18. 解 $\because$ $X_i \sim N(0, 0.3^2)$ 且 $X_1, X_2 \cdots X_{10}$ 相互独立

$\therefore$ $y_i = \frac{X_i}{0.3} \sim N(0,1)$ 且 $y_1, y_2 \cdots y_{10}$ 相互独立

$\therefore$ $\sum_{i=1}^{10} y_i^2 = \frac{1}{0.3^2} \sum_{i=1}^{10} X_i^2 \sim \chi^2(10)$

$\therefore$ $P\{\sum_{i=1}^{10} X_i^2 > 1.44\} = P\{\frac{1}{0.3^2} \sum_{i=1}^{10} X_i^2 > 16\} = 1 - P\{\frac{1}{0.3^2} \sum_{i=1}^{10} X_i^2 \le 16\} = 0.1$

19. 解 $\because$ $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2 \cdots n$ 且各 X_i 相互独立

$\therefore$ $\frac{X_i - \mu}{\sigma} \sim N(0,1)$, $i = 1, 2 \cdots n$

$\therefore$ $\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \sim \chi^2(n)$

令 $Y = \sum_{i=1}^{n} (X_i - \mu)^2$ $\cancel{= \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2}$

$$\Rightarrow f_Y(y) = \frac{1}{\sigma^2} f_2\left(\frac{y}{\sigma^2}\right) = \begin{cases} \dfrac{\left(\frac{y}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{y}{2\sigma^2}}}{2^{\frac{n}{2}} \sigma^2 \Gamma\left(\frac{n}{2}\right)} & y > 0 \\ 0 & y \le 0 \end{cases}$$

30. 解 $X_i \sim N(0,1)$, $i = 1, 2 \cdots 6$ 且相互独立

$\therefore$ $X_1 + X_2 + X_3 \sim N(0,3)$ $\qquad$ $X_4 + X_5 + X_6 \sim N(0,3)$

$\Rightarrow$ $\frac{X_1 + X_2 + X_3}{\sqrt{3}} \sim N(0,1)$ $\qquad$ $\frac{X_4 + X_5 + X_6}{\sqrt{3}} \sim N(0,1)$

$\therefore$ $\left(\frac{X_1 + X_2 + X_3}{\sqrt{3}}\right)^2 + \left(\frac{X_4 + X_5 + X_6}{\sqrt{3}}\right)^2 \sim \chi^2(2)$

$\Rightarrow$ $C = \frac{1}{3}$

2.

$\because \sum_{i=1}^{m} x_i \sim N(0, m\sigma^2) \quad \therefore \dfrac{\sum_{i=1}^{m} x_i}{\sqrt{m\sigma^2}} \sim N(0,1)$

又 $\because \dfrac{x_i}{\sigma} \sim N(0,1) \quad i=m+1, \cdots n$

$\dfrac{x_{m+1}}{\sigma}, \dfrac{x_{m+2}}{\sigma} \cdots \dfrac{x_n}{\sigma}$ 相互独立

$\therefore$

$\dfrac{1}{\sigma^2}\sum_{i=m+1}^{n} x_i^2 \sim \chi^2(n-m)$

$\therefore \dfrac{\sum_{i=1}^{m} x_i / \sqrt{m\sigma^2}}{\sqrt{\dfrac{\frac{1}{\sigma^2}\sum_{i=m+1}^{n} x_i^2}{n-m}}} = \sqrt{\dfrac{n-m}{m}} \dfrac{\sum_{i=1}^{m} x_i}{\sqrt{\sum_{i=m+1}^{n} x_i^2}} \sim t(n-m)$

$\therefore c = \sqrt{\dfrac{n-m}{m}}$

42. 证明

$t \sim t(n) \Rightarrow t^2 \sim F(1,n)$

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$\therefore P\{t^2 \le F_{1-\alpha}(1,n)\} = 1-\alpha$

$\Rightarrow P\{-\sqrt{F_{1-\alpha}(1,n)} \le t \le \sqrt{F_{1-\alpha}(1,n)}\} = 1-\alpha$

又 $P\{t \le -\sqrt{F_{1-\alpha}(1,n)}\} = 1 - P\{t \le \sqrt{F_{1-\alpha}(1-n)}\}$

$\therefore P\{t \le \sqrt{F_{1-\alpha}(1,n)}\} = 1 - \dfrac{\alpha}{2}$

$\Rightarrow [t_{1-\frac{\alpha}{2}}(n)]^2 = F_{1-\alpha}(1,n)$

7. 矩估计:

密度函数 $f(x;\theta)=\begin{cases}\theta x^{-\theta-1}, & x>1\\ 0, & x\le 1\end{cases}$ $\Rightarrow \mu=E(X)=\theta\int_1^{+\infty}x\cdot x^{-\theta-1}dx=\frac{\theta}{\theta-1}$

$\therefore \hat{\theta}=\frac{\bar{X}}{\bar{X}-1}$

极大似然估计

$L(\theta)=\prod_{i=1}^{n}\theta x_i^{-\theta-1}I_{\{x_i>1\}}(x_i)$　($x_1, x_2\cdots x_n$ 均 >1)

$\Rightarrow \ln L(\theta)=n\ln\theta-(\theta+1)\sum_{i=1}^{n}\ln x_i$

令 $\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\ln x_i=0$ $\Rightarrow \theta=\frac{n}{\sum_{i=1}^{n}\ln x_i}$

① $\frac{n}{\sum_{i=1}^{n}\ln x_i}>1$ 时 $\hat{\theta}=\frac{n}{\sum_{i=1}^{n}\ln x_i}$

② $\frac{n}{\sum_{i=1}^{n}\ln x_i}\le 1$ 时　由 $\theta>1$ 及 $\sum_{i=1}^{n}\ln x_i\ge n$ $\Rightarrow \frac{d\ln L(\theta)}{d\theta}<0$

即 $\ln L(\theta)$ 为 $\theta(\theta>1)$ 的单减函数　$\hat{\theta}=1$

$\Rightarrow \hat{\theta}=\max\{1, \frac{n}{\sum_{i=1}^{n}\ln x_i}\}$

10. (1) $L(\theta)=\theta^n e^{[-n\theta(\bar{x}-\alpha)]}$ $\Rightarrow \ln L(\theta)=n\ln\theta-n\theta(\bar{x}-\alpha)$　($x_1,\cdots x_n>\alpha$)

$\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}-n(\bar{x}-\alpha)$ $\Rightarrow \hat{\theta}=\frac{1}{\bar{x}-\alpha}$

(2) $L(\alpha)=\prod_{i=1}^{n}\theta I_{\{x_i>\alpha\}}(x_i)e^{-\theta(x_i-\alpha)}$ 为 α 的单增函数 $\Rightarrow \hat{\alpha}=x_{(1)}$

(3) 固定 θ，$L(\alpha,\theta)$ 是 α 的单增函数 $\Rightarrow \hat{\alpha}=x_{(1)}$

$L(\hat{\alpha},\theta)$ 看成 θ 的函数 $\Rightarrow \hat{\theta}=\frac{1}{\bar{x}-x_{(1)}}$　故 $\begin{cases}\hat{\alpha}=x_{(1)}\\ \hat{\theta}=\frac{1}{\bar{x}-x_{(1)}}\end{cases}$

13. 解

似然函数

$$L(\mu_1,\mu_2,\sigma^2)=(2\pi)^{-\frac{n_1+n_2}{2}}(\sigma^2)^{-\frac{n_1+n_2}{2}}e^{\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n_1}(x_i-\mu_1)^2+\sum_{i=1}^{n_2}(y_i-\mu_2)^2\right]\right)}$$

$$\ln L(\mu_1,\mu_2,\sigma^2)=-\frac{n_1+n_2}{2}\ln 2\pi-\frac{n_1+n_2}{2}\ln\sigma^2-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n_1}(x_i-\mu_1)^2+\sum_{i=1}^{n_2}(y_i-\mu_2)^2\right]$$

$$\Rightarrow\begin{cases}\frac{\partial\ln L}{\partial\mu_1}=\frac{n_1}{\sigma^2}(\bar{x}-\mu_1)\\ \frac{\partial\ln L}{\partial\mu_2}=\frac{n_2}{\sigma^2}(\bar{y}-\mu_2)\\ \frac{\partial\ln L}{\partial(\sigma^2)}=-\frac{n_1+n_2}{2\sigma^2}+\frac{1}{2\sigma^4}\left[\sum_{i=1}^{n}(x_i-\mu_1)^2+\sum_{i=1}^{m}(y_i-\mu_2)^2\right]\end{cases}$$

令 $\frac{\partial\ln L}{\partial\mu_1}=0$，$\frac{\partial\ln L}{\partial\mu_2}=0$，$\frac{\partial\ln L}{\partial(\sigma^2)}=0$

$$\Rightarrow\quad\hat{\mu}_1=\bar{x},\ \hat{\mu}_2=\bar{y},\quad\hat{\sigma}^2=\frac{1}{n_1+n_2}\left[\sum_{i=1}^{n_1}(x_i-\bar{x})^2+\sum_{i=1}^{n_2}(y_i-\bar{y})^2\right]$$

48. 证明

$$E(T_n)=\frac{2}{n(n+1)}\sum_{i=1}^{n}i\,E(X_i)=\mu$$

$$Var(T_n)=\frac{4}{n^2(n+1)^2}\sum_{i=1}^{n}i^2 Var(X_i)$$

$$=\frac{4\sigma^2}{n^2(n+1)^2}\sum_{i=1}^{n}i^2=\frac{4\sigma^2}{n^2(n+1)^2}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{2(2n+1)}{3n(n+1)}\sigma^2$$

由 Chebyshev 不等式

对 $\forall\varepsilon>0$ 有 $P\{|T_n-\mu|\geq\varepsilon\}\leq\frac{Var\theta(T_n)}{\varepsilon^2}=\frac{2(2n+1)}{3n(n+1)\varepsilon^2}\sigma^2$

$\lim_{n\to\infty}P\{|T_n-\mu|\geq\varepsilon\}=0\Rightarrow T_n$ 是 μ 的 相合估计