5. Lec 7. Q 6.

6 Decomposition of the projection matrix Show that $H - H_1 = \tilde{H}_2$.

Let $X = (X_1, X_2)$. $H = X(X^TX)^{-1}X^T$ $H_1 = X_1(X_1^TX_1)^{-1}X_1^T$.

From Lemma $(X^TX)^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$

$$H = (X_1, X_2)\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} = X_1S_{11}X_1^T + X_2S_{21}X_1^T + X_1S_{12}X_2^T + X_2S_{22}X_2^T$$

① $X_1S_{11}X_1 = X_1(X_1^TX_1)^{-1}X_1^T + X_1(X_1^TX_1)^{-1}X_1^TX_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^TX_1(X_1^TX_1)^{-1}X_1^T$

$= H_1 + H_1X_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^TH_1$

② $X_2S_{21}X_1^T = -X_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^TX_1(X_1^TX_1)^{-1}X_1^T = -X_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^TH_1$

③ $X_1S_{12}X_2^T = -X_1(X_1^TX_1)^{-1}X_1^TX_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^T = -H_1X_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^T$

④ $X_2S_{22}X_2^T = X_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^T$

Hence ① + ② + ③ + ④ $= H_1 + (I_n - H_1)X_2(\tilde{X}_2^T\tilde{X}_2)^{-1}X_2^T(I_n - H_1)$

$= H_1 + \tilde{H}_2$.

6. Lec 8. Q1.

1 Testing linear hypotheses under heteroskedasticity Under the heteroskedastic linear model, how to test the hypotheses

$$H_0 : c^{\mathrm{T}}\beta = 0, \quad c \in \mathbb{R}^p$$

and

$$H_0 : C\beta = 0, \quad C \in \mathbb{R}^{l\times p}?$$

Under heteroskedasticity.

$$\hat{V} = n^{-1}\left(n^{-1}\sum_{i=1}^{n} x_i x_i^T\right)^{-1}\left(n^{-1}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 x_i x_i^T\right)\left(n^{-1}\sum_{i=1}^{n} x_i x_i^T\right)^{-1}$$

$$= (X^TX)^{-1}(X^T\hat{\Omega}X)(X^TX)^{-1} \quad \text{where } \hat{\Omega} = \operatorname{diag}(\hat{\varepsilon}_1^2, \cdots \hat{\varepsilon}_n^2).$$

Then $(\hat{\beta}-\beta) \xrightarrow{L} N(0, \hat{V})$.

① $H_0: c^T\beta = 0$

$$c^T\hat{\beta} - c^T\beta \xrightarrow{L} N(0, c^T\hat{V}c).$$

Use t-test. $t_c = \dfrac{c^T\hat{\beta} - c^T\beta}{\sqrt{c^T\hat{V}c}} \sim t_{n-p}$.

② $H_0: C\beta = 0$.

$$C\hat{\beta} - C\beta \xrightarrow{L} N(0, C\hat{V}C^T)$$

Use F-test $F_c = \dfrac{(C\hat{\beta}-C\beta)^T(C\hat{V}C^T)(C\hat{\beta}-C\beta)}{l} \sim F_{l,n-p}$.