1. $\tilde{X} = X\tau \Rightarrow X = \tilde{X}\tau^{-1} = \tilde{X}\tau'$

$\hat{\beta} = (X'X)^{-1}X'Y = (\tau\tilde{X}'\tilde{X}\tau')^{-1}\tau\tilde{X}'Y = \tau(\tilde{X}'\tilde{X})^{-1}\tau'\tau\tilde{X}'Y$

$= \tau(\tilde{X}'\tilde{X})^{-1}\tilde{X}'Y = \tau\tilde{\beta} \Rightarrow \tilde{\beta} = \tau'\hat{\beta}$

$\tilde{Y} = \tilde{X}\tilde{\beta} = \tilde{X}\tau'\tau(\tilde{X}'\tilde{X})^{-1}\tilde{X}'Y = \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}'Y = \hat{Y}$

$\therefore$ $\tilde{\beta}$ depends on τ, but $\tilde{Y}$ doesn't depends on τ.

2. $Y = \begin{pmatrix} y_{11} & \cdots & y_{1q} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nq} \end{pmatrix} = \begin{pmatrix} y_1^T \\ \vdots \\ y_n^T \end{pmatrix} = (Y_1, \cdots, Y_q)$, $X = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix} = \begin{pmatrix} x_1^T \\ \vdots \\ x_n^T \end{pmatrix} = (X_1, \cdots, X_p)$

$B^T = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \vdots & \vdots & & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix}$, For $i = 1, 2, \cdots, n$, we get:

$$\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iq} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \vdots & \vdots & & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix} \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{pmatrix} = \begin{pmatrix} \beta_{11}x_{i1} + \cdots + \beta_{1p}x_{ip} \\ \vdots \\ \beta_{q1}x_{i1} + \cdots + \beta_{qp}x_{ip} \end{pmatrix}$$

$\|y_i - B^T x_i\|^2 = [(y_{i1} - \beta_{11}x_{i1} - \cdots - \beta_{1p}x_{ip})^2 + \cdots + (y_{iq} - \beta_{q1}x_{i1} - \cdots - \beta_{qp}x_{ip})^2]$

$\therefore \min_{B \in R^{p\times q}} \sum_{i=1}^{n} \|y_i - B^T x_i\|^2 = \min_{B \in R^{p\times q}} \sum_{i=1}^{n} \sum_{j=1}^{q} (y_{ij} - \beta_{j1}x_{i1} - \cdots - \beta_{qj}x_{ij})^2$

$= \min_{B \in R^{p\times q}} \sum_{j=1}^{q} \sum_{i=1}^{n} (y_{ij} - \beta_{1j}x_{i1} - \cdots - \beta_{qj}x_{ij})^2 = \min_{B \in R^{p\times q}} \sum_{j=1}^{q} \|Y_j - XB_j\|^2$

For each term $\|Y_j - XB_j\|^2$, we know: $\hat{B}_j = (X'X)^{-1}X'Y_j = \arg\min_{B_j \in R_{q\times 1}} \|Y_j - XB_j\|^2$

$\therefore$ Hence, $(\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_q) = \hat{B} = \arg\min_{B \in R^{p\times q}} \sum_{i=1}^{n} \|y_i - B^T x_i\|^2$, where $\hat{B}_i = (X'X)^{-1}X'Y_i$, $i = 1, 2, \cdots, q$

3. $H = X(X'X)^{-1}X'$, $H^2 = X\cancel{(X'X)^{-1}X'X}(X'X)^{-1}X' = X(X'X)^{-1}X' = H$

$HX = X(X'X)^{-1}X'X = X$

$(I_n - H)X = X - HX = X - X = 0$.

$H(I_n - H) = H - H^2 = 0$

$(I_n - H)H = H - H^2 = 0$

$\therefore$ H and $(I_n - H)$ are orthogonal

4. $E(\tilde{Y}) = E(\tilde{H}Y) = \tilde{H}E(Y) = \tilde{H}X\beta = X\beta$ $\quad \because \tilde{H}X = X$

$\mathrm{cov}(\tilde{Y}) = \mathrm{cov}(\tilde{H}Y) = \mathrm{cov}(\tilde{H}Y + HY - HY)$

$= \mathrm{cov}(HY) + \mathrm{cov}(HY, (\tilde{H} - H)Y) + \mathrm{cov}((\tilde{H} - H)Y, HY) + \mathrm{cov}((\tilde{H} - H)Y)$

$= \mathrm{cov}(\hat{Y}) + \mathrm{cov}((\tilde{H} - H)Y) + \mathrm{cov}(HY, (\tilde{H} - H)Y) + \mathrm{cov}((\tilde{H} - H)Y, HY)$

$\mathrm{cov}(HY, (\tilde{H} - H)Y) = H\,\mathrm{cov}(Y)(\tilde{H}' - H')$

$= \sigma^2 H(\tilde{H}' - H') = \sigma^2[H\tilde{H}' - HH'] = \sigma^2[H\tilde{H}' - H]$

$= \sigma^2[X(X'X)^{-1}\underbrace{X'\tilde{H}'}_{X'} - H] = \sigma^2[X(X'X)^{-1}X' - H] = 0$

$\therefore \mathrm{cov}(\tilde{Y}) - \mathrm{cov}(\hat{Y}) = \mathrm{cov}((\tilde{H} - H)Y) \succeq 0$

$\Rightarrow \mathrm{cov}(\hat{Y}) \preceq \mathrm{cov}(\tilde{Y})$

5. Under Gaussian Linear Model, we know: $y_i \overset{i.n.d}{\sim} N(x_i^T\beta, \sigma^2)$

$$L(\beta,\sigma^2;Y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y_i - x_i^T\beta)^2}{2\sigma^2}\right\} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\sum_{i=1}^{n} \frac{(y_i - x_i^T\beta)^2}{2\sigma^2}\right\}$$

$$l(\beta,\sigma^2;Y) = -\frac{n}{2}\log 2\pi - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2$$

$$\frac{\partial l(\beta,\sigma^2;Y)}{\partial\beta} = \frac{2}{2\sigma^2}\sum_{i=1}^{n} x_i(y_i - x_i^T\beta) = 0 \Rightarrow X^T(Y - X\beta) = 0 \quad \therefore \beta_{MLE} = (X^TX)^{-1}X^TY$$

$$\frac{\partial l(\beta,\sigma^2;Y)}{\partial\sigma^2} = -\frac{n}{\sigma} + \sum_{i=1}^{n}(y_i - x_i^T\beta)^2\frac{1}{\sigma^3} = 0 \Rightarrow \sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - x_i^T\beta_{MLE})$$

$\therefore \beta_{MLE} = \beta_{OLS} = (X'X)^{-1}X'Y, \quad \sigma^2_{MLE} = RSS/n$

$$MSE(\hat\sigma^2) = var(\hat\sigma^2) + bias^2(\hat\sigma^2) = var(\hat\sigma^2) = \frac{2\sigma^4}{n-p}$$

$$MSE(\tilde\sigma^2) = var(\tilde\sigma^2) + bias^2(\tilde\sigma^2) = \frac{n-p}{n^2}\cdot 2\sigma^4 + \frac{p^2}{n^2}\sigma^4 = \frac{p^2 + 2(n-p)}{n^2}\sigma^4$$

If $\frac{2\sigma^4}{n-p} \geq \frac{p^2+2(n-p)}{n^2}\sigma^4$, $p^2 - (n+2)p + 4n > 0$

$\therefore$ when $p^2 - (n+2)p + 4n \geq 0$, $MSE(\hat\sigma^2) \geq MSE(\tilde\sigma^2)$

otherwise $MSE(\hat\sigma^2) < MSE(\tilde\sigma^2)$

6. $H_0: \beta_1 = \beta_2 = \cdots = \beta_J \iff H_0: \begin{pmatrix}\beta_1-\beta_2\\ \beta_1-\beta_3\\ \cdots\\ \beta_1-\beta_J\end{pmatrix} = 0$

$$\begin{pmatrix}\beta_1-\beta_2\\ \beta_1-\beta_3\\ \cdots\\ \beta_1-\beta_J\end{pmatrix} = \begin{pmatrix}1 & -1 & 0 & \cdots & 0\\ 1 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & -1\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_J\end{pmatrix} = C\beta \qquad C := \begin{pmatrix}1 & -1 & 0 & \cdots & 0\\ 1 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & -1\end{pmatrix}_{(J-1)\times J}$$

$$F_C = \frac{(C\hat\beta - C\beta)^T\{C(X^TX)^{-1}C^T\}^{-1}(C\hat\beta - C\beta)}{(J-1)\hat\sigma^2} \sim F_{J-1,n-J} \xrightarrow{\text{under } H_0} \frac{(C\hat\beta)'[C(X^TX)^{-1}C^T]^{-1}(C\hat\beta)}{(J-1)\hat\sigma^2} \sim F_{J-1,n-J}$$

$$X = \left.\begin{pmatrix}1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\begin{matrix}\}n_1\\ \\ \\ \}n_2\\ \\ \\ \vdots\\ \}n_J\\ \\ \\ \end{matrix}\right. \quad X'X = \begin{pmatrix}\overbrace{1\cdots1}^{n_1} & \overbrace{0\cdots0}^{n_2} & \cdots & \overbrace{0\cdots0}^{n_J}\\ 0\cdots0 & 1\cdots1 & \cdots & 0\cdots0\\ \vdots & \vdots & & \vdots\\ 0\cdots0 & 0\cdots0 & \cdots & 1\cdots1\end{pmatrix}\begin{pmatrix}1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix} = \begin{pmatrix}n_1 & 0 & \cdots & 0\\ 0 & n_2 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & n_J\end{pmatrix}, \quad (X'X)^{-1} = \begin{pmatrix}1/n_1 & \cdots & 0\\ \vdots & 1/n_2 & \vdots\\ 0 & \cdots & 1/n_J\end{pmatrix}$$

$$C(X'X)^{-1}C' = \begin{pmatrix}1 & -1 & 0 & \cdots & 0\\ 1 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & -1\end{pmatrix}\begin{pmatrix}1/n_1 & \cdots & 0\\ \vdots & 1/n_2 & \vdots\\ 0 & \cdots & 1/n_J\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\\ -1 & 0 & \cdots & 0\\ 0 & -1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1\end{pmatrix} = \begin{pmatrix}1/n_1 & -1/n_2 & 0 & \cdots & 0\\ 1/n_1 & 0 & -1/n_3 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 1/n_1 & 0 & 0 & \cdots & -1/n_J\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\\ -1 & 0 & \cdots & 0\\ 0 & -1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1\end{pmatrix}$$

$$= \begin{pmatrix}1/n_1+1/n_2 & 1/n_1 & \cdots & 1/n_1\\ 1/n_1 & 1/n_1+1/n_3 & \cdots & 1/n_1\\ \vdots & \vdots & & \vdots\\ 1/n_1 & 1/n_1 & \cdots & 1/n_1+1/n_J\end{pmatrix} = \begin{pmatrix}1/n_2 & & \\ & \ddots & \\ & & 1/n_J\end{pmatrix} + \begin{pmatrix}1/n_1 & 1/n_1 & \cdots & 1/n_1\\ \vdots & \vdots & & \vdots\\ 1/n_1 & 1/n_1 & \cdots & 1/n_1\end{pmatrix}$$

$$= \begin{pmatrix}1/n_2 & & \\ & \ddots & \\ & & 1/n_J\end{pmatrix} + \begin{pmatrix}1/n_1\\ \vdots\\ 1/n_1\end{pmatrix}(1,\cdots,1) := A + uv^T, \quad A = \begin{pmatrix}1/n_2 & & \\ & \ddots & \\ & & 1/n_J\end{pmatrix}, u = \begin{pmatrix}1/n_1\\ \vdots\\ 1/n_1\end{pmatrix}, v = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

$$\therefore [C(X'X)^{-1}C']^{-1} = A^{-1} - (1 + v^TA^{-1}u)^{-1}A^{-1}uv^TA^{-1}$$

$$= \begin{pmatrix}n_2 & & \\ & \ddots & \\ & & n_J\end{pmatrix} - \left(1 + \frac{n_2+\cdots+n_J}{n_1}\right)^{-1}\begin{pmatrix}n_2 & & \\ & \ddots & \\ & & n_J\end{pmatrix}\begin{pmatrix}1/n_1 & \cdots & 1/n_1\\ \vdots & & \vdots\\ 1/n_1 & \cdots & 1/n_1\end{pmatrix}\begin{pmatrix}n_2 & & \\ & \ddots & \\ & & n_J\end{pmatrix}$$

$$= \begin{pmatrix} n_1 & & & \\ & n_2 & & \\ & & \ddots & \\ & & & n_J \end{pmatrix} - \frac{1}{n_1+\cdots+n_J}\begin{pmatrix} n_2^2 & \cdots & n_2 n_J \\ n_2 n_3 & \cdots & n_3 n_J \\ \vdots & & \vdots \\ n_2 n_J & \cdots & n_J^2 \end{pmatrix} = \frac{1}{n_1+\cdots+n_J}\begin{pmatrix} n_1(n_1+\cdots+n_J)-n_2^2 & \cdots & -n_2 n_J \\ -n_2 n_3 & \cdots & -n_3 n_J \\ \vdots & & \\ -n_2 n_J & \cdots & n_J(n_1+\cdots+n_J)-n_J^2 \end{pmatrix}$$

$$\hat{\beta} = (X'X)^{-1}X'Y = \begin{pmatrix} 1/n_1 & & \\ & \ddots & \\ & & 1/n_J \end{pmatrix} \begin{pmatrix} \overbrace{1 \cdots 1}^{n_1} & \overbrace{0 \cdots 0}^{n_2} & \overbrace{0 \cdots 0}^{n_J} \\ 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 1 \end{pmatrix} Y = \begin{pmatrix} \overbrace{1/n_1 \cdots 1/n_1}^{n_1} & \overbrace{0 \cdots 0}^{n_2} & \overbrace{0 \cdots 0}^{n_J} \\ 0 \cdots 0 & 1/n_2 \cdots 1/n_2 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots 0 & 0 \cdots 0 & 1/n_J \cdots 1/n_J \end{pmatrix} Y = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_J \end{pmatrix}$$

$$C\hat{\beta} = \begin{pmatrix} \bar{y}_1 - \bar{y}_2 \\ \bar{y}_1 - \bar{y}_3 \\ \vdots \\ \bar{y}_1 - \bar{y}_J \end{pmatrix}$$

$$(C\hat{\beta})'[C(X'X)^{-1}C']^{-1}(C\hat{\beta}) = \frac{1}{n_1+\cdots+n_J}(\bar{y}_1-\bar{y}_2, \bar{y}_1-\bar{y}_3, \cdots, \bar{y}_1-\bar{y}_J)\begin{pmatrix} n_1(n_1+\cdots+n_J)-n_2^2 & \cdots & -n_2 n_J \\ -n_2 n_3 & \cdots & -n_3 n_J \\ \vdots & & \\ -n_2 n_J & \cdots & n_J(n_1+\cdots+n_J)-n_J^2 \end{pmatrix}\begin{pmatrix} \bar{y}_1 - \bar{y}_2 \\ \bar{y}_1 - \bar{y}_3 \\ \vdots \\ \bar{y}_1 - \bar{y}_J \end{pmatrix}$$

$$= \sum_{j=1}^{J} n_j \left(\bar{y}_j - \frac{\sum_{j=1}^{J} \bar{y}_j n_j}{n_1+\cdots+n_J}\right)^2$$

$$= \sum_{j=1}^{J} n_j (\bar{y}_j - \bar{y})^2$$

$$\therefore (C\hat{\beta})'[C(X'X)^{-1}C']^{-1}(C\hat{\beta}) / (J-1) = \sum_{j=1}^{J} n_j (\bar{y}_j - \bar{y})^2 / (J-1)$$

$$\hat{\sigma}^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})/(n-J), \quad Y - X\hat{\beta} = \begin{pmatrix} y_1 - \bar{y}_1 \\ \vdots \\ y_{n_1} - \bar{y}_1 \\ y_{n_1+1} - \bar{y}_2 \\ \vdots \\ y_{n_2} - \bar{y}_2 \\ \vdots \\ y_{n_1+\cdots+n_{J-1}+1} - \bar{y}_J \\ \vdots \\ y_{n_1+\cdots+n_J} - \bar{y}_J \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} n_2 \\ \vdots \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} n_J \end{matrix}$$

$$\therefore \hat{\sigma}^2 = \sum_{j=1}^{J} \sum_{i \in I_j} (y_i - \bar{y}_j)^2 / (n-J)$$

$$\Rightarrow F_C = \frac{\sum_{j=1}^{J} n_j (\bar{y}_j - \bar{y})^2 / (J-1)}{\sum_{j=1}^{J} \sum_{i \in I_j} (y_i - \bar{y}_j)^2 / (n-J)} \sim F_{J-1, n-J}$$