Lecture 2

August 28, 2018

Simple Linear Regression

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Why can we also say E(Y)=Xb,ignoring conditional expectation?

- We can alternatively write the simple linear regression model as

$$y_i = \beta_0 + \beta_1 x_i + e_i, \quad \text{where } \mathbb{E}(e_i|x_i) = 0.$$

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- These estimates can be computed by simply differentiating Q with respect to b_0 and b_1 and setting the partial derivatives equal to zero. This gives

$$\frac{\partial Q}{\partial b_0} = -2 \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) = 0,$$

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- These equations can be equivalently written as

$$nb_0 + b_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \quad \text{and} \quad b_0 \sum_{i=1}^{n} x_i + b_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i.$$

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- The above two equations are known as the normal equations. These can be easily solved to give
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})y_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$
and
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x},$$
where we used the notation $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

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- Note that for these estimates to make sense, we need $\sum_{i=1}^{n}(x_i - \bar{x})^2 > 0$ which means that the x-variables are not all the same.

The OLS Regression Line

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- There is a very interesting alternative way of writing the regression line. This uses the following notation. Let

$$s_x := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad \text{and} \quad s_y := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}.$$

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- Also let

$$r := \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right),$$

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- Note that the regression line passes through the point $(\bar{y}, \bar{x})$.

Expected values and variances of the OLS estimators

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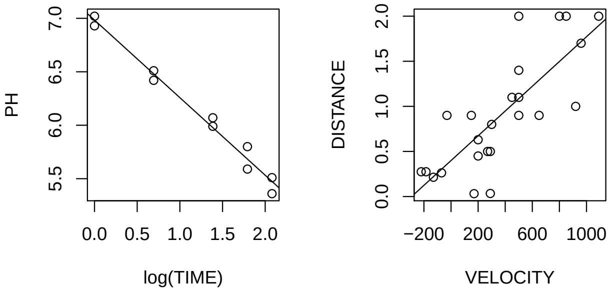
- Show that the last expression equals β_0.

Examples



PH
7.0
6.5
6.0
5.5
0.0
0.5
1.0
1.5
2.0
log(TIME)
DISTANCE
2.0
1.5
1.0
0.5
0.0
−200
200
600
1000
VELOCITY