1. (a) the set V consisting of all ~~linear~~ combinations of elements of S

(b) The space spanned by the column vectors

(c) The maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the space generated by its columns

(d) A square matrix that is not invertible is called singular. If and only if its determinant is 0

(e) An orthogonal matrix is a square matrix whose columns and rows are orthogonal unit vectors. $Q^TQ = QQ^T = I$, $Q^T = Q^{-1}$

(f) the number of column vectors are equals to its rank, or all rows and columns are linearly independent.

(g) $x^TAx = x^Taa^Tx = (a^Tx)^T(a^Tx) = \|a^Tx\|_2^2 \geq 0$ hence is a PSD

(h) rank(A) = 1 suppose $a = (a_1, a_2, a_3)$ Then $A = aa^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (a_1, a_2, a_3)$

$$= \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{pmatrix} = \left(a_1\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad a_2\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad a_3\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right)$$

We can see that all column vectors are in the same direction as a. So the maximum linearly independent columns is 1

(i) $x^TCx = x^TCC^Tx = (C^Tx)^T(C^Tx) = \|C^Tx\|_2^2 \geq 0$ for any $x \in \mathbb{R}^d$

hence C is PSD.