tan 2\varphi=\frac{2a_{12}^{(4)}}{a_{11}^{(4)}-a_{22}^{(4)}}=\frac{1}{c}$$

$$c=\frac{0.4672+0.9340}{-2\times 0.1598}=-4.3842,\ \ t=\frac{-1}{4.3842+\sqrt{4.3842^2+1}}=-0.1126$$

$$\cos\varphi=\frac{1}{\sqrt{1+t^2}}=0.9937,\quad \sin\varphi=\frac{t}{\sqrt{1+t^2}}=-0.1119,$$

误差为$\varepsilon=10^{-3}$

$$A_5=U_{12}^T A_4 U_{12}=\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 0.4672 & -0.1598 & -0.0017 \\ -0.1598 & -0.9340 & 0 \\ -0.0017 & 0 & 20.9669 \end{bmatrix}\cdot\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{pmatrix} 0.4851 & 0 & \boxed{0.0017} \\ 0 & -0.9520 & \boxed{-0.0002} \\ 0.0017 & -0.0002 & 20.9669 \end{pmatrix}$$

$$\therefore \lambda_1\approx 0.4851,\quad \lambda_2\approx -0.9520,$$

$$\lambda_3\approx 20.9669$$

$$U = U_1 U_2 U_3 U_4 U_5 = \begin{pmatrix} 0.7998 & -0.3139 & 0.5117 \\ -0.2257 & 0.6325 & 0.7408 \\ -0.6159 & -0.7147 & 0.4351 \end{pmatrix}$$

故, 对应的一组单位特征向量分别为

$$x_1 = \begin{pmatrix} 0.7998 \\ -0.2257 \\ -0.6159 \end{pmatrix}, \; x_2 = \begin{pmatrix} -0.3139 \\ 0.6325 \\ -0.7147 \end{pmatrix}, \; x_3 = \begin{pmatrix} 0.5117 \\ 0.7408 \\ 0.4351 \end{pmatrix}$$

小　结

Jacobi法的算法

1. 给定矩阵A，收敛条件ε

2. 找按模最大的元素a_{pq}

3. 计算ϕ,$\sin\phi$和$\cos\phi$,其中ϕ满足$\tan 2\phi = \dfrac{2a_{pq}}{a_{pp} - a_{qq}}$

4. 用(**)式子计算$A^{(1)}$中的元素$a_{ij}^{(1)}$,

5. 如果$\sum\limits_{i<j}^{n} |a_{ij}^{(1)}| < \varepsilon$,则停止，否则返回第 2 步.

停止计算时,得特征值　$\lambda_i \approx a_{ii}^{(1)}$.

作 业

Page 65-66:

7， 8