1. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}$ (1)

$= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$ (2)

From (1), we get:

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}^{-1}\begin{pmatrix} (A-BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^{-1}$

$= \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix}\begin{pmatrix} (A-BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}$

$= \begin{pmatrix} (A-BD^{-1}C)^{-1} & 0 \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1} \end{pmatrix}\begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1}+D^{-1} \end{pmatrix}$

From (2), we get:

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}$

$= \begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$

By the uniqueness of inverse matrix, we know:

$(A-BD^{-1}C)^{-1} = A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1}$

$(D-CA^{-1}B)^{-1} = D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1}$

2. Sherman – Morrison formula: $(A+uv^T)^{-1} = A^{-1}-(1+v^TA^{-1}u)^{-1}A^{-1}uv^TA^{-1}$

proof: $A^{-1}-(1+v^TA^{-1}u)^{-1}A^{-1}uv^TA^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$

$\therefore (A+uv^T)(A^{-1}-(1+v^TA^{-1}u)^{-1}A^{-1}uv^TA^{-1}) = (A+uv^T)(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u})$

$= I - \frac{uv^TA^{-1}}{1+v^TA^{-1}u} + uv^TA^{-1} - \frac{uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$

$= I + uv^TA^{-1} - \frac{u\cancel{(1+v^TA^{-1}u)}v^TA^{-1}}{\cancel{1+v^TA^{-1}u}}$

$= I + uv^TA^{-1} - uv^TA^{-1} = I$

$(A^{-1}(1+v^TA^{-1}u)^{-1}A^{-1}uv^TA^{-1})(A+uv^T) = (A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u})(A+uv^T)$

$= I + A^{-1}uv^T - \frac{A^{-1}uv^T+A^{-1}uv^TA^{-1}uv^T}{1+v^TA^{-1}u} = I + A^{-1}uv^T - \frac{A^{-1}u\cancel{(1+v^TA^{-1}u)}v^T}{\cancel{1+v^TA^{-1}u}} = I$

$\therefore (A+uv^T)^{-1} = A^{-1}-(1+v^TA^{-1}u)^{-1}A^{-1}uv^TA^{-1}$

3. It's obvious that independent $\Rightarrow$ uncorrelated

then we proof uncorrelated $\Rightarrow$ independent

if $\Sigma_{12}=0$, $\Sigma = cov\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$, $a^T\Sigma a = a_1^T\Sigma_{11}a_1 + a_2^T\Sigma_{22}a_2$

$M_Y(a) = e^{a^T\mu+\frac{1}{2}a^T\Sigma a} = e^{a^T\mu+\frac{1}{2}(a_1^T\Sigma_{11}a_1+a_2^T\Sigma_{22}a_2)}$, $e^{a^T\mu} = e^{a_1^T\mu_1}e^{a_2^T\mu_2}$

$M_{Y_1}(a_1)\cdot M_{Y_2}(a_2) = e^{a_1^T\mu_1+\frac{1}{2}a_1^T\Sigma_{11}a_1}e^{a_2^T\mu_2+\frac{1}{2}a_2^T\Sigma_{22}a_2}$

$\Rightarrow M_Y(a) = M_{Y_1}(a_1)M_{Y_2}(a_2)$

$\therefore$ Y_1 and Y_2 are independent

4. $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$

$\operatorname{cov}(Y_1+Y_2, Y_1-Y_2) = \operatorname{var} Y_1 - \operatorname{cov}(Y_1, Y_2) + \operatorname{cov}(Y_2, Y_1) - \operatorname{var} Y_2$

$= 1-\rho+\rho-1 = 0 \Rightarrow Y_1+Y_2$ & Y_1-Y_2 are uncorrelated

Because (Y_1+Y_2, Y_1-Y_2) is multivariate normal

$\therefore Y_1+Y_2 \perp\!\!\!\perp Y_1-Y_2$

5. (a) $Y^TAY = Y^TAA^-AY = (AY)^TA^-AY$

$\operatorname{cov}(a^TY, AY) = a^T \operatorname{cov}(Y) A = \sigma^2 a^TA = 0$

$\therefore a^TY$ & AY are uncorrelated.

Since $\begin{pmatrix} a^TY \\ AY \end{pmatrix} = \begin{pmatrix} a^T \\ A \end{pmatrix} Y \sim$ multivariate normal distribution

$\therefore a^TY$ & AY are independent

Y^TAY is the function of AY, thus $a^TY \perp\!\!\!\perp Y^TAY$

(b) $Y^TAY = Y^TAA^-AY = (AY)^TA^-AY$, $Y^TBY = (BY)^TB^-BY$

$\operatorname{cov}(AY, BY) = A \operatorname{cov}(Y) B^T = \sigma^2 AB = 0$, $\operatorname{cov}(BY, AY) = 0$

$\therefore AY$ & BY are uncorrelated $\Rightarrow AY \perp\!\!\!\perp BY$

Since Y^TAY & Y^TBY are function of AY and BY

$\therefore Y^TAY \perp\!\!\!\perp Y^TBY$

6. $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2$

$\operatorname{cov}(\bar{X}, X_j-\bar{X}) = \operatorname{cov}(X_j, \bar{X}) - \operatorname{cov}(\bar{X}, \bar{X}) = \frac{1}{n} - \frac{1}{n} = 0 \quad j=1,2,\cdots,n$

def: $Y = (X_1-\bar{X}, \cdots, X_n-\bar{X})' = (Y_1, \cdots, Y_n)'$, thus $\operatorname{cov}(\bar{X}, Y) = 0 \Rightarrow \bar{X}$ & Y are uncorrelated

$\operatorname{cov}(\bar{X}, Y) = \begin{pmatrix} 1/n & 0 \\ 0 & \operatorname{cov}(Y) \end{pmatrix}$ and $(\bar{X}, Y)$ follows multivariate normal distribution

$\therefore \bar{X}$ and Y are independent

$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^n Y_i^2 = \frac{1}{n-1}Y'Y \Rightarrow \bar{X}$ and S^2 are independent

$\therefore \bar{X} \perp\!\!\!\perp S^2$

7. Get the idea from Peng Yan.

①. MGF of Y^TAY

$M_{Y^TAY}(t) = E(e^{tY^TAY}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-ty^TAy} \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(y-\mu)^T\Sigma^{-1}(y-\mu)\}\, dy$

$= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\{-\frac{1}{2}[y^T(I-2tA\Sigma)\Sigma^{-1}y - 2\mu^T\Sigma^{-1}y + \mu^T\Sigma^{-1}\mu]\}\, dy$

$= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\{-\frac{1}{2}[\mu^T\Sigma^{-1}\mu - \theta^TV^{-1}\theta]\} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\{-\frac{1}{2}[(y-\theta)^TV^{-1}(y-\theta)]\}\, dy$

Where $V^{-1} = (I-2tA\Sigma)\Sigma^{-1}$ and $\theta^T = \mu^T(I-2tA\Sigma)^{-1}$

$= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\{-\frac{1}{2}[\mu^T\Sigma^{-1}\mu - \theta^TV^{-1}\theta]\}$

$= |I-2tA\Sigma|^{-1/2} \exp\{-\frac{1}{2}[\mu^T(I-(I-2tA\Sigma)^{-1})\Sigma^{-1}\mu\}$

$\log[M_{Y^TAY}(t)] = -\frac{1}{2}\log|C| - \frac{1}{2}\mu^T(I-C^{-1})\Sigma^{-1}\mu$, where $C = I-2tA\Sigma$

$$\frac{\partial^2 \log[M_{Y^TAY}(t)]}{\partial t^2} = \frac{1}{2}\frac{1}{|C|^2}\left(\frac{d|C|}{dt}\right)^2 - \frac{1}{2}\frac{1}{|C|}\frac{d^2|C|}{dt^2} - \frac{1}{2}\mu^TC^{-1}\frac{d^2C}{dt^2}C^{-1}\Sigma^{-1}\mu + \mu\left(C^{-1}\frac{dC}{dt}\right)^2C^{-1}\Sigma^{-1}\mu$$

suppose the eigenvalues of $A\Sigma$ are λ_i, then

$$|C| = \prod_{i=1}^{p}(1-2t\lambda_i) = 1-2t\sum_i \lambda_i + 4t^2 \sum_{i\neq j}\lambda_i\lambda_j - \cdots + (-1)^p 2^p t^p \prod_{i=1}^{p}\lambda_i$$

$$\frac{d|C|}{dt} = -2\sum_i \lambda_i + 8t\sum_{i\neq j}\lambda_i\lambda_j + \cdots \qquad \frac{d^2|C|}{dt^2} = 8\sum_{i\neq j}\lambda_i\lambda_j + \cdots$$

when $t=0$,

$$\begin{cases} |C| = 1 \\ \frac{d|C|}{dt} = -2\sum_i \lambda_i = -2\,\mathrm{trace}(A\Sigma) \\ \frac{d^2|C|}{dt^2} = 8\sum_{i\neq j}\lambda_i\lambda_j \end{cases}$$

Hence,

$$\mathrm{var}(Y^TAY) = \frac{d^2}{dt^2}\log M_{Y^TAY}(0) = 2[\mathrm{trace}(A\Sigma)]^2 - 4\sum_{i\neq j}\lambda_i\lambda_j + 4\mu^T A\Sigma A\mu$$

$$= 2\left\{[\mathrm{trace}(A\Sigma)]^2 - 2\sum_{i\neq j}\lambda_i\lambda_j\right\} + 4\mu^T A\Sigma A\mu$$

$$= 2\left\{\mathrm{trace}[(A\Sigma)]^2 - 2\sum_{i\neq j}\lambda_i\lambda_j\right\} + 4\mu^T A\Sigma A\mu$$

$$= 2\,\mathrm{trace}[(A\Sigma)^2] + 4\mu^T A\Sigma A\mu, \quad \text{since } [\mathrm{trace}(A\Sigma)]^2 = \mathrm{trace}[(A\Sigma)^2] + 2\sum_{i\neq j}\lambda_i\lambda_j$$

$$= 2\,\mathrm{trace}(A\Sigma A\Sigma) + 4\mu^T A\Sigma\mu$$