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Local Hardy spaces associated with inhomogeneous higher order elliptic operators

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Let L be a divergence form inhomogeneous higher order elliptic operator with complex bounded measurable coefficients. In this paper, for all $p \in (0, \infty)$ and L satisfying a weak ellipticity condition, the authors introduce the local Hardy spaces $h^p_L(\mathbb{R}^n)$ associated with L, which coincide with Goldberg's local Hardy spaces $h^p(\mathbb{R}^n)$ for all $p \in (0, \infty)$ when $L \equiv -\Delta$ (the Laplace operator). The authors also establish a real-variable theory of $h^p_L(\mathbb{R}^n)$, which includes their characterizations in terms of the local molecules, the square functions or the maximal functions, the complex interpolation and dual spaces. These real-variable characterizations on the local Hardy spaces are new even when $L \equiv -\mathrm{div}(A\nabla)$ (the divergence form homogeneous second-order elliptic operator). Moreover, the authors show that $h^p_L(\mathbb{R}^n)$ coincides with the Hardy space $H^p_{L+\delta}(\mathbb{R}^n)$ associated with the operator $L+\delta$ for all $p \in (0, \infty)$, where δ is some positive constant depending on the ellipticity and the off-diagonal estimates of L. As an application, the authors establish some mapping properties for the local Riesz transforms $\nabla^k(L+\delta)^{-1/2}$ on $H^p_{L+\delta}(\mathbb{R}^n)$, where $k \in \{0, \ldots, m\}$ and $p \in (0, 2]$.

Keywords: Higher order elliptic operator; ellipticity condition; off-diagonal estimate; parabolic Caccioppoli inequality; local Hardy space; tent space; Lipschitz space; square function; maximal function; molecule; complex interpolation; Riesz transform.

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1. Introduction

Let L be an inhomogeneous higher order elliptic operator of the form

$$L := \sum_{0\le|\alpha|,\,|\beta|\le m} (-1)^{|\alpha|}\partial^\alpha(a_{\alpha,\,\beta}\partial^\beta) \tag{1.1}$$

interpreted in the weak sense via a sesquilinear form (see Sec. 2.1 below for its exact definition), where $\{a_{\alpha,\,\beta}\}_{0\le|\alpha|,\,|\beta|\le m}$ are bounded measurable functions mapping $\mathbb{R}^n$ into $\mathbb{C}$. The aim of this paper is to establish a real-variable theory of the associated Hardy space adapted to L, when L is endowed with different ellipticity conditions. Recently, the study of Hardy spaces associated with different differential operators has attracted lots of attentions since they are good substitutes of the Lebesgue spaces $L^p(\mathbb{R}^n)$, for smaller p, in many fields of harmonic analysis and partial differential equations (see, for example, [18–21, 28, 52, 13, 29–31, 47, 58, 2, 45]). We now give a brief overview of the progress known so far in the research of Hardy spaces associated with elliptic operators. For more related results, we refer the reader to [29–32, 6, 40, 42, 46, 47, 43] and their references.

If $L \equiv -\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator, Fefferman and Stein [35] first introduced the Hardy space $H^p(\mathbb{R}^n)$ associated with $-\Delta$ and extensively studied its various real-variable characterizations. For example, $H^p(\mathbb{R}^n)$ can be characterized as the collection of all $f \in S'(\mathbb{R}^n)$ (the *space of Schwartz distributions*) such that their *square functions*

$$S_{-\Delta}(f)(x) := \left\{\iint_{\Gamma(x)} |t^2(-\Delta)e^{t^2\Delta}(f)(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\right\}^{\frac{1}{2}}, \quad x\in\mathbb{R}^n$$

belong to $L^p(\mathbb{R}^n)$. Here and hereafter, for all $x \in \mathbb{R}^n$,

$$\Gamma(x) := \left\{(y,\,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n\times(0,\,\infty) : |y-x|<t\right\} \tag{1.2}$$

denotes the *cone with vertex* x. Later, to make the Hardy space more suitable to the study on problems from partial differential equations, Goldberg [38] introduced a local version of the Hardy space $h^p(\mathbb{R}^n)$, which can be characterized as the collection of all $f \in S'(\mathbb{R}^n)$ such that their *local square functions*

$$S_{-\Delta,\,\mathrm{loc}}(f)(x) := \left\{\int_0^1 \int_{|y-x|<t} |t^2(-\Delta)e^{t^2\Delta}(f)(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\right\}^{\frac{1}{2}}, \quad x\in\mathbb{R}^n \tag{1.3}$$

belong to $L^p(\mathbb{R}^n)$. Observe that, since the heat semigroup $e^{t\Delta}$ can be represented via the convolution with the Gaussian kernel $t^{-n/2}e^{-\pi|x|^2/t}$, the study of the Hardy space associated with $-\Delta$ falls under the scope of the classical Calderón–Zygmund theory.

Let $L \equiv -\mathrm{div}(\mathrm{A}\nabla)$ be the divergence form homogeneous second-order elliptic operator, where $A := A(x)$ is an $n\times n$ matrix of complex bounded measurable coefficients defined on $\mathbb{R}^n$ which satisfies the *ellipticity condition*

$$\lambda|\xi|^2 \le \Re e(A\xi \cdot \overline{\xi}) \quad \text{and} \quad |A\xi \cdot \overline{\zeta}| \le \Lambda|\xi||\zeta|$$

for all $\xi,\, \zeta \in \mathbb{C}^n$ and for some positive constants $0 < \lambda \le \Lambda < \infty$ independent of ξ and ζ. Due to the absence of the regularity of the coefficients, the heat kernels of these operators may not have the following Gaussian upper bound: there exist positive constants C and c such that, for all $t \in (0,\, \infty)$ and $x,\, y \in \mathbb{R}^n$,

$$|e^{-tL}(x,\, y)| \le Ct^{-n/2} \exp\left\{-c\frac{|x-y|^2}{t}\right\} \tag{1.4}$$

and hence the classical Calderón–Zygmund theory fails in this case (see, for example, [3, Chap. 1] for an excellent survey on this topic). However, in many situations, the above Gaussian upper bound (1.4) can be replaced by the so-called $L^p - L^q$ off-diagonal estimates of semigroups [36, 25]. To be precise, recall that, for $1 \le p \le q \le \infty$, a family $\{T_t\}_{t>0}$ of operators is said to satisfy the $L^p - L^q$ *off-diagonal estimates* if there exist positive constants C and c such that, for all $t \in (0,\, \infty)$, closed sets $E,\, F \subset \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ supported in E,

$$\|T_t(f)\|_{L^q(F)} \le Ct^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \exp\left\{-c\frac{[d(E,\, F)]^2}{t}\right\} \|f\|_{L^p(E)},$$

where

$$d(E,\, F) := \inf_{x\in E,\, y\in F} |x-y|$$

denotes the *Euclidean distance* between E and F and $\|f\|_{L^p(E)} := \|f\chi_E\|_{L^p(\mathbb{R}^n)}$. Notice also that the Gaussian upper bound is just equivalent to an $L^1 - L^\infty$ off-diagonal estimate (see, for example, [5, p. 266]). Motivated by the pioneering work of Auscher *et al.* (see, for example, [6]) and resting on the theory of tent spaces in [22], the bounded holomorphic functional calculus in [54] and the above $L^p - L^q$ off-diagonal estimates, Hofmann–Mayboroda [42], Hofmann–Mayboroda–McIntosh [43] and Jiang–Yang [46] introduced the Hardy spaces $H^p_L(\mathbb{R}^n)$ associated to L via the following *square function*

$$S_L(f)(x) := \left\{\iint_{\Gamma(x)} |t^2Le^{-t^2L}(f)(y)|^2\, \frac{dy\, dt}{t^{n+1}}\right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^n, \tag{1.5}$$

where $\Gamma(x)$ is as in (1.2), and established their various real-variable characterizations, including those characterizations, respectively, in terms of the molecule, the generalized square function, the nontangential maximal function or the Riesz transform. Recall that, in particular, the Riesz transform $\nabla L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for all $p \in (n/(n+1),\, 2+\epsilon)$, where $\epsilon \in (0,\, \infty)$ depends on L.

Now, let $m \in \mathbb{N} := \{1, 2, \ldots\}$ and L_0 be a homogeneous $2m$th-order elliptic operator of the form

$$L_0 := \sum_{|\alpha|=m=|\beta|} (-1)^m \partial^\alpha(a_{\alpha,\,\beta}\partial^\beta), \tag{1.6}$$

where $\{a_{\alpha,\,\beta}\}_{|\alpha|=m=|\beta|}$ are bounded measurable functions mapping $\mathbb{R}^n$ into $\mathbb{C}$ and satisfy the following ellipticity condition.

Ellipticity condition $(\mathcal{E}_0)$**.** There exist positive constants $0 < \lambda \le \Lambda < \infty$ such that, for all multi-indices α and β satisfying $|\alpha| = m = |\beta|$, and $f \in \dot{W}^{m,\,2}(\mathbb{R}^n)$,

$$\|a_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^n)} \le \Lambda \quad \text{and} \quad \Re e(\mathfrak{a}(f,\, f)) \ge \lambda \|\nabla^m f\|^2_{L^2(\mathbb{R}^n)}, \tag{1.7}$$

where $\dot{W}^{m,\,2}(\mathbb{R}^n)$ denotes the *m-order homogeneous Sobolev space* equipped with the usual *norm*

$$\|f\|_{\dot{W}^{m,\,2}(\mathbb{R}^n)} := \left[\sum_{|\alpha|=m} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}\right]^{1/2},$$

and $\mathfrak{a}$ denotes the associated *sesquilinear form* of L_0.

This kind of operators is the homogenous version of the operators L defined in (1.1). They naturally arise in many problems in physics and mathematics (see, for example, [9, 26, 24, 7, 4, 51, 37, 53] and their references). As for the theory of Hardy spaces associated with higher order elliptic operators, in [27, 16], the authors independently introduced the Hardy spaces $H^p_{L_0}(\mathbb{R}^n)$ associated with the homogeneous higher order elliptic operator L_0, for $p \in (0,\, 1]$, as in (1.6). Moreover, in [16], the authors established various characterizations of $H^p_{L_0}(\mathbb{R}^n)$, respectively, in terms of the molecule, the generalized square function or the Riesz transform. See also [27] for many other interesting characterizations of these Hardy spaces in the case of $p = 1$. In [15], the authors considered the Hardy spaces $H^p_{L_0}(\mathbb{R}^n)$ for all $p \in (0,\, \infty)$. In particular, if L_0 satisfies a strong ellipticity condition (which is equivalent to the ellipticity condition $(\mathcal{E}_0)$ when $m = 1$), the authors established the maximal function characterization of $H^p_{L_0}(\mathbb{R}^n)$.

Motivated by the above results, the main purpose of this article is to study the Hardy space associated with an inhomogeneous higher order elliptic operator L of the form (1.1) under the ellipticity condition as weak as possible. Intuitively, the operator L defined in (1.1) can be viewed as a homogeneous $2m$th-order elliptic operator L_0 in (1.6) perturbed by the lower order terms. Thus, it is reasonable to imagine that the Hardy space $H^p_L(\mathbb{R}^n)$ associated with L as in (1.1) should be the "local" version of the Hardy space associated with L_0. A typical example is that, when $L \equiv -\Delta + I$, the Hardy space $H^p_{-\Delta+I}(\mathbb{R}^n)$ associated with the Schrödinger operator $-\Delta + I$ coincides with the local Hardy space $h^p(\mathbb{R}^n)$ of Goldberg (see Remark 3.15(iii) below for more details), which is just the local version of the Hardy space associated with $-\Delta$. Observe also that in the aforementioned works concerning the second-order elliptic operator L, L is always assumed to satisfying the "stronger" elliptic condition such as (1.7). The main reason for this assumption is that the "stronger" elliptic condition makes L an operator of type ω, which then implies that L has a bounded functional calculus in $L^2(\mathbb{R}^n)$. However, this result may no longer hold true when L just satisfies the weak ellipticity condition. Thus, to overcome this difficulty, instead of using the bounded functional calculus of L

in $L^2(\mathbb{R}^n)$, we consider the bounded functional calculus of the operator $L + \delta_0$, where $\delta_0 \in [0, \infty)$ is as in (2.7) below (see also (3.13) below for the corresponding Calderón reproducing formula in this case).

We now sketch the main results of this article.

Let L be an inhomogeneous higher order elliptic operator as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ in Sec. 2.1. We first introduce the local Hardy spaces $h^p_L(\mathbb{R}^n)$ associated with L for all $p \in (0, \infty)$ (see Definition 3.14 below), which coincide with Goldberg's local Hardy spaces $h^p(\mathbb{R}^n)$ for all $p \in (0, \infty)$ when $L \equiv -\Delta$ (the Laplace operator). Then we establish a real-variable theory of $h^p_L(\mathbb{R}^n)$, which includes their local molecular (see Theorem 3.18 below), square function (see Theorem 3.30 below) and nontangential maximal function characterizations (see Theorem 6.6 below), complex interpolation (see Theorem 3.31 below), and duality (see Theorem 5.3 below). These results extend the real-variable theory of Goldberg's local Hardy spaces to all local Hardy spaces associated with higher order elliptic operators, and are new even when $L \equiv -\mathrm{div}(A\nabla)$ (the divergence form homogeneous second-order elliptic operator).

Moreover, let $\delta \in (0, \infty)$ be a constant depending on the ellipticity and the off-diagonal estimates of L (see (4.1) below). We introduce the Hardy spaces $H^p_{L+\delta}(\mathbb{R}^n)$ and show that, for all $p \in (0, \infty)$, the local Hardy spaces $h^p_L(\mathbb{R}^n)$ coincide with the "global" Hardy spaces $H^p_{L+\delta}(\mathbb{R}^n)$ (see Corollary 4.9 below).

We point out that our study on the Hardy spaces $H^p_{L+\delta}(\mathbb{R}^n)$ has been preceded by several partial results. In [25], for L satisfying the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ and being symmetric, Davies proved that there exists a positive constant κ such that $\{e^{-t(L+\kappa)}\}_{t>0}$ satisfies "global" off-diagonal estimates. Auscher [3] also considered the $L^p(\mathbb{R}^n)$ theory of $L + s$, for L being a divergence form homogeneous second-order elliptic operator and $s \in (0, \infty)$. Also, in [48], for an abstract operator L satisfying a Poisson upper estimate, Jiang *et al.* introduced a local Hardy space $h^1_L(\mathbb{R}^n)$ in a way different from Definition 3.14 below and proved that $h^1_L(\mathbb{R}^n) = H^1_{L+I}(\mathbb{R}^n)$, where the latter Hardy space was introduced by Duong and Yang [31, 32]. Thus, Corollary 4.9 generalizes the corresponding result in [48] to the cases of inhomogeneous higher order elliptic operators and $p \neq 1$. In particular, if $L := -\Delta$, Corollary 4.9 implies that, for all $p \in (0, \infty)$,

$$h^p(\mathbb{R}^n) = h^p_{-\Delta}(\mathbb{R}^n) = H^p_{-\Delta+I}(\mathbb{R}^n),$$

where $h^p(\mathbb{R}^n)$ denotes the local Hardy space introduced by Goldberg [38] and $H^p_{-\Delta+I}(\mathbb{R}^n)$ denotes the Hardy space associated with the Schrödinger operator $-\Delta + I$ (see Remark 3.15(iii) for a more detailed discussion). Also, Corollary 4.9 makes us reduce a question of the local Hardy $h^p_L(\mathbb{R}^n)$ to that of the Hardy space $H^p_{L+\delta}(\mathbb{R}^n)$. As an application, we prove that, for all $k \in \{0, \ldots, m\}$ and $p \in (0, 2]$, the local Riesz transforms $\nabla^k(L + \delta)^{-1/2}$ are bounded from $H^p_{L+\delta}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and, for all $k \in \{1, \ldots, m\}$ and $p \in (\frac{n}{n+k}, 2]$, $\nabla^k(L + \delta)^{-1/2}$ are bounded from $H^p_{L+\delta}(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ (see Corollary 4.11 below).

As was seen from Definition 3.14 below, the quasi-norm of $h^p_L(\mathbb{R}^n)$ contains three terms: $\|S_{L,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)}$, $\|S_{L,\,\delta_0,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)}$ and $\|e^{-L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}$ (see Definition 3.14 below for their exact definitions). We point out that the term $\|S_{L,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)}$ is necessary when defining local Hardy spaces (see, for example, (1.3)), which has also naturally appeared in [48, 39] to define their local Hardy spaces. The term $\|S_{L,\,\delta_0,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)}$ is new, which arises because of the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ (see the proof of Theorem 3.18 below for more details). The term $\|e^{-L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}$ is needed when we consider the case that the heat semigroup of L may have no pointwise estimate or that $p < 1$ (see Remark 3.15(ii) below for more details). Recall that the space $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ was first introduced by Carbonaro *et al.* [17] in the case $p \in [1, \infty)$, where the authors introduced a kind of local Hardy spaces of differential forms on Riemannian manifolds and established many interesting real-variable properties of these local Hardy spaces. In this article, we further show that $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ coincide with the Hardy spaces $H^p_I(\mathbb{R}^n)$ associated with the identity operator I for all $p \in (0, \infty)$ (see Proposition 3.8 below), which have already naturally appeared in [14].

We now make some notes on the methods used in this article to establish the real-variable theory of $h^p_L(\mathbb{R}^n)$. Concretely speaking, when establishing the local molecular characterization of $h^p_L(\mathbb{R}^n)$, to show $h^p_L(\mathbb{R}^n)$ is contained in the local molecular Hardy space, we need a local Calderón reproducing formula of the operator $L + \delta_0$ (see (3.13) below) to project the elements in $h^p_L(\mathbb{R}^n)$ into the local tent space $t^p(\mathbb{R}^n \times (0, 1])$ or the space $L^p_{\mathcal{Q}}(\mathbb{R}^n)$. Also, the atomic decompositions of the local tent space $t^p(\mathbb{R}^n \times (0, 1])$ and $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ are needed. Here, we use the local Calderón reproducing formula for the operator $L + \delta_0$ (instead of L) is because $L + \delta_0$ (rather than L) satisfies the bounded functional calculus, when L satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. On the other hand, to show that the local molecular Hardy space is contained in $h^p_L(\mathbb{R}^n)$, we need to prove that each molecule m is uniformly bounded in $h^p_L(\mathbb{R}^n)$. This, together with Definitions 3.14 and 3.16 below, implies that we need to estimate $\|S_{L,\,\mathrm{loc}}(m)\|_{L^p(\mathbb{R}^n)}$, $\|S_{L,\,\delta_0,\,\mathrm{loc}}(m)\|_{L^p(\mathbb{R}^n)}$ and $\|e^{-L}(m)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}$ by considering two different cases of m based on the size of the radius of the associated ball of m. All the above estimates need the local off-diagonal estimates of the semigroup generated by L, which are established in Sec. 2.2 (see, for example, Proposition 2.10 below).

To obtain the square function characterization of $h^p_L(\mathbb{R}^n)$, we introduce the direct sum spaces

$$t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$$

for all $p \in (0, \infty)$, and establish a retract relation between $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ and $h^p_L(\mathbb{R}^n)$ via the local Calderón reproducing formula (3.13). We point out that, although Theorem 3.30 and Definition 3.14 look similar (in particularly in the case $p \in (0, 2]$), the main difference lies in the case $p \in (2, \infty)$, where, Definition 3.14 uses the duality to define $h^p_L(\mathbb{R}^n)$, while Theorem 3.30 still uses square functions. Notice also that the retract relation between

$t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ and $h^p_L(\mathbb{R}^n)$ plays an important role in the complex interpolation of $h^p_L(\mathbb{R}^n)$.

Also, to obtain the boundedness of the local Riesz transforms $\nabla^k(L+\delta)^{-1/2}$, $k \in \{0, \ldots, m\}$, we establish some local off-diagonal estimates of the gradient semigroups

$$\{(t^{1/(2m)}\nabla)^k e^{-tL}\}_{t>0}, \quad k \in \{0, \ldots, m\}$$

(see Theorem 2.11 below).

Finally, to establish the maximal function characterization of $H^p_{L+\delta}(\mathbb{R}^n)$, we prove the following inequalities that, for all $f \in L^2(\mathbb{R}^n)$,

$$\|S_{L+\delta_1}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|S_{h,\, L+\delta_2}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{N}^{\gamma}_{h,\, L+\delta}(f)\|_{L^p(\mathbb{R}^n)}$$

with $0 < \delta < \delta_1 < \delta_2 < \infty$ and the implicit positive constants independent of f, where $S_{L+\delta_1}(f)$, $S_{h,\, L+\delta_2}(f)$ and $\mathcal{N}^{\gamma}_{h,\, L+\delta}(f)$ are, respectively, the square function, the Lusin-area function and the nontangential maximal function defined as in (4.2), (6.18) and (6.25) (see Propositions 6.3 and 6.4 below). To this end, we prove two kinds of parabolic Caccioppoli's inequalities for L (see Lemmas 6.1 and 6.2 below). We point out that the proof of the first kind of parabolic Caccioppoli's inequalities is inspired by Auscher *et al.* [7], while the proof of the second one borrows an idea of induction from Barton [10], which enables us to remove all the gradient terms on the right-hand side of the obtained inequality (6.10) below.

The organization of this article is as follows. First, in Sec. 2.1, we recall some basic facts on the inhomogeneous operator L; then, in Sec. 2.2, we establish some off-diagonal estimates of L. In Sec. 3.1, we compare two kinds of spaces $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ and $H^p_I(\mathbb{R}^n)$, which are respectively introduced in [17] and [14], and are necessary in the notion of $h^p_L(\mathbb{R}^n)$. In Secs. 3.2 and 3.3, we introduce the local Hardy spaces $h^p_L(\mathbb{R}^n)$ associated with L for all $p \in (0, \infty)$ and characterize them via molecules for $p \in (0, 1]$, or the square function and prove the complex interpolation for $p \in (0, \infty)$. In Sec. 4, we introduce another Hardy space $H^p_{L+\delta}(\mathbb{R}^n)$ with $\delta \in (0, \infty)$ depending on the ellipticity and the off-diagonal estimates of L. We also establish the equivalence between the spaces $H^p_{L+\delta}(\mathbb{R}^n)$ and $h^p_L(\mathbb{R}^n)$ for all $p \in (0, \infty)$ in this section. Finally, in Secs. 5 and 6, we study the duality and the maximal function characterizations of $h^p_L(\mathbb{R}^n)$.

We end this section by making some conventions on the notation. Throughout the article, we always let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Denote the *differential operator* $\frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ simply by ∂^α, where $\alpha := (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We use C to denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line, and $C_{(\alpha,\ldots)}$ to denote a *positive constant* depending on the parameters $\alpha, \ldots$. *Constants with subscripts*, such as C_1, do not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\lambda \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and $\lambda B := B(x, \lambda r)$. Also, for any set $E \subset \mathbb{R}^n$,

χ_E denotes its *characteristic function* and, for all $z \in \mathbb{C}$, $\Re e z$ and $\Im m\, z$ denote its *real part* and *imaginary part*, respectively.

2. Inhomogeneous Higher Order Elliptic Operators

In this section, we study the inhomogeneous higher order elliptic operator L as in (1.1). To this end, we first collect some basic facts on L in Sec. 2.1; then, in Sec. 2.2, we establish some local off-diagonal estimates for L.

2.1. *Inhomogeneous higher order elliptic operators*

Let $W^{m,\,2}(\mathbb{R}^n)$ be the *m-order inhomogeneous Sobolev space* equipped with the usual *norm*

$$\|f\|_{W^{m,\,2}(\mathbb{R}^n)} := \left[\sum_{0\le|\alpha|\le m} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^2\right]^{1/2}.$$

For all multi-indices α and β satisfying $0 \le |\alpha|,\, |\beta| \le m$, let $a_{\alpha,\,\beta}$ be the complex-valued L^∞-function on $\mathbb{R}^n$. For all f and $g \in W^{m,\,2}(\mathbb{R}^n)$, define the *sesquilinear form* $\mathfrak{a}$, mapping $W^{m,\,2}(\mathbb{R}^n) \times W^{m,\,2}(\mathbb{R}^n)$ into $\mathbb{C}$, by setting

$$\mathfrak{a}(f,\, g) := \sum_{0\le|\alpha|,\,|\beta|\le m} \int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha g(x)}\, dx. \tag{2.1}$$

We endow $\mathfrak{a}$ with the following ellipticity conditions.

Ellipticity condition $(\mathcal{E})$**.** There exist constants $0 < \lambda \le \Lambda < \infty$ such that, for all $f,\, g \in W^{m,\,2}(\mathbb{R}^n)$,

$$\left|\sum_{0\le|\alpha|,\,|\beta|\le m} \int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha g(x)}\, dx\right| \le \Lambda\|f\|_{W^{m,\,2}(\mathbb{R}^n)}\|g\|_{W^{m,\,2}(\mathbb{R}^n)}$$

and

$$\Re e\left\{\sum_{0\le|\alpha|,\,|\beta|\le m} \int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha f(x)}\, dx\right\} \ge \lambda\|f\|_{W^{m,\,2}(\mathbb{R}^n)}^2.$$

Weak ellipticity condition $(\widetilde{\mathcal{E}})$**.** For all $f,\, g \in W^{m,\,2}(\mathbb{R}^n)$, define the *leading part* of $\mathfrak{a}$, mapping $W^{m,\,2}(\mathbb{R}^n) \times W^{m,\,2}(\mathbb{R}^n)$ into $\mathbb{C}$, by setting

$$\mathfrak{a}_0(f,\, g) := \sum_{|\alpha|=m=|\beta|} \int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha g(x)}\, dx. \tag{2.2}$$

We say that $\mathfrak{a}$ satisfies the *Weak ellipticity condition* $(\widetilde{\mathcal{E}})$ if $\mathfrak{a}_0$ satisfies the Ellipticity condition $(\mathcal{E}_0)$ in (1.7) and the coefficients $\{a_{\alpha,\,\beta}\}_{0\le|\alpha|,\,|\beta|\le m}$ are bounded.

For all $k \in \{1, \ldots, 2m\}$, let

$$M_k := \sup\{\|a_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^n)} : 0 \le |\alpha|, |\beta| \le m,\ |\alpha| + |\beta| = 2m - k\} \tag{2.3}$$

and $M := \sum_{k=1}^{2m} M_k^{\frac{2m}{k}}$. Then, for all α and β satisfying $0 \le |\alpha| + |\beta| < 2m$, we have $\|a_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^n)} \le M$.

Remark 2.1. (i) We first observe that, from [1, Theorem 5.2(3)] with a slight modification, we easily deduce the following *interpolation inequality*: there exists a positive constant $C_{(n,m)}$, depending only on n and m, such that, for all balls B, $f \in W^{m,p}(B)$ and $k \in \{0, \ldots, m\}$,

$$\|\nabla^k f\|_{L^2(B)} \le C_{(n,m)} \|\nabla^m f\|_{L^2(B)}^{k/m} \|f\|_{L^2(B)}^{1-k/m}. \tag{2.4}$$

In particular, when $B \equiv \mathbb{R}^n$, (2.4) still holds true. We also point out that (2.4) can be extended to the following weighted version, namely, there exists a positive constant $C_{(n,m)}$, depending only on n and m, such that, for all balls B, $f \in C_c^\infty(B)$ and $k \in \{0, \ldots, m\}$,

$$\|\nabla^k f\|_{L^2_w(B)} \le C_{(n,m)} \|\nabla^m f\|_{L^2_w(B)}^{k/m} \|f\|_{L^2_w(B)}^{1-k/m}, \tag{2.5}$$

where w is a nonnegative measurable weight function and, for any function g, $\|g\|_{L^2_w(B)}$ denotes the *weighted Lebesgue norm* defined by

$$\|g\|_{L^2_w(B)} := \left\{\int_B |g(x)|^2 w(x)\, dx\right\}^{\frac{1}{2}}.$$

Inequality (2.5) can be proved by following the same line of the proof of (2.4) (see, for example, the proof of [1, Theorem 5.2]), the details being omitted. Moreover, if $w \le 1$, by a density argument, we know that (2.5) also holds true for all $f \in W^{m,2}(\mathbb{R}^n)$.

(ii) Let $\mathfrak{a}$ satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. Then, by the interpolation inequality (2.4) and Cauchy's inequality with ϵ, we see immediately that, for all $f \in W^{m,p}(\mathbb{R}^n)$,

$$\begin{aligned}\|\nabla^m f\|_{L^2(\mathbb{R}^n)}^2 &\lesssim \Re e\{\mathfrak{a}(f, f)\} + [\Re e\{\mathfrak{a}_0(f, f)\} - \Re e\{\mathfrak{a}(f, f)\}]\\ &\lesssim \Re e\{\mathfrak{a}(f, f)\} + \sum_{0 \le |\alpha|+|\beta| < 2m} M \int_{\mathbb{R}^n} |\partial^\beta f(x) \overline{\partial^\alpha f(x)}|\, dx\\ &\lesssim \Re e\{\mathfrak{a}(f, f)\} + \epsilon \|\nabla^m f\|_{L^2(\mathbb{R}^n)}^2 + C_{(\epsilon)} M \|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

which implies the following *weak Garding's inequality*: there exist positive constants δ_1 and δ_2 such that, for all $f \in W^{m,p}(\mathbb{R}^n)$,

$$\Re e\{\mathfrak{a}(f, f)\} \ge \delta_1 \|\nabla^m f\|_{L^2(\mathbb{R}^n)}^2 - \delta_2 \|f\|_{L^2(\mathbb{R}^n)}^2. \tag{2.6}$$

See also [7, p. 314] for a similar argument.

(iii) If $\mathfrak{a}$ satisfies the Ellipticity condition $(\mathcal{E})$, we know that, for all α and β, $\|a_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^n)} \le \Lambda$ and (2.6) holds true with $\delta_1 \equiv \lambda$ and $\delta_2 \equiv 0$. Hence, in this sense, the Ellipticity condition $(\mathcal{E})$ is stronger than the Weak ellipticity condition $(\widetilde{\mathcal{E}})$.

Let $\mathfrak{a}$ satisfy the Ellipticity condition $(\mathcal{E})$. By the facts that $W^{m,2}(\mathbb{R}^n)$ is complete and dense in $L^2(\mathbb{R}^n)$, together with the Ellipticity condition $(\mathcal{E})$, we conclude that $\mathfrak{a}$ is a densely defined, accretive, continuous and closed sesquilinear form (see [56, Definition 1.4] for their definitions). Moreover, from [56, Proposition 1.22], we deduce that there exists a densely defined operator L associated with $\mathfrak{a}$ such that, for all $s \in (0,\infty)$, the range of $sI + L$ is dense in $L^2(\mathbb{R}^n)$. We write L formally as in (1.1).

Moreover, by the Ellipticity condition $(\mathcal{E})$, we know that, for all $f \in D(L)$ (the *definition domain* of L),

$$|\tan(\arg(L(f),\, f))_{L^2(\mathbb{R}^n)}| = \left|\frac{\Im m(L(f),\, f)_{L^2(\mathbb{R}^n)}}{\Re e(L(f),\, f)_{L^2(\mathbb{R}^n)}}\right| \le \frac{\Lambda}{\lambda},$$

which implies that

$$|\arg(L(f),\, f)_{L^2(\mathbb{R}^n)}| \le \arctan\frac{\Lambda}{\lambda}.$$

Here and hereafter, $(\arg(L(f),\, f))_{L^2(\mathbb{R}^n)}$ denotes the *inner product* of $L^2(\mathbb{R}^n)$ and $\arg z$ for any $z \in \mathbb{C}\backslash\{0\}$ denotes the *argument* of z. Thus, L is an m-arctan $\frac{\Lambda}{\lambda}$-*accretive* operator in $L^2(\mathbb{R}^n)$. This implies that L is of type $\arctan\frac{\Lambda}{\lambda}$ and hence $-L$ generates a bounded holomorphic semigroup $\{e^{-zL}\}_{z\in S^0_{\pi/2-\arctan\frac{\Lambda}{\lambda}}}$ (see [54] for the definition of types of operators). Here and hereafter, for any $\omega \in [0,\pi)$,

$$S_\omega := \{z \in \mathbb{C}\backslash\{(0,0)\} : |\arg z| \le \omega\} \quad\text{and}\quad S^0_\omega := \{z \in \mathbb{C}\backslash\{(0,0)\} : |\arg z| < \omega\}.$$

Moreover, L is one-to-one in $L^2(\mathbb{R}^n)$, which implies that L has a bounded functional calculus in $L^2(\mathbb{R}^n)$ (see [8, p. 8]).

For $\mathfrak{a}$ satisfying the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ and $\delta_0 \in [\delta_2,\infty)$ with δ_2 as in (2.6), let $\mathfrak{a}_{\delta_0}$ be the sesquilinear form defined by setting, for all $f, g \in W^{m,2}(\mathbb{R}^n)$,

$$\mathfrak{a}_{\delta_0}(f,\, g) := \mathfrak{a}(f,\, g) + \delta_0(f,\, g)_{L^2(\mathbb{R}^n)}. \tag{2.7}$$

Similar to the case of the Ellipticity condition $(\mathcal{E})$, we know that $\mathfrak{a}_{\delta_0}$ is a densely defined, accretive, continuous and closed sesquilinear form. Moreover, the operator L_{δ_0} associated with $\mathfrak{a}_{\delta_0}$ can be written formally as $L + \delta_0$.

As before, by the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ and (2.6), we know that there exists a positive constant c_0 such that, for all $f \in D(L)$,

$$\begin{aligned}|\tan(\arg((L+\delta_0)(f),\, f))_{L^2(\mathbb{R}^n)}| &= \left|\frac{\Im m((L+\delta_0)(f),\, f)_{L^2(\mathbb{R}^n)}}{\Re e((L+\delta_0)(f),\, f)_{L^2(\mathbb{R}^n)}}\right| \\ &\le \frac{(\delta_0 + M + \Lambda_0)\|f\|^2_{W^{2,m}(\mathbb{R}^n)}}{\delta_1\|\nabla^m f\|^2_{L^2(\mathbb{R}^n)} + (\delta_0 - \delta_2)\|f\|^2_{L^2(\mathbb{R}^n)}} \le c_0,\end{aligned}$$

where Λ_0 and M are, respectively, as in the Ellipticity condition $(\mathcal{E}_0)$ and (2.3). This implies that $-(L+\delta_0)$ generates a bounded holomorphic semigroup $\{e^{-z(L+\delta_0)}\}_{z\in S_{\pi/2-\arctan c_0}}$. Thus, we can define $L := (L+\delta_0)-\delta_0$ and the associated semigroup e^{-tL} by setting $e^{-tL} := e^{-t(L+\delta_0)}e^{t\delta_0}$. Moreover, $L_0+\delta_0$ has a bounded functional calculus in $L^2(\mathbb{R}^n)$.

For the inhomogeneous operator L, Auscher *et al.* proved the following result on Kato's square root problem (see [4, Theorem 1.3]).

Proposition 2.2 ([4]). *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E})$. *Then the domain of the square root of L is given by the inhomogeneous Sobolev space $W^{m,2}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for all $f\in W^{m,2}(\mathbb{R}^n)$,*

$$\frac{1}{C}\|\sqrt{L}(f)\|_{L^2(\mathbb{R}^n)} \le \|f\|_{W^{m,2}(\mathbb{R}^n)} \le C\|\sqrt{L}(f)\|_{L^2(\mathbb{R}^n)}.$$

Let $\delta_0\in(0,\infty)$ be as in (2.7) and $\delta\in(\delta_0,\infty)$. If L satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, then, since $L+\delta$ satisfies the Ellipticity condition $(\mathcal{E})$, we have the following conclusion.

Corollary 2.3. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$. *Assume that $\delta_0\in[\delta_2,\infty)$ is as in* (2.7) *and $\delta\in(\delta_0,\infty)$. Then the domain of the square root of $L+\delta$ is given by the inhomogeneous Sobolev space $W^{m,2}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for all $f\in W^{m,2}(\mathbb{R}^n)$,*

$$\frac{1}{C}\|\sqrt{L+\delta}(f)\|_{L^2(\mathbb{R}^n)} \le \|f\|_{W^{m,2}(\mathbb{R}^n)} \le C\|\sqrt{L+\delta}(f)\|_{L^2(\mathbb{R}^n)}.$$

2.2. *Local off-diagonal estimates of L*

In this subsection, we establish some local off-diagonal estimates for operators generated by L, which play an essential role in the theory of local Hardy spaces associated with L. To begin with, we first recall a result from [25, Theorem 8], which states that, if L satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ and is symmetric, then $\{e^{-tL}\}_{t>0}$ satisfies the following L^2 off-diagonal estimates.

Proposition 2.4 ([25]). *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$ *and assume that the coefficient matrix $\{a_{\alpha,\beta}\}_{\alpha,\beta}$ is conjugate symmetric, namely, for all α and β, $a_{\alpha,\beta}=\overline{a_{\beta,\alpha}}$. Then the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the following local L^2 off-diagonal estimates: there exist positive constants κ and C such that, for all $t\in(0,\infty)$, compact convex sets $E,F\subset\mathbb{R}^n$ and $f\in L^2(\mathbb{R}^n)$ supported in E,*

$$\|e^{-tL}(f)\|_{L^2(F)} \le \exp\left\{\kappa t - C\frac{[d(E,F)]^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\}\|f\|_{L^2(E)}. \tag{2.8}$$

For completeness, we now reprove (2.8) by presenting more details and removing the assumptions that the coefficient matrix is symmetric, and that the sets E and

F are compact convex. Our method is essentially based on the proofs of [25, Theorem 8] and [11, Proposition 3.1].

Let $\mathscr{E}_m$ be the *set* of all bounded real-valued C^∞ functions ϕ on $\mathbb{R}^n$ satisfying that, for all multi-indices $\alpha \in \mathbb{N}^n$ satisfying $1 \le |\alpha| \le m$, $\|D^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \le 1$. For $\mathscr{E}_m$, the following conclusion plays an important role in this subsection.

Lemma 2.5 ([25]). *Let E, F be two disjoint closed subsets in $\mathbb{R}^n$ and*

$$\widetilde{d}(E,\, F) := \sup_{\phi \in \mathscr{E}_m} \inf\{\phi(x) - \phi(y) \colon x \in E,\, y \in F\}.$$

Then $d(E,\, F) \sim \widetilde{d}(E,\, F)$, where $d(E,\, F)$ denotes the Euclidean distance between E and F.

Remark 2.6. We point out that Lemma 2.5 was first proved by Davies [25, Lemma 4] in the case that E and F are compact convex, however, it still holds true for all closed sets (see [62, Lemma 2.3] for more details).

Let $\phi \in \mathscr{E}_m$, $\lambda \in [0, \infty)$ and $f,\, g \in W^{m,\,2}(\mathbb{R}^n)$. Define the *twist sesquilinear form $\mathfrak{a}_{\lambda,\,\phi}$ under exponential perturbation*, mapping $W^{m,\,2}(\mathbb{R}^n) \times W^{m,\,2}(\mathbb{R}^n)$ into $\mathbb{C}$, by setting

$$\begin{aligned}\mathfrak{a}_{\lambda,\,\phi}(f,\, g) &:= \sum_{0 \le |\alpha|,\,|\beta| \le m} \int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x) e^{\lambda\phi} \partial^\beta (e^{-\lambda\phi} f)(x) \overline{e^{-\lambda\phi} \partial^\alpha (e^{\lambda\phi} g)(x)}\, dx \\ &= \mathfrak{a}(e^{-\lambda\phi} f,\, e^{\lambda\phi} g), \end{aligned} \tag{2.9}$$

where $\mathfrak{a}$ is as in (2.1). We remark that this kind of twist sesquilinear forms first appeared in [25, p. 143] and was generalized by Blunck and Kunstmann [12] to the space of homogeneous type.

Although the twist sesquilinear form $\mathfrak{a}_{\lambda,\,\phi}$ may not be accretive, we can define the operator $L_{\lambda,\,\phi} := e^{\lambda\phi} L e^{-\lambda\phi}$, with the *domain*

$$\mathcal{D}(L_{\lambda,\,\phi}) := \{f \in L^2(\mathbb{R}^n) \colon e^{-\lambda\phi} f \in \mathcal{D}(L)\},$$

associated with $\mathfrak{a}_{\lambda,\,\phi}$ in the following way: for all $f \in \mathcal{D}(L_{\lambda,\,\phi})$ and $g \in W^{m,\,2}(\mathbb{R}^n)$, we have $e^{-\lambda\phi} f \in \mathcal{D}(L)$, $e^{\lambda\phi} g \in W^{m,\,2}(\mathbb{R}^n)$ and hence, we can write

$$\mathfrak{a}_{\lambda,\,\phi}(f,\, g) = \mathfrak{a}(e^{-\lambda\phi} f,\, e^{\lambda\phi} g) = (L(e^{-\lambda\phi} f),\, e^{\lambda\phi} g)_{L^2(\mathbb{R}^n)} = (L_{\lambda,\,\phi} f,\, g)_{L^2(\mathbb{R}^n)}.$$

Recall that $L + \delta_0$ is of type ω in $L^2(\mathbb{R}^n)$ for some $\omega \in [0, \frac{\pi}{2})$. Observe that, for all $\xi \in \mathbb{C}$,

$$(\xi I - [L_{\lambda,\,\phi} + \delta_0])^{-1} = (\xi I - [e^{\lambda\phi} L e^{-\lambda\phi} + \delta_0])^{-1} = e^{\lambda\phi} (\xi I - [L + \delta_0])^{-1} e^{-\lambda\phi},$$

which implies that the spectrum of $L + \delta_0$ and $L_{\lambda,\,\phi} + \delta_0$ are the same. Moreover, we see that $L_{\lambda,\,\phi} + \delta_0$ is also of type ω. Thus, $-(L_{\lambda,\,\phi} + \delta_0)$ generates a bounded holomorphic semigroup $\{e^{-z(L_{\lambda,\,\phi}+\delta_0)}\}_{z\in S^0_{\frac{\pi}{2}-\omega}}$. By [50, (5.1), p. 489] (or [23, Proposition 3.6] for a more detailed description), we obtain the following formula

$$\begin{aligned} e^{-t(L_{\lambda,\,\phi}+\delta_0)}(f) &= \frac{1}{2\pi i}\int_\Gamma e^{z\xi}(\xi I + [L_{\lambda,\,\phi}+\delta_0])^{-1}(f)\,d\xi \\ &= e^{\lambda\phi}\left[\frac{1}{2\pi i}\int_\Gamma e^{z\xi}(\xi I + [L+\delta_0])^{-1}(f)\,d\xi\right]e^{-\lambda\phi}, \qquad (2.10)\end{aligned}$$

where, for $\theta\in(\pi/2+|\arg(z)|,\,\pi-\omega)$, the path Γ is the union of the rays

$$\gamma^\pm := \{z\in\mathbb{C}\colon z = re^{\pm i\theta},\, r\in[R,\,\infty)\}$$

and the arc $\gamma_0 := \{z\in\mathbb{C}\colon z = Re^{i\psi}, |\psi|\le\theta\}$, going around the origin counter-clockwise. By formula (2.10), we see that the corresponding semigroup $\{e^{-t(L_{\lambda,\,\phi}+\delta_0)}\}_{t>0}$ can be formally written as $\{e^{\lambda\phi}e^{-t(L+\delta_0)}e^{-\lambda\phi}\}_{t>0}$. Hence, we can write, for all $t\in(0,\,\infty)$,

$$e^{-tL_{\lambda,\,\phi}} = e^{\lambda\phi}e^{-tL}e^{-\lambda\phi}. \qquad (2.11)$$

We first have the following perturbation estimate for the sesquilinear forms, which is an analog of [25, Lemma 2].

Lemma 2.7. *Let* $\mathfrak{a}$ *and* $\mathfrak{a}_{\lambda,\,\phi}$ *be respectively as in* (2.1) *and* (2.9), *where* $\mathfrak{a}$ *satisfies the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$. *Then there exists a positive constant* C_0 *such that, for all* $f\in W^{m,\,2}(\mathbb{R}^n)$,

$$|\mathfrak{a}_{\lambda,\,\phi}(f,\,f)-\mathfrak{a}(f,\,f)| \le \frac{1}{4}\Re e\{\mathfrak{a}(f,\,f)\} + C_0(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. Although the proof of this lemma is similar to that of [25, Lemma 2], in this article, we still give some details, since this perturbation estimate is important to establish the off-diagonal estimates for $\{e^{-tL}\}_{t>0}$.

Observe that, for each multi-index α satisfying $0\le|\alpha|\le m$, by Leibniz's rule and properties of ϕ, we know that there exists a family $\{b_{\alpha,\,\gamma}\}_\gamma$ of uniformly bounded functions such that

$$e^{\lambda\phi}\partial^\alpha(e^{-\lambda\phi}f) = \partial^\alpha f + \sum_{\substack{0\le|\gamma|<|\alpha|\\ 0\le q\le|\alpha|-|\gamma|}} b_{\alpha,\,\gamma}\lambda^q\partial^\gamma f.$$

Thus, by an elementary calculation, we conclude that

$$\begin{aligned}\mathfrak{a}_{\lambda,\,\phi}(f,\,f) = &\sum_{0\le|\alpha|,\,|\beta|\le m}\int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha f(x)}\,dx \\ &+ \sum_{0\le|\alpha|,\,|\beta|\le m}\sum_{\substack{0\le|\gamma_1|<|\alpha|\\ 0\le q_1\le|\alpha|-|\gamma_1|}}\int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)b_{\alpha,\,\gamma_1}(x)\partial^\beta f(x)\lambda^{q_1}\overline{\partial^{\gamma_1}f(x)}\,dx \\ &+ \sum_{0\le|\alpha|,\,|\beta|\le m}\sum_{\substack{0\le|\gamma_2|<|\beta|\\ 0\le q_2\le|\beta|-|\gamma_2|}}\int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x)b_{\beta,\,\gamma_2}(x)\partial^{\gamma_2} f(x)\lambda^{q_2}\overline{\partial^{\alpha}f(x)}\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0\le|\alpha|,\,|\beta|\le m} \sum_{\substack{0\le|\gamma_1|<|\alpha|\\ 0\le q_1\le|\alpha|-|\gamma_1|}} \sum_{\substack{0\le|\gamma_2|<|\beta|\\ 0\le q_2\le|\beta|-|\gamma_2|}} \\
&\times \int_{\mathbb{R}^n} a_{\alpha,\,\beta}(x) b_{\alpha,\,\gamma_1}(x) b_{\beta,\,\gamma_2}(x) \partial^{\gamma_2} f(x) \\
&\times \lambda^{q_1+q_2} \overline{\partial^{\gamma_1} f(x)}\, dx =: \mathfrak{a}(f,\, f) + \mathcal{A} + \mathcal{B} + \mathcal{C}.
\end{aligned}
$$

For $\mathcal{A}$, by the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, the uniformly boundedness of $\{b_{\alpha,\,\gamma}\}_{\alpha,\,\gamma}$, Hölder's inequality and Cauchy's inequality with ϵ, we find that, for every $\epsilon \in (0,\,\infty)$, there exists a positive constant $C_{(\epsilon)}$ such that

$$
\begin{aligned}
\mathcal{A} &\lesssim \sum_{0\le|\alpha|,\,|\beta|\le m} \sum_{\substack{0\le|\gamma_1|<|\alpha|\\ 0\le q_1\le|\alpha|-|\gamma_1|}} \int_{\mathbb{R}^n} \lambda^{q_1} |\partial^\beta f(x)||\partial^{\gamma_1} f(x)|\, dx \\
&\lesssim \sum_{0\le|\alpha|,\,|\beta|\le m} \sum_{\substack{0\le|\gamma_1|<|\alpha|\\ 0\le q_1\le|\alpha|-|\gamma_1|}} \|\partial^\beta f\|_{L^2(\mathbb{R}^n)} \|\lambda^{q_1}\partial^{\gamma_1} f\|_{L^2(\mathbb{R}^n)} \\
&\le \sum_{0\le|\alpha|,\,|\beta|\le m} \sum_{\substack{0\le|\gamma_1|<|\alpha|\\ 0\le q_1\le|\alpha|-|\gamma_1|}} [\epsilon\|\partial^\beta f\|^2_{L^2(\mathbb{R}^n)} + C_{(\epsilon)}\|\lambda^{q_1}\partial^{\gamma_1} f\|^2_{L^2(\mathbb{R}^n)}] \\
&=: \sum_{0\le|\alpha|,\,|\beta|\le m} \sum_{\substack{0\le|\gamma_1|<|\alpha|\\ 0\le q_1\le|\alpha|-|\gamma_1|}} [\epsilon\|\partial^\beta f\|^2_{L^2(\mathbb{R}^n)} + C_{(\epsilon)}\widetilde{\mathcal{A}}_{q_1,\,\gamma_1}]. \qquad (2.12)
\end{aligned}
$$

By Plancherel's theorem and Young's inequality with ϵ, we know that, for every $\widetilde{\epsilon} \in (0,\,\infty)$, there exists a positive constant $C_{(\widetilde{\epsilon})}$ such that

$$
\begin{aligned}
\widetilde{\mathcal{A}}_{q_1,\,\gamma_1} &\sim \int_{\mathbb{R}^n} |[|\xi|^{|\gamma_1|}\lambda^{q_1}]\widehat{f}(\xi)|^2\, d\xi \lesssim \int_{\mathbb{R}^n} [\widetilde{\epsilon}|\xi|^{2m} + C_{(\widetilde{\epsilon})}(1+\lambda^{2m})]|\widehat{f}(\xi)|^2\, d\xi \\
&\le \widetilde{\epsilon}\|\nabla^m f\|^2_{L^2(\mathbb{R}^n)} + C_{(\widetilde{\epsilon})}(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

which, together with (2.12), implies that

$$
\mathcal{A} \le \sum_{0\le|\beta|\le m} [\epsilon\|\partial^\beta f\|^2_{L^2(\mathbb{R}^n)} + C_{(\epsilon)}(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)}].
$$

Using the interpolation inequality (2.4), Young's inequality with ϵ and the weak Garding's inequality (2.6), we see that

$$
\begin{aligned}
\mathcal{A} &\le \epsilon\|\nabla^m f\|^2_{L^2(\mathbb{R}^n)} + C_{(\epsilon)}(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)} \\
&\le \epsilon\Re e\{\mathfrak{a}(f,\, f)\} + C_{(\epsilon)}(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

which is desired.

The estimates for $\mathcal{B}$ and $\mathcal{C}$ are similar, the details being omitted. Thus, we conclude that

$$
|\mathfrak{a}_{\lambda,\,\phi}(f,\, f) - \mathfrak{a}(f,\, f)| \le \epsilon\Re e\{\mathfrak{a}(f,\, f)\} + C_{(\epsilon)}(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)},
$$

which completes the proof of Lemma 2.7, by letting ϵ sufficiently small. □

With the help of Lemma 2.7, we are able to prove the following conclusion.

Lemma 2.8. *Let $\mathfrak{a}$ and $\mathfrak{a}_{\lambda,\phi}$ be respectively as in* (2.1) *and* (2.9), *where $\mathfrak{a}$ satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. Then there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\|e^{-tL_{\lambda,\phi}}(f)\|^2_{L^2(\mathbb{R}^n)} \leq \exp\{C(1+\lambda^{2m})t\}\|f\|^2_{L^2(\mathbb{R}^n)} \tag{2.13}$$

and

$$\|tL_{\lambda,\phi}e^{-tL_{\lambda,\phi}}(f)\|^2_{L^2(\mathbb{R}^n)} \leq \exp\{C(1+\lambda^{2m})t\}\|f\|^2_{L^2(\mathbb{R}^n)}. \tag{2.14}$$

Proof. This lemma can be proved by a way similar to that used in the proof of [25, Lemma 6]. For all $t \in (0,\infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $f_t(x) := e^{-tL_{\lambda,\phi}}(f)(x)$. From the semigroup property of $e^{-tL_{\lambda,\phi}}$, Lemma 2.7 and the weak Garding's inequality (2.6), we deduce that there exists a positive constant $\widetilde{C}$ such that

$$\begin{aligned}
\frac{d}{dt}\|f_t\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{d}{dt}f_t(x)\overline{f_t(x)}\,dx + \int_{\mathbb{R}^n} f_t(x)\overline{\frac{d}{dt}f_t(x)}\,dx \\
&= 2\Re e\left\{\int_{\mathbb{R}^n} \frac{d}{dt}f_t(x)\overline{f_t(x)}\,dx\right\} = -2\Re e\{\mathfrak{a}_{\lambda,\phi}(f_t, f_t)\} \\
&= 2[\Re e\{\mathfrak{a}(f, f) - \mathfrak{a}_{\lambda,\phi}(f, f)\} - \Re e\{\mathfrak{a}(f, f)\}] \\
&\leq 2\left[C_0(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)} - \frac{3}{4}\Re e\{\mathfrak{a}(f, f)\}\right] \\
&\leq 2\left[C_0(1+\lambda^{2m}) + \frac{3}{4}\delta_2\right]\|f\|^2_{L^2(\mathbb{R}^n)} \leq 2\widetilde{C}(1+\lambda^{2m})\|f\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where

$$\widetilde{C} := \max\{C_0, 3\delta_2/4\}, \tag{2.15}$$

C_0 and δ_2 are as in Lemma 2.7 and (2.6), respectively.

Solving the differential inequality above, we conclude that

$$\|f_t\|^2_{L^2(\mathbb{R}^n)} \leq \exp\{2\widetilde{C}(1+\lambda^{2m})t\}\|f\|^2_{L^2(\mathbb{R}^n)},$$

which completes the proof of (2.13).

We now turn to the proof of (2.14). Let $\theta \in (0, \frac{\pi}{2})$ be such that

$$\tan\theta < \frac{1}{\sqrt{(\Lambda_0/\lambda_0)^2 - 1}},$$

where λ_0 and Λ_0 are, respectively, the ellipticity constants for the leading part L_0 of L. Then, by the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, we know that, for all $f \in W^{m,2}(\mathbb{R}^n)$

$$\begin{aligned}
&[\Re e(e^{i\theta}L_0(f), f)_{L^2(\mathbb{R}^n)}]^2 + [\Im m(e^{i\theta}L_0(f), f)_{L^2(\mathbb{R}^n)}]^2 \\
&\quad = |(e^{i\theta}L_0(f), f)_{L^2(\mathbb{R}^n)}|^2 \leq \Lambda_0^2\|f\|^4_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, by the Weak ellipticity condition $(\widetilde{\epsilon})$, we have

$$\begin{aligned}\Re e(e^{i\theta}L_0(f),\, f)_{L^2(\mathbb{R}^n)} &= (\cos\theta)\Re e(e^{i\theta}L_0(f),\, f)_{L^2(\mathbb{R}^n)}\\ &\quad - (\sin\theta)\Im m(e^{i\theta}L_0(f),\, f)_{L^2(\mathbb{R}^n)}\\ &\geq (\cos\theta)\lambda_0\|f\|^2_{L^2(\mathbb{R}^n)} - (\sin\theta)\sqrt{\Lambda_0^2-\lambda_0^2}\|f\|^2_{L^2(\mathbb{R}^n)}\\ &= \lambda_0\|f\|^2_{L^2(\mathbb{R}^n)}[\cos\theta - (\sin\theta)\sqrt{(\Lambda_0/\lambda_0)^2-1}],\end{aligned}$$

which, together with the assumption that $\tan\theta < \frac{1}{\sqrt{(\Lambda_0/\lambda_0)^2-1}}$, shows that

$$\cos\theta - (\sin\theta)\sqrt{(\Lambda_0/\lambda_0)^2-1} > 0.$$

Hence, $e^{i\theta}L$ also satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$.

Now, for $r\in(0,\infty)$ and θ as above, using an argument same as that used in the proof of (2.13), we obtain

$$\|e^{-re^{i\theta}L_{\lambda,\phi}}(f)\|^2_{L^2(\mathbb{R}^n)} \leq \exp\{2\widetilde{C}(1+\lambda^{2m})r\}\|f\|^2_{L^2(\mathbb{R}^n)},$$

where $\widetilde{C}$ is as in (2.15). Moreover, by the following Cauchy's formula that, for all $k\in\mathbb{N}$,

$$\begin{aligned}&\{t(L_{\lambda,\phi}+\widetilde{C}[1+\lambda^{2m}])\}^k e^{-t(L_{\lambda,\phi}+\widetilde{C}[1+\lambda^{2m}])}\\ &\quad = (-1)^k k!\frac{t^k}{2\pi i}\int_{\{z\in\mathbb{C}:\,|z-t|=\eta t\}} e^{-z(L_{\lambda,\phi}+\widetilde{C}[1+\lambda^{2m}])}\frac{dz}{(z-t)^{k+1}}\end{aligned}\tag{2.16}$$

(see, for example, [40, p. 14]) with $\eta\in(0,\infty)$ small enough, we see that

$$\|t(L_{\lambda,\phi}+\widetilde{C}[1+\lambda^{2m}])e^{-t(L_{\lambda,\phi}+\widetilde{C}[1+\lambda^{2m}])}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \lesssim 1,$$

which, combined with (2.13), implies that (2.14) holds true. Here and hereafter, for any bounded linear operator T on $L^2(\mathbb{R}^n)$, $\|T\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ denotes its *operator norm*. This finishes the proof of Lemma 2.8. □

We now turn to the proof of L^2 off-diagonal estimates of $\{e^{-tL}\}_{t>0}$.

Theorem 2.9. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$. *Then the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the following local L^2 off-diagonal estimates: there exist positive constants κ, C and $\widetilde{C}$ such that, for all $t\in(0,\infty)$, closed sets E, F in $\mathbb{R}^n$ and $f\in L^2(\mathbb{R}^n)$ supported in E,*

$$\|e^{-tL}(f)\|_{L^2(F)} \leq C\exp\left\{\kappa t - \widetilde{C}\frac{[d(E,\,F)]^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\}\|f\|_{L^2(E)}.\tag{2.17}$$

Proof. Let E and F be two closed sets in $\mathbb{R}^n$. Without loss of generality, we may assume that $d(E,\,F)>0$. Take $\phi\in\mathscr{E}_m$ satisfying $\phi|_E\geq 0$ and $\phi|_F\leq -\frac{d(E,F)}{1+\epsilon}$, where ϵ is some suitable positive constant (see [25, p. 151] for the existence of the function ϕ satisfying these conditions). From this and (2.11), we deduce that, for all $\lambda \in (0, \infty)$,

$$\begin{aligned}\|\chi_E e^{-tL}\chi_F\|_{\mathcal{L}(L^2(\mathbb{R}^n))} &= \|e^{-\lambda\phi}\chi_E e^{\lambda\phi} e^{-tL} e^{-\lambda\phi}\chi_F e^{\lambda\phi}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\\ &\le \|e^{-\lambda\phi}\chi_E\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|e^{-tL_{\lambda,\,\phi}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|\chi_F e^{\lambda\phi}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.\end{aligned}$$

By properties of ϕ and Lemma 2.8, we conclude that there exists a positive constant C_0 such that

$$\|\chi_E e^{-tL}\chi_F\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \exp\left\{C_0(1+\lambda^{2m})t - \lambda\frac{d(E,\,F)}{1+\epsilon}\right\},$$

which implies (2.17) by letting $\lambda := \widetilde{C}\frac{[d(E,\,F)]^{\frac{1}{2m-1}}}{t^{\frac{1}{2m-1}}}$ with $\widetilde{C} \in (0,\,[\frac{1}{(1+\epsilon)C_0}]^{\frac{1}{2m-1}})$. This finishes the proof of Theorem 2.9. $\square$

We now consider the $L^p(\mathbb{R}^n)$ theory of $\{e^{-tL}\}_{t>0}$. Let $\kappa \in (0, \infty)$ and $(p_-(L+\kappa),\, p_+(L+\kappa))$ be the *maximal interval of exponents* $p \in [1, \infty]$ such that $\{e^{-t(L+\kappa)}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$.

Proposition 2.10. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$. *Then there exists a positive constant κ such that*

(i)
$$\begin{cases}(p_-(L+\kappa),\, p_+(L+\kappa)) = (1,\,\infty), & \text{when } n < 2m,\\[2ex] \left[\dfrac{n}{n+2m},\, \dfrac{n}{n-2m}\right] \subset (p_-(L+\kappa),\, p_+(L+\kappa)), & \text{when } n > 2m.\end{cases}$$

(ii) *For all $k \in \mathbb{Z}_+$ and $p_-(L+\kappa) < p \le q < p_+(L+\kappa)$, the family $\{[t(L+\kappa)]^k e^{-t(L+\kappa)}\}_{t>0}$ of operators satisfies the following m-L^p-L^q off-diagonal estimates*: *there exist positive constants C and $\widetilde{C}$ such that, for all $t \in (0, \infty)$, closed sets E, F in $\mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ supported in E,*

$$\begin{aligned}&\|[t(L+\kappa)]^k e^{-t(L+\kappa)}(f)\|_{L^q(F)}\\ &\quad\le Ct^{\frac{n}{2m}(\frac{1}{q}-\frac{1}{p})}\exp\left\{-\widetilde{C}\frac{[d(E,\,F)]^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\}\|f\|_{L^p(\mathbb{R}^n)}. \qquad (2.18)\end{aligned}$$

(iii) *For all $k \in \mathbb{Z}_+$ and $p_-(L+\kappa) < p \le q < p_+(L+\kappa)$, the family $\{(tL)^k e^{-tL}\}_{t>0}$ of operators satisfies the following local m-L^p-L^q off-diagonal estimates*: *there exist positive constants C and $\widetilde{C}$ such that, for all $t \in (0, 1]$, closed sets E, F in $\mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ supported in E,*

$$\begin{aligned}&\|(tL)^k e^{-tL}(f)\|_{L^q(F)}\\ &\quad\le Ct^{\frac{n}{2m}(\frac{1}{q}-\frac{1}{p})}\exp\left\{\kappa t-\widetilde{C}\frac{[d(E,\,F)]^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\}\|f\|_{L^p(\mathbb{R}^n)}. \qquad (2.19)\end{aligned}$$

Proof. Part (i) was essentially proved in [25, Theorems 20 and 25], where, although L is assumed to be symmetric, it can be extended to the present setting with some minor modifications as in the proofs of Lemmas 2.7 and 2.8, the details being omitted.

Part (iii) is an easy consequence of (ii), the details being omitted.

We now turn to the proof of (ii). If $k = 0$, the proof of (ii) is similar to those of [3, Proposition 3.2(1) and (2)] with Gagliardo–Nirenberg's inequality used therein replaced by the following higher order Gagliardo–Nirenberg's inequality from [55, p. 125] that, for all $\alpha \in [0, 1]$ satisfying $[1 + \frac{n}{m}(\frac{1}{p} - \frac{1}{2})]\alpha = \frac{n}{m}(\frac{1}{p} - \frac{1}{2})$,

$$\|f\|_{L^2(\mathbb{R}^n)} \lesssim \|\nabla^m f\|^{\alpha}_{L^2(\mathbb{R}^n)} \|f\|^{1-\alpha}_{L^p(\mathbb{R}^n)},$$

where the implicit positive constant depends on m, but is independent of f; see also [27, Theorem 3.1] for a similar argument. Observe that, by (2.6), we can choose κ large enough so that $L + \kappa$ satisfies the Ellipticity condition $(\mathcal{E})$. Thus, we have $\|\nabla^m f\|_{L^2(\mathbb{R}^n)} \lesssim \Re e\{((L + \kappa)(f), f)_{L^2(\mathbb{R}^n)}\}$, which makes the argument used in the proof of [3, Proposition 3.2(i)] still be valid in the present setting.

The proof of (2.18) for $k \in \mathbb{N}$ is similar to that of [40, Proposition 3.1]. We first need to extend the off-diagonal estimates (2.18) with $k = 0$ to the complex time; then using Cauchy's formula as in (2.16), we obtain (2.18) for $k \in \mathbb{N}$, the technical details being omitted; see also the proof of (2.14) of Lemma 2.8 for a similar argument. This finishes the proof of Proposition 2.10. $\square$

We now turn to gradient estimates of $\{e^{-tL}\}_{t>0}$. Our method is from Blunck and Kunstmann (see [11, Proposition 3.1]), where the authors studied the off-diagonal estimates of the associated gradient semigroup when L is homogeneous and of the form (1.6).

Theorem 2.11. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$. *Then there exists a positive constant κ such that, for all $k \in \{0, \ldots, m\}$, the family $\{(t^{1/(2m)}\nabla)^k e^{-tL}\}_{t>0}$ of operators satisfies the local L^2 off-diagonal estimates as in* (2.17).

Proof. Observe that, if $k = 0$, then Theorem 2.11 is just Theorem 2.9. Thus, to show Theorem 2.11, we only need to consider the case $k \in \{1, \ldots, m\}$. We first prove Theorem 2.11 in the case when $k = m$. In this case, let $\lambda \in (0, \infty)$, $\phi \in \mathscr{E}_m$ and $\mathfrak{a}_{\lambda,\phi}$ be the twist sesquilinear form as in (2.9). Let $\widetilde{\mathfrak{a}}_{\lambda,\phi} := \mathfrak{a}_{\lambda,\phi} + \mu(1 + \lambda^{2m})$, where $\mu \in (0, \infty)$ and $\mu(1 + \lambda^{2m})$ is defined as a sesquilinear form $\mathfrak{b}$ by multiplication, namely, for all $f, g \in L^2(\mathbb{R}^n)$,

$$\mathfrak{b}(f, g) := \int_{\mathbb{R}^n} \mu(1 + \lambda^{2m}) f(x)\, \overline{g(x)}\, dx.$$

Let $\widetilde{L}_{\lambda,\phi}$ be the operator associated with $\widetilde{\mathfrak{a}}_{\lambda,\phi}$. Then $\widetilde{L}_{\lambda,\phi}$ can be written formally as $L_{\lambda,\phi} + \mu(1 + \lambda^{2m})$, where $L_{\lambda,\phi}$ is the operator associated with $\mathfrak{a}_{\lambda,\phi}$. By letting

μ be sufficiently large and using Lemma 2.8, we conclude that

$$\|e^{-t\widetilde{L}_{\lambda,\,\phi}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \lesssim 1 \tag{2.20}$$

and

$$\|t\widetilde{L}_{\lambda,\,\phi}e^{-t\widetilde{L}_{\lambda,\,\phi}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \lesssim 1. \tag{2.21}$$

For all $t \in (0,\,\infty)$ and $f \in L^2(\mathbb{R}^n)$, let $u(t) := e^{-t\widetilde{L}_{\lambda,\,\phi}}(f)$. From the interpolation inequality (2.4), the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, Lemma 2.7, Hölder's inequality, (2.20) and (2.21), we deduce that, for all $k \in \{0, \ldots, m\}$,

$$\begin{aligned}
\|\nabla^k u(t)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \Re e\{\mathfrak{a}(u(t),\,u(t))\} + \delta_2\|u(t)\|^2_{L^2(\mathbb{R}^n)}\\
&\lesssim \Re e\{\mathfrak{a}_{\lambda,\,\phi}(u(t),\,u(t))\} + \mu(1+\lambda^{2m})\|u(t)\|^2_{L^2(\mathbb{R}^n)}\\
&\lesssim |\Re e\{\widetilde{\mathfrak{a}}_{\lambda,\,\phi}(u(t),\,u(t))\}|\\
&\lesssim |(\widetilde{L}_{\lambda,\,\phi}e^{-t\widetilde{L}_{\lambda,\,\phi}}(f), e^{-t\widetilde{L}_{\lambda,\,\phi}}(f))_{L^2(\mathbb{R}^n)}| \lesssim \frac{1}{t}\|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{2.22}$$

Moreover, by the interpolation inequality (2.4) again, we find that, for all $l \in \{1, \ldots, k\}$ and $g \in W^{m,\,2}(\mathbb{R}^n)$,

$$\|\nabla^l g\|_{L^2(\mathbb{R}^n)} \lesssim \|\nabla^k g\|^{l/k}_{L^2(\mathbb{R}^n)}\|g\|^{1-l/k}_{L^2(\mathbb{R}^n)}, \tag{2.23}$$

which, together with (2.22) and (2.20), immediately implies that

$$\|t^{l/(2k)}\nabla^l e^{-t\widetilde{L}_{\lambda,\,\phi}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \lesssim 1.$$

Thus, by the relationships between $L_{\lambda,\,\phi}$, $\widetilde{L}_{\lambda,\,\phi}$ and L (see (2.11)), we further know that, for all $l \in \{1, \ldots, k\}$ and $f \in L^2(\mathbb{R}^n)$,

$$\|t^{1/2}\nabla^k(e^{\lambda\phi}e^{-tL}e^{-\lambda\phi}(f))\|_{L^2(\mathbb{R}^n)} \lesssim e^{\mu(1+\lambda^{2m})t}\|f\|_{L^2(\mathbb{R}^n)} \tag{2.24}$$

and

$$\|t^{l/(2k)}\nabla^l(e^{\lambda\phi}e^{-tL}e^{-\lambda\phi}(f))\|_{L^2(\mathbb{R}^n)} \lesssim e^{\mu(1+\lambda^{2m})t}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.25}$$

Furthermore, as in the proof of [11, Proposition 3.1, p. 147], we have the following elementary formula

$$e^{\lambda\phi}(\nabla^m e^{-tL})e^{-\lambda\phi} = \sum_{i=1}^m \psi_i \sum_{j=0}^i (-\lambda)^{i-j}\nabla^j(e^{-tL_{\lambda,\,\phi}}(f)),$$

where $\{\psi_i\}_{i=1}^m$, depending on ϕ, is a family of bounded functions on $\mathbb{R}^n$. Thus, if $t \in [1, \infty)$, from (2.24), we deduce that

$$\begin{aligned}
\|e^{\lambda\phi}t^{1/2}(\nabla^m e^{-tL})e^{-\lambda\phi}(f)\|_{L^2(\mathbb{R}^n)} &\lesssim \sum_{i=1}^m\sum_{j=0}^i |\lambda|^{i-j}\|t^{1/2}\nabla^j(e^{-tL_{\lambda,\,\phi}}(f))\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \sum_{i=1}^m\sum_{j=0}^i |\lambda|^{i-j}t^{\frac{i-j}{2m}}e^{\mu(1+\lambda^{2m})t}\|f\|_{L^2(\mathbb{R}^n)}\\
&\lesssim e^{(1+\mu)(1+\lambda^{2m})t}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

If $t \in (0, 1)$, from (2.20) and (2.25), we deduce that

$$\begin{aligned}
&\|e^{\lambda\phi}t^{1/2}(\nabla^m e^{-tL})e^{-\lambda\phi}(f)\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \left\| \sum_{i=1}^m \psi_i \sum_{j=0}^i (-\lambda)^{i-j} t^{\frac{i-j}{2m}} t^{\frac{j}{2m}} \nabla^j (e^{-tL_{\lambda,\,\phi}}(f)) \right\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{i=1}^m \sum_{j=0}^i |\lambda|^{i-j} t^{\frac{i-j}{2m}} e^{\mu(1+\lambda^{2m})t} \|f\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim e^{(1+\mu)(1+\lambda^{2m})t} \|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Combining the above two cases, we conclude that

$$\|e^{\lambda\phi}t^{1/2}(\nabla^m e^{-tL})e^{-\lambda\phi}(f)\|_{L^2(\mathbb{R}^n)} \lesssim e^{(1+\mu)(1+\lambda^{2m})t} \|f\|_{L^2(\mathbb{R}^n)}.$$

The remainder of the proof is similar to that of Theorem 2.9 (see (2.18)), the details being omitted. This shows that Theorem 2.11 holds true when $k = m$.

Now, we consider the case when $k \in \{1, \ldots, m-1\}$. In this case, we first observe that the interpolation inequality (2.4) still holds true when the balls used therein replaced by annuli (see [1, Theorem 5.2(3)] with a slight modification). This, combined with the already proved cases of $k = 0$ and $k = m$, shows that, for all $k \in \{1, \ldots, m-1\}$, the family $\{(t^{1/(2m)}\nabla)^k e^{-tL}\}_{t>0}$ of operators also satisfies the local L^2 off-diagonal estimates as in (2.17) by restricting E and F being balls or annuli (except both being annuli), namely, for all $k \in \{1, \ldots, m-1\}$, $\{(t^{1/(2m)}\nabla)^k e^{-tL}\}_{t>0}$ satisfies the so-called off-diagonal estimates on balls introduced in [5]. Moreover, by [5, Proposition 3.2], we further conclude that, for all $k \in \{0, \ldots, m\}$, the family $\{(t^{1/(2m)}\nabla)^k e^{-tL}\}_{t>0}$ of operators also satisfies the local L^2 off-diagonal estimates as in (2.17) for any closed sets E and F. This implies that Theorem 2.11 holds true in the case when $k \in \{1, \ldots, m-1\}$ and hence finishes the proof of Theorem 2.11. □

3. The Local Hardy Space $h^p_L(\mathbb{R}^n)$

In this section, let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. For all $p \in (0, \infty)$, we study the local Hardy spaces $h^p_L(\mathbb{R}^n)$ associated with L. To this end, we first introduce two spaces $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ and $H^p_I(\mathbb{R}^n)$, for $p \in (0, \infty)$, in Sec. 3.1, which are necessary when defining $h^p_L(\mathbb{R}^n)$; then, in Sec. 3.2, we give the definitions of $h^p_L(\mathbb{R}^n)$ and their local molecular characterizations for $p \in (0, 1]$; finally, in Sec. 3.3, we establish the square function characterization and the complex interpolation of $h^p_L(\mathbb{R}^n)$ for $p \in (0, \infty)$.

3.1. *The spaces $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ and $H^p_I(\mathbb{R}^n)$*

In this subsection, we study the spaces $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ and $H^p_I(\mathbb{R}^n)$ for all $p \in (0, \infty)$, which play important roles in the notion of $h^p_L(\mathbb{R}^n)$. Recall that the space $L^p_{\mathcal{Q}}(\mathbb{R}^n)$

was first introduced in [17], and $H_I^p(\mathbb{R}^n)$ is the Hardy space associated with the identity operator first appeared in [14].

We first recall the notion of unit cube structures on $\mathbb{R}^n$ from [17, Definition 4.1].

Definition 3.1 ([17]). Let $\mathcal{Q} := \{Q_j\}_j$ be a countable collection of disjoint cubes that cover $\mathbb{R}^n$. The collection $\mathcal{Q}$ is called a *unit cube structure on* $\mathbb{R}^n$ if there exist $\delta \in (0, 1]$ and a sequence $\{B_j\}_j$ of balls with radius 1 such that $\delta B_j \subset Q_j \subset B_j$ for all j.

Remark 3.2. One of the typical examples of unit cube structures on $\mathbb{R}^n$ is the following family of cubes, $\{k/\sqrt{n} + [0, 1/\sqrt{n})^n\}_{k\in\mathbb{Z}^n}$.

The following notion of the space $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ was first introduced by Carbonaro *et al.* [17, Definition 4.3] when $p \in [1, \infty)$.

Definition 3.3 ([17]). Let $\mathcal{Q}$ be a unit cube structure on $\mathbb{R}^n$. For all $p \in (0, \infty)$, the *space* $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ is defined as the collection of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} := \left\{\sum_{Q_j\in\mathcal{Q}} [|Q_j|^{\frac{1}{p}-\frac{1}{2}} \|\chi_{Q_j} f\|_{L^2(\mathbb{R}^n)}]^p\right\}^{1/p} < \infty.$$

It follows from [17] that $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ is a quasi-Banach space, which is independent of the choice of unit cube structures $\mathcal{Q}$. For more details on $L^p_{\mathcal{Q}}(\mathbb{R}^n)$, we refer the reader to [17, Sec. 4].

To obtain the atomic characterization of $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ with $p \in (0, 1]$, we first introduce the notion of $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-atoms, which when $p = 1$ was introduced by Carbonaro *et al.* [17, Definition 4.5].

Definition 3.4. Let $p \in (0, 1]$ and B be a ball in $\mathbb{R}^n$ with its radius $r_B \geq 1$. A measurable function a on $\mathbb{R}^n$ is called an $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-*atom associated with* B if $\operatorname{supp} a \subset B$ and $\|a\|_{L^2(\mathbb{R}^n)} \leq |B|^{\frac{1}{2}-\frac{1}{p}}$.

Recall that the following atomic characterization of $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ when $p = 1$ was obtained in [17, Theorem 4.6].

Proposition 3.5. *Let* $p \in (0, 1]$.

(i) *If* $\{\lambda_j\}_j \in l^p$ *and* $\{a_j\}_j$ *is a sequence of* $L^p_{\mathcal{Q}}(\mathbb{R}^n)$*-atoms, then* $\sum_j \lambda_j a_j$ *converges in* $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* $\{\lambda_j\}_j$ *and* $\{a_j\}_j$, *such that*

$$\left\|\sum_j \lambda_j a_j\right\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \leq C\|\{\lambda_j\}_j\|_{l^p}.$$

(ii) *If* $f \in L^p_{\mathcal{Q}}(\mathbb{R}^n)$, *then there exist* $\{\lambda_j\}_j \in l^p$ *and a sequence* $\{a_j\}_j$ *of* $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-*atoms such that* $f = \sum_j \lambda_j a_j$ *in* $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ *and almost everywhere in* $\mathbb{R}^n$. *Moreover,*

$$\|f\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \sim \inf\left\{\|\{\lambda_j\}_j\|_{l^p} : f = \sum_j \lambda_j a_j\right\},$$

where the infimum is taken over all the decompositions of f *as above.*

(iii) *For any* $q \in (0, \infty)$, *if* $f \in L^p_{\mathcal{Q}}(\mathbb{R}^n) \cap L^q_{\mathcal{Q}}(\mathbb{R}^n)$, *then the summation* $f = \sum_j \lambda_j a_j$ *in* (i) *and* (ii) *also converges in* $L^q_{\mathcal{Q}}(\mathbb{R}^n)$.

Proof. Notice first that the case when $p = 1$ of Proposition 3.5 has already been proved in [17, Theorem 4.6]. Thus, to finish the proof of Proposition 3.5, we only need to consider the case of $p \in (0, 1)$ by borrowing some ideas from [17].

We now first prove (i). Let a be an $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-atom associated with the ball B. Let

$$\mathcal{Q}_B := \{Q_j \in \mathcal{Q} : Q_j \cap B \neq \emptyset\}.$$

From the facts that B is bounded with its radius $r_B \geq 1$, $\{Q_j\}_j$ are disjoint and $l(Q_j) \sim 1$ for all j, we deduce that $|B| \sim \sum_{Q_j \in \mathcal{Q}_B} |Q_j|$, which, combined with Hölder's inequality and Definition 3.4, implies that

$$\begin{aligned}\|a\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} &= \left[\sum_{Q_j \in \mathcal{Q}_B} |Q_j|^{\frac{2-p}{2}} \|\chi_{Q_j} a\|^p_{L^2(\mathbb{R}^n)}\right]^{1/p} \\ &\leq \left(\sum_{Q_j \in \mathcal{Q}_B} |Q_j|\right)^{\frac{2-p}{2p}} \left(\sum_{Q_j \in \mathcal{Q}_B} \|\chi_{Q_j} a\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}} \lesssim 1.\end{aligned}$$

Thus, we conclude that (i) holds true.

We now turn to the proofs of (ii) and (iii). Let $f \in L^p_{\mathcal{Q}}(\mathbb{R}^n)$. For each $Q_j \in \mathcal{Q}$ and $\|\chi_{Q_j} f\|_{L^2(\mathbb{R}^n)} \neq 0$, let

$$a_j := \frac{\chi_{Q_j} f}{|Q_j|^{\frac{1}{p}-\frac{1}{2}} \|\chi_{Q_j} f\|_{L^2(\mathbb{R}^n)}} \quad \text{and} \quad \lambda_j := |Q_j|^{\frac{1}{p}-\frac{1}{2}} \|\chi_{Q_j} f\|_{L^2(\mathbb{R}^n)};$$

otherwise, let $a_j \equiv 0 \equiv \lambda_j$. Then it is easy to see that, for each j, a_j is an $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-atom and

$$\left\{\sum_j |\lambda_j|^p\right\}^{\frac{1}{p}} = \left\{\sum_{Q_j \in \mathcal{Q}} [|Q_j|^{\frac{1}{p}-\frac{1}{2}} \|\chi_{Q_j} f\|_{L^2(\mathbb{R}^n)}]^p\right\}^{\frac{1}{p}} = \|f\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}.$$

Moreover, from the disjointness of $\{Q_j\}_j$, we deduce that the summation $f = \sum_j \lambda_j a_j$ converges almost everywhere in $\mathbb{R}^n$, which, together with (i), implies (ii).

Now, we claim that, for any $q \in (0, \infty)$, if $f \in L^q_{\mathcal{Q}}(\mathbb{R}^n)$, then the above summation converges also in $L^q_{\mathcal{Q}}(\mathbb{R}^n)$. Indeed, letting $N \in \mathbb{N}$, from the fact that

$f = \sum_j \lambda_j a_j$ converges almost everywhere in $\mathbb{R}^n$ and the disjointness of $\{Q_j\}_j$, we deduce that

$$\left\| f - \sum_{j \le N} \lambda_j a_j \right\|_{L^q_{\mathcal{Q}}(\mathbb{R}^n)} = \left[\sum_{Q_k \in \mathcal{Q}} |Q_k|^{\frac{2-q}{2}} \left\| \chi_{Q_k} \sum_{j=N+1}^{\infty} \lambda_j a_j \right\|^q_{L^2(\mathbb{R}^n)} \right]^{1/q}$$
$$= \left[\sum_{j=N+1}^{\infty} |Q_j|^{\frac{2-q}{2}} \|\chi_{Q_j} f\|^q_{L^2(\mathbb{R}^n)} \right]^{1/q}.$$

By letting $N \to \infty$, we see that the claim holds true, which immediately implies the validity of (iii) and hence completes the proof of Proposition 3.5. □

Remark 3.6. From the proof of Proposition 3.5, we deduce that it is possible to require that the radii of the associated balls of the atoms in Proposition 3.5 equal to 1 (see also [17, Remark 4.7]).

Now, we introduce the notion of the Hardy space $H^p_I(\mathbb{R}^n)$ associated with the identity operator I. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $S_I(f)(x)$ be the *square function* defined by setting,

$$S_I(f)(x) := \left[\iint_{\Gamma(x)} |t^{2m} e^{-t^{2m}} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right]^{\frac{1}{2}}$$
$$= \left\{ \int_0^{\infty} \left[\frac{1}{t^n} \int_{B(x,\,t)} |f(y)|^2\,dy \right] t^{4m} e^{-2t^{2m}} \frac{dt}{t} \right\}^{\frac{1}{2}}. \tag{3.1}$$

Definition 3.7. Let I be the identity operator. If $p \in (0, 2]$, then the *Hardy space* $H^p_I(\mathbb{R}^n)$ *associated with* I is defined as the completion of the set

$$\{f \in L^2(\mathbb{R}^n) : \|f\|_{H^p_I(\mathbb{R}^n)} := \|S_I(f)\|_{L^p(\mathbb{R}^n)} < \infty\}$$

with respect to the quasi-norm $\|\cdot\|_{H^p_I(\mathbb{R}^n)}$.

If $p \in (2, \infty)$, the *Hardy space* $H^p_I(\mathbb{R}^n)$ *associated with* I is defined as the dual space of the Hardy space $H^{p'}_I(\mathbb{R}^n)$, where $p' := \frac{p}{p-1} \in (1, 2)$ denotes the *conjugate exponent* of p.

Our main result of this subsection is the following proposition.

Proposition 3.8. *For* $p \in (0, \infty)$, $H^p_I(\mathbb{R}^n) = L^p_{\mathcal{Q}}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

Before proving Proposition 3.8, we first make the following remarks on the Hardy space $H^p_I(\mathbb{R}^n)$.

Remark 3.9. (i) The space $H^p_I(\mathbb{R}^n)$ has appeared in [14, Remark 3.8].

(ii) It is easy to see that the identity operator I satisfies the following assumptions:

(a) I is one-to-one, nonnegative and self-adjoint in $L^2(\mathbb{R}^n)$;

(b) for all $q \in [1, \infty]$, the semigroup $\{e^{-tI}\}_{t>0}$ satisfies the following off-diagonal estimates: for all $t \in (0, \infty)$, closed sets E, F in $\mathbb{R}^n$ and $f \in L^q(\mathbb{R}^n)$ supported in E,

$$\|e^{-tI}(f)\|_{L^q(F)} \leq \begin{cases} 0, & \text{when } d(E,\, F) > 0, \\ e^{-t}\|f\|_{L^q(E)}, & \text{when } d(E,\, F) = 0. \end{cases} \tag{3.2}$$

In particular, $\{e^{-tI}\}_{t>0}$ satisfies the so-called Davies–Gaffney estimates as in [25, 40, 42, 43, 46]. Thus, by [40, Corollary 5.3; 47, Theorem 5.1], we know that, for $p \in (0, 1]$, $H^p_I(\mathbb{R}^n)$ has the atomic and the molecular characterizations (see Definition 3.12 below for the notions of $H^p_I(\mathbb{R}^n)$-atoms and molecules).

(iii) Compared with the Davies–Gaffney estimates, the off-diagonal estimates (3.2) have many advantages. For example, the off-diagonal estimates in (3.3) vanish whenever $d(E, F) > 0$, which enable us to obtain some better norm estimates. For example, let $\sigma, \tau \in (0, \infty)$, $\mu \in (0, \pi/2)$ and

$$\Psi_{\sigma,\,\tau}(S^0_\mu) := \Big\{\psi :\ S^0_\mu \to \mathbb{C} : \psi \text{ is analytic and there exists a positive constant } C$$
$$\text{such that, for every } \xi \in S^0_\mu,\ |\psi(\xi)| \leq C\frac{|\xi|^\sigma}{1+|\xi|^{\sigma+\tau}}\Big\}.$$

Assuming $\psi \in \Psi_{\sigma,\,\tau}(S^0_\mu)$ and

$$f \in H^\infty(S^0_\mu) := \{\psi :\ S^0_\mu \to \mathbb{C} : \psi \text{ is bounded and analytic}\},$$

following the proof of [43, Lemma 2.28] with the Davies–Gaffney estimates used therein replaced by (3.2), we find that there exists a positive constant C such that, for all $t \in (0, \infty)$, closed sets E, F in $\mathbb{R}^n$ and $g \in L^2(\mathbb{R}^n)$ supported in E,

$$\|\psi(tI)f(I)(g)\|_{L^2(F)} \leq \begin{cases} 0, & \text{when } d(E,\, F) > 0, \\ C\|f\|_{L^\infty(S^0_\mu)}\|g\|_{L^2(E)}, & \text{when } d(E,\, F) = 0. \end{cases} \tag{3.3}$$

For all $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0, \infty)$ and $\mu \in (0, \pi/2)$, let $\psi \in \Psi_{\sigma_1,\,\tau_1}(S^0_\mu)$, $\widetilde{\psi} \in \Psi_{\sigma_2,\,\tau_2}(S^0_\mu)$ and $f \in H^\infty(S^0_\mu)$. Following [43, (4.3)] (see also [16, (5.5)] for $m \geq 1$), the operator Q^f is defined by setting

$$Q^f(F)(x,\, s) := \int_0^\infty (\psi(s^{2m}I)f(I)\widetilde{\psi}(t^{2m}I)(F(\cdot,\, t)))(x)\,\frac{dt}{t},$$

where $x \in \mathbb{R}^n$, $s \in (0, \infty)$ and F is in the tent space $T^2(\mathbb{R}^{n+1}_+)$ as in [22]. Using some arguments similar to those used in the proofs of [43, Proposition 4.4] and [16, (5.10)], we conclude that, for all $p \in (0, \infty)$ and $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0, \infty)$, Q^f is bounded on the tent space $T^p(\mathbb{R}^{n+1}_+)$. Here, comparing with [43, Proposition 4.4], we much enlarge the ranges of σ_1, σ_2, τ_1 and τ_2.

With the help of the boundedness of Q^f on $T^p(\mathbb{R}^{n+1}_+)$, following the proof of [43, Proposition 4.9] (via the definition of $H^p_I(\mathbb{R}^n)$ and the Calderón reproducing formula), we see that, for all $\sigma, \tau \in (0, \infty)$, $\psi \in \Psi_{\sigma,\tau}(S^0_\mu)$ and $p \in (0, \infty)$, the *operator* $Q_{\psi, I}$ defined by setting, for all $t \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$Q_{\psi, I}(f)(x, t) := \psi(t^{2m} I)(f)(x) \tag{3.4}$$

is bounded from $H^p_I(\mathbb{R}^n)$ to $T^p(\mathbb{R}^{n+1}_+)$ and the *operator* $\pi_{\psi, I}$ defined by setting, for all $F \in T^2(\mathbb{R}^{n+1}_+)$ and $x \in \mathbb{R}^n$,

$$\pi_{\psi, I}(F)(x) := \int_0^\infty \psi(t^{2m} I)(F(\cdot, t))(x)\, \frac{dt}{t}$$

is bounded from $T^p(\mathbb{R}^{n+1}_+)$ to $H^p_I(\mathbb{R}^n)$.

Proposition 3.10. *Let $p \in (0, \infty)$ and I be the identity operator. For every $f \in L^2(\mathbb{R}^n)$, let*

$$\|f\|_{\widetilde{H}^p_I(\mathbb{R}^n)} := \|S_I(f)\|_{L^p(\mathbb{R}^n)},$$

where S_I is as in (3.1)*. Then there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{H^p_I(\mathbb{R}^n)} \le \|f\|_{\widetilde{H}^p_I(\mathbb{R}^n)} \le C\|f\|_{H^p_I(\mathbb{R}^n)}. \tag{3.5}$$

In particular, $\|\cdot\|_{\widetilde{H}^p_I(\mathbb{R}^n)}$ constitutes a quasi-norm in $H^p_I(\mathbb{R}^n)$, which is equivalent to the quasi-norm $\|\cdot\|_{H^p_I(\mathbb{R}^n)}$.

Proof. We prove this proposition by borrowing some ideas from the proof of [43, Corollary 4.17]. If $p \in (0, 2]$, Proposition 3.10 is merely Definition 3.7. Thus, we only need to consider the case $p \in (2, \infty)$. Indeed, let $f \in L^2(\mathbb{R}^n)$. From Remark 3.9(iii) on the boundedness of the operator $Q_{\psi, I}$ as in (3.4) from $H^p_I(\mathbb{R}^n)$ to $T^p(\mathbb{R}^{n+1}_+)$, we deduce that

$$\|f\|_{\widetilde{H}^p_I(\mathbb{R}^n)} = \|S_I(f)\|_{L^p(\mathbb{R}^n)} = \|t^{2m} e^{-t^{2m} I}(f)\|_{T^p(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{H^p_I(\mathbb{R}^n)},$$

which proves the second inequality of (3.5).

To prove the first inequality of (3.5), using duality and the fact $p' := \frac{p}{p-1} \in (1, 2)$, we see that, for all $g \in H^{p'}_I(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx\right| &\sim \left|\int_{\mathbb{R}^n} \left[\int_0^\infty t^{4m} e^{-2t^{2m}}\, \frac{dt}{t} f(x)\right] \overline{g(x)}\, dx\right| \\ &\sim \left|\int_{\mathbb{R}^n} \int_0^\infty t^{2m} e^{-t^{2m}} f(x) \overline{t^{2m} e^{-t^{2m}} g(x)}\, \frac{dt\, dx}{t}\right| \\ &\lesssim \|f\|_{\widetilde{H}^p_I(\mathbb{R}^n)} \|g\|_{\widetilde{H}^{p'}_I(\mathbb{R}^n)} \sim \|f\|_{\widetilde{H}^p_I(\mathbb{R}^n)} \|g\|_{H^{p'}_I(\mathbb{R}^n)},\end{aligned}$$

which, together with the arbitrariness of g, implies $\|f\|_{H^p_I(\mathbb{R}^n)} \lesssim \|f\|_{\widetilde{H}^p_I(\mathbb{R}^n)}$. Thus, (3.5) holds true.

To finish the proof of Proposition 3.10, it suffices to show that $L^2(\mathbb{R}^n) \cap H^p_I(\mathbb{R}^n)$ is dense in $H^p_I(\mathbb{R}^n)$ for $p \in (2, \infty)$. Recall that the following Calderón reproducing formula

$$f = C_{(m)} \int_0^\infty t^{4m} e^{-2t^{2m}} f \,\frac{dt}{t} \tag{3.6}$$

holds true in $L^2(\mathbb{R}^n)$ with $C_{(m)}$ being a positive constant, depending on m, such that

$$C_{(m)} \int_0^\infty t^{4m} e^{-2t^{2m}} \,\frac{dt}{t} = 1.$$

From Remark 3.9(iii) on the boundedness of $Q_{\psi,\,I}$ and $\pi_{\psi,\,I}$, respectively, from $H_I^p(\mathbb{R}^n)$ to $T^p(\mathbb{R}^{n+1}_+)$ and from $T^p(\mathbb{R}^{n+1}_+)$ to $H_I^p(\mathbb{R}^n)$, and the density of $L^2(\mathbb{R}^n) \cap H_I^{p'}(\mathbb{R}^n)$ in $H_I^{p'}(\mathbb{R}^n)$, we deduce that the Calderón reproducing formula (3.6) also holds true in $H_I^{p'}(\mathbb{R}^n)$. By duality, we conclude that (3.6) holds true in $H_I^p(\mathbb{R}^n)$.

Moreover, let $K \in \mathbb{N}$, χ_{U_K} be the *characteristic function* of the set

$$U_K := \left\{(x,\,t) \in \mathbb{R}^{n+1}_+ : |x| < K,\ \frac{1}{K} < t < K\right\}$$

and $F \in T^p(\mathbb{R}^{n+1}_+)$. It is easy to see that $\{\chi_{U_K} F\}_{K\in\mathbb{N}} \subset (T^2(\mathbb{R}^{n+1}_+) \cap T^p(\mathbb{R}^{n+1}_+))$ and

$$\lim_{K\to\infty} \chi_{U_K} F = F \quad \text{in } T^p(\mathbb{R}^{n+1}_+).$$

Now, let $f \in H_I^p(\mathbb{R}^n)$. By Remark 3.9(iii) and (3.6), we have

$$f = C_{(m)} \int_0^\infty t^{4m} e^{-2t^{2m}} f \,\frac{dt}{t} = \lim_{K\to\infty} C_{(m)} \int_0^\infty t^{2m} e^{-t^{2m}} (\chi_{U_K} t^{2m} e^{-t^{2m}} f) \,\frac{dt}{t}$$

in $H_I^p(\mathbb{R}^n)$ and, for all $K \in \mathbb{N}$, $\int_0^\infty t^{2m} e^{-t^{2m}} (\chi_{U_K} t^{2m} e^{-t^{2m}} f) \frac{dt}{t} \in L^2(\mathbb{R}^n) \cap H_I^p(\mathbb{R}^n)$, where $C_{(m)}$ is as in (3.6). By letting $K \to \infty$, we conclude the density of $L^2(\mathbb{R}^n) \cap H_I^p(\mathbb{R}^n)$ in $H_I^p(\mathbb{R}^n)$. This finishes the proof of Proposition 3.10. □

Proposition 3.11. (i) *Let $p \in (1,\,\infty)$. Then $[H_I^p(\mathbb{R}^n)]^* = H_I^{p'}(\mathbb{R}^n)$, where $p' := \frac{p}{p-1}$ is the conjugate exponent of p.*

(ii) *Let $0 < p_1 < p_2 < \infty, \theta \in (0,\,1)$ and $p \in (0,\,\infty)$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$[H_I^{p_1}(\mathbb{R}^n),\, H_I^{p_2}(\mathbb{R}^n)]_\theta = H_I^p(\mathbb{R}^n).$$

Proof. Statement (i) follows from Definition 3.7. The proof of (ii) is similar to that of [43, Lemma 4.20], since $H_I^p(\mathbb{R}^n)$ is a retract of the tent space $T^p(\mathbb{R}^{n+1}_+)$, the details being omitted. This finishes the proof of Proposition 3.11. □

We now establish the relation between $H_I^p(\mathbb{R}^n)$ and $L_{\mathcal{Q}}^p(\mathbb{R}^n)$. To this end, we need the following notions of $(p,\,2,\,m)_I$-atoms and $(p,\,2,\,m)_I$-molecules based on [40, Definitions 2.1 and 2.3; 47, Definitions 4.2 and 4.3].

Definition 3.12. Let $p \in (0,\,1]$, $M \in \mathbb{N}$ and $B := B(x_B,\,r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\,\infty)$ be a ball in $\mathbb{R}^n$. A function $a \in L^2(\mathbb{R}^n)$ is called a $(p,\,2,\,M)_I$-*atom* *associated with* B if $\operatorname{supp} a \subset B$ and, for all $\ell \in \{0, \dots, M\}$, $\|r_B^{-2m\ell} a\|_{L^2(\mathbb{R}^n)} \le |B|^{\frac{1}{2}-\frac{1}{p}}$.

For all $\epsilon \in (0, \infty)$, a function $\alpha \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M, \epsilon)_I$-*molecule associated with* B if, for each $\ell \in \{0, \dots, M\}$ and $i \in \mathbb{Z}_+$, $\|r_B^{-2m\ell} \alpha\|_{L^2(S_i(B))} \le 2^{-i\epsilon}|2^i B|^{\frac{1}{2}-\frac{1}{p}}$, where, for all $i \in \mathbb{N}$, $S_i(B) := 2^i B \backslash (2^{i-1}B)$ and $S_0(B) := B$.

From their size conditions, we deduce that, for each $H_I^p(\mathbb{R}^n)$-atom or molecule, the radius r_B of the associated ball B satisfies $r_B \ge 1$ when M is large enough.

Remark 3.13. Since I is a one-to-one operator of type $\omega \in (0, \pi/2]$ having a bounded H_∞ functional calculus and satisfying the Davies–Gaffney estimates as in (2.8) with $\kappa = 0$, by [16, Theorem 4.5], we know that, for all $p \in (0, 1]$, $H_I^p(\mathbb{R}^n)$ can be characterized by the molecules defined as in Definition 3.12.

Moreover, since, for all $p \in (0, 1]$, $M \in \mathbb{N}$ and $\epsilon \in (0, \infty)$, it is easy to see that, for any $j \in \mathbb{Z}_+$ and $(p, 2, M, \epsilon)_I$-molecule α associated with a ball B, $2^{j\epsilon}\chi_{S_j(B)}\alpha$ is a $(p, 2, M)_I$-atom associated with the ball $2^j B$, where $S_0(B) := B$ and, for all $j \in \mathbb{N}$, $S_j(B) := 2^j B \backslash 2^{j-1} B$. Thus, $H_I^p(\mathbb{R}^n)$ can be characterized by the atoms defined as in Definition 3.12.

We now turn to the proof of Proposition 3.8.

Proof of Proposition 3.8. We prove this proposition by considering three cases on the size of p. If $p \in (0, 1]$, from Definitions 3.4 and 3.12, we deduce that each $H_I^p(\mathbb{R}^n)$-atom is an $L_{\mathcal{Q}}^p(\mathbb{R}^n)$-atom, which, together with Proposition 3.5 and Remark 3.9(ii), implies that $H_I^p(\mathbb{R}^n) \subset L_{\mathcal{Q}}^p(\mathbb{R}^n)$.

On the other hand, observe that each $L_{\mathcal{Q}}^p(\mathbb{R}^n)$-atom, with the radius of the associated ball equal to 1, is also an $H_I^p(\mathbb{R}^n)$-atom. Let $f \in L_{\mathcal{Q}}^p(\mathbb{R}^n)$. By Proposition 3.5 and Remark 3.6, we know that f has an $L_{\mathcal{Q}}^p(\mathbb{R}^n)$-atomic decomposition with the radii of the associated balls equaling to 1, which, together with Proposition 3.5 and Remark 3.9(ii), implies that $L_{\mathcal{Q}}^p(\mathbb{R}^n) \subset H_I^p(\mathbb{R}^n)$. This proves Proposition 3.8 for $p \in (0, 1]$.

If $p \in (1, 2]$, by [17, Proposition 4.4; 14, Proposition 3.9], we know that $H_I^2(\mathbb{R}^n) = L^2(\mathbb{R}^n) = L_{\mathcal{Q}}^2(\mathbb{R}^n)$, which, together with the fact $H_I^1(\mathbb{R}^n) = L_{\mathcal{Q}}^1(\mathbb{R}^n)$, Proposition 3.11(ii) and the complex interpolation of $L_{\mathcal{Q}}^p(\mathbb{R}^n)$ (see [17, Theorem 4.10]), implies that, for all $p \in (1, 2]$, $H_I^p(\mathbb{R}^n) = L_{\mathcal{Q}}^p(\mathbb{R}^n)$.

The case $p \in (2, \infty)$ follows immediately from Proposition 3.11(i) and the duality of $L_{\mathcal{Q}}^p(\mathbb{R}^n)$ (see [17, Theorem 4.8]). This finishes the proof of Proposition 3.8. $\square$

3.2. *The local Hardy space $h_L^p(\mathbb{R}^n)$ associated with L*

In this subsection, let L be the inhomogeneous higher order elliptic operator in divergence form as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. We introduce the local Hardy space $h_L^p(\mathbb{R}^n)$ associated with L. As was stated in the introduction, since L is inhomogeneous, it is natural to consider the local Hardy space associated with L. We refer the reader to [38, 57, 17, 28, 13, 59, 60, 39] and the references cited therein for more related properties about some other local Hardy spaces.

Definition 3.14. Let L be an inhomogeneous higher order elliptic operator as in (1.1) satisfying the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ and $\delta_0 \in [0, \infty)$ as in (2.7). For all $p \in (0, 2]$, the *local Hardy space* $h^p_L(\mathbb{R}^n)$ *associated with* L is defined as the completion of the space

$$\{f \in L^2(\mathbb{R}^n) : \|f\|_{h^p_L(\mathbb{R}^n)} := \|S_{L,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)} + \|S_{L,\,\delta_0,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)} + \|e^{-L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} < \infty\}$$

with respect to the *quasi-norm* $\|\cdot\|_{h^p_L(\mathbb{R}^n)}$, where $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ is as in Definition 3.3 and, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *local square function* $S_{L,\,\mathrm{loc}}(f)$ is defined by setting

$$S_{L,\,\mathrm{loc}}(f)(x) := \left[\int_0^1 \int_{|y-x|<t} |t^{2m} L e^{-t^{2m}L}(f)(y)|^2 \, \frac{dy\,dt}{t^{n+1}}\right]^{1/2} \tag{3.7}$$

and the *local square function* $S_{L,\,\delta_0,\,\mathrm{loc}}(f)$ by setting

$$S_{L,\,\delta_0,\,\mathrm{loc}}(f)(x) := \left[\int_0^1 \int_{|y-x|<t} |t^{2m} \delta_0 e^{-t^{2m}L}(f)(y)|^2 \, \frac{dy\,dt}{t^{n+1}}\right]^{1/2}.$$

For $p \in (2, \infty)$, $h^p_L(\mathbb{R}^n)$ is defined as the dual space of $h^{p'}_{L^*}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$ and $p' := \frac{p}{p-1}$ the *conjugate exponent* of p.

Observe that, if L satisfies the Ellipticity condition $(\mathcal{E})$, then L has a bounded functional calculus in $L^2(\mathbb{R}^n)$. Thus, in this case, we can choose $\delta_0 \equiv 0$ in (2.7) (see (2.6) and the arguments above Proposition 2.2) and hence $S_{L,\,\delta_0,\,\mathrm{loc}}(f) \equiv 0$.

Remark 3.15. (i) We point out that, in the above definition of $h^p_L(\mathbb{R}^n)$, the space $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ can be replaced by $H^p_I(\mathbb{R}^n)$ defined as in Definition 3.7, since the two spaces coincide with each other for all $p \in (0, \infty)$.

(ii) We also point out that the local operator-adapted Hardy spaces have appeared before; see [48] in the case when the considered operator L satisfies the Poisson estimates and [39] when L is self-adjoint and satisfies the second-order Gaussian estimates; in both cases, the authors defined the local Hardy space $h^1_L(\mathbb{R}^n)$ via the norm $\|f\|_{h^1_L(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \|S_{L,\,\mathrm{loc}}(f)\|_{L^1(\mathbb{R}^n)}$, where $S_{L,\,\mathrm{loc}}(f)$ is as in (3.7). Comparing the three terms in the quasi-norm of Definition 3.14, $\|f\|_{h^1_L(\mathbb{R}^n)}$ has only two terms. The main reason of this difference is that, since L in [48, 39] is self-adjoint, so one can do the functional calculus on L itself, while, in the present case, we have to do the functional calculus on the operator $L + \delta_0$, which leads to the term $\|S_{L,\,\delta_0,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)}$ (see the proof of Theorem 3.18 for more details). Moreover, recalling in [48, 39], to obtain the local molecular characterization of

$h^1_L(\mathbb{R}^n)$, by the local Calderón formula (see also (3.12) for a variant of the local Calderón formula), one was reduced to establishing an $L^1 - L^2$ off-diagonal estimate for e^{-L} (see, for example, the proof of [48, Lemma 3.3]). This holds true when L has pointwise estimates of the associated heat kernel, but may not hold true for general operators or the case $p \in (0, 1)$. Thus, instead of the term $\|f\|_{L^1(\mathbb{R}^n)}$, we use the term $\|e^{-L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}$. Recall also that, in [17], the space $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ was first introduced to define the local Hardy spaces of differential forms on Riemannian manifolds.

(iii) Let $C_1 \in \mathbb{R}$ and $-\Delta$ be the Laplace operator. Observe that $-\Delta + C_1 I$ is a special example of the inhomogeneous elliptic operator as in (1.1) and satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. For all $p \in (0, \infty)$, let $h^p_{-\Delta+C_1 I}(\mathbb{R}^n)$ be the local Hardy space defined as in Definition 3.14. From the fact that, for all $C \in [0, \infty)$, $\{e^{t(\Delta - CI)}\}_{t>0}$ satisfies the off-diagonal estimates as in (2.18) and Theorem 4.8 below, we deduce that, for all $C_2 \in (\max\{0, C_1\}, \infty)$ and $p \in (0, \infty)$,

$$h^p_{-\Delta+C_1 I}(\mathbb{R}^n) = H^p_{-\Delta+C_2 I}(\mathbb{R}^n),$$

where $H^p_{-\Delta+C_2 I}(\mathbb{R}^n)$ denotes the Hardy space associated with the Schrödinger operator $-\Delta + C_2 I$ (see also Definition 4.1 below). Recall that this kind of Hardy spaces was first introduced by Dziubański and Zienkiewicz [33, 34]. From the atomic characterization, the interpolation and the duality of both $h^p(\mathbb{R}^n)$ and $H^p_{-\Delta+C_2 I}(\mathbb{R}^n)$ (see [34, 38] and Theorem 3.31 below), we deduce that, for all $p \in (0, \infty)$, $H^p_{-\Delta+C_2 I}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$, where $h^p(\mathbb{R}^n)$ is the classical local Hardy space introduced by Goldberg [38]. Thus, for all $C_1 \in \mathbb{R}$ and $p \in (0, \infty)$, $h^p_{-\Delta+C_1 I}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$, which implies that our local Hardy space $h^p_L(\mathbb{R}^n)$ defined as in Definition 3.14 goes back to the classical local Hardy space when $L \equiv -\Delta + C_1 I$ for all $C_1 \in \mathbb{R}$.

We now establish the local molecular characterization of $h^p_L(\mathbb{R}^n)$ for $p \in (0, 1]$.

Definition 3.16. Let $p \in (0, 1]$, L be an inhomogeneous higher order elliptic operator as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let $B := B(x_B, r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ be a ball in $\mathbb{R}^n$. For all $M \in \mathbb{N}$ and $\epsilon \in (0, \infty)$, a function $\alpha \in L^2(\mathbb{R}^n)$ is called a *local* $(p, 2, M, \epsilon)_L$-*molecule associated with* B if, for each $\ell \in \{0, \dots, M\}$, α belongs to the range of L^ℓ in $L^2(\mathbb{R}^n)$. Moreover,

(i) when $r_B \geq 1$, for all $i \in \mathbb{Z}_+$, $\|\alpha\|_{L^2(S_i(B))} \leq 2^{-i\epsilon}|2^i B|^{\frac{1}{2}-\frac{1}{p}}$;
(ii) when $r_B < 1$, for all $\ell \in \{0, \dots, M\}$ and $i \in \mathbb{Z}_+$,

$$\|(r_B^{2m} L)^{-\ell}(\alpha)\|_{L^2(S_i(B))} \leq 2^{-i\epsilon}|2^i B|^{\frac{1}{2}-\frac{1}{p}}. \tag{3.8}$$

Assume that $\{\alpha_j\}_{j\in\mathbb{N}}$ is a sequence of local $(p, 2, M, \epsilon)_L$-molecules and $\{\lambda_j\}_{j\in\mathbb{N}} \in l^p$. For any $f \in L^2(\mathbb{R}^n)$, if $f = \sum_{j\in\mathbb{N}} \lambda_j \alpha_j$ holds true in $L^2(\mathbb{R}^n)$, then $\sum_{j\in\mathbb{N}} \lambda_j \alpha_j$ is called a *local molecular* $(p, 2, M, \epsilon)_L$-*representation* of f.

Definition 3.17. Let $p \in (0, 1]$, L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let $M \in \mathbb{N}$ and $\epsilon \in (0, \infty)$. The *local molecular Hardy space*

$h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)$ is defined as the completion of the space

$$\widetilde{h}^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n) := \{f : f \text{ has a local molecular } (p,\,2,\,M,\,\epsilon)_L\text{-representation}\}$$

with respect to the *quasi-norm*

$$\|f\|_{h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)} := \inf\left\{\|\{\lambda_j\}_{j\in\mathbb{N}}\|_{l^p} : f = \sum_{j\in\mathbb{N}} \lambda_j\alpha_j \text{ is a local molecular } (p,\,2,\,M,\,\epsilon)_L\text{-representation}\right\},$$

where the infimum is taken over all the local molecular $(p,\,2,\,M,\,\epsilon)_L$-representations of f as above.

To simplify the notation, we sometimes write $h^p_{L,\,\mathrm{mol}}(\mathbb{R}^n)$ instead of $h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)$, when there exists no confusion.

Theorem 3.18. *Let $p \in (0,\,1], L$ be as in* (1.1) *and satisfy the Weak ellipticity condition* ($\widetilde{\mathcal{E}}$), *and let $\epsilon \in (0,\,\infty)$ and $M \in \mathbb{N}$ satisfy $M > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$. Then $h^p_L(\mathbb{R}^n) = h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)$ with equivalent quasi-norms.*

To prove Theorem 3.18, we need some basic results on the local tent space $t^p(\mathbb{R}^n \times (0,\,1])$ for $p \in (0,\,\infty)$.

Definition 3.19 ([17]). Let $p \in (0,\,\infty)$. The *local tent space* $t^p(\mathbb{R}^n \times (0,\,1])$ is defined as the collection

$$\{F : \mathbb{R}^n \times (0,\,1] \to \mathbb{C} : \|F\|_{t^p(\mathbb{R}^n\times(0,\,1])} := \|\mathcal{A}_{\mathrm{loc}}(F)\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where, for all measurable functions F defined on $\mathbb{R}^n \times (0,\,1]$ and $x \in \mathbb{R}^n$, its *local $\mathcal{A}$-functional* $\mathcal{A}_{\mathrm{loc}}(F)$ is defined by setting

$$\mathcal{A}_{\mathrm{loc}}(F)(x) := \left\{\iint_{\Gamma_{\mathrm{loc}}(x)} |F(y,\,t)|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$

with

$$\Gamma_{\mathrm{loc}}(x) := \{(y,\,t) \in \mathbb{R}^n \times (0,\,1] : |y - x| < t\} \tag{3.9}$$

being the *truncated cone with vertex x*.

For $p \in (0,\,1]$, $t^p(\mathbb{R}^n \times (0,\,1])$ can be characterized by the local tent atoms.

Definition 3.20. Let $p \in (0,\,1]$ and B be a ball in $\mathbb{R}^n$. A function A on $\mathbb{R}^n \times (0,\,1]$ is called a $(t^p,\,2)$-*atom associated with B* if

(i) $\operatorname{supp} A \subset T^{\mathrm{loc}}(B)$, where, for any measurable set E, the *local tent over E*, $T^{\mathrm{loc}}(E)$, is defined by $T^{\mathrm{loc}}(E) := \{(x,\,t) \in \mathbb{R}^n \times (0,\,1] : d(x,\,E^{\complement}) \ge t\}$;

(ii)

$$\left\{\iint_{\mathbb{R}^{n+1}_+} |A(y,\, t)|^2 \,\frac{dy\,dt}{t}\right\}^{1/2} \le |B|^{\frac{1}{2}-\frac{1}{p}}.$$

The following local atomic decomposition of $t^p(\mathbb{R}^n \times (0,\, 1])$ was first proved in [17, Theorem 3.6] for $p = 1$, which can be extended to the case $p \in (0,\, 1]$ with some minor modifications in the proof of [17, Theorem 3.6].

Proposition 3.21 ([17]). *Let $p \in (0,\, 1]$. Then, for all $F \in t^p(\mathbb{R}^n \times (0,\, 1])$, there exist sequences $\{\lambda_j\}_j$ of complex numbers, $\{B_j\}_j$ of balls and $\{A_j\}_j$ of $(t^p,\, 2)$-atoms associated, respectively, to $\{B_j\}_j$ such that, for almost every $(x,\, t) \in \mathbb{R}^n \times (0,\, 1]$,*

$$F(x,\, t) = \sum_j \lambda_j A_j(x,\, t) \tag{3.10}$$

and $\sum_j |\lambda_j|^p \sim \|F\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}$, where the implicit equivalent positive constants depend only on n. Moreover, if $F \in t^p(\mathbb{R}^n \times (0,\, 1]) \cap t^2(\mathbb{R}^n \times (0,\, 1])$, the decomposition in (3.10) *also converges in $t^2(\mathbb{R}^n \times (0,\, 1])$.*

Let $M \in \mathbb{N}$. For all $F \in t^2(\mathbb{R}^n \times (0,\, 1])$, $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, the *local projection operator* $\pi^{\mathrm{loc}}_{M,\,k,\,L}$ is defined by

$$\pi^{\mathrm{loc}}_{M,\,k,\,L}(F)(x) := \int_0^1 (t^{2m}L)^{M+1}(t^{2m}\delta_0)^k e^{-t^{2m}(L+\delta_0)}(F(\cdot,\, t))(x)\,\frac{dt}{t},$$

where $\delta_0 \in (0,\, \infty)$ is the same as in (2.7).

Proposition 3.22. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let $p \in (0,\, 1]$, $\epsilon \in (0,\, \infty)$ and $M_0, k \in \mathbb{N}$. Then, for each $M \in \{M_0, M_0+1, \ldots\}$ and $(t^p,\, 2)$-atom A, associated with the ball B, $\pi^{\mathrm{loc}}_{M,\,k,\,L}(A)$ is a local $(p,\, 2,\, M_0,\, \epsilon)_L$-molecule associated with B, up to a harmless positive constant multiple. Moreover, $\pi^{\mathrm{loc}}_{M,\,k,\,L}$ can be extended to a bounded linear operator from $t^p(\mathbb{R}^n \times (0,\, 1])$ to $h^p_{L,\,\mathrm{mol},\,M_0,\,\epsilon}(\mathbb{R}^n)$.*

Proof. We first claim that, to prove this proposition, we only need to consider the operator $\pi^{\mathrm{loc}}_{M,\,1,\,L}$. Indeed, if this proposition holds true for $\pi^{\mathrm{loc}}_{M,\,1,\,L}$, then, for any $k \in \mathbb{N}$, observing the fact that there exists a positive constant C such that, for all $F \in t^2(\mathbb{R}^n \times (0,\, 1])$ and $x \in \mathbb{R}^n$

$$|\pi^{\mathrm{loc}}_{M,\,k,\,L}(F)(x)| \le C|\pi^{\mathrm{loc}}_{M,\,1,\,L}(F)(x)|,$$

we see that all the estimates concerning $\pi^{\mathrm{loc}}_{M,\,k,\,L}$ can be reduced to those concerning $\pi^{\mathrm{loc}}_{M,\,1,\,L}$. This shows that the above claim holds true.

Now, let A be a $(t^p,\, 2)$-atom associated with B and $\alpha := \pi^{\mathrm{loc}}_{M,\,1,\,L}(A)$. We now prove that α is a local $(p,\, 2,\, M_0,\, \epsilon)_L$-molecule, up to a harmless positive constant multiple, by considering the following two cases.

If $j \in \{3, 4, \ldots\}$, then, for all $g \in L^2(\mathbb{R}^n)$ satisfying $\operatorname{supp} g \subset S_j(B)$ and $\|g\|_{L^2(\mathbb{R}^n)} \le 1$, by duality, Hölder's inequality, Definition 3.20 and Proposition 2.10

(with L replaced by its adjoint operator L^*), we conclude that there exists a sufficiently large constant μ such that, for all $\ell \in \{0, \ldots, M_0\}$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} (r_B^{2m} L)^{-\ell}(\alpha)(x)\overline{g(x)}\,dx\right| \\
&\quad \le \int_0^{r_B}\int_{\mathbb{R}^n} |A(x,\,t)\overline{(r_B^{2m}L^*)^{-\ell}(t^{2m}L^*)^{M+1}(t^{2m}\delta_0)e^{-t^{2m}(L^*+\delta_0)}(g)(x)}|\,\frac{dxdt}{t} \\
&\quad \lesssim \left\{\int_0^{r_B}\int_B |A(x,\,t)|^2\,\frac{dxdt}{t}\right\}^{\frac12} \\
&\qquad \times \left\{\int_0^{r_B}\int_B |(t^{2m}L^*)^{M+1-\ell}(t^{2m}\delta_0)e^{-t^{2m}(L^*+\delta_0)}(g)(x)|^2\,\frac{dxdt}{t}\right\}^{1/2} \\
&\quad \lesssim |B|^{\frac12-\frac1p}\left\{\int_0^{r_B}\left[\exp\left\{-\frac{(2^jr_B)^{2m/(2m-1)}}{t^{2m/(2m-1)}}\right\}\right]^2\frac{dt}{t}\right\}^{1/2}\|g\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim 2^{-j\mu}|2^jB|^{\frac12-\frac1p},
\end{aligned}$$

which, together with the arbitrariness of g, implies that, for all $\ell \in \{0, \ldots, M_0\}$ and $j \in \{3, 4, \ldots\}$,

$$\|(r_B^{2m}L)^{-\ell}(\alpha)\|_{L^2(S_j(B))} \lesssim 2^{-j\mu}|2^jB|^{\frac12-\frac1p}.$$

The proof for the case when $j \in \{0,\,1,\,2\}$ is similar, we only need to replace Proposition 2.10, used in the case $j \ge 3$, by the following local quadratic estimates

$$\left\{\int_0^1\int_{\mathbb{R}^n} |(t^{2m}L^*)^{M+1-\ell}(t^{2m}\delta_0)e^{-t^{2m}(L^*+\delta_0)}(g)(x)|^2\,\frac{dxdt}{t}\right\}^{1/2} \lesssim \|g\|_{L^2(\mathbb{R}^n)}, \tag{3.11}$$

which follows immediately from the corresponding quadratic estimates of the operators $L^* + \delta_0 I$ and $\delta_0 I$ (see (3.26) below for a similar argument). Thus, α is a local $(p,\,2,\,M_0,\,\epsilon)_L$-molecule associated with B, up to a harmless positive constant multiple, which completes the proof of Proposition 3.22. $\square$

Remark 3.23. We point out that, from the proof of Proposition 3.22, we actually deduce that $\pi^{\mathrm{loc}}_{M,\,k,\,L}$ maps each $(t^p,\,2)$-atom to a "global" $(p,\,2,\,M_0,\,\epsilon)_L$-molecule up to a harmless positive constant multiple, namely, (3.8) always holds true despite the size of the associated ball of the molecule.

We now turn to the proof of Theorem 3.18.

Proof of Theorem 3.18. We prove this theorem by showing that

$$\left[h^p_L(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right] = \left[h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right]$$

with equivalent quasi-norms. We first prove that

$$\left[h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right] \subset \left[h^p_{L,\,M,\,\mathrm{mol},\,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right].$$

Let $\delta_0 \in (0, \infty)$ be as in (2.7). From the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, it follows that $L + \delta_0$ is an operator of type ω in $L^2(\mathbb{R}^n)$ for some $\omega \in [0, \pi/2)$. Moreover, $L + \delta_0$ has a bounded H_∞ functional calculus in $L^2(\mathbb{R}^n)$.

Now, let $\widetilde{C}$ and $C_{(k)}$ be the constants such that

$$1 = \widetilde{C} \int_0^1 (t^{2m} z)^{2(M+1)} e^{-2t^{2m} z} \,\frac{dt}{t} + \sum_{k=0}^{2(M+1)} C_{(k)} z^k e^{-2z} \tag{3.12}$$

for all $z \in \mathbb{C}$ satisfying $\Re e\, z > 0$. The following local Calderón reproducing formula

$$\begin{aligned} f &= \widetilde{C} \int_0^1 [t^{2m}(L+\delta_0)]^{2(M+1)} e^{-2t^{2m}(L+\delta_0)}(f)\,\frac{dt}{t} \\ &\quad + \sum_{k=0}^{2(M+1)} C_{(k)} (L+\delta_0)^k e^{-2(L+\delta_0)}(f) \\ &= \sum_{M+2 \le j \le 2(M+1)} \widetilde{C}_{(j)} \int_0^1 (t^{2m}L)^j (t^{2m}\delta_0)^{2(M+1)-j} e^{-2t^{2m}(L+\delta_0)}(f)\,\frac{dt}{t} \\ &\quad + \sum_{0 \le j \le M+1} \widetilde{C}_{(j)} \int_0^1 \cdots \frac{dt}{t} + \sum_{k=0}^{2(M+1)} C_{(k)} (L+\delta_0)^k e^{-2(L+\delta_0)}(f) \\ &=: f_1 + f_2 + f_3 \end{aligned} \tag{3.13}$$

holds true for all $f \in L^2(\mathbb{R}^n)$, where $\widetilde{C}$ and $C_{(k)}$ are the same as in (3.12) and $\widetilde{C}_{(j)}$ is a positive constant depending on j (see also [48, (3.11); 28, Lemma 3.9] for some other Calderón reproducing formulae of similar nature).

Now, let $f \in h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Using (3.13), we write $f := f_1 + f_2 + f_3$ in $L^2(\mathbb{R}^n)$.

To estimate f_1, since $f \in h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, using the fact that, for all $t \in (0, \infty)$, $e^{-t^{2m}\delta_0} \le 1$, Definition 3.14 and some local quadratic estimates as in (3.11) via replacing

$$(t^{2m}L^*)^{M+1-\ell}(t^{2m}\delta_0) e^{-t^{2m}(L^*+\delta_0)}$$

by $(t^{2m}L)^j (t^{2m}\delta_0)^{2(M+1)-j} e^{-t^{2m}(L+\delta_0)}$, we know that

$$t^{2m} L e^{-t^{2m}(L+\delta_0)}(f) \in t^p(\mathbb{R}^n \times (0, 1]) \cap t^2(\mathbb{R}^n \times (0, 1]),$$

which, combined with Proposition 3.21, implies that there exist $\{\lambda_l\}_l \in l^p$ and a sequence $\{A_l\}_l$ of $(t^p, 2)$-atoms such that $t^{2m} L e^{-t^{2m}(L+\delta_0)}(f) = \sum_l \lambda_l A_l$ in

$t^p(\mathbb{R}^n\times(0,\,1])\cap t^2(\mathbb{R}^n\times(0,\,1])$ and $\|\{\lambda_l\}_l\|_{l^p}\lesssim\|t^{2m}Le^{-t^{2m}(L+\delta_0)}(f)\|_{t^p(\mathbb{R}^n\times(0,\,1])}\lesssim\|f\|_{h^p_L(\mathbb{R}^n)}$. This, together with Proposition 3.22, shows that

$$\begin{aligned}
f_1 &= \sum_{M+2\le j\le 2(M+1)}\widetilde{C}_{(j)}\int_0^1(t^{2m}L)^{j-1}\\
&\quad\times(t^{2m}\delta_0)^{2(M+1)-j}e^{-t^{2m}(L+\delta_0)}(t^{2m}Le^{-t^{2m}(L+\delta_0)}(f))\,\frac{dt}{t}\\
&=\sum_l\lambda_l\sum_{M+2\le j\le 2(M+1)}\widetilde{C}_{(j)}\int_0^1(t^{2m}L)^{j-1}\\
&\quad\times(t^{2m}\delta_0)^{2(M+1)-j}e^{-t^{2m}(L+\delta_0)}(A_l)\,\frac{dt}{t}\\
&=:\sum_l\lambda_l\alpha_l,
\end{aligned}$$

where, for each l, α_l is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule and hence

$$\begin{aligned}
\|f_1\|_{h^p_{L,\,\mathrm{mol},\,\mathrm{M},\,\epsilon}(\mathbb{R}^n)}&\lesssim\|\{\lambda_l\}_l\|_{l^p}\lesssim\|t^{2m}Le^{-t^{2m}(L+\delta_0)}(f)\|_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\lesssim\|f\|_{h^p_L(\mathbb{R}^n)}.
\end{aligned}\tag{3.14}$$

To estimate f_2, since $f\in h^p_L(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, we know that, for all $t\in(0,\,1]$,

$$t^{2m}\delta_0e^{-t^{2m}(L+\delta_0)}(f)\in t^p(\mathbb{R}^n\times(0,\,1])\cap t^2(\mathbb{R}^n\times(0,\,1]).$$

This, combined with Proposition 3.21, implies that there exist $\{\widetilde{\lambda}_i\}_i$ and a sequence $\{\widetilde{A}_i\}_i$ of $(t^p,2)$-atoms associated, respectively, with $\{B_i\}_i$ such that $t^{2m}\delta_0e^{-t^{2m}(L+\delta_0)}(f)=\sum_i\widetilde{\lambda}_i\widetilde{A}_i$ in $t^p(\mathbb{R}^n\times(0,\,1])\cap t^2(\mathbb{R}^n\times(0,\,1])$ and

$$\|\{\widetilde{\lambda}_i\}_i\|_{l^p}\sim\|t^{2m}\delta_0e^{-t^{2m}(L+\delta_0)}(f)\|_{t^p(\mathbb{R}^n\times(0,\,1])}\lesssim\|f\|_{h^p_L(\mathbb{R}^n)}.\tag{3.15}$$

Thus, we have

$$\begin{aligned}
f_2&=\sum_{0\le j\le M+1}\widetilde{C}_{(j)}\int_0^1(t^{2m}L)^j(t^{2m}\delta_0)^{2(M+1)-j}e^{-2t^{2m}(L+\delta_0)}(f)\,\frac{dt}{t}\\
&=\sum_{0\le j\le M+1}\widetilde{C}_{(j)}\sum_i\widetilde{\lambda}_i\int_0^1(t^{2m}L)^j(t^{2m}\delta_0)^{2(M+1)-j-1}e^{-t^{2m}(L+\delta_0)}(\widetilde{A}_i)\,\frac{dt}{t}
\end{aligned}$$

holds true in $L^2(\mathbb{R}^n)$.

Now, let A be a $(t^p,\,2)$-atom associated with $B:=B(x_B,\,r_B)$ for some $x_B\in\mathbb{R}^n$ and $r_B\in(0,\,\infty)$, $j\in\{0,\ldots,M+1\}$ and

$$\alpha_j:=\int_0^1(t^{2m}L)^j(t^{2m}\delta_0)^{2(M+1)-j-1}e^{-t^{2m}(L+\delta_0)}(A(\cdot,\,t))(x)\,\frac{dt}{t}.$$

We estimate α_j based on two cases on the size of r_B.

If $r_B \ge 1$, then, for all $l \in \mathbb{Z}_+$, let $g \in L^2(\mathbb{R}^n)$ satisfy $\operatorname{supp} g \subset S_l(B)$ and $\|g\|_{L^2(\mathbb{R}^n)} \le 1$. By duality, Hölder's inequality and Proposition 2.10, we see that there exists a sufficiently large positive constant η such that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \alpha_j(x)\overline{g(x)}\,dx\right| \\
&\quad = \left|\int_0^1 \int_{\mathbb{R}^n} A(x,\,t)\overline{(t^{2m}L^*)^j(t^{2m}\delta_0)^{2(M+1)-j-1}e^{-t^{2m}(L^*+\delta_0)}(g)(x)}\,\frac{dxdt}{t}\right| \\
&\quad \lesssim \left\{\int_0^1 \int_{\mathbb{R}^n} |A(x,\,t)|^2\,\frac{dxdt}{t}\right\}^{\frac{1}{2}} \\
&\qquad \times \left\{\int_0^1 \int_B |(t^{2m}L^*)^j(t^{2m}\delta_0)^{2(M+1)-j-1}e^{-t^{2m}(L^*+\delta_0)}(g)(x)|^2\,\frac{dxdt}{t}\right\}^{\frac{1}{2}} \\
&\quad \lesssim |B|^{\frac{1}{2}-\frac{1}{p}}\left[\int_0^1 \exp\left\{-C\frac{[d(B,\,S_l(B))]^{2m/(2m-1)}}{t^{2m/(2m-1)}}\right\}\,\frac{dt}{t}\right]^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-l\eta}.
\end{aligned}$$

This shows that α_j is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule associated with B, up to a harmless positive constant multiple.

If $r_B < 1$, let $B^* := B(x_B,\,1)$. Then, for all $l \in \mathbb{Z}_+$ and $g \in L^2(\mathbb{R}^n)$ satisfying $\operatorname{supp} g \subset S_l(B^*)$, and $\|g\|_{L^2(\mathbb{R}^n)} \le 1$, by an argument similar to that used in the proof for the case $r_B \ge 1$ and the assumption that $M > \frac{n}{2m}(\frac{1}{p}-\frac{1}{2})$, we conclude that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \alpha_j(x)\overline{g(x)}\,dx\right| \\
&\quad = \left|\int_0^{r_B} \int_{\mathbb{R}^n} A(x,\,t)\overline{(t^{2m}L^*)^j(t^{2m}\delta_0)^{2(M+1)-j-1}e^{-t^{2m}(L^*+\delta_0)}(g)(x)}\,\frac{dxdt}{t}\right| \\
&\quad \lesssim \left\{\int_0^{r_B} \int_{\mathbb{R}^n} |A(x,\,t)|^2\,\frac{dxdt}{t}\right\}^{\frac{1}{2}} r_B^{2mM} \\
&\qquad \times \left\{\int_0^{r_B} \int_B |(t^{2m}L^*)^j(t^{2m}\delta_0)^{M+1-j}e^{-t^{2m}(L^*+\delta_0)}(g)(x)|^2\,\frac{dxdt}{t}\right\}^{\frac{1}{2}} \\
&\quad \lesssim |B|^{\frac{1}{2}-\frac{1}{p}}r_B^{2mM}\left\{\int_0^1 \exp\left\{-\frac{[d(B,\,S_l(B^*))]^{2m/(2m-1)}}{t^{2m/(2m-1)}}\right\}\,\frac{dt}{t}\right\}^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim 2^{-l\eta},
\end{aligned}$$

where η is the positive constant as in the case $r_B > 1$. This shows that α_j is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule associated with B^* up to a harmless constant multiple.

Thus, we know that f_2 has a local molecular $(p,\,2,\,M,\,\epsilon)_L$-representation. This implies that $f_2 \in h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Moreover, from (3.15), we deduce that

$$\|f_2\|_{h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)} \lesssim \|f\|_{h^p_L(\mathbb{R}^n)}. \tag{3.16}$$

We now turn to the estimates of f_3. From $f \in h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, Definition 3.14 and Proposition 2.10, it follows that $e^{-L}(f) \in L^p_{\mathcal{Q}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Proposition 3.5, we know that there exist $\{\lambda_j\}_j \in l^p$ and a sequence $\{A_j\}_j$ of $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-atoms such that $e^{-L}(f) = \sum_j \lambda_j A_j$ in $L^p_{\mathcal{Q}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\|\{\lambda_j\}_j\|_{l^p} \lesssim \|e^{-L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \lesssim \|f\|_{h^p_L(\mathbb{R}^n)}$, which implies that

$$f_3 = \sum_j \lambda_j \sum_{k=0}^{2(M+1)} C_{(k)}(L+\delta_0)^k e^{-(L+2\delta_0)}(A_j) =: \sum_j \lambda_j \sum_{k=0}^{2(M+1)} C_{(k)}\alpha_j \tag{3.17}$$

in $L^2(\mathbb{R}^n)$.

For all j, let $B_j := B(x_{B_j}, r_{B_j})$ for some $x_{B_j} \in \mathbb{R}^n$ and $r_{B_j} \in (0, \infty)$ be the ball associated with A_j. From Definition 3.4, it follows that $r_{B_j} \geq 1$, which, combined with Proposition 2.10, implies that

$$\|\alpha_j\|_{L^2(S_i(B_j))} \lesssim \exp\{-[d(S_i(B_j), B_j)]^{2m/(2m-1)}\}\|A_j\|_{L^2(B_j)} \lesssim 2^{-i\eta},$$

where η is the positive constant as above. This shows that α_j is a local $(p, 2, M, \epsilon)_L$-molecule associated with B_j up to a harmless constant multiple, which, together with (3.17), implies that $f_3 \in h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\|f_3\|_{h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)} \lesssim \|f\|_{h^p_L(\mathbb{R}^n)}$.

Combining the estimates of f_1, f_2 and f_3, we conclude that

$$\left[h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right] \subset \left[h^p_{L,\,M,\,\mathrm{mol},\,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right].$$

We now prove the inclusion

$$\left[h^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right] \subset \left[h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right]. \tag{3.18}$$

To this end, it suffices to show that, for each local $(p, 2, M, \epsilon)_L$-molecule α associated with $B := B(x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,

$$\|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim 1, \tag{3.19}$$

$$\|S_{L,\,\mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim 1 \tag{3.20}$$

and

$$\|e^{-L}(\alpha)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \lesssim 1. \tag{3.21}$$

To prove (3.19), we consider two cases. If $r_B \geq 1$, we first write

$$\|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j\in\mathbb{Z}_+} \|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|_{L^2(S_j(B))}|S_j(B)|^{\frac{1}{p}-\frac{1}{2}}. \tag{3.22}$$

For $j \in \mathbb{Z}_+$ and $j \leq 5$, using some local quadratic estimates as in (3.11) via replacing

$$(t^{2m}L^*)^{M+1-\ell}(t^{2m}\delta_0)e^{-t^{2m}(L^*+\delta_0)}$$

by $t^{2m}\delta_0 e^{-t^{2m}(L+\delta_0)}$, we see that $S_{L,\,\delta_0,\,\mathrm{loc}}$ is bounded on $L^2(\mathbb{R}^n)$, which, together with Definition 3.16, implies that

$$\sum_{j=0}^{5} \|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|_{L^2(S_j(B))}|S_j(B)|^{\frac{1}{p}-\frac{1}{2}} \lesssim \sum_{j\in\mathbb{Z}_+} \|\alpha\|_{L^2(\mathbb{R}^n)}|B|^{\frac{1}{p}-\frac{1}{2}} \lesssim 1.$$

For $j \geq 6$, let $\widetilde{S}_j(B) := 2^{j+1}B\backslash(2^{j-2}B)$ and $\widetilde{\widetilde{S}}_j(B) := 2^{j+2}B\backslash(2^{j-3}B)$. By Theorem 2.10 and Definition 3.16, we know that

$$\begin{aligned}
&\|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|^2_{L^2(S_j(B))}\\
&\quad= \int_{S_j(B)} \left[\int_0^1 \int_{|y-x|<t} |t^{2m}\delta_0 e^{-t^{2m}(L+\delta_0)}(\alpha)(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\right] dx\\
&\quad= \int_0^1 \int_{\widetilde{S}_j(B)} |t^{2m}\delta_0 e^{-t^{2m}(L+\delta_0)}[\chi_{\widetilde{\widetilde{S}}_j(B)} + \chi_{\mathbb{R}^n\backslash\widetilde{\widetilde{S}}_j(B)}](\alpha)(y)|^2 \,\frac{dy\,dt}{t}\\
&\quad\lesssim \|\alpha\|^2_{L^2(\widetilde{\widetilde{S}}_j(B))} + \int_0^1 t^{4m} \exp\left\{-C\frac{[d(\widetilde{S}_j(B),\,\mathbb{R}^n\backslash\widetilde{\widetilde{S}}_j(B))]^{2m/(2m-1)}}{t^{2m/(2m-1)}}\right\}\\
&\quad\quad\times\frac{dt}{t}\|\alpha\|^2_{L^2(\mathbb{R}^n)} \lesssim 2^{-j2\epsilon}|2^jB|^{1-\frac{2}{p}}, \qquad (3.23)
\end{aligned}$$

which implies that

$$\sum_{j=6}^{\infty} \|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|_{L^2(S_j(B))}|S_j(B)|^{\frac{1}{p}-\frac{1}{2}} \lesssim 1. \qquad (3.24)$$

Thus, we conclude that (3.19) holds true in the case when $r_B \geq 1$.

We now consider the case $r_B < 1$. Let $\theta \in (\frac{n}{2mM}(\frac{1}{p}-\frac{1}{2}),\,1)$. We first write

$$\begin{aligned}
&\|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|^2_{L^2(S_j(B))}\\
&\quad= \int_{S_j(B)} \left\{\left[\int_0^{c_j} + \int_{c_j}^1\right] \int_{|y-x|<t} |t^{2m}\delta_0 e^{-t^{2m}(L+\delta_0)}(\alpha)(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\right\} dx\\
&\quad=: \mathcal{A}_j + \mathcal{B}_j,
\end{aligned}$$

where $c_j := \min\{2^{\theta(j-5)}r_B,\,1\}$.

The estimate of $\mathcal{A}_j$ is similar to the case $r_B \geq 1$. Using an argument similar to that used in (3.23), we see that

$$\mathcal{A}_j \lesssim 2^{-j2\epsilon}|2^jB|^{1-\frac{2}{p}}. \qquad (3.25)$$

To estimate $\mathcal{B}_j$, we may assume that $2^{\theta(j-5)}r_B < 1$; otherwise, $\mathcal{B}_j \equiv 0$. Then, by Fubini's theorem, Theorem 2.10, the assumption $r_B < 1$ and Definition 3.16, we obtain

$$\begin{aligned}
\mathcal{B}_j &\lesssim \int_{2^{\theta(j-5)}r_B}^1 \|t^{2m}\delta_0(t^{2m}L)^M e^{-t^{2m}(L+\delta_0)}(L^{-M}(\alpha))\|^2_{L^2(\mathbb{R}^n)} \,\frac{dt}{t^{4mM+1}}\\
&\lesssim \int_{2^{\theta(j-5)}r_B}^1 \|(r_B^{2m}L)^{-M}(\alpha)\|^2_{L^2(\mathbb{R}^n)} \,\frac{r_B^{4mM}\,dt}{t^{4mM+1}} \lesssim 2^{-\theta(j-5)4mM}|B|^{1-\frac{2}{p}},
\end{aligned}$$

which, combined with the assumption that $\theta \in (\frac{n}{2mM}(\frac{1}{p}-\frac{1}{2}),\,1)$, implies that there exists a positive constant ϵ_0 such that $\mathcal{B}_j \lesssim 2^{-2j\epsilon_0}|2^jB|^{1-\frac{2}{p}}$. This, together with (3.25) and an argument similar to that used in (3.22) and (3.24), implies that $\|S_{L,\,\delta_0,\,\mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim 1$ when $r_B < 1$. Thus, (3.19) holds true.

The proof of (3.20) is similar to that of (3.19). Observe that the local off-diagonal estimates as in Theorem 2.10 are still valid in this case. Moreover, notice that $L + \delta_0$ has a bounded holomorphic functional calculus in $L^2(\mathbb{R}^n)$. By the quadratic estimates of $L + \delta_0$, we know that $S_{L+\delta_0}$, defined as in (3.1) with I replaced by $L + \delta_0$, is bounded on $L^2(\mathbb{R}^n)$. Thus, for any $g \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\|S_{L,\,\mathrm{loc}}(g)\|^2_{L^2(\mathbb{R}^n)}\\
&\quad = \int_{\mathbb{R}^n}\left[\int_0^1\int_{|y-x|<t}|t^{2m}Le^{-t^{2m}L}(g)(y)|^2\frac{dy\,dt}{t^{n+1}}\right]dx\\
&\quad = \int_{\mathbb{R}^n}\left[\int_0^1\int_{|y-x|<t}|t^{2m}[(L+\delta_0)-\delta_0]e^{-t^{2m}[(L+\delta_0)-\delta_0]}(g)(y)|^2\frac{dy\,dt}{t^{n+1}}\right]dx\\
&\quad \lesssim \int_{\mathbb{R}^n}\left[\int_0^1\int_{|y-x|<t}|t^{2m}(L+\delta_0)e^{-t^{2m}(L+\delta_0)}e^{t^{2m}\delta_0}(g)(y)|^2\frac{dy\,dt}{t^{n+1}}\right]dx\\
&\qquad + \int_{\mathbb{R}^n}\left[\int_0^1\int_{|y-x|<t}|t^{2m}\delta_0e^{t^{2m}\delta_0}e^{-t^{2m}(L+\delta_0)}(g)(y)|^2\frac{dy\,dt}{t^{n+1}}\right]dx\\
&\quad \lesssim \|S_{L+\delta_0}(g)\|^2_{L^2(\mathbb{R}^n)} + \int_0^1(t^{2m}\delta_0)^2e^{2t^{2m}\delta_0}\|e^{-t^{2m}(L+\delta_0)}(g)\|_{L^2(\mathbb{R}^n)}\frac{dt}{t}\\
&\quad \lesssim \|g\|^2_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{3.26}$$

which implies that $S_{L,\,\mathrm{loc}}$ is bounded on $L^2(\mathbb{R}^n)$. Thus, all the tools used in the proof of (3.19) are still valid, which further lead to the estimate of (3.20).

To prove (3.21), if $r_B \geq 1$, by Theorem 2.9, we know that there exists a positive constant ϵ such that, for all $i \in \mathbb{Z}_+$, $\|\chi_{S_i(B)}e^{-L}(\alpha)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-i\epsilon}|2^iB|^{\frac{1}{2}-\frac{1}{p}}$, which immediately implies that $e^{-L}(\alpha)$ is an $H^p_I(\mathbb{R}^n)$-molecule associated with B up to a harmless constant multiple. This, combined with Proposition 3.8, implies that $\|e^{-L}(\alpha)\|_{L^p_{\mathbb{Q}}(\mathbb{R}^n)} \lesssim 1$.

If $r_B < 1$, let $b := L^{-M}(\alpha)$ and $B^* := B(x_B, 1)$. For $i \in \mathbb{N} \cap [3, \infty)$, we first write

$$\begin{aligned}
\|\chi_{S_i(B^*)}e^{-L}(\alpha)\|_{L^2(\mathbb{R}^n)} &= \left\|\chi_{S_i(B^*)}L^Me^{-L}\left(\sum_{l\in\mathbb{Z}_+}(\chi_{S_l(B)}b)\right)\right\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \left\|\chi_{S_i(B^*)}L^Me^{-L}\left(\sum_{2^l<\frac{2^{i-2}}{r_B}\ \text{or}\ 2^l\geq\frac{2^{i+2}}{r_B}}(\chi_{S_l(B)}b)\right)\right\|_{L^2(\mathbb{R}^n)}\\
&\quad + \sum_{\frac{2^{i-2}}{r_B}\leq 2^l<\frac{2^{i+2}}{r_B}}\|\chi_{S_i(B^*)}L^Me^{-L}(\chi_{S_l(B)}b)\|_{L^2(\mathbb{R}^n)}\\
&=: \mathrm{I} + \mathrm{J}.
\end{aligned} \tag{3.27}$$

For I, let $\mathrm{E}_i := \bigcup_{2^l < \frac{2^{i-2}}{r_B} \text{ or } 2^l \geq \frac{2^{i+2}}{r_B}} S_l(B)$. It is easy to see that $\mathrm{E}_i \subset \mathbb{R}^n \setminus \widetilde{\widetilde{S}}_i(B^*)$. Thus, using Proposition 2.10, $r_B < 1$, Definition 3.16 and the assumption that $M > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$, we see that there exists a positive constant ϵ such that

$$\begin{aligned} \mathrm{I} &\lesssim \exp\{-2^{i\frac{2m}{2m-1}}\}\|b\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \exp\{-2^{i\frac{2m}{2m-1}}\} r_B^{2mM}|B|^{\frac{1}{2}-\frac{1}{p}} \lesssim 2^{-i[\epsilon+n(\frac{1}{p}-\frac{1}{2})]}, \end{aligned} \tag{3.28}$$

which is desired.

For J, by the boundedness of $L^M e^{-L}$ on $L^2(\mathbb{R}^n)$, Definition 3.16 and $(2^l r_B)^{-1} \leq 2^{-i+2}$, we conclude that

$$\begin{aligned} \mathrm{J} &\lesssim \sum_{\frac{2^{i-2}}{r_B} \leq 2^l < \frac{2^{i+2}}{r_B}} \|\chi_{S_l(B)} b\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \sum_{\frac{2^{i-2}}{r_B} \leq 2^l < \frac{2^{i+2}}{r_B}} 2^{-\frac{\epsilon}{2}l} r_B^{2mM+\frac{\epsilon}{2}} (2^l r_B)^{-[n(\frac{1}{p}-\frac{1}{2})+\frac{\epsilon}{2}]} \\ &\lesssim 2^{-i[n(\frac{1}{p}-\frac{1}{2})+\frac{\epsilon}{2}]} \sum_{l\in\mathbb{Z}_+} 2^{-\frac{\epsilon}{2}l} \sim 2^{-i[n(\frac{1}{p}-\frac{1}{2})+\frac{\epsilon}{2}]}, \end{aligned}$$

which, together with (3.27) and (3.28), implies that there exists a positive constant ϵ such that, for all $i \in \mathbb{N} \cap [3, \infty)$, $\|\chi_{S_i(B^*)} e^{-L}(\alpha)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-i[n(\frac{1}{p}-\frac{1}{2})+\frac{\epsilon}{2}]}$.

The estimates for the case $i \in \{0, 1, 2\}$ are similar and, in this case, we use the boundedness of $L^M e^{-L}$ on $L^2(\mathbb{R}^n)$ to replace the off-diagonal estimates used in (3.28), the details being omitted. Thus, $e^{-L}(\alpha)$ is an $H_I^p(\mathbb{R}^n)$-molecule associated with B^* up to a harmless constant multiple. Thus, by Proposition 3.8, we obtain $\|e^{-L}(\alpha)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \lesssim 1$ when $r_B < 1$, which, combined the case $r_B \geq 1$, implies that (3.21) holds true. This finishes the proof of (3.18) and hence Theorem 3.18. □

3.3. *The square function characterization and the complex interpolation of $h_L^p(\mathbb{R}^n)$*

In this subsection, we establish the square function characterization of $h_L^p(\mathbb{R}^n)$ for all $p \in (0, \infty)$. By Definition 3.14, we know that $h_L^p(\mathbb{R}^n)$ has the square function characterization for $p \in (0, 2]$. For $p \in (2, \infty)$, to establish the square function characterization, we need to establish a retract relationship between $h_L^p(\mathbb{R}^n)$ and the direct sum space $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$, from which, we further deduce the complex interpolation of $h_L^p(\mathbb{R}^n)$. Here, for all $p \in (0, \infty)$, a triple of functions, (F, G, h), is said to be in the *direct sum space* $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ if $F \in t^p(\mathbb{R}^n \times (0, 1])$, $G \in t^p(\mathbb{R}^n \times (0, 1])$ and $h \in L^p_{\mathcal{Q}}(\mathbb{R}^n)$. Moreover, let

$$\begin{aligned} &\|(F, G, h)\|_{t^p(\mathbb{R}^n \times (0,1]) \oplus t^p(\mathbb{R}^n \times (0,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)} \\ &\quad := \|f\|_{t^p(\mathbb{R}^n \times (0,1])} + \|g\|_{t^p(\mathbb{R}^n \times (0,1])} + \|h\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}. \end{aligned}$$

We point out that the idea of using the direct sum space is inspired by [17], where the authors studied the direct sum spaces $t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ for all $p \in [1, \infty)$.

Now, let $\{M_i\}_{i=1}^5 \subset \mathbb{Z}_+$ and $f \in L^2(\mathbb{R}^n)$. For all $(x, t) \in \mathbb{R}^{n+1}_+$, the *operator* $\mathcal{Q}_{\{M_i\}_i, L}$ is defined by setting

$$\begin{aligned}
&\mathcal{Q}_{\{M_i\}_i, L}(f)(x, t)\\
&\quad := ((t^{2m}L)^{M_1}(t^{2m}\delta_0)^{M_2}e^{-t^{2m}(L+\delta_0)}(f)(x),\\
&\qquad (t^{2m}L)^{M_3}(t^{2m}\delta_0)^{M_4}e^{-t^{2m}(L+\delta_0)}(f)(x),\ (L+\delta_0)^{M_5}e^{-(L+\delta_0)}(f)(x))\\
&\quad =: (T_{M_1, M_2, t, L}(f)(x),\ T_{M_3, M_4, t, L}(f)(x),\ S_{M_5, L}(f)(x)), \qquad (3.29)
\end{aligned}$$

where δ_0 is as in (2.7).

Let $(F, G, h) \in t^2(\mathbb{R}^n \times (0, 1]) \oplus t^2(\mathbb{R}^n \times (0, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$, the *operator* $\Pi_{\{M_i\}_i, L}$ is defined by setting

$$\begin{aligned}
\Pi_{\{M_i\}_i, L}(F, G, h)(x) &:= \int_0^1 T_{M_1, M_2, t, L}(F(\cdot, t))(x)\,\frac{dt}{t}\\
&\quad + \int_0^1 T_{M_3, M_4, t, L}(G(\cdot, t))(x)\,\frac{dt}{t} + S_{M_5, L}(h)(x), \qquad (3.30)
\end{aligned}$$

where $T_{M_1, M_2, t, L}$, $T_{M_3, M_4, t, L}$ and $S_{M_5, L}$ are defined as in (3.29).

From its definition, the local quadratic estimates as in (3.11) via replacing

$$(t^{2m}L^*)^{M+1-\ell}(t^{2m}\delta_0)e^{-t^{2m}(L^*+\delta_0)},$$

respectively, by

$$(t^{2m}L)^{M_1}(t^{2m}\delta_0)^{M_2}e^{-t^{2m}(L+\delta_0)} \quad \text{and} \quad (t^{2m}L)^{M_3}(t^{2m}\delta_0)^{M_4}e^{-t^{2m}(L+\delta_0)},$$

and Proposition 2.10, we deduce that, for all $\{M_i\}_{i=1}^5 \subset \mathbb{Z}_+$, $\mathcal{Q}_{\{M_i\}_i, L}$ is bounded from $L^2(\mathbb{R}^n)$ to $t^2(\mathbb{R}^n \times (0, 1]) \oplus t^2(\mathbb{R}^n \times (0, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$. This, together with the fact that $\Pi_{\{M_i\}_i, L}$ is the adjoint of $\mathcal{Q}_{\{M_i\}, L^*}$, shows that $\Pi_{\{M_i\}_i, L}$ is bounded from $t^2(\mathbb{R}^n \times (0, 1]) \oplus t^2(\mathbb{R}^n \times (0, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Indeed, let $(F, G, h) \in t^2(\mathbb{R}^n \times (0, 1]) \oplus t^2(\mathbb{R}^n \times (0, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$. For all $\varphi \in L^2(\mathbb{R}^n)$, by duality, the boundedness of $\mathcal{Q}_{\{M_i\}_i, L^*}$ from $L^2(\mathbb{R}^n)$ to $t^2(\mathbb{R}^n \times (0, 1]) \oplus t^2(\mathbb{R}^n \times (0, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$ and Proposition 3.8, we see that

$$\begin{aligned}
&|\langle \Pi_{\{M_i\}_i, L}(F, G, h), \varphi\rangle_{L^2(\mathbb{R}^n)}|\\
&\quad = \left| \left\langle \int_0^1 T_{M_1, M_2, t, L}(F(\cdot, t))\,\frac{dt}{t}, \varphi \right\rangle_{L^2(\mathbb{R}^n)} \right.\\
&\qquad + \left\langle \int_0^1 T_{M_3, M_4, t, L}(G(\cdot, t))\,\frac{dt}{t}, \varphi \right\rangle_{L^2(\mathbb{R}^n)}\\
&\qquad \left. + \langle S_{M_5, L}(h), \varphi\rangle_{L^2(\mathbb{R}^n)} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_0^1 \int_{\mathbb{R}^n} F(x,\, t)\overline{T_{M_1,\, M_2,\, t,\, L^*}(\varphi)(x)}\, \frac{dxdt}{t} \right.\\
&\quad + \int_0^1 \int_{\mathbb{R}^n} G(x,\, t)\overline{T_{M_3,\, M_4,\, t,\, L^*}(\varphi)(x)}\, \frac{dxdt}{t}\\
&\quad \left. + \langle h,\, S_{M_5,\, L^*}(\varphi)\rangle_{L^2(\mathbb{R}^n)} \right|\\
&\lesssim [\|F\|_{t^2(\mathbb{R}^n\times(0,\,1))} + \|G\|_{t^2(\mathbb{R}^n\times(0,\,1))} + \|h\|_{L^2_{\mathcal{Q}}(\mathbb{R}^n)}]\|\varphi\|_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

which, combined with the arbitrariness of φ, implies that $\Pi_{\{M_i\}_i,\, L}$ is bounded from

$$
t^2(\mathbb{R}^n \times (0,\, 1]) \oplus t^2(\mathbb{R}^n \times (0,\, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n) \quad \text{to} \quad L^2(\mathbb{R}^n).
$$

Now, let

$$
\mathcal{T} := \mathcal{Q}_{\{M_i\}_i,\, L} \circ \Pi_{\{\widetilde{M_i}\}_i,\, L}. \tag{3.31}
$$

For all $(F,\, G,\, h) \in t^2(\mathbb{R}^n \times (0,\, 1]) \oplus t^2(\mathbb{R}^n \times (0,\, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$, it is easy to see that

$$
\begin{aligned}
&\mathcal{Q}_{\{M_i\}_i,\, L} \circ \Pi_{\{\widetilde{M_i}\}_i,\, L}(F,\, G,\, h)\\
&\quad = (T_{M_1,\, M_2,\, s,\, L},\, T_{M_3,\, M_4,\, s,\, L},\, S_{M_5,\, L})\\
&\qquad \circ \left[\int_0^1 T_{\widetilde{M_1},\, \widetilde{M_2},\, t,\, L}(F(\cdot,\, t))\, \frac{dt}{t} + \int_0^1 T_{\widetilde{M_3},\, \widetilde{M_4},\, t,\, L}(G(\cdot,\, t))\, \frac{dt}{t} + S_{\widetilde{M_5},\, L}(h)\right]\\
&\quad = \left(\int_0^1 T_{M_1,\, M_2,\, s,\, L} T_{\widetilde{M_1},\, \widetilde{M_2},\, t,\, L}(F(\cdot,\, t))\, \frac{dt}{t}\right.\\
&\qquad + \int_0^1 T_{M_1,\, M_2,\, s,\, L} T_{\widetilde{M_3},\, \widetilde{M_4},\, t,\, L}(G(\cdot,\, t))\, \frac{dt}{t} + T_{M_1,\, M_2,\, s,\, L} S_{\widetilde{M_5} L}(h),\\
&\qquad \int_0^1 T_{M_3,\, M_4,\, s,\, L} T_{\widetilde{M_1},\, \widetilde{M_2},\, t,\, L}(F(\cdot,\, t))\, \frac{dt}{t}\\
&\qquad + \int_0^1 T_{M_3,\, M_4,\, s,\, L} T_{\widetilde{M_3},\, \widetilde{M_4},\, t,\, L}(G(\cdot,\, t))\, \frac{dt}{t} + T_{M_3,\, M_4,\, s,\, L} S_{\widetilde{M_5} L}(h),\\
&\qquad \int_0^1 S_{M_5,\, L} T_{\widetilde{M_1},\, \widetilde{M_2},\, t,\, L}(F(\cdot,\, t))\, \frac{dt}{t}\\
&\qquad \left. + \int_0^1 S_{M_5,\, L} T_{\widetilde{M_3},\, \widetilde{M_4},\, t,\, L}(G(\cdot,\, t))\, \frac{dt}{t} + S_{M_5,\, L} S_{\widetilde{M_5},\, L}(h)\right)\\
&\quad =: (\mathcal{T}_{1,\,1}(F) + \mathcal{T}_{1,\,2}(G) + \mathcal{T}_{1,\,3}(h),\, \mathcal{T}_{2,\,1}(F) + \mathcal{T}_{2,\,2}(G) + \mathcal{T}_{2,\,3}(h),\\
&\qquad \mathcal{T}_{3,\,1}(F) + \mathcal{T}_{3,\,2}(G) + \mathcal{T}_{3,\,3}(h))\\
&\quad =: \begin{pmatrix} \mathcal{T}_{1,\,1}, & \mathcal{T}_{1,\,2}, & \mathcal{T}_{1,\,3} \\ \mathcal{T}_{2,\,1}, & \mathcal{T}_{2,\,2}, & \mathcal{T}_{2,\,3} \\ \mathcal{T}_{3,\,1}, & \mathcal{T}_{3,\,2}, & \mathcal{T}_{3,\,3} \end{pmatrix} \begin{pmatrix} F \\ G \\ h \end{pmatrix}.
\end{aligned} \tag{3.32}
$$

The next proposition establishes the boundedness of $\mathcal{T}$ on $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ for $p \in (0, 1]$.

Proposition 3.24. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let $p \in (0, 1], \{M_i\}_{i=1}^5, \{\widetilde{M_i}\}_{i=1}^5 \subset \mathbb{Z}_+$ satisfy $M_1 + M_2 > 0, M_3 + M_4 > 0, \widetilde{M_1} + \widetilde{M_2} > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$ and $\widetilde{M_3} + \widetilde{M_4} > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$. Then the operator $\mathcal{T}$, defined as in* (3.31), *is bounded on $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$.*

Proof. By (3.32), to prove this proposition, it suffices to show that, for $j \in \{1, 2\}$, $k \in \{1, 2\}$, each $(t^p, 2)$-atom A and $L^p_{\mathcal{Q}}(\mathbb{R}^n)$-atom $\widetilde{A}$,

$$\|\mathcal{T}_{j,\,k}(A)\|_{t^p(\mathbb{R}^n\times(0,\,1])} \lesssim 1, \tag{3.33}$$

$$\|\mathcal{T}_{j,\,3}(\widetilde{A})\|_{t^p(\mathbb{R}^n\times(0,\,1])} \lesssim 1, \tag{3.34}$$

$$\|\mathcal{T}_{3,\,j}(A)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \lesssim 1 \tag{3.35}$$

and

$$\|\mathcal{T}_{3,\,3}(\widetilde{A})\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \lesssim 1. \tag{3.36}$$

We first show (3.33). By symmetry, we only consider the case $j = 1$ and $k = 1$. Let $B := B(x_B, r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ be the ball associated with A. If $r_B \ge 1$, for $l \in \mathbb{N}$ with $l \ge 5$, let $S_l(T^{\mathrm{loc}}(B)) := T^{\mathrm{loc}}(2^l B)\backslash T^{\mathrm{loc}}(2^{l-1}B)$. We write

$$\begin{aligned}
&\|\mathcal{T}_{1,\,1}(A)\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\quad= \left\|\int_0^1 T_{M_1,\,M_2,\,s,\,L} T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)\,\frac{dt}{t}\right\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\quad\le \left\|\chi_{T^{\mathrm{loc}}(2^5B)}\int_0^1 T_{M_1,\,M_2,\,s,\,L} T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)\,\frac{dt}{t}\right\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\qquad+\sum_{l=6}^\infty \left\|\chi_{S_l(T^{\mathrm{loc}}(B))}\int_0^1 T_{M_1,\,M_2,\,s,\,L} T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)\,\frac{dt}{t}\right\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\quad=: (\mathrm{I}_5)^p + \sum_{l=6}^\infty (\mathrm{I}_l)^p.
\end{aligned} \tag{3.37}$$

We first estimate I_5. Observe that, for all $(y, s) \in T^{\mathrm{loc}}(2^5B)$, $|y - x| < t$ and $t \in (0, 1]$, we have $x \in 2^5B$. From this, Hölder's inequality, the boundedness of the operator

$$S := \int_0^1 T_{M_1,\,M_2,\,s,\,L} T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}\,\frac{dt}{t}$$

on $t^2(\mathbb{R}^n \times (0, 1])$ (via the local quadratic estimates as in (3.11) with replacing

$$(t^{2m}L^*)^{M+1-\ell}(t^{2m}\delta_0)e^{-t^{2m}(L^*+\delta_0)}$$

by S and a dual argument) and Definition 3.20, we deduce that

$$\begin{aligned}
\mathrm{I}_5 &= \left\{\int_{\mathbb{R}^n}\left[\iint_{\Gamma_{\mathrm{loc}}(x)\cap T^{\mathrm{loc}}(2^5B)}\left|\int_0^1 T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)\,\frac{dt}{t}\right|^2\right.\right.\\
&\qquad\left.\left.\times\frac{dy\,ds}{s^{n+1}}\right]^{\frac{p}{2}}dx\right\}^{\frac{1}{p}}\\
&\lesssim |B|^{\frac{1}{p}-\frac{1}{2}}\left\{\int_{2^5B}\left[\iint_{\Gamma_{\mathrm{loc}}(x)\cap T^{\mathrm{loc}}(2^5B)}\left|\int_0^1 T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)\,\frac{dt}{t}\right|^2\right.\right.\\
&\qquad\left.\left.\times\frac{dy\,ds}{s^{n+1}}\right]dx\right\}^{\frac{1}{2}}\lesssim |B|^{\frac{1}{p}-\frac{1}{2}}\|A\|_{t^2(\mathbb{R}^n\times(0,\,1])}\lesssim 1,
\end{aligned}\tag{3.38}$$

where $\Gamma_{\mathrm{loc}}(x)$ for $x\in\mathbb{R}^n$ is as in (3.9). Observe that the estimates in (3.38) are independent of whether $r_B\geq 1$ or $r_B<1$.

For $l\geq 6$, observe that, for all $(y,\,s)\in S_l(T^{\mathrm{loc}}(B))$, $|y-x|<t$ and $t\in(0,\,1]$, we have $x\in\widetilde{S}_l(B):=2^{l+1}B\backslash(2^{l-2}B)$. Similar to the estimate of I_5, we know that

$$\begin{aligned}
\mathrm{I}_l &\lesssim \left\{\int_{\mathbb{R}^n}\left[\iint_{\Gamma_{\mathrm{loc}}(x)\cap S_l(T^{\mathrm{loc}}(B))}\left|\int_0^1 T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)\frac{dt}{t}\right|^2\right.\right.\\
&\qquad\left.\left.\times\frac{dy\,ds}{s^{n+1}}\right]^{\frac{p}{2}}dx\right\}^{\frac{1}{p}}\\
&\lesssim |2^lB|^{\frac{1}{p}-\frac{1}{2}}\left\{\int_{\widetilde{S}_l(B)}\left[\iint_{\Gamma_{\mathrm{loc}}(x)\cap S_l(T^{\mathrm{loc}}(B))}\left|\int_0^1 T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)\right.\right.\right.\\
&\qquad\left.\left.\left.\times\frac{dt}{t}\right|^2\frac{dy\,ds}{s^{n+1}}\right]dx\right\}^{\frac{1}{2}}=:|2^lB|^{\frac{1}{p}-\frac{1}{2}}\times\widetilde{\mathrm{I}}_l.
\end{aligned}\tag{3.39}$$

To bound $\widetilde{\mathrm{I}}_l$, using Minkowski's integral inequality, Fubini's theorem and the assumption $r_B\geq 1$, we write it into

$$\begin{aligned}
\widetilde{\mathrm{I}}_l &= \left\{\int_{\widetilde{S}_l(B)}\left[\iint_{\Gamma_{\mathrm{loc}}(x)\cap S_l(T^{\mathrm{loc}}(B))}\left|\int_0^1 T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)\,\frac{dt}{t}\right|^2\right.\right.\\
&\qquad\left.\left.\times\frac{dy\,ds}{s^{n+1}}\right]dx\right\}^{\frac{1}{2}}\\
&\lesssim \int_0^1\left\{\left[\int_0^t+\int_t^1\right]\int_{\widetilde{S}_l(B)}|T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)|^2\,\frac{dy\,ds}{s}\right\}^{1/2}\frac{dt}{t}\\
&=:\widetilde{\mathrm{I}}_{l,\,1}+\widetilde{\mathrm{I}}_{l,\,2}.
\end{aligned}$$

For $\widetilde{\mathrm{I}}_{l,\,1}$, let $a_1 \in [0,\, M_1]$ and $a_2 \in [0,\, M_2]$ such that $a := a_1 + a_2 \in (0,\, M_1 + M_2]$. By the assumption $r_B \geq 1$, Proposition 2.10 and the composition rule of off-diagonal estimates (see, for example, [41, Lemma 2.3] or [16, Lemma 3.2]), we find that

$$\begin{aligned}
\widetilde{\mathrm{I}}_{l,\,1} &\lesssim \int_0^1 \left\{ \int_0^t \int_{\widetilde{S}_l(B)} \left(\frac{s}{t}\right)^{4ma} \left| \left(\frac{t}{s}\right)^{2ma} T_{M_1,\,M_2,\,s,\,L} T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t) \right|^2 \right. \\
&\qquad \left. \times \frac{dy\,ds}{s} \right\}^{1/2} \frac{dt}{t} \\
&\lesssim \int_0^1 \left\{ \int_0^t \int_{\widetilde{S}_l(B)} \left(\frac{s}{t}\right)^{4ma} |T_{M_1-a_1,\,M_2-a_2,\,s,\,L} T_{\widetilde{M}_1+a_1,\,\widetilde{M}_2+a_2,\,t,\,L}(A)(y,\,t)|^2 \right. \\
&\qquad \left. \times \frac{dy\,ds}{s} \right\}^{1/2} \frac{dt}{t} \\
&\lesssim \int_0^1 \left[\int_0^t \left(\frac{s}{t}\right)^{4ma} \exp\left\{ -\frac{[d(\widetilde{S}_l(B),\,B)]^{2m/(2m-1)}}{t^{2m/(2m-1)}} \right\} \frac{ds}{s} \right]^{1/2} \\
&\qquad \times \|A(\cdot,\,t)\|_{L^2(\mathbb{R}^n)} \frac{dt}{t} \\
&\lesssim \left[\int_0^1 \int_0^t \left(\frac{s}{t}\right)^{4ma} \exp\left\{ -\frac{[d(\widetilde{S}_l(B),\,B)]^{2m/(2m-1)}}{t^{2m/(2m-1)}} \right\} \frac{ds}{s} \frac{dt}{t} \right]^{1/2} \\
&\qquad \times \|A\|_{t^2(\mathbb{R}^n \times (0,\,1])} \lesssim 2^{-l\eta} |2^l B|^{\frac{1}{2}-\frac{1}{p}},
\end{aligned}$$

where $\eta \in (0,\,\infty)$, arising from the exponential decay, is large enough.

For $\widetilde{\mathrm{I}}_{l,\,2}$, let $b_1 \in [0,\, \widetilde{M}_1]$, $b_2 \in [0,\, \widetilde{M}_2]$ such that $b := b_1 + b_2 \in (\widetilde{b},\, \widetilde{M}_1 + \widetilde{M}_2)$, where $\widetilde{b} \in (\frac{n}{2m}(\frac{1}{p} - \frac{1}{2}),\, \widetilde{M}_1 + \widetilde{M}_2)$. By an argument similar to that used in the estimate of $\widetilde{\mathrm{I}}_{l,\,1}$, we have

$$\begin{aligned}
\widetilde{\mathrm{I}}_{l,\,2} &\lesssim \int_0^1 \left\{ \int_t^1 \int_{\widetilde{S}_l(B)} \left(\frac{t}{s}\right)^{4mb} |T_{M_1+b_1,\,M_2+b_2,\,s,\,L} T_{\widetilde{M}_1-b_1,\,\widetilde{M}_2-b_2,\,t,\,L}(A)(y,\,t)|^2 \right. \\
&\qquad \left. \times \frac{dy\,ds}{s} \right\}^{1/2} \frac{dt}{t} \\
&\lesssim \int_0^1 \left[\int_t^1 \left(\frac{t}{s}\right)^{4mb} \exp\left\{ -\frac{[d(\widetilde{S}_l(B),\,B)]^{2m/(2m-1)}}{s^{2m/(2m-1)}} \right\} \frac{ds}{s} \right]^{1/2} \|A(\cdot,\,t)\|_{L^2(\mathbb{R}^n)} \frac{dt}{t} \\
&\lesssim \left[\int_0^1 \int_t^1 \left(\frac{t}{s}\right)^{4mb} \exp\left\{ -\frac{[d(\widetilde{S}_l(B),\,B)]^{2m/(2m-1)}}{s^{2m/(2m-1)}} \right\} \frac{ds}{s} \frac{dt}{t} \right]^{1/2} \|A\|_{t^2(\mathbb{R}^n \times (0,\,1])}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\{\int_0^1 \int_t^1 \left(\frac{t}{s}\right)^{4mb} \left(\frac{s}{2^l r_B}\right)^{4m\widetilde{b}} \frac{ds}{s}\frac{dt}{t}\right\}^{1/2} \|A\|_{t^2(\mathbb{R}^n\times(0,\,1])} \\
&\lesssim 2^{-l[2m\widetilde{b}-n(\frac{1}{p}-\frac{1}{2})]}|2^l B|^{\frac{1}{2}-\frac{1}{p}}.
\end{aligned}
$$

Combining the estimates of $\widetilde{\mathrm{I}}_{l,\,1}$ and $\widetilde{\mathrm{I}}_{l,\,2}$, we conclude that

$$
\sum_{l=6}^{\infty}(\mathrm{I}_l)^p \lesssim \sum_{l=6}^{\infty}|2^l B|^{1-\frac{p}{2}}(\widetilde{\mathrm{I}}_l)^p \lesssim \sum_{l=6}^{\infty}|2^l B|^{1-\frac{p}{2}}(\widetilde{\mathrm{I}}_{l,\,1}+\widetilde{\mathrm{I}}_{l,\,2})^p \lesssim 1,
$$

which, together with the estimates of I_5, implies that, for $j=1$ and $k=1$, (3.33) holds true in the case of $r_B \ge 1$.

If $r_B < 1$, let $\{\mathrm{I}_l\}_{l=5}^{\infty}$ be as in (3.37) again. The estimate of I_5 is the same as in (3.38). To bound I_l for $l \ge 6$, similar to the case $r_B \ge 1$, let $\widetilde{\mathrm{I}}_l$ be as in (3.39). By Definition 3.20, Minkowski's integral inequality and Fubini's theorem, we further write

$$
\begin{aligned}
\widetilde{\mathrm{I}}_l &\lesssim \int_0^{r_B}\left\{\int_{2^l B}\left[\iint_{\Gamma_{\mathrm{loc}}(x)\cap S_l(T^{\mathrm{loc}}(B))} |T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)|^2\right.\right. \\
&\qquad \left.\left.\times\frac{dy\,ds}{s^{n+1}}\right]dx\right\}^{1/2}\frac{dt}{t} \\
&\lesssim \int_0^{r_B}\left\{\left[\int_0^{2^{l-2}r_B}+\int_{2^{l-2}r_B}^{2^l r_B}\right]\int_{\{y\in\mathbb{R}^n:\,2^{l-1}r_B-s\le|y-x_B|<2^l r_B-s\}}\right. \\
&\qquad \left.\times|T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)|^2\frac{dy\,ds}{s}\right\}^{1/2}\frac{dt}{t} =: \mathcal{A}_l+\mathcal{B}_l.
\end{aligned}
$$

Observe that, for all $s \in (0,\,2^{l-2}r_B)$ and $2^{l-1}r_B - s \le |y-x_B| < 2^l r_B - s$, we have $y \in \widetilde{S}_l(B) := 2^{l+1}B\backslash(2^{l-2}B)$. This implies that the estimate of $\mathcal{A}_l$ is similar to that of $\widetilde{\mathrm{I}}_l$ in the case $r_B \ge 1$. Thus, there exists a positive constant η such that $\mathcal{A}_l \lesssim 2^{-l\eta}|2^l B|^{\frac{1}{2}-\frac{1}{p}}$.

For $\mathcal{B}_l$, let $\widetilde{b}_1 \in (0,\,\widetilde{M}_1]$, $\widetilde{b}_2 \in (0,\,\widetilde{M}_2]$ and $\widetilde{b} := \widetilde{b}_1 + \widetilde{b}_2$ satisfying $\widetilde{M}_1 + \widetilde{M}_2 > \widetilde{b} > \frac{n}{2m}(\frac{1}{p}-\frac{1}{2})$. By $s > t$, Proposition 2.10, Hölder's inequality and Definition 3.20, we obtain

$$
\begin{aligned}
\mathcal{B}_l &\lesssim \int_0^{r_B}\left\{\int_{2^{l-2}r_B}^{2^l r_B}\int_{2^l B}|T_{M_1,\,M_2,\,s,\,L}T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A)(y,\,t)|^2\frac{dy\,ds}{s}\right\}^{1/2}\frac{dt}{t} \\
&\lesssim \int_0^{r_B}\left\{\int_{2^{l-2}r_B}^{2^l r_B}\left(\frac{t}{s}\right)^{4m\widetilde{b}}\int_{2^l B}|T_{M_1+\widetilde{b}_1,\,M_2+\widetilde{b}_2,\,s,\,L}T_{\widetilde{M}_1-\widetilde{b}_1,\,\widetilde{M}_2-\widetilde{b}_2,\,t,\,L}(A)(y,\,t)|^2\right. \\
&\qquad \left.\times\frac{dy\,ds}{s}\right\}^{1/2}\frac{dt}{t}
\end{aligned}
$$

$$\lesssim \left\{ \int_0^{r_B} \int_{2^{l-2}r_B}^{2^l r_B} \left(\frac{t}{s}\right)^{4m\widetilde{b}} \frac{ds}{s}\frac{dt}{t} \right\}^{1/2} \|A\|_{t^2(\mathbb{R}^n\times(0,\,1])}$$
$$\lesssim 2^{-l\eta}|2^l B|^{\frac{1}{2}-\frac{1}{p}},$$

where $\eta := 2m\widetilde{b} - n(\frac{1}{p}-\frac{1}{2}) > 0$. This, combined with the estimate of $\mathcal{A}_l$, implies that $\widetilde{\mathrm{I}}_l$ satisfies the same estimates as in the case $r_B \geq 1$. Thus, (3.33) holds true for $j = 1$ and $k = 1$.

Now, we turn to the proof of (3.34). By symmetry, we only consider the case $j = 1$. Let $B := (x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ be the ball associated with $\widetilde{A}$. Using Remark 3.6, we may assume $r_B = 1$. Moreover, we write

$$\begin{aligned}
&\|T_{M_1,\,M_2,\,s,\,L}S_{\widetilde{M}_5,\,L}(\widetilde{A})\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\quad\lesssim \|\chi_{T^{\mathrm{loc}}(2^5B)}T_{M_1,\,M_2,\,s,\,L}S_{\widetilde{M}_5,\,L}(\widetilde{A})\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\qquad+\sum_{l=6}^{\infty}\|\chi_{S_l(T^{\mathrm{loc}}(B))}T_{M_1,\,M_2,\,s,\,L}S_{\widetilde{M}_5,\,L}(\widetilde{A})\|^p_{t^p(\mathbb{R}^n\times(0,\,1])}\\
&\quad=: (\mathrm{J}_5)^p + \sum_{l=6}^{\infty}(\mathrm{J}_l)^p.
\end{aligned}$$

To control J_5, by the local quadratic estimates as in (3.11), we know that the operator

$$T_{M_1,\,M_2,\,s,\,L}S_{\widetilde{M}_5,\,L}$$

is bounded from $L^2(\mathbb{R}^n)$ to $t^2(\mathbb{R}^n\times(0,\,1])$, which, combined with Hölder's inequality and Definition 3.4, implies that $\mathrm{J}_5 \lesssim |B|^{\frac{1}{p}-\frac{1}{2}}\|\widetilde{A}\|_{L^2(\mathbb{R}^n)} \lesssim 1$.

The estimates of J_l for $l \geq 6$ are similar to that of I_l in the proof of (3.33), but the corresponding argument is more simple, since we only need to consider the case $r_B \geq 1$ and $t = 1$ when following the argument used in the estimate for I_l, the details being omitted. Thus, (3.34) holds true.

To prove (3.35) for $j = 1$, let $B := (x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ be the ball associated with A. If $r_B \geq 1$, using an argument similar to that used in the proof of (3.33) (for the case $r_B \geq 1$ and $s = 1$), we see that there exists a positive constant η such that, for all $l \in \mathbb{Z}_+$,

$$\|\chi_{S_l(B)}\mathcal{T}_{3,\,1}(A)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-l\eta}|2^l B|^{\frac{1}{2}-\frac{1}{p}},$$

which, together with Definition 3.4, implies that $\mathcal{T}_{3,\,1}(A)$ is an $H^p_I(\mathbb{R}^n)$-molecule associated with B. From this and Proposition 3.8, we deduce that (3.35) holds true for $j = 1$ and $r_B \geq 1$.

For $r_B < 1$, let $B^* := B(x_B, 1)$. By Definition 3.20, we write

$$\begin{aligned}
&S_{M_5,\,L}\int_0^1 T_{\widetilde{M}_1,\,\widetilde{M}_2,\,t,\,L}(A(\cdot,\,t))\,\frac{dt}{t}\\
&\quad= L^{\widetilde{M}_1}\delta_0^{\widetilde{M}_2}S_{M_5,\,L}\int_0^{r_B} t^{2m(\widetilde{M}_1+\widetilde{M}_2)}T_{0,\,0,\,t,\,L}(A(\cdot,\,t))\,\frac{dt}{t}\\
&\quad= L^{\widetilde{M}_1}\delta_0^{\widetilde{M}_2}S_{M_5,\,L}\left(\sum_{l\in\mathbb{Z}_+}\chi_{S_l(B)}\int_0^{r_B} t^{2m(\widetilde{M}_1+\widetilde{M}_2)}T_{0,\,0,\,t,\,L}(A(\cdot,\,t))\,\frac{dt}{t}\right).
\end{aligned}$$

Via an argument similar to that used in (3.27), we know that $\mathcal{T}_{3,\,1}(A)$ is an $H^p_I(\mathbb{R}^n)$-molecule associated with B^*. From this and Proposition 3.8, we deduce that (3.35) holds true for $j = 1$ and $r_B < 1$. This proves (3.35). The case $j = 2$ is similar, the details being omitted.

Also, one can prove (3.36) by using Proposition 2.10, the details being omitted. This finishes the proof of Proposition 3.24. □

Corollary 3.25. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let* $\{M_i\}_{i=1}^5, \{\widetilde{M}_i\}_{i=1}^5 \subset \mathbb{Z}_+$ *and $\mathcal{T}$ be defined as in* (3.31).

(i) *For all $p \in (0, 2]$, if $M_1 + M_2 > 0, M_3 + M_4 > 0, \widetilde{M}_1 + \widetilde{M}_2 > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$ and $\widetilde{M}_3 + \widetilde{M}_4 > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$, then the operator $\mathcal{T}$ is bounded on $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$.*

(ii) *For all $p \in (2, \infty)$, if $\widetilde{M}_1 + \widetilde{M}_2 > 0$, $\widetilde{M}_3 + \widetilde{M}_4 > 0$, $M_1 + M_2 > \frac{n}{2m}(\frac{1}{2} - \frac{1}{p})$ and $M_3 + M_4 > \frac{n}{2m}(\frac{1}{2} - \frac{1}{p})$, then the operator $\mathcal{T}$ is bounded on $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$.*

Proof. For $p \in (0, 1]$, Corollary 3.25 is just Proposition 3.24. For $p \in (1, \infty)$, the proof of Corollary 3.25 is an immediate consequence of Proposition 3.24, the already known boundedness for $p = 2$, and the interpolation and the duality of $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ (see [17, Theorems 3.9, 3.10, 4.8 and 4.10] for the dualities and the interpolations of $t^p(\mathbb{R}^n \times (0, 1])$ and $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ with $p \in [1, \infty)$), which completes the proof of Corollary 3.25. □

Proposition 3.26. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let* $\{M_i\}_{i=1}^5, \{\widetilde{M}_i\}_{i=1}^5 \subset \mathbb{Z}_+$ *satisfy $M_1 + M_2 > 0, M_3 + M_4 > 0, \widetilde{M}_1 + \widetilde{M}_1 > 0$ and $\widetilde{M}_3 + \widetilde{M}_4 > 0$.*

(i) *If $p \in (0, 2]$, then the operator $\mathcal{Q}_{\{M_i\}_i,\,L}$, defined as in* (3.29), *is bounded from $h^p_L(\mathbb{R}^n)$ to $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$, where $h^p_L(\mathbb{R}^n)$ is the local Hardy space defined as in Definition* 3.14. *Moreover, if $\widetilde{M}_1 + \widetilde{M}_2 > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$ and $\widetilde{M}_3 + \widetilde{M}_4 > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$, then the operator $\Pi_{\{\widetilde{M}_i\}_i,\,L}$, defined as in* (3.30), *is bounded from $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ to $h^p_L(\mathbb{R}^n)$.*

(ii) *If $p \in (2, \infty)$, then the operator $\Pi_{\{\widetilde{M}_i\}_i, L}$ is bounded from $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ to $h^p_L(\mathbb{R}^n)$. Moreover, if $M_1 + M_2 > \frac{n}{2m}(\frac{1}{2} - \frac{1}{p})$ and $M_3 + M_4 > \frac{n}{2m}(\frac{1}{2} - \frac{1}{p})$, then the operator $\mathcal{Q}_{\{M_i\}_i, L}$ is bounded from $h^p_L(\mathbb{R}^n)$ to $t^p(\mathbb{R}^n \times (0, 1]) \oplus t^p(\mathbb{R}^n \times (0, 1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$.*

Proof. We first consider the case $p \in (0, 2]$. For all $t \in (0, 1]$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\widetilde{\mathcal{Q}}(f)(x, t) := (t^{2m} L e^{-t^{2m} L}(f),\, t^{2m} \delta_0 e^{-t^{2m}(L+\delta_0)}(f),\, e^{-L}(f)), \tag{3.40}$$

where δ_0 is as in (2.7). From the fact that, for all $t \in (0, 1]$, $e^{-t^{2m}\delta_0} \le 1$, we deduce that, for all $p \in (0, 2]$ and $f \in h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\begin{aligned} &\|\mathcal{Q}_{\{1,0,0,1,0\}, L}(f)\|_{t^p(\mathbb{R}^n\times(0,1])\oplus t^p(\mathbb{R}^n\times(0,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)} \\ &\quad \lesssim \|\widetilde{\mathcal{Q}}(f)\|_{t^p(\mathbb{R}^n\times(0,1])\oplus t^p(\mathbb{R}^n\times(0,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)}, \end{aligned}$$

which, combined with Definition 3.14, implies the boundedness of $\mathcal{Q}_{\{M_i\}_i, L}$, for $M_1 = 1$, $M_2 = 0$, $M_3 = 0$, $M_4 = 1$ and $M_5 = 0$, from $h^p_L(\mathbb{R}^n)$ to $t^p(\mathbb{R}^n \times (0, 1])$.

For general $\{M_i\}_i$, let $f \in L^2(\mathbb{R}^n) \cap h^p_L(\mathbb{R}^n)$. By the local Calderón reproducing formula (3.13) with $M > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$, the fact that, for all $t \in (0, 1]$, $e^{-t^{2m}\delta_0} \le 1$ and Corollary 3.25, we have

$$\begin{aligned} &\|T_{M_1, M_2, s, L}(f)\|_{t^p(\mathbb{R}^n\times(0,1])} \\ &\quad \lesssim \sum_{j=0}^{M+1} \left\| \int_0^1 T_{M_1, M_2, s, L} T_{j, 2(M+1)-j-1, t, L}[t^{2m}\delta_0 e^{-t^{2m}(L+\delta_0)}(f)] \frac{dt}{t} \right\|_{t^p(\mathbb{R}^n\times(0,1])} \\ &\qquad + \sum_{j=M+2}^{2(M+1)} \left\| \int_0^1 T_{M_1, M_2, s, L} T_{j-1, 2(M+1)-j, t, L} e^{-t^{2m}\delta_0} \right. \\ &\qquad \left. \times [t^{2m} L e^{-t^{2m}L}(f)] \frac{dt}{t} \right\|_{t^p(\mathbb{R}^n\times(0,1])} \\ &\qquad + \sum_{k=0}^{2(M+1)} \|T_{M_1, M_2, s, L}(L+\delta_0)^k e^{-(L+2\delta_0)}(e^{-L}(f))\|_{t^p(\mathbb{R}^n\times(0,1])} \\ &\quad \lesssim \|S_{L,\delta_0,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)} + \|S_{L,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)} + \|e^{-L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \\ &\quad \sim \|f\|_{h^p_L(\mathbb{R}^n)}, \end{aligned}$$

which implies that $T_{M_1, M_2, s, L}$ is bounded from $h^p_L(\mathbb{R}^n)$ to $t^p(\mathbb{R}^n \times (0, 1])$. By using the same method, we know that $T_{M_3, M_4, s, L}$ is also bounded from $h^p_L(\mathbb{R}^n)$ to $t^p(\mathbb{R}^n \times (0, 1])$.

We now prove that $S_{M_5,\,L}$ is bounded from $h^p_L(\mathbb{R}^n)$ to $L^p_{\mathcal{Q}}(\mathbb{R}^n)$. Using again the local Calderón reproducing formula (3.13), we find that

$$\begin{aligned}
&\|S_{M_5,\,L}(f)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}\\
&\quad\lesssim \sum_{j=0}^{M+1}\left\|\int_0^1 S_{M_5,\,L}T_{j,\,2(M+1)-j-1,\,t,\,L}[(t^{2m}\delta_0)e^{-t^{2m}(L+\delta_0)}(f)]\,\frac{dt}{t}\right\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}\\
&\qquad+\sum_{j=M+2}^{2(M+1)}\left\|\int_0^1 S_{M_5,\,L}T_{j-1,\,2(M+1)-j,\,t,\,L}e^{-t^{2m}\delta_0}[(t^{2m}L)e^{-t^{2m}L}(f)]\,\frac{dt}{t}\right\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)}\\
&\qquad+\sum_{k=0}^{2(M+1)}\|S_{M_5,\,L}(L+\delta_0)^k e^{-L-2\delta_0}(e^{-L}(f))\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)},
\end{aligned}$$

which, together with Corollary 3.25, implies that $S_{M_5,\,L}$ is bounded from $h^p_L(\mathbb{R}^n)$ to $L^p_{\mathcal{Q}}(\mathbb{R}^n)$. Thus, from its definition, we deduce that $\mathcal{Q}_{\{M_i\}_i,\,L}$ is bounded from $h^p_L(\mathbb{R}^n)$ to $t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ for all $p\in(0,\,2]$.

For $\Pi_{\{\widetilde{M}_i\}_i,\,L}$, let

$$\begin{aligned}
(F,\,G,\,h)\in{}&\big[t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)\big]\\
&\cap\big[t^2(\mathbb{R}^n\times(0,\,1])\oplus t^2(\mathbb{R}^n\times(0,\,1])\oplus L^2(\mathbb{R}^n)\big].
\end{aligned}$$

From (3.30) and the definition of $h^p_L(\mathbb{R}^n)$, we deduce that

$$\begin{aligned}
&\|\Pi_{\{\widetilde{M}_i\}_i,\,L}(F,\,G,\,h)\|_{h^p_L(\mathbb{R}^n)}\\
&\quad=\|\widetilde{\mathcal{Q}}\circ\Pi_{\{\widetilde{M}_i\}_i,\,L}(F,\,G,\,h)\|_{t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)},
\end{aligned}$$

where $\widetilde{\mathcal{Q}}$ is as in (3.40). Since $\widetilde{M}_1+\widetilde{M}_2>\frac{n}{2m}(\frac1p-\frac12)$ and $\widetilde{M}_3+\widetilde{M}_4>\frac{n}{2m}(\frac1p-\frac12)$, by using an argument similar to that used in the proof of Proposition 3.24, and Corollary 3.25 (observe that, since $t\in(0,\,1]$, the term $e^{-t^{2m}\delta_0}$ does not affect the argument), we know that $\widetilde{\mathcal{Q}}\circ\Pi_{\{\widetilde{M}_i\}_i,\,L}$ is bounded on $t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ for all $p\in(0,\,2]$, which, combined with Definition 3.14, implies the boundedness of $\Pi_{\{\widetilde{M}_i\}_i,\,L}$ from $t^p(\mathbb{R}^n\times(0,\,1])$ to $h^p_L(\mathbb{R}^n)$.

The case $p\in(2,\,\infty)$ follows from the dualities of the spaces $h^p_L(\mathbb{R}^n)$, $t^p(\mathbb{R}^n\times(0,\,1])$ and $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ for $p\in(1,\,\infty)$, and the duality between the operators $\mathcal{Q}_{\{M_i\},\,L}$ and $\Pi_{\{M_i\},\,L^*}$, the details being omitted. This finishes the proof of Proposition 3.26. □

Remark 3.27. Let $\{M_i\}_{i=1}^5\subset\mathbb{Z}_+$ and $f\in L^2(\mathbb{R}^n)$. For all $(x,\,t)\in\mathbb{R}^{n+1}_+$, define the *operator* $\widetilde{\mathcal{Q}}_{\{M_i\}_i,\,L}$ by setting

$$\begin{aligned}
\widetilde{\mathcal{Q}}_{\{M_i\}_i,\,L}(f)(x,\,t):={}&((t^{2m}L)^{M_1}(t^{2m}\delta_0)^{M_2}e^{-t^{2m}L}(f)(x),\,(t^{2m}L)^{M_3}\\
&\times(t^{2m}\delta_0)^{M_4}e^{-t^{2m}(L+\delta_0)}(f)(x),(L+\delta_0)^{M_5}e^{-L}(f)(x))\\
=:{}&(\widetilde{T}_{M_1,\,M_2,\,t,\,L}(f)(x),\,T_{M_3,\,M_4,\,t,\,L}(f)(x),\,\widetilde{S}_{M_5,\,L}(f)(x)),
\end{aligned}\tag{3.41}$$

where δ_0 is as in (2.7).

Let $(F, G, h) \in t^2(\mathbb{R}^n \times (0, 1]) \oplus t^2(\mathbb{R}^n \times (0, 1]) \oplus L^2_{\mathcal{Q}}(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$, define the *operator* $\widetilde{\Pi}_{\{M_i\}_i, L}$ by setting

$$\begin{aligned}
&\widetilde{\Pi}_{\{M_i\}_i, L}(F, G, h)(x) \\
&\quad := \int_0^1 \widetilde{T}_{M_1, M_2, t, L}(F(\cdot, t))(x) \frac{dt}{t} \\
&\qquad + \int_0^1 T_{M_3, M_4, t, L}(G(\cdot, t))(x) \frac{dt}{t} + \widetilde{S}_{M_5, L}(h)(x),
\end{aligned}$$

where $\widetilde{T}_{M_1, M_2, t, L}$, $T_{M_3, M_4, t, L}$ and $\widetilde{S}_{M_5, L}$ are defined as in (3.41).

We point out that the operators $\widetilde{\mathcal{Q}}_{\{M_i\}_i, L}$ and $\widetilde{\Pi}_{\{M_i\}_i, L}$ satisfy the same bounded properties as those of $\Pi_{\{\widetilde{M}_i\}_i, L}$ and $\mathcal{Q}_{\{M_i\}_i, L}$ in Proposition 3.26, since the term $e^{-t^{2m}\delta_0}$ does not affect the arguments when $t \in (0, 1]$.

Remark 3.28. We also point out that the local Calderón reproducing formula (3.13), originally holding true in $L^2(\mathbb{R}^n)$, can be extended to $h^p_L(\mathbb{R}^n)$ for all $p \in (0, \infty)$. Indeed, rewrite the local Calderón reproducing formula (3.13) into

$$\begin{aligned}
f = &\sum_{M+2 \le j \le 2(M+1)} \widetilde{C}_{(j)} \int_0^1 (t^{2m}L)^{j-M}(t^{2m}\delta_0)^{2(M+1)-j} e^{-t^{2m}(L+\delta_0)} \\
&\times [(t^{2m}L)^M e^{-t^{2m}(L+\delta_0)}(f)] \frac{dt}{t} \\
&+ \sum_{0 \le j \le M+1} \widetilde{C}_{(j)} \int_0^1 (t^{2m}L)^j (t^{2m}\delta_0)^{2(M+1)-j-M} \\
&\times e^{-t^{2m}(L+\delta_0)}[(t^{2m}\delta_0)^M e^{-t^{2m}(L+\delta_0)}(f)] \frac{dt}{t} \\
&+ \sum_{k=0}^{2(M+1)} C_{(k)}(L+\delta_0)^k e^{-(L+\delta_0)}[e^{-(L+\delta_0)}(f)],
\end{aligned}$$

where $M > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$, and $\widetilde{C}_{(j)}$ and $C_{(k)}$ are as in (3.13). This, together with Proposition 3.26 and the density of $L^2(\mathbb{R}^n) \cap h^p_L(\mathbb{R}^n)$ in $h^p_L(\mathbb{R}^n)$ with $p \in (0, 2]$, implies that the above local Calderón reproducing formula holds true in $h^p_L(\mathbb{R}^n)$ for all $p \in (0, 2]$. Using duality, we know that the local Calderón formula can be extended to $h^p_L(\mathbb{R}^n)$ for all $p \in (2, \infty)$.

We now establish the square function characterization of $h^p_L(\mathbb{R}^n)$.

Definition 3.29. Let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let $\delta_0 \in (0, \infty)$ be as in Definition 3.14 and $M_1, M_2 \in \mathbb{Z}_+$.

For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $S_{M_1,\,L,\,\mathrm{loc}}(f)$ be the *local square function* of f defined by setting

$$S_{M_1,\,L,\,\mathrm{loc}}(f)(x) := \left[\int_0^1 \int_{|y-x|<t} |(t^{2m}L)^{M_1} e^{-t^{2m}L}(f)(y)|^2 \, \frac{dy\,dt}{t^{n+1}}\right]^{1/2}$$

and $S_{M_2,\,L,\,\delta_0,\,\mathrm{loc}}(f)$ the *local square function* of f defined by setting

$$S_{M_2,\,L,\,\delta_0,\,\mathrm{loc}}(f)(x) := \left[\int_0^1 \int_{|y-x|<t} |(t^{2m}\delta_0)^{M_2} e^{-t^{2m}(L+\delta_0)}(f)(y)|^2 \, \frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

where δ_0 is as in (2.7). Then the *local Hardy space* $h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)$ is defined as the completion of the space

$$\begin{aligned}\{f \in L^2(\mathbb{R}^n) : &\|f\|_{h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)} \\ &:= \|S_{M_1,\,L,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)} + \|S_{M_2,\,L,\,\delta_0,\,\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n)} \\ &\quad + \|e^{-L}(f)\|_{L^p_{\mathbb{Q}}(\mathbb{R}^n)} < \infty\}\end{aligned}$$

with respect to the *quasi-norm* $\|\cdot\|_{h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)}$, where p, M_1 and M_2 satisfy

$$\begin{cases} M_1,\, M_2 \in \mathbb{N}, & \text{when } p \in (0,\,2], \\ M_1,\, M_2 > \dfrac{n}{2m}\left(\dfrac{1}{2} - \dfrac{1}{p}\right), & \text{when } p \in (2,\,\infty). \end{cases}$$

Observe that, if L satisfies the Ellipticity condition $(\mathcal{E})$, then we can choose $\delta_0 \equiv 0$ and hence $S_{M_2,\,L,\,\delta_0,\,\mathrm{loc}}(f) \equiv 0$.

Theorem 3.30. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let M_1, $M_2 \in \mathbb{N}$ be as in Definition* 3.29. *Then, for all $p \in (0,\,\infty)$, $h^p_L(\mathbb{R}^n) = h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. We first consider the case $p \in (0,\,2]$. Let $f \in L^2(\mathbb{R}^n) \cap h^p_L(\mathbb{R}^n)$. By Definition 3.29 and Remark 3.27, we know that

$$\begin{aligned}\|f\|_{h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)} &= \|((t^{2m}L)^{M_1} e^{-t^{2m}L}(f),\, (t^{2m}\delta_0)^{M_2} e^{-t^{2m}(L+\delta_0)}(f), \\ &\qquad e^{-L}(f))\|_{t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathbb{Q}}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{h^p_L(\mathbb{R}^n)},\end{aligned} \tag{3.42}$$

which implies that

$$\left[L^2(\mathbb{R}^n) \cap h^p_L(\mathbb{R}^n)\right] \subset \left[L^2(\mathbb{R}^n) \cap h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)\right].$$

On the other hand, let $f \in L^2(\mathbb{R}^n) \cap h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)$. By the local Calderón reproducing formula (3.13) with the same positive constants $\widetilde{C}_{(j)}$ and $C_{(k)}$ as in (3.13), we write

$$
\begin{aligned}
f &= \sum_{M+2\le j\le 2(M+1)} \widetilde{C}_{(j)} \int_0^1 (t^{2m}L)^j (t^{2m}\delta_0)^{2(M+1)-j} e^{-2t^{2m}(L+\delta_0)}(f)\,\frac{dt}{t} \\
&\quad + \sum_{0\le j\le M+1} \widetilde{C}_{(j)} \int_0^1 \cdots \frac{dt}{t} + \sum_{k=0}^{2(M+1)} C_{(k)}(L+\delta_0)^k e^{-2(L+\delta_0)}(f) \\
&= \int_0^1 \sum_{M+2\le j\le 2(M+1)} \widetilde{C}_{(j)} (t^{2m}L)^{j-M_1} (t^{2m}\delta_0)^{2(M+1)-j} e^{-t^{2m}(L+\delta_0)} e^{-t^{2m}\delta_0} \\
&\quad \times [(t^{2m}L)^{M_1} e^{-t^{2m}L}(f)]\,\frac{dt}{t} \\
&\quad + \int_0^1 \sum_{0\le j\le M+1} \widetilde{C}_{(j)} (t^{2m}L)^j (t^{2m}\delta_0)^{2(M+1)-j-M_2} e^{-t^{2m}(L+\delta_0)} \\
&\quad \times [(t^{2m}\delta_0)^{M_2} e^{-t^{2m}(L+\delta_0)}(f)]\,\frac{dt}{t} + \sum_{k=0}^{2(M+1)} C_{(k)}(L+\delta_0)^k e^{-(L+\delta_0)} e^{-\delta_0}[e^{-L}(f)] \\
&=: \Pi_{M_1,\,M_2} \circ \mathcal{Q}_{M_1,\,M_2}(f), \qquad (3.43)
\end{aligned}
$$

where, for all $f \in L^2(\mathbb{R}^n)$ and $(x,\,t) \in \mathbb{R}^n \times (0,\,1]$,

$$
\begin{aligned}
&\mathcal{Q}_{M_1,\,M_2}(f)(x,\,t) \\
&\quad := ((t^{2m}L)^{M_1} e^{-t^{2m}L}(f)(x),\, (t^{2m}\delta_0)^{M_2} e^{-t^{2m}(L+\delta_0)}(f)(x),\, e^{-L}(f)(x)) \qquad (3.44)
\end{aligned}
$$

and, for all $(F,\,G,\,h) \in t^2(\mathbb{R}^n \times (0,\,1]) \oplus t^2(\mathbb{R}^n \times (0,\,1]) \oplus L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$
\begin{aligned}
&\Pi_{M_1,\,M_2}(F,\,G,\,h)(x) \\
&\quad := \int_0^1 \sum_{M+2\le j\le 2(M+1)} \widetilde{C}_{(j)} (t^{2m}L)^{j-M_1} (t^{2m}\delta_0)^{2(M+1)-j} \\
&\qquad \times e^{-t^{2m}(L+\delta_0)} e^{-t^{2m}\delta_0} (F(\cdot,\,t))(x)\,\frac{dt}{t} \\
&\qquad + \int_0^1 \sum_{0<j\le M+1} \widetilde{C}_{(j)} (t^{2m}L)^j (t^{2m}\delta_0)^{2(M+1)-j-M_2} e^{-t^{2m}(L+\delta_0)} (G(\cdot,\,t))(x)\,\frac{dt}{t} \\
&\qquad + \sum_{k=0}^{2(M+1)} C_{(k)}(L+\delta_0)^k e^{-(L+\delta_0)} e^{-\delta_0}(h)(x). \qquad (3.45)
\end{aligned}
$$

Let $\mathcal{Q}_0 := (t^{2m}Le^{-t^{2m}L},\, t^{2m}\delta_0 e^{-t^{2m}(L+\delta_0)},\, e^{-L})$. By (3.43) and Definition 3.14, we have

$$\begin{aligned}\|f\|_{h^p_L(\mathbb{R}^n)} &= \|\Pi_{M_1,\,M_2}\circ \mathcal{Q}_{M_1,\,M_2}(f)\|_{h^p_L(\mathbb{R}^n)}\\ &= \|\mathcal{Q}_0\circ\Pi_{M_1,\,M_2}\circ \mathcal{Q}_{M_1,\,M_2}(f)\|_{t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)}.\end{aligned}$$

By Remark 3.27, we know that, for all $p\in(0,\,2]$, $\mathcal{Q}_0$ is bounded from $h^p_L(\mathbb{R}^n)$ to

$$t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$$

and $\Pi_{M_1,\,M_2}$ bounded from $t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ to $h^p_L(\mathbb{R}^n)$. Thus, $\mathcal{Q}_0\circ\Pi_{M_1,\,M_2}$ is bounded on $t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$, which, combined with Definition 3.29, shows that

$$\|f\|_{h^p_L(\mathbb{R}^n)}\lesssim \|\mathcal{Q}_{M_1,\,M_2}(f)\|_{t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)}\sim\|f\|_{h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)}.$$

This implies that $\left[L^2(\mathbb{R}^n)\cap h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)\right]\subset\left[L^2(\mathbb{R}^n)\cap H^p_L(\mathbb{R}^n)\right]$. Hence, by a density argument, we know that, for all $p\in(0,\,2]$, $h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)=H^p_L(\mathbb{R}^n)$ with equivalent quasi-norms.

We now consider the case when $p\in(2,\,\infty)$. To this end, we first claim that $L^2(\mathbb{R}^n)\cap h^p_L(\mathbb{R}^n)$ is dense in $h^p_L(\mathbb{R}^n)$. Indeed, let $f\in h^p_L(\mathbb{R}^n)$. By Remark 3.28, we know that the local Calderón reproducing formula $f=\Pi_{M_1,\,M_2}\circ\mathcal{Q}_{M_1,\,M_2}(f)$ also holds true in $h^p_L(\mathbb{R}^n)$. For any $N\in(0,\,\infty)$ and

$$(F,\,G,\,h)\in t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n),$$

let

$$\chi_N(F,\,G,\,h):=(\chi_{\{|x|<N,\,\frac{1}{N}<t<1\}}F,\,\chi_{\{|x|<N,\,\frac{1}{N}<t<1\}}G,\,\chi_{\{|x|<N\}}h).$$

It is easy to see that

$$\begin{aligned}\chi_N(F,\,G,\,h)\in&\left[t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)\right]\\ &\cap\left[t^2(\mathbb{R}^n\times(0,\,1])\oplus t^2(\mathbb{R}^n\times(0,\,1])\oplus L^2(\mathbb{R}^n)\right].\end{aligned}$$

Thus, let $f_N:=\Pi_{M_1,\,M_2}\circ(\chi_N\mathcal{Q}_{M_1,\,M_2}(f))$. Since $\mathcal{Q}_{M_1,\,M_2}(f)\in t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$, we have $f_N\in h^p_L(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $\lim_{N\to\infty}f_N=f$ in $h^p_L(\mathbb{R}^n)$. This implies the density of $L^2(\mathbb{R}^n)\cap h^p_L(\mathbb{R}^n)$ in $h^p_L(\mathbb{R}^n)$.

Now, we turn to the proof of $h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)=h^p_L(\mathbb{R}^n)$ with equivalent quasi-norms for $p\in(2,\,\infty)$. The inclusion that $\left[L^2(\mathbb{R}^n)\cap h^p_L(\mathbb{R}^n)\right]\subset\left[L^2(\mathbb{R}^n)\cap h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)\right]$ is similar to the case when $p\in(0,\,2]$ (see estimate (3.42) for some details; here, we need the assumption $M_1,\,M_2>\frac{n}{2m}(\frac12-\frac1p)$), the details being omitted.

To prove the converse inclusion, using $h^p_L(\mathbb{R}^n)=(h^{p'}_{L^*}(\mathbb{R}^n))^*$, the local Calderón reproducing formula (3.43), $M_1,\,M_2>\frac{n}{2m}(\frac12-\frac1p)$ and Remark 3.27, we know that there exists $g \in h^{p'}_{L^*}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $\|g\|_{h^{p'}_{L^*}(\mathbb{R}^n)} \le 1$ such that

$$\begin{aligned}
\|f\|_{h^p_L(\mathbb{R}^n)} &\sim \left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx\right| \sim \left|\int_{\mathbb{R}^n} \Pi_{M_1,\,M_2}\circ \mathcal{Q}_{M_1,\,M_2}(f)(x)\overline{g(x)}\,dx\right| \\
&\sim \left|\int_{\mathbb{R}^n} \mathcal{Q}_{M_1,\,M_2}(f)(x)\overline{(\Pi_{M_1,\,M_2})^*(g)(x)}\,dx\right| \\
&\lesssim \|\mathcal{Q}_{M_1,\,M_2}(f)\|_{t^p(\mathbb{R}^n\times(0,\,1])\oplus t^p(\mathbb{R}^n\times(0,\,1])\oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)} \\
&\quad \times \|(\Pi_{M_1,\,M_2})^*(g)\|_{t^{p'}(\mathbb{R}^n\times(0,\,1])\oplus t^{p'}(\mathbb{R}^n\times(0,\,1])\oplus L^{p'}_{\mathcal{Q}}(\mathbb{R}^n)} \\
&\lesssim \|f\|_{h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)}\|g\|_{h^{p'}_{L^*}(\mathbb{R}^n)} \sim \|f\|_{h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n)},
\end{aligned}$$

which implies that $\left[h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right] \subset \left[h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right]$ for $p \in (2,\,\infty)$. Thus, by the density of $h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $h^p_L(\mathbb{R}^n)$, we conclude that $h^p_{\{M_1,\,M_2\},\,L}(\mathbb{R}^n) = h^p_L(\mathbb{R}^n)$ with equivalent quasi-norms. This finishes the proof of Theorem 3.30. □

The square function characterization of $h^p_L(\mathbb{R}^n)$ enables us to establish the following complex interpolation of $h^p_L(\mathbb{R}^n)$. In what follows, $[\cdot,\,\cdot]_\theta$ for $\theta \in (0,\,1)$ denotes the complex interpolation (see, for example, [49, Sec. 7; 61]).

Theorem 3.31. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let* $0 < p_1 < p_2 < \infty, \theta \in (0,\,1)$ *and* $p \in (0,\,\infty)$ *satisfy* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then* $[h^{p_1}_L(\mathbb{R}^n),\, h^{p_2}_L(\mathbb{R}^n)]_\theta = h^p_L(\mathbb{R}^n)$.

Proof. Let $\mathcal{Q}_{M_1,\,M_2}$ and $\Pi_{M_1,\,M_2}$ be, respectively, as in (3.44) and (3.45). By Theorem 3.30 and the local Calderón reproducing formula (3.43), we know that $h^p_L(\mathbb{R}^n)$ is a retract of $t^p(\mathbb{R}^n \times (0,\,1]) \oplus t^p(\mathbb{R}^n \times (0,\,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ via $\mathcal{Q}_{M_1,\,M_2}$ and $\Pi_{M_1,\,M_2}$. Thus, Theorem 3.31 follows immediately from the complex interpolation of $t^p(\mathbb{R}^n \times (0,\,1]) \oplus t^p(\mathbb{R}^n \times (0,\,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$.

To finish the proof of Theorem 3.31, we give a brief proof of the complex interpolation of $t^p(\mathbb{R}^n \times (0,\,1]) \oplus t^p(\mathbb{R}^n \times (0,\,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$. Indeed, from their definitions, it is easy to see that, for all $p \in (0,\,\infty)$, the spaces $t^p(\mathbb{R}^n \times (0,\,1])$ and $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ are analytically convex (see [49] for the notions of analytically convex spaces and also [43, (8.26)] for a similar argument). This implies that $t^p(\mathbb{R}^n \times (0,\,1]) \oplus t^p(\mathbb{R}^n \times (0,\,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$ is also analytically convex.

Now, for all $(F,\,G,\,h) \in t^p(\mathbb{R}^n \times (0,\,1]) \oplus t^p(\mathbb{R}^n \times (0,\,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$, let $P_1(F,\,G,\,h) := F$, $P_2(F,\,G,\,h) := G$ and $P_3(F,\,G,\,h) := h$. By [49, Lemma 7.11], the complex interpolations of $t^p(\mathbb{R}^n \times (0,\,1])$ and $L^p_{\mathcal{Q}}(\mathbb{R}^n)$ (see [17, Theorems 3.10 and 4.10]), we know that, for all $0 < p_1 < p_2 < \infty$, $\theta \in (0,\,1)$ and $p \in (0,\,\infty)$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$,

$$t^p = [t^{p_0},\, t^{p_1}]_\theta = P_1([t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}},\, t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}}]_\theta),$$

$$t^p = [t^{p_0},\, t^{p_1}]_\theta = P_2([t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}},\, t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}}]_\theta)$$

and

$$L^p_{\mathcal{Q}} = [L^{p_0}_{\mathcal{Q}},\, L^{p_1}_{\mathcal{Q}}]_\theta = P_3([t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}},\, t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}}]_\theta),$$

which, together with the facts that $(P_1,\, P_2,\, P_3) \equiv I$ in $(t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}}) + (t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}})$ and that $[t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}},\, t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}}]_\theta \subset \big[(t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}}) + (t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}})\big]$, imply that

$$[t^{p_0} \oplus t^{p_0} \oplus L^{p_0}_{\mathcal{Q}},\, t^{p_1} \oplus t^{p_1} \oplus L^{p_1}_{\mathcal{Q}}]_\theta = t^p \oplus t^p \oplus L^p_{\mathcal{Q}}.$$

This finishes the proof of Theorem 3.31. □

4. The Hardy Space $H^p_{L+\delta}(\mathbb{R}^n)$ Associated with $L + \delta$

In this section, let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. Let $\delta_0 \in [0, \infty)$ be as in (2.7). From the arguments below (2.7), we deduce that $L+\delta_0$ is of type ω with $\omega \in [0, \frac{\pi}{2})$ and satisfies the bounded functional calculus in $L^2(\mathbb{R}^n)$. Moreover, by Proposition 2.10, we know that there exists a positive constant κ such that, for all $k \in \mathbb{Z}_+$, the family $\{[t(L+\kappa)]^k e^{-t(L+\kappa)}\}_{t>0}$ of operators satisfies the m-L^p-L^q off-diagonal estimates (2.18) for all $p_-(L+\kappa) < p \le q < p_+(L+\kappa)$, where $(p_-(L+\kappa),\, p_+(L+\kappa))$ denotes the *maximal interval* of exponents $p \in [1, \infty]$ such that $\{e^{-t(L+\kappa)}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Let

$$\delta \in (\max\{\delta_0,\, \kappa\},\, \infty). \tag{4.1}$$

Here, we need the strict inequality $\delta > \max\{\delta_0,\, \kappa\}$ because we need to use Corollary 2.3 in this section.

In particular, if $L \equiv -\Delta$, we can take both κ in Proposition 2.10 and δ_0 in (2.7) to be 0. Thus, in this case, $\delta \in (0, \infty)$.

We know that the operator $L+\delta$ has all the above properties, which indicates that it is possible to introduce the following Hardy spaces associated with the operator $L+\delta$.

Definition 4.1. Let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let $\delta \in (0, \infty)$ be as in (4.1).

For all $p \in (0, 2]$, the *Hardy space* $H^p_{L+\delta}(\mathbb{R}^n)$ is defined as the completion of the space

$$\mathbb{H}^p_{L+\delta}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^p_{L+\delta}(\mathbb{R}^n)} := \|S_{L+\delta}(f)\|_{L^p(\mathbb{R}^n)} < \infty\}$$

with respect to the *quasi-norm* $\|\cdot\|_{H^p_{L+\delta}(\mathbb{R}^n)}$, where, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *square function* $S_{L+\delta}(f)$ is defined by setting

$$S_{L+\delta}(f)(x) := \left[\iint_{\Gamma(x)} |t^{2m}(L+\delta)e^{-t^{2m}(L+\delta)}(f)(y)|^2\, \frac{dy\, dt}{t^{n+1}}\right]^{1/2} \tag{4.2}$$

and $\Gamma(x)$ for all $x \in \mathbb{R}^n$ is as in (1.2).

For $p \in (2, \infty)$, $H^p_{L+\delta}(\mathbb{R}^n)$ is defined as the dual space of $H^{p'}_{L^*+\delta}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$ and p' the *conjugate exponent* of p.

Remark 4.2. (i) We point out that, from the arguments below (2.7), it follows that $L+\delta$ is a one-to-one operator of type $\omega \in [0, \frac{\pi}{2})$ having a bounded H_∞ functional calculus in $L^2(\mathbb{R}^n)$ and satisfying the off-diagonal estimates as in (2.18). This implies that the Hardy space $H^p_{L+\delta}(\mathbb{R}^n)$, for $p \in (0, 1]$, falls into the scope of [16], where the authors introduced the Hardy spaces associated with some abstract operators satisfying the above assumptions. Thus, by [16, Theorems 4.5 and 5.2], we know that $H^p_{L+\delta}(\mathbb{R}^n)$ for $p \in (0, 1]$ has characterizations in terms of the molecule and the square function, respectively (see Definition 4.3 below for the notion of $H^p_{L+\delta}(\mathbb{R}^n)$-molecules).

(ii) We also point out that, for all $p \in (0, \infty)$, $H^p_{L+\delta}(\mathbb{R}^n)$ forms a complex interpolation scale as in Theorem 3.31, since $H^p_{L+\delta}(\mathbb{R}^n)$ is a retract of the tent space $T^p(\mathbb{R}^{n+1}_+)$.

(iii) Let $(p_-(L+\delta), p_+(L+\delta))$ denote the *maximal interval* of exponents $p \in [1, \infty]$ such that $\{e^{-t(L+\delta)}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Since $L+\delta$ has a bounded functional calculus in $L^2(\mathbb{R}^n)$, we know $2 \in (p_-(L+\delta), p_+(L+\delta))$. Similar to the proof of [43, Proposition 9.1(v)], we obtain $H^p_{L+\delta}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all $p \in (p_-(L+\delta), p_+(L+\delta))$ with equivalent quasi-norms.

We now state the molecular characterization of $H^p_{L+\delta}(\mathbb{R}^n)$.

Definition 4.3. Let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let δ be as in Definition 4.1, $p \in (0, 1]$, $\epsilon \in (0, \infty)$ and $M \in \mathbb{N}$. A function $\alpha \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M, \epsilon)_{L+\delta}$-*molecule* if there exists a ball $B \subset \mathbb{R}^n$ such that, for each $\ell \in \{0, \ldots, M\}$, α belongs to the range of $(L+\delta)^\ell$ in $L^2(\mathbb{R}^n)$ and, for all $i \in \mathbb{Z}_+$ and $\ell \in \{0, \ldots, M\}$,

$$\|(r_B^{2m}[L+\delta])^{-\ell}(\alpha)\|_{L^2(S_i(B))} \le 2^{-i\epsilon}|2^iB|^{\frac{1}{2}-\frac{1}{p}}.$$

Assume that $\{\alpha_j\}_j$ is a sequence of $(p, 2, M, \epsilon)_{L+\delta}$-molecules and $\{\lambda_j\}_j \in l^p$. For any $f \in L^2(\mathbb{R}^n)$, if $f = \sum_j \lambda_j\alpha_j$ holds true in $L^2(\mathbb{R}^n)$, then $\sum_j \lambda_j\alpha_j$ is called a *molecular* $(p, 2, M, \epsilon)_{L+\delta}$-*representation* of f.

Definition 4.4. Let L be as in (1.6) and satisfy the Ellipticity condition $(\mathcal{E}_0)$, and let $p \in (0, 1]$, $\epsilon \in (0, \infty)$ and $M \in \mathbb{N}$. The *molecular Hardy space* $H^p_{L+\delta,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)$ is defined as the completion of the space

$$\begin{aligned}&\mathbb{H}^p_{L+\delta,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)\\ &\qquad := \{f \in L^2(\mathbb{R}^n) : f \text{ has a molecular } (p, 2, M, \epsilon)_{L+\delta}\text{-representation}\}\end{aligned}$$

with respect to the *quasi-norm*

$$\begin{aligned}\|f\|_{H^p_{L+\delta,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)} := \inf\Bigg\{\left(\sum_j |\lambda_j|^p\right)^{1/p} : f = \sum_j \lambda_j\alpha_j \text{ is a molecular}\\ (p, 2, M, \epsilon)_{L+\delta}\text{-representation}\Bigg\},\end{aligned}$$

where the infimum is taken over all the molecular $(p,\, 2,\, M,\, \epsilon)_{L+\delta}$-representations of f as above.

Theorem 4.5 ([16]). *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let δ be as in Definition* 4.1, $p \in (0,\, 1]$, $\epsilon \in (0,\, \infty)$ *and $M \in \mathbb{N}$ such that $M > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$. Then $H^p_{L+\delta}(\mathbb{R}^n) = H^p_{L+\delta,\, \mathrm{mol},\, M,\, \epsilon}(\mathbb{R}^n)$ with equivalent quasi-norms.*

We now establish the equivalence between $h^p_L(\mathbb{R}^n)$ and $H^p_{L+\delta}(\mathbb{R}^n)$ for $p \in (0,\, 1]$.

Theorem 4.6. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let δ be as in Definition* 4.1 *and $h^p_L(\mathbb{R}^n)$ as in Definition* 3.14. *Then, for all $p \in (0,\, 1]$, $h^p_L(\mathbb{R}^n) = H^p_{L+\delta}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. We first prove $\left[H^p_{L+\delta}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right] \subset \left[h^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right]$. To this end, let $f \in H^p_{L+\delta}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Theorem 4.5 and Definition 3.14 together with Fatou's lemma, we see that it suffices to show, for each $(p,\, 2,\, M,\, \epsilon)_{L+\delta}$-molecule α associated with the ball $B := B(x_B,\, r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\, \infty)$,

$$\|S_{L,\, \mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim 1, \tag{4.3}$$

$$\|S_{L,\, \delta,\, \mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim 1 \tag{4.4}$$

and

$$\|e^{-L}(\alpha)\|_{L^p_{\mathbb{Q}}(\mathbb{R}^n)} \lesssim 1. \tag{4.5}$$

We first prove (4.3) by considering two cases on the size of r_B. If $r_B \geq 1$, using Proposition 2.10 and an argument same as that used in (3.23), we know that there exists a positive constant η such that

$$\|S_{L,\, \mathrm{loc}}(\alpha)\|^2_{L^2(S_j(B))} \lesssim 2^{-j\eta} |2^j B|^{1-\frac{2}{p}}, \tag{4.6}$$

which, together with Hölder's inequality, implies that

$$\|S_{L,\, \mathrm{loc}}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j \in \mathbb{Z}_+} \|S_{L,\, \mathrm{loc}}(\alpha)\|_{L^2(S_j(B))} |2^j B|^{\frac{1}{p} - \frac{1}{2}} \lesssim 1.$$

That is, (4.3) holds true in this case.

If $r_B < 1$, then, by Fubini's theorem, we first write

$$\begin{aligned} \|S_{L,\, \mathrm{loc}}(\alpha)\|^2_{L^2(S_j(B))} &= \int_{S_j(B)} \left[\int_0^1 \int_{|y-x|<t} |t^{2m} L e^{-t^{2m} L}(\alpha)(y)|^2 \, \frac{dy\, dt}{t^{n+1}} \right] dx \\ &= \int_{S_j(B)} \left[\left(\int_0^{c_j} + \int_{c_j}^1 \right) \int_{|y-x|<t} |t^{2m} L e^{-t^{2m} L}(\alpha)(y)|^2 \right. \\ &\quad \left. \times \frac{dy\, dt}{t^{n+1}} \right] dx \\ &=: \mathcal{A}_j + \mathcal{B}_j, \end{aligned}$$

where $c_j := \min\{2^{j-3}r_B, 1\}$. If $j \le 3$, we only need to use the local quadratic estimates as in (3.11) to replace the off-diagonal estimates used in the case $j > 3$ and, if $2^{j-3}r_B \ge 1$, we only need to estimate $\mathcal{A}_j$.

We first estimate $\mathcal{A}_j$. By Hölder's inequality, Fubini's theorem, Proposition 2.10 and Definition 4.3, we conclude that there exists a positive constant η such that

$$\mathcal{A}_j \lesssim \int_0^{2^{j-3}r_B} \int_{\widetilde{S}_j(B)} |t^{2m} L e^{-t^{2m}L}(\alpha)(y)|^2 \,\frac{dy\,dt}{t} \lesssim 2^{-j\eta}|2^j B|^{1-\frac{2}{p}}. \tag{4.7}$$

To bound $\mathcal{B}_j$ in the case of $2^{j-2}r_B < 1$, from the semigroup property of e^{-tL}, the fact that the family $\{t^{2m}Le^{-\frac{1}{2}t^{2m}L}[t^{2m}(L+\delta)e^{-\frac{1}{2}t^{2m}L}]^M\}_{t\in(0,1]}$ of operators is bounded on $L^2(\mathbb{R}^n)$, Definition 4.3 and $M > \frac{n}{2m}(\frac{1}{p}-\frac{1}{2})$, we deduce that

$$\begin{aligned}\mathcal{B}_j &\lesssim \int_{2^{j-3}r_B}^1 \int_{\mathbb{R}^n} |t^{2m}Le^{-\frac{1}{2}t^{2m}L}[t^{2m}(L+\delta)e^{-\frac{1}{2}t^{2m}L}]^M (L+\delta)^{-M}(\alpha)(y)|^2 \,\frac{dy\,dt}{t^{4mM+1}}\\ &\lesssim 2^{-j[4mMn-(\frac{2}{p}-1)]}|2^jB|^{1-\frac{2}{p}}.\end{aligned}$$

This, combined with (4.7), shows that (4.6) also holds true when $r_B < 1$. Thus, (4.3) also holds true in this case.

The proof of (4.4) is similar to that of (4.3), the details being omitted.

We now turn to the proof of (4.5). If $r_B \ge 1$, using Proposition 2.10 and Definition 4.3, we see that there exists a positive constant η such that, for all $j \in \mathbb{N}$,

$$\begin{aligned}\|\chi_{S_j(B)}e^{-L}(\alpha)\|_{L^2(\mathbb{R}^n)} &\le \|\chi_{S_j(B)}[e^{-L}(\chi_{\widetilde{S}_j(B)}\alpha)]\|_{L^2(\mathbb{R}^n)}\\ &\quad + \|\chi_{S_j(B)}[e^{-L}(\chi_{\mathbb{R}^n\setminus\widetilde{S}_j(B)}\alpha)]\|_{L^2(\mathbb{R}^n)}\\ &\lesssim 2^{-j\eta}|2^jB|^{\frac{1}{2}-\frac{1}{p}}\end{aligned}$$

(see, for example, the estimate (3.23) for more details). This shows that $e^{-L}(\alpha)$ is an $H_I^p(\mathbb{R}^n)$-molecule associated with B up to a harmless positive constant multiple, which, together with Proposition 3.8, implies that (4.5) holds true in this case.

If $r_B < 1$, let $B^* := \frac{1}{r_B}B := B(x_B, 1)$. For all $j \in \mathbb{Z}_+$, we write

$$\begin{aligned}&\|\chi_{S_j(B^*)}e^{-L}(\alpha)\|_{L^2(\mathbb{R}^n)}\\ &\quad = \left\|\chi_{S_i(B^*)}(L+\delta)^M e^{-L}\left(\sum_{l\in\mathbb{N}}[\chi_{S_l(B)}(L+\delta)^{-M}(\alpha)]\right)\right\|_{L^2(\mathbb{R}^n)}\\ &\quad \le \left\|\chi_{S_i(B^*)}(L+\delta)^M e^{-L}\left(\sum_{l\in\mathbb{N},\,2^l<\frac{2^{i-2}}{r_B}\text{ or }2^l\ge\frac{2^{i+2}}{r_B}}[\chi_{S_l(B)}(L+\delta)^{-M}(\alpha)]\right)\right\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

$$+ \sum_{l\in\mathbb{N},\, \frac{2^{i-2}}{r_B}\le 2^l<\frac{2^{i+2}}{r_B}} \|\chi_{S_i(B^*)}(L+\delta)^M e^{-L}(\chi_{S_l(B)}(L+\delta)^{-M}(\alpha))\|_{L^2(\mathbb{R}^n)}$$
$$=: \mathrm{I}+\mathrm{J}.$$

Since $(L+\delta)^M e^{-L}$ satisfies the off-diagonal estimates as in (2.19), the estimates for I and J are similar to those for I and J in (3.27), respectively. Thus, we conclude that $e^{-L}(\alpha)$ is an $H^p_I(\mathbb{R}^n)$-molecule associated with B^* up to a harmless positive constant multiple, which, together with Proposition 3.8, implies that $\|e^{-L}(\alpha)\|_{L^p_{\mathcal{Q}}(\mathbb{R}^n)} \lesssim 1$. This shows that (4.5) holds true when $r_B<1$, which, combined with (4.3) and (4.4), implies $\left[H^p_{L+\delta}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right]\subset\left[h^p_L(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right]$.

We now turn to the proof of the converse inclusion. To this end, by Theorem 3.18, we only need to show that, for all $\epsilon\in(0,\infty)$, $M\in\mathbb{N}$ satisfying $M>\frac{n}{2m}(\frac{1}{p}-\frac{1}{2})$ and any local $(p,2,M,\epsilon)_L$-molecule $\widetilde{\alpha}$ associated with the ball $\widetilde{B}:=B(x_{\widetilde{B}}, r_{\widetilde{B}})$ for some $x_{\widetilde{B}}\in\mathbb{R}^n$ and $r_{\widetilde{B}}\in(0,\infty)$,

$$\|S_{L+\delta}(\widetilde{\alpha})\|_{L^p(\mathbb{R}^n)}\lesssim 1. \tag{4.8}$$

If $r_{\widetilde{B}}\ge 1$, for all $j\in\mathbb{Z}_+$, we first write

$$\begin{aligned}
&\|S_{L+\delta}(\widetilde{\alpha})\|^2_{L^2(S_j(\widetilde{B}))}\\
&\quad=\int_{S_j(\widetilde{B})}\left[\int_0^\infty\int_{|y-x|<t}|t^{2m}(L+\delta)e^{-t^{2m}(L+\delta)}(\widetilde{\alpha})(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right]dx\\
&\quad=\int_{S_j(\widetilde{B})}\left[\left(\int_0^{2^{j-3}r_{\widetilde{B}}}+\int_{2^{j-3}r_{\widetilde{B}}}^\infty\right)\int_{|y-x|<t}|t^{2m}(L+\delta)\right.\\
&\qquad\left.\times e^{-t^{2m}(L+\delta)}(\widetilde{\alpha})(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right]dx=:\widetilde{\mathcal{A}}_j+\widetilde{\mathcal{B}}_j,
\end{aligned}\tag{4.9}$$

where, without loss of generality, we may assume that $j>3$. If $j\le 3$, we only need to use the quadratic estimates of $L+\delta$ to replace the off-diagonal estimates used in the case $j>3$ as follows.

We first bound $\widetilde{\mathcal{B}}_j$. Let $\widetilde{b}:=(L+\delta)^{-M}(\widetilde{\alpha})$. From the Weak ellipticity condition $(\widetilde{\mathcal{E}})$ and Corollary 2.3, we deduce that the operator

$$(L+\delta)^{-1}=(L+\delta)^{-\frac{1}{2}}(L+\delta)^{-\frac{1}{2}}$$

is bounded on $L^2(\mathbb{R}^n)$. This shows that $\|\widetilde{b}\|_{L^2(\mathbb{R}^n)}\lesssim\|\widetilde{\alpha}\|_{L^2(\mathbb{R}^n)}$. Moreover, by Fubini's theorem, the boundedness of the family $\{[t^{2m}(L+\delta)]^{M+1}e^{-t^{2m}(L+\delta)}\}_{t>0}$ of operators on $L^2(\mathbb{R}^n)$, Definition 3.16 and the fact that $r_{\widetilde{B}}\ge 1$, we find that

$$\begin{aligned}
\widetilde{\mathcal{B}}_j&\lesssim\int_{2^{j-3}r_{\widetilde{B}}}^\infty\int_{\mathbb{R}^n}|[t^{2m}(L+\delta)]^{M+1}e^{-t^{2m}(L+\delta)}(\widetilde{b})(y)|^2\,\frac{dy\,dt}{t^{4mM+1}}\\
&\lesssim\|\widetilde{b}\|^2_{L^2(\mathbb{R}^n)}\int_{2^{j-3}r_{\widetilde{B}}}^\infty\frac{dt}{t^{4mM+1}}\lesssim 2^{-j[4mM-n(\frac{2}{p}-1)]}|2^j\widetilde{B}|^{1-\frac{2}{p}},
\end{aligned}\tag{4.10}$$

which is desired.

The estimate of $\widetilde{\mathcal{A}}_j$ is similar to that of $\mathcal{A}_j$ in (4.7), the details being omitted. Thus, (4.8) holds true when $r_{\widetilde{B}} \geq 1$.

For $r_{\widetilde{B}} < 1$, using an argument similar to that used in (4.9), we are reduced to the estimates of $\widetilde{\mathcal{A}}_j$ and $\widetilde{\mathcal{B}}_j$. The estimate of $\widetilde{\mathcal{A}}_j$ is independent of the size of $r_{\widetilde{B}}$. We only need to bound $\widetilde{\mathcal{B}}_j$. Indeed, similar to (4.10), by the boundedness of the family $\{(t^{2m}L)^M[t^{2m}(L+\delta)]e^{-t^{2m}(L+\delta)}\}_{t>0}$ of operators on $L^2(\mathbb{R}^n)$ (which can be deduced from the fact that, for all $k \in \mathbb{Z}_+$, the family $\{[t^{2m}(L+\delta)]^{k+1}e^{-t^{2m}(L+\delta)}\}_{t>0}$ of operators is bounded on $L^2(\mathbb{R}^n)$ via mathematical induction) and Definition 3.16, we conclude that

$$\begin{aligned}\widetilde{\mathcal{B}}_j &\lesssim \int_{2^{j-3}r_{\widetilde{B}}}^{\infty}\int_{\mathbb{R}^n} |(t^{2m}L)^M[t^{2m}(L+\delta)]e^{-t^{2m}(L+\delta)}L^{-M}(\widetilde{\alpha})(y)|^2\,\frac{dy\,dt}{t^{4mM+1}}\\ &\lesssim \|L^{-M}(\widetilde{\alpha})\|^2_{L^2(\mathbb{R}^n)}\int_{2^{j-3}r_{\widetilde{B}}}^{\infty}\frac{dt}{t^{4mM+1}} \lesssim 2^{-j[4mM-n(\frac{2}{p}-1)]}|2^j\widetilde{B}|^{1-\frac{2}{p}},\end{aligned}$$

which implies that (4.8) holds true for $r_{\widetilde{B}} < 1$. This proves

$$\left[h^p_L(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right] \subset \left[H^p_{L+\delta}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right].$$

Thus, $\left[h^p_L(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right] = \left[H^p_{L+\delta}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right]$ with equivalent quasi-norms, which, combined with a density argument, completes the proof of Theorem 4.6. □

We now establish the equivalence between $h^p_L(\mathbb{R}^n) = H^p_{L+\delta}(\mathbb{R}^n)$ for $p \in (1, \infty)$. To this end, we need the following result.

Lemma 4.7. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* ($\widetilde{\mathcal{E}}$). *Then $h^2_L(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. We point out that the inclusion $L^2(\mathbb{R}^n) \subset h^2_L(\mathbb{R}^n)$ follows immediately from Definition 3.14 and the boundedness of $S_{L,\,\mathrm{loc}}$, $S_{L,\,\delta,\,\mathrm{loc}}$ and e^{-L} on $L^2(\mathbb{R}^n)$, while the converse inclusion follows from the local Calderón reproducing formula (3.43) and Remark 3.27 (see the proof of [43, Proposition 9.1(v)] for a similar argument with more details). This finishes the proof of Lemma 4.7. □

Combining Theorem 4.6, Lemma 4.7, Definition 3.14, Theorem 3.31, and the interpolation and the dual of $H^p_{L+\delta}(\mathbb{R}^n)$ (see (ii) and (iii) of Remark 4.2), we immediately obtain the following conclusion, the details being omitted.

Theorem 4.8. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* ($\widetilde{\mathcal{E}}$), *and let δ be as in Definition* 4.1 *and $h^p_L(\mathbb{R}^n)$ as in Definition* 3.14. *Then, for all $p \in (0, \infty)$, $h^p_L(\mathbb{R}^n) = H^p_{L+\delta}(\mathbb{R}^n)$ with equivalent quasi-norms.*

From Theorem 4.8 and Remark 4.2, we deduce the following conclusion, the details being omitted.

Corollary 4.9. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* ($\widetilde{\mathcal{E}}$), *and let δ be as in Definition* 4.1 *and $(p_-(L+\delta),\, p_+(L+\delta))$ as in Proposition* 2.10.

Then, for all $p \in (p_-(L+\delta),\, p_+(L+\delta)), h^p_L(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ *with equivalent quasi-norms.*

Theorem 4.8 also implies the following corollary, the details being omitted.

Corollary 4.10. *Let* L *be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let* $\delta \in (\max\{\delta_0,\, \kappa\},\, \infty)$ *be as in* (4.1). *Then, for all* $\delta_1, \delta_2 \in [\delta,\, \infty)$ *and* $p \in (0,\, \infty), H^p_{L+\delta_1}(\mathbb{R}^n) = H^p_{L+\delta_2}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

As an application, we also have the following boundedness on the Riesz transforms

$$\{\nabla^k(L+\delta)^{-1/2}\}_{k\in\{0,\ldots,m\}}.$$

Corollary 4.11. *Let* L *be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and* δ *be as in Definition* 4.1.

(i) *If* $p \in (1,\, 2]$, *then, for all* $k \in \{0, \ldots, m\}$, *the Riesz transform* $\nabla^k(L+\delta)^{-1/2}$ *is bounded from* $H^p_{L+\delta}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$;

(ii) *If* $p \in (\frac{n}{n+k},\, 1]$, *then, for all* $k \in \{1, \ldots, m\}, \nabla^k(L+\delta)^{-1/2}$ *is bounded from* $H^p_{L+\delta}(\mathbb{R}^n)$ *to the classical Hardy space* $H^p(\mathbb{R}^n)$.

Proof. The case $p = 2$ follows immediately from Corollary 2.3 and the fact that

$$H^2_{L+\delta}(\mathbb{R}^n) = L^2(\mathbb{R}^n) = H^2(\mathbb{R}^n).$$

We now turn to the case $p \in (0,\, 1]$. In this case, from Theorem 2.11 and an argument similar to that used in the proof of [16, Lemma 6.1], we deduce that, for $k \in \{0, \ldots, m\}$, $t \in (0,\, \infty)$ and all closed sets E, F in $\mathbb{R}^n$ with $\operatorname{dist}(E,\, F) > 0$, $f \in L^2(\mathbb{R}^n)$ supported in E,

$$\|\nabla^k(L+\delta)^{-1/2}(I - e^{-t[L+\delta]})^M(f)\|_{L^2(F)} \lesssim \left(\frac{t}{[\operatorname{dist}(E,\, F)]^{2m}}\right)^M \|f\|_{L^2(E)}$$

and

$$\|\nabla^k(L+\delta)^{-1/2}(t[L+\delta]e^{-t[L+\delta]})^M(f)\|_{L^2(F)} \lesssim \left(\frac{t}{[\operatorname{dist}(E,\, F)]^{2m}}\right)^M \|f\|_{L^2(E)},$$

which, together with an argument similar to that used in the proof of [46, Theorem 7.1], shows that (i) holds true for all $p \in (0,\, 1]$. Also, from the proofs of [46, Theorem 7.4], [43, Theorem 5.2] or [16, Theorem 6.1]), we deduce that (ii) holds true for all $p \in (\frac{n}{n+k},\, 1]$.

The case $p \in (1,\, 2)$ follows from the interpolation of operators (see Theorem 3.31 together with [44, Theorem, p. 52]), which completes the proof of Corollary 4.11.

□

5. The Local Lipschitz Space Associated with L

Let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. In this section, we characterize the dual space of $h_L^p(\mathbb{R}^n)$ via some local Lipschitz space associated with L^*, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$.

Let $\alpha \in [0, \infty)$ and $M \in \mathbb{N}$ satisfy $M > \frac{1}{2}(\alpha + \frac{n}{2})$. For all $\epsilon \in (0, \infty)$, we say a function μ on $\mathbb{R}^n$ belongs to the *space* $M_{\alpha, L}^{\epsilon, M}(\mathbb{R}^n)$ if, for all $k \in \{0, \ldots, M\}$, μ belongs to the range of L^k in $L^2(\mathbb{R}^n)$ and

$$\|\mu\|_{M_{\alpha, L}^{\epsilon, M}(\mathbb{R}^n)} := \sup_{j \in \mathbb{Z}_+} \left\{ 2^{j(\frac{n}{2}+\alpha+\epsilon)} \sum_{k=0}^{M} \|L^{-k}(\mu)\|_{L^2(S_j(B_0))} \right\} < \infty,$$

where $B_0 := B(0, 1)$ is the unit ball of $\mathbb{R}^n$.

Let

$$f \in M_{\alpha, L}^{M, *}(\mathbb{R}^n) := \bigcap_{\epsilon > 0} (M_{\alpha, L}^{\epsilon, M}(\mathbb{R}^n))^*. \tag{5.1}$$

A function $f \in M_{\alpha, L}^{M, *}(\mathbb{R}^n)$ is said to be in the *local Lipschitz space* $\Lambda_{L^*, \mathrm{loc}}^{\alpha}(\mathbb{R}^n)$ if

$$\begin{aligned} \|f\|_{\Lambda_{L^*, \mathrm{loc}}^{\alpha}(\mathbb{R}^n)} := {} & \sup_{B \subset \mathbb{R}^n, r_B \le 1} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{ \frac{1}{|B|} \int_B |(I - e^{-r_B^{2m} L^*})^M (f)(x)|^2 \, dx \right\}^{\frac{1}{2}} \\ & + \sup_{B \subset \mathbb{R}^n, r_B \le 1} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{ \frac{1}{|B|} \int_B |(I - e^{-r_B^{2m} \delta})^M (f)(x)|^2 \, dx \right\}^{\frac{1}{2}} \\ & + \sup_{B \subset \mathbb{R}^n, r_B \ge 1} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{ \frac{1}{|B|} \int_B |f(x)|^2 \, dx \right\}^{\frac{1}{2}} < \infty, \end{aligned} \tag{5.2}$$

where $\delta \in (0, \infty)$ is as in Definition 4.1, B is a ball of $\mathbb{R}^n$ and r_B denotes its radius. We point out that the formula $(I - e^{-r_B^{2m}\delta})^M$ in the second integral in (5.2) can be moved out from the integral, since it is just a constant. We put it in the present position because this brings some convenience in the calculation (see, for example, (5.8)).

Lemma 5.1. *Let $\alpha \in [0, \infty), M \in \mathbb{N}$ satisfy $M > \frac{1}{2}(\alpha + \frac{n}{2})$ and $f \in M_{\alpha, L}^{M, *}(\mathbb{R}^n)$. Then $f \in \Lambda_{L^*, \mathrm{loc}}^{\alpha}(\mathbb{R}^n)$ if and only if*

$$\begin{aligned} & \|f\|_{\Lambda_{L^*, \mathrm{loc}, \mathrm{Res}}^{\alpha}(\mathbb{R}^n)} \\ & := \sup_{B \subset \mathbb{R}^n, r_B \le 1} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{ \frac{1}{|B|} \int_B |[I - (I + r_B^{2m} L^*)^{-1}]^M (f)(x)|^2 \, dx \right\}^{\frac{1}{2}} \\ & \quad + \sup_{B \subset \mathbb{R}^n, r_B \le 1} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{ \frac{1}{|B|} \int_B |[I - (I + r_B^{2m} \delta)^{-1}]^M (f)(x)|^2 \, dx \right\}^{\frac{1}{2}} \\ & \quad + \sup_{B \subset \mathbb{R}^n, r_B \ge 1} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{ \frac{1}{|B|} \int_B |f(x)|^2 \, dx \right\}^{\frac{1}{2}} < \infty, \end{aligned}$$

where $\delta \in (0, \infty)$ is as in Definition 4.1, *B is a ball of $\mathbb{R}^n$ and r_B denotes its radius.*

Proof. The proof of Lemma 5.1 is similar to that of [42, Lemma 8.1], the details being omitted. □

Lemma 5.2. *Let* $f \in \Lambda^{\alpha}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$. *Then* f *satisfies the following control growth estimates: there exists a positive constant* $\epsilon_0 \in (0, \infty)$, *independent of* f *and* M, *such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\epsilon_0}}\,dx < \infty \tag{5.3}$$

and

$$\int_{\mathbb{R}^n} \frac{|[I-(I+L^*)^{-1}]^M(f)(x)|^2}{1+|x|^{n+\epsilon_0}}\,dx < \infty, \tag{5.4}$$

where $M \in \mathbb{N}$ *satisfies* $M > \frac{1}{2}(\alpha + \frac{n}{2})$.

Proof. Let $B_0 := B(0, 1)$ be the unit ball of $\mathbb{R}^n$ and n_0 the least natural number satisfying that, for all $j \in \mathbb{N}$, $S_j(B_0) \subset \bigcup_{k=1}^{2^{j(n+n_0)}} B_{k,\,j}$, where $\{B_{k,\,j}\}_k$ ranges over all families of balls that cover $S_j(B_0) := 2^j B_0 \backslash (2^{j-1}B_0)$ with radius $r_{B_{k,\,j}} \in (1, 2]$. It is easy to see that $n_0 < \infty$ and n_0 is independent of $j \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\epsilon_0}}\,dx \\
&\quad \lesssim \sum_{j=0}^{\infty} \int_{S_j(B_0)} \frac{|f(x)|^2}{1+|x|^{n+\epsilon_0}}\,dx \lesssim \sum_{j=0}^{\infty} \sum_{k=1}^{2^{j(n+n_0)}} \int_{B_{k,\,j}} 2^{-j(n+\epsilon_0)}|f(x)|^2\,dx \\
&\quad \lesssim \sum_{j=0}^{\infty} \sum_{k=1}^{2^{j(n+n_0)}} 2^{-j(n+\epsilon_0)} \|f\|^2_{\Lambda^{\alpha}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)}.
\end{aligned}$$

By letting $\epsilon_0 > n_0$ in the above formulae, we obtain (5.3). Using a similar argument and Lemma 5.1, we see that (5.4) also holds true. This finishes the proof of Lemma 5.2. □

We now show that the local Calderón reproducing formula (3.13) also holds true in some "dual" sense.

Lemma 5.3. *Let* L *be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let* $p \in (0, 1]$. *Then, for all* $f \in \Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$ *and* g *being a finite linear combination of* $h^p_L(\mathbb{R}^n)$*-molecules, it holds true that*

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx \\
&\quad = \sum_{j=0}^{M+1} \widetilde{C}_{(j)} \int_{\mathbb{R}^n} \int_0^1 (t^{2m}L^*)^j (t^{2m}\delta)^{2(M+1)-j-1} e^{-t^{2m}(L^*+\delta)}(f)(x) \\
&\qquad \times \overline{[t^{2m}\delta e^{-t^{2m}(L+\delta)}(g)(x)]}\,\frac{dx\,dt}{t}
\end{aligned}$$

$$+\sum_{j=M+2}^{2(M+1)}\widetilde{C}_{(j)}\int_{\mathbb{R}^n}\int_0^1(t^{2m}L^*)^{j-1}(t^{2m}\delta)^{2(M+1)-j}e^{-t^{2m}(L^*+\delta)}e^{-t^{2m}\delta}(f)(x)$$
$$\times\overline{[t^{2m}Le^{-t^{2m}L}(g)(x)]}\,\frac{dxdt}{t}$$
$$+\sum_{k=0}^{2(M+1)}C_{(k)}\int_{\mathbb{R}^n}(L^*+\delta)^k e^{-\delta}e^{-(L^*+\delta)}(f)(x)\overline{[e^{-L}(g)(x)]}\,dx, \tag{5.5}$$

where $\widetilde{C}_{(j)}$ and $C_{(k)}$ are the positive constants as in (3.13).

Proof. Let $f\in\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$. Without loss of generality, we may assume that g is an $h^p_L(\mathbb{R}^n)$-molecule associated with $B:=B(0,\,r_B)$ for some $r_B\in(0,\infty)$.

If $r_B\geq 1$, by Hölder's inequality and Definition 3.16, we see that there exists a positive constant $\epsilon\in(0,\infty)$ such that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}f(x)\overline{g(x)}\,dx\right| &\leq\sum_{j\in\mathbb{Z}_+}\left\{\int_{S_j(B)}|f(x)|^2\,dx\right\}^{\frac12}\left\{\int_{S_j(B)}|g(x)|^2\,dx\right\}^{\frac12}\\ &\leq\sum_{j\in\mathbb{Z}_+}\frac{1}{|2^jB|^{\frac1p-1}}\left\{\frac{1}{|2^jB|}\int_{2^jB}|f(x)|^2\,dx\right\}^{\frac12}2^{-j\epsilon}\\ &\lesssim\sum_{j\in\mathbb{Z}_+}2^{-j\epsilon}\|f\|_{\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)}\lesssim\|f\|_{\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)},\end{aligned}$$

which, combined with the fact that the local Calderón reproducing formula (3.13), with δ_0 replaced by δ, holds true in $h^p_L(\mathbb{R}^n)$ and a duality argument, implies that (5.5) holds true.

If $r_B<1$, from the local Calderón reproducing formula (3.13), with δ_0 replaced by δ, and the following formula

$$f=\sum_{l=0}^{M}\frac{M!}{(M-l)!l!}(L^*)^{-l}[I-(I+L^*)^{-1}]^M(f)$$

holding true in $L^2(\mathbb{R}^n)$, we deduce that

$$\begin{aligned}\langle f,\,g\rangle &=\left\langle f,\,\widetilde{C}\int_0^1[t^{2m}(L+\delta)]^{2(M+1)}e^{-2t^{2m}(L+\delta)}(g)\,\frac{dt}{t}\right.\\ &\qquad\left.+\sum_{k=0}^{2(M+1)}C_{(k)}(L+\delta)^k e^{-2(L+\delta)}(g)\right\rangle\\ &=\sum_{l=0}^{M}\frac{M!}{(M-l)!l!}\left\langle[I-(I+L^*)^{-1}]^M(f),\right.\end{aligned}$$

$$\widetilde{C}\int_0^1 [t^{2m}(L+\delta)]^{2(M+1)}e^{-2t^{2m}(L+\delta)}L^{-l}(g)\,\frac{dt}{t}\Big\rangle$$

$$+\left\langle f,\ \sum_{k=0}^{2(M+1)} C_{(k)}(L+\delta)^k e^{-2(L+\delta)}(g)\right\rangle =: \sum_{l=0}^{M}\mathcal{A}_l+\mathcal{B},$$

where $\widetilde{C}$ and $C_{(k)}$ are as in (3.13).

Now, for any $\eta\in(0,\infty)$, let

$$\mathcal{A}_{l,\,\eta} := \left\langle [I-(I+L^*)^{-1}]^M(f),\ \widetilde{C}\int_0^\eta [t^{2m}(L+\delta)]^{2(M+1)}e^{-2t^{2m}(L+\delta)}L^{-l}(g)\,\frac{dt}{t}\right\rangle.$$

To finish the proof of Lemma 5.3 for $r_B<1$, we only need to show

$$\lim_{\eta\to 0^+}\mathcal{A}_{l,\,\eta}=0, \tag{5.6}$$

here and hereafter, $\eta\to 0^+$ means that $\eta>0$ and $\eta\to 0$. Indeed, observe that, for all $i\in\{1,\ldots,2M+1\}$, the operators $[\eta^{2m}(L+\delta)]^i e^{-2\eta^{2m}(L+\delta)}\to 0$ and $e^{-2\eta^{2m}(L+\delta)}\to I$ in the strong operator topology as $\eta\to 0^+$. These, together with Definition 3.16 and an argument similar to that used in the proof of [42, (8.36) through (8.40)], imply that (5.6) holds true, which completes the proof of Lemma 5.3. $\square$

Theorem 5.4. *Let L be as in* (1.1) *and satisfy the Weak ellipticity condition* $(\widetilde{\mathcal{E}})$, *and let $p\in(0,1]$. Then $\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)=(h^p_L(\mathbb{R}^n))^*$ in the following sense: every $f\in(h^p_L(\mathbb{R}^n))^*$ belongs to the space $\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\|f\|_{\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)}\le C\|f\|_{(h^p_L(\mathbb{R}^n))^*},$$

where C is a positive constant independent of f; conversely, every $f\in\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$, initially defined as a bounded linear functional on finite linear combinations of $(p,2,\epsilon,M)_L$-molecules, with $\epsilon>0$ and $M>\frac n2(\frac1p-\frac12)$, via the pairing of the space $M^{\epsilon,\,M}_{n(\frac1p-1),\,L}(\mathbb{R}^n)$ with its dual, has a unique bounded extension to $h^p_L(\mathbb{R}^n)$ and

$$\|f\|_{(h^p_L(\mathbb{R}^n))^*}\le C\|f\|_{\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)},$$

where C is a positive constant independent of f.

Proof. We first prove $(h^p_L(\mathbb{R}^n))^*\subset\Lambda^{n(\frac1p-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$. Let $f\in(h^p_L(\mathbb{R}^n))^*$. Then, for any $g\in h^p_L(\mathbb{R}^n)$, it holds true that

$$|\langle f,\,g\rangle|\lesssim\|f\|_{(h^p_L(\mathbb{R}^n))^*}\|g\|_{h^p_L(\mathbb{R}^n)}.$$

Moreover, since, for any $\epsilon\in(0,\infty)$, $M\in\mathbb{N}$ satisfies $M>\frac12[n(\frac1p-1)+\frac n2]$ and any $\mu\in M^{\epsilon,\,M}_{n(\frac1p-1),\,L}(\mathbb{R}^n)$ is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule associated with $B_0:=B(0,\,1)$ up to a harmless positive constant multiple, it follows that $f\in M^{M,\,*}_{n(\frac1p-1),\,L}(\mathbb{R}^n)$.

Thus, to prove $f \in \Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$, we only need to show that

$$\|f\|_{\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)} \lesssim \|f\|_{(h^p_L(\mathbb{R}^n))^*}. \tag{5.7}$$

Indeed, for any ball B in $\mathbb{R}^n$ with its radius $r_B \le 1$ and $\varphi \in L^2(B)$ with $\operatorname{supp}\varphi \subset B$, using an argument similar to that used in the proof of [43, (3.56)], we see that $\frac{1}{|B|^{\frac{1}{p}-\frac{1}{2}}}(I - e^{-r_B^{2m}L})^M(\varphi)$ is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule associated with B up to a harmless positive constant multiple.

Moreover, let $B^* := B(x_B,\,1)$. For all $j \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&\left\|\frac{1}{|B|^{\frac{1}{p}-\frac{1}{2}}}[I - e^{-r_B^{2m}\delta}]^M(\varphi)\right\|_{L^2(S_j(B^*))} \\
&\quad = \left\|\frac{r_B^{2mM}}{|B|^{\frac{1}{p}-\frac{1}{2}}}\left[\int_0^{r_B}\frac{t^{2m-1}}{r_B^{2m}}e^{-t^{2m}\delta}\,dt\right]^M(\varphi)\right\|_{L^2(S_j(B^*))} \\
&\quad \lesssim 2^{-j\eta}r_B^{2mM-n(\frac{1}{p}-\frac{1}{2})}\|\varphi\|_{L^2(B)} \lesssim 2^{-j\eta},
\end{aligned} \tag{5.8}$$

which implies that $\frac{1}{|B|^{\frac{1}{p}-\frac{1}{2}}}(I - e^{-r_B^{2m}\delta})^M(\varphi)$ is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule associated with B^* up to a harmless positive constant multiple.

If $r_B \ge 1$, we see that $\frac{1}{|B|^{\frac{1}{p}-\frac{1}{2}}}\varphi$ is a local $(p,\,2,\,M,\,\epsilon)_L$-molecule associated with B up to a harmless positive constant multiple. Combining these facts, we conclude that (5.7) holds true, which implies that the inclusion $(h^p_L(\mathbb{R}^n))^* \subset \Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$ holds true.

We now turn to the proof of $\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n) \subset (h^p_L(\mathbb{R}^n))^*$. Let $f \in \Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)$ and g be a finite linear combination of local $(p,\,2,\,M,\,\epsilon)_L$-molecules up to a harmless positive constant multiple. By Lemma 5.3, we have

$$\begin{aligned}
|\langle f,\,g\rangle| \lesssim{}& \sum_{j=0}^{M+1}\left|\int_{\mathbb{R}^n}\int_0^1 (t^{2m}L^*)^j(t^{2m}\delta)^{2(M+1)-j-1}e^{-t^{2m}(L^*+\delta)}e^{-t^{2m}\delta}(f)(x)\right. \\
&\left.\times\overline{[t^{2m}\delta e^{-t^{2m}(L+\delta)}(g)(x)]}\,\frac{dxdt}{t}\right| \\
&+ \sum_{j=M+2}^{2(M+1)}\left|\int_{\mathbb{R}^n}\int_0^1 (t^{2m}L^*)^{j-1}(t^{2m}\delta)^{2(M+1)-j}e^{-t^{2m}(L^*+\delta)}e^{-t^{2m}\delta}(f)(x)\right. \\
&\left.\times\overline{[t^{2m}Le^{-t^{2m}L}(g)(x)]}\,\frac{dxdt}{t}\right| \\
&+ \sum_{k=0}^{2(M+1)}\left|\int_{\mathbb{R}^n}(L^*+\delta)^k e^{-(L^*+\delta)}e^{-\delta}(f)(x)\overline{e^{-L}(g)(x)}\,dx\right| \\
=:{}& \sum_{j=0}^{M+1}\mathcal{A}_j + \sum_{j=M+2}^{2(M+1)}\widetilde{\mathcal{A}}_j + \sum_{k=0}^{2(M+1)}\mathcal{B}_k.
\end{aligned}$$

To estimate $\mathcal{A}_j$, by Definition 3.14 and Proposition 3.21, we know that

$$\begin{aligned}
\mathcal{A}_j &\lesssim \sum_i |\lambda_{j,\,i,\,1}| \left| \int_{\mathbb{R}^n} \int_0^1 (t^{2m}L^*)^j (t^{2m}\delta)^{2(M+1)-j-1} e^{-t^{2m}(L^*+\delta)} \right. \\
&\quad \left. \times e^{-t^{2m}\delta}(f)(x)\overline{A_{j,\,i,\,1}(x)}\,\frac{dxdt}{t} \right| \\
&\quad + \sum_i |\lambda_{j,\,i,\,2}| \left| \int_{\mathbb{R}^n} \int_0^1 (t^{2m}L^*)^j (t^{2m}\delta)^{2(M+1)-j-1} e^{-t^{2m}(L^*+\delta)} \right. \\
&\quad \left. \times e^{-t^{2m}\delta}(f)(x)\overline{A_{j,\,i,\,2}(x)}\,\frac{dxdt}{t} \right| \\
&=: \sum_i |\lambda_{j,\,i,\,1}|\mathcal{A}_{j,\,i,\,1} + \sum_i |\lambda_{j,\,i,\,2}|\mathcal{A}_{j,\,i,\,2},
\end{aligned}$$

where $\{\lambda_{j,\,i,\,1}\}_i$, $\{\lambda_{j,\,i,\,2}\}_i \subset \mathbb{C}$ satisfy

$$\sum_i |\lambda_{j,\,i,\,1}|^p + \sum_i |\lambda_{j,\,i,\,2}|^p \lesssim \|g\|^p_{h^p_L(\mathbb{R}^n)},$$

$A_{j,\,i,\,1}$ is a $(t^p,\,2)$-atom associated with the ball $B_{j,\,i,\,1}$ satisfying its radius $r_{B_{j,\,i,\,1}} < 1$ and $A_{j,\,i,\,2}$ a $(t^p,\,2)$-atom associated with the ball $B_{j,\,i,\,2}$ satisfying its radius $r_{B_{j,\,i,\,1}} \geq 1$.

We first bound $\mathcal{A}_{j,\,i,\,1}$. By Hölder's inequality and Definition 3.20, we see that

$$\begin{aligned}
\mathcal{A}_{j,\,i,\,1} &\lesssim |B_{j,\,i,\,1}|^{\frac{1}{p}-\frac{1}{2}} \left\{ \iint_{T^{\mathrm{loc}}(B_{j,\,i,\,1})} |(t^{2m}L^*)^j (t^{2m}\delta)^{2(M+1)-j-1} \right. \\
&\quad \left. \times e^{-t^{2m}(L^*+\delta)}(f)(x)|^2\,\frac{dxdt}{t} \right\}^{\frac{1}{2}},
\end{aligned}$$

where $T^{\mathrm{loc}}(B_{j,\,i,\,1})$ is as in Definition 3.20(i) with E replaced by $B_{j,\,i,\,1}$. By using some calculations similar to those used in the proof of [42, Lemma 8.3] and Lemma 5.1, we further obtain

$$\begin{aligned}
\mathcal{A}_{j,\,i,\,1} &\lesssim \sup_{B\subset\mathbb{R}^n,\, r_B\leq 1} \frac{1}{|B|^{\frac{1}{p}-1}} \left\{ \frac{1}{|B|} \int_{\mathbb{R}^n} |(I - e^{-r_B^{2m}\delta})^M (f)(x)|^2\,dx \right\}^{\frac{1}{2}} \\
&\lesssim \|f\|_{\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)}.
\end{aligned}$$

For $\mathcal{A}_{j,\,i,\,2}$, by Hölder's inequality and Proposition 2.10, we conclude that

$$\begin{aligned}
\mathcal{A}_{j,\,i,\,2} &\lesssim |B_{j,\,i,\,2}|^{\frac{1}{p}-\frac{1}{2}} \left\{ \int_0^{r_B} \int_{B_{j,\,i,\,2}} \left| (t^{2m}L^*)^j (t^{2m}\delta)^{2(M+1)-j-1} e^{-t^{2m}(L^*+\delta)} \right. \right. \\
&\quad \left. \left. \times \left(\sum_{l\in\mathbb{Z}_+} \chi_{S_l(B_{j,\,i,\,2})} f \right)(x) \right|^2 \frac{dxdt}{t} \right\}^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \sum_{l\in\mathbb{Z}_+} |B_{j,\,i,\,2}|^{\frac{1}{p}-\frac{1}{2}} \left[\int_0^{r_{B_{j,\,i,\,2}}} \exp\left\{-\frac{[d(B_{j,\,i,\,2},\, S_l(B_{j,\,i,\,2}))]^{2m/(2m-1)}}{t^{2m/(2m-1)}}\right\}\frac{dt}{t}\right]^{1/2}$$
$$\times \left\{\int_{S_l(B_{j,\,i,\,2})} |f(x)|^2\,dx\right\}^{1/2}$$
$$\lesssim \sup_{B\subset\mathbb{R}^n,\, r_B\ge 1} \frac{1}{|B|^{\frac{1}{p}-1}} \left\{\frac{1}{|B|}\int_B |f(x)|^2\,dx\right\}^{1/2} \lesssim \|f\|_{\Lambda^{n(\frac{1}{p}-2)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)}. \tag{5.9}$$

Thus, we have

$$\sum_{j=0}^{M+1} \mathcal{A}_j \lesssim \|f\|_{\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)} \|g\|_{h^p_L(\mathbb{R}^n)}.$$

Similarly, we also obtain

$$\sum_{j=M+2}^{2(M+1)} \widetilde{\mathcal{A}}_j \lesssim \|f\|_{\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)} \|g\|_{h^p_L(\mathbb{R}^n)}.$$

The estimate of $\mathcal{B}_k$ is similar to (5.9), the details being omitted. Combining the estimates of $\mathcal{A}_j$, $\widetilde{\mathcal{A}}_j$ and $\mathcal{B}_k$, we obtain

$$|\langle f,\, g\rangle| \lesssim \|f\|_{\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n)} \|g\|_{h^p_L(\mathbb{R}^n)},$$

which implies that f can be extended to a bounded linear functional on $h^p_L(\mathbb{R}^n)$. Thus,

$$\Lambda^{n(\frac{1}{p}-1)}_{L^*,\,\mathrm{loc}}(\mathbb{R}^n) \subset (h^p_L(\mathbb{R}^n))^*,$$

which completes the proof of Theorem 5.4. □

Remark 5.5. Let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$, and let δ be as in Definition 4.1. For all $\alpha \in [0,\,\infty)$ and $M \in \mathbb{N}$ satisfying $M > \frac{1}{2}(\alpha+\frac{n}{2})$, let $M^{M,\,*}_{\alpha,\,L+\delta}(\mathbb{R}^n)$ be defined as in (5.1). The *local Lipschitz space* $\Lambda^\alpha_{L^*+\delta}(\mathbb{R}^n)$ associated with $L+\delta$ is defined as the collection of all $f \in M^{M,\,*}_{\alpha,\,L+\delta}(\mathbb{R}^n)$ such that

$$\|f\|_{\Lambda^\alpha_{L^*+\delta}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^{\frac{\alpha}{n}}} \left\{\frac{1}{|B|}\int_B |(I-e^{-r_B^{2m}(L^*+\delta)})^M(f)(x)|^2\,dx\right\}^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$. Similar to [43, Theorem 3.52], we know that, for all $p \in (0,\,1]$, $\Lambda^{n(\frac{1}{p}-1)}_{L^*+\delta}(\mathbb{R}^n) = (H^p_{L+\delta}(\mathbb{R}^n))^*$, which, combined with Theorem 4.8, implies that, for all $\alpha \in [0,\,\infty)$, $\Lambda^\alpha_{L^*,\,\mathrm{loc}}(\mathbb{R}^n) = \Lambda^\alpha_{L^*+\delta}(\mathbb{R}^n)$.

6. The Maximal Function Characterization of $H^p_{L+\delta}(\mathbb{R}^n)$

In this section, letting L be the inhomogeneous higher order elliptic operator as in (1.1), we study the maximal function characterization of the Hardy space $H^p_{L+\delta}(\mathbb{R}^n)$. To this end, we need the following ellipticity condition.

Ellipticity condition $(\widetilde{\mathcal{E}_1})$**.** Let $\mathfrak{a}_0$ be the leading part of the sesquilinear form of L as in (2.2). Then there exists a positive constant λ_1 such that, for all $\xi := \{\xi_\alpha\}_{|\alpha|=m}$ with $\xi_\alpha \in \mathbb{C}$ and almost every $x \in \mathbb{R}^n$,

$$\Re e\left\{\sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta}(x)\xi_\alpha\overline{\xi_\beta}\right\} \geq \lambda_1|\xi|^2 = \lambda_1\left\{\sum_{|\alpha|=m}|\xi_\alpha|^2\right\}. \tag{6.1}$$

Moreover, the coefficients $\{a_{\alpha,\,\beta}\}_{0\leq|\alpha|,\,|\beta|\leq m}$ are bounded.

It is easy to see that, if L also satisfies the Ellipticity condition $(\widetilde{\mathcal{E}_1})$, then L also satisfies the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. Moreover, if the coefficients $\{a_{\alpha,\,\beta}\}_{|\alpha|=m=|\beta|}$ are bounded, then the equivalence between (6.1) and the Ellipticity condition $(\mathcal{E}_0)$ is a specific feature of second-order operators (see, for example, [8, p. 15]). For more relationships on these two kinds of ellipticity conditions, we refer the reader to [4, p. 365].

Let $\delta \in (0,\,\infty)$ be as in Definition 4.1. We present the nontangential maximal function characterization of $H^p_{L+\delta}(\mathbb{R}^n)$ for $p \in (0,\,1]$. Recall that, in [15], for the homogeneous operator as in (1.6), the authors established the nontangential maximal function characterization of the associated Hardy space. We point out that the basic idea of our method here follows that of [15]. But we need to make some modifications because of the perturbation of the lower order terms of $L+\delta$.

As in the case of homogeneous operators, we need to establish two parabolic Caccioppoli's inequalities for L as follows.

Lemma 6.1. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\widetilde{\mathcal{E}_1})$, $t \in (0,\,\infty)$, $f \in L^2(\mathbb{R}^n)$ *and* $u(x,\,t) := e^{-t^{2m}L}(f)(x)$ *for all* $x \in \mathbb{R}^n$. *Then there exists a positive constant C, independent of f, such that, for all* $x_0 \in \mathbb{R}^n$, $r \in (0,\,\infty)$ *and* $t_0 \in (3r,\,\infty)$,

$$\begin{aligned}
&\int_{t_0-r}^{t_0+r}\int_{B(x_0,\,r)}|\nabla^m u(x,\,t)|^2\,dxdt\\
&\quad\leq C\left\{\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)}|u(x,\,t)|^2\,dxdt\right.\\
&\qquad\left.+\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)}|\nabla^{m-1}u(x,\,t)|^2\,dxdt\right\}\\
&\qquad+\sum_{k=0}^{m}\sum_{l=0}^{k-1}\frac{C}{r^{2(k-l)}}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)}|\nabla^l u(x,\,t)|^2\,dxdt.
\end{aligned}\tag{6.2}$$

Moreover, for any $\epsilon \in (0, \infty)$, there exists a positive constant $C_{(\epsilon)}$, depending on ϵ, but independent of f, such that

$$\int_{t_0-r}^{t_0+r}\int_{B(x_0,\,r)} |\nabla^m u(x,\,t)|^2\,dxdt$$
$$\leq \epsilon \int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)} |\nabla^m u(x,\,t)|^2\,dxdt$$
$$+ C_{(\epsilon)} \sum_{k=0}^{m} \frac{1}{r^{2k}} \int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)} |u(x,\,t)|^2\,dxdt. \tag{6.3}$$

Proof. Let $\eta \in C_c^\infty(B(x_0,\,2r))$ satisfy $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B(x_0,\,r)$ and, for all $k \in \{0, \ldots, m\}$,

$$\|\nabla^k \eta\|_{L^\infty(\mathbb{R}^n)} \lesssim r^{-k}.$$

Let $\gamma \in C_c^\infty(t_0 - 2r,\, t_0 + 2r)$ satisfy $0 \leq \gamma \leq 1$, $\gamma \equiv 1$ on $(t_0 - r,\, t_0 + r)$ and $\|\partial_t \gamma\|_{L^\infty(\mathbb{R})} \lesssim \frac{1}{r}$. By properties of η, we first write

$$\int_{t_0-r}^{t_0+r}\left[\int_{B(x_0,\,r)} |\nabla^m u(x,\,t)|^2\,dx\right] dt$$
$$\lesssim \int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)} |\nabla^m u(x,\,t)[\eta(x)]^m|^2\,dx\right]\gamma(t)\,dt =: \mathcal{A}. \tag{6.4}$$

To bound $\mathcal{A}$, by the Ellipticity condition $(\widetilde{\mathcal{E}}_1)$ and properties of γ, we know that

$$\mathcal{A} \lesssim \Re e\left(\int_{t_0-2r}^{t_0+2r} \sum_{|\alpha|=m=|\beta|}\left\{\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t)\overline{\partial^\alpha u(x,\,t)}[\eta(x)]^{2m}\,dx\right\}\right.$$
$$\left.\times\,\gamma(t)\,dt\right)$$
$$\lesssim \left|\Re e\left(\int_{t_0-2r}^{t_0+2r} \sum_{0\leq|\alpha|,\,|\beta|\leq m}\left\{\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t)\overline{\partial^\alpha u(x,\,t)}[\eta(x)]^{2m}\,dx\right\}\right.\right.$$
$$\left.\left.\times\,\gamma(t)\,dt\right)\right|$$
$$+\left|\Re e\left(\int_{t_0-2r}^{t_0+2r} \sum_{0\leq|\alpha|,\,|\beta|\leq m,\,|\alpha|+|\beta|<2m}\left\{\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t)\overline{\partial^\alpha u(x,\,t)}\right.\right.\right.$$
$$\left.\left.\left.\times\,[\eta(x)]^{2m}\,dx\right\}\gamma(t)\,dt\right)\right| =: \mathrm{I} + \mathrm{II}. \tag{6.5}$$

We first bound II, by writing

$$\begin{aligned}
\mathrm{II} \leq &\left[\sum_{0\le|\beta|<m,\,|\alpha|=m} + \sum_{0\le|\alpha|<m,\,|\beta|=m} + \sum_{0\le|\alpha|<m,\,0\le|\beta|<m}\right] \\
&\times \left|\Re e\left(\int_{t_0-2r}^{t_0+2r}\left\{\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t)\overline{\partial^\alpha u(x,\,t)}[\eta(x)]^{2m}\,dx\right\}\gamma(t)\,dt\right)\right| \\
=: &\ \mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3. \qquad (6.6)
\end{aligned}$$

By Cauchy's inequality with ϵ, the Ellipticity condition $(\widetilde{\mathcal{E}_1})$ and the interpolation inequality (2.4), we know that, for all $\epsilon \in (0, \infty)$, there exists a positive constant $C_{(\epsilon)}$ such that

$$\begin{aligned}
\mathrm{II}_1 &\lesssim \epsilon\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|\nabla^m u(x,\,t)|^2[\eta(x)]^{2m}\,dx\right]\gamma(t)\,dt \\
&\quad + C_{(\epsilon)}\left\{\sum_{k=0}^{m-1}\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|\nabla^k u(x,\,t)|^2\,dx\right]dt\right\} \\
&\lesssim \epsilon\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|\nabla^m u(x,\,t)|^2[\eta(x)]^{2m}\,dx\right]\gamma(t)\,dt \\
&\quad + C_{(\epsilon)}\left\{\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|u(x,\,t)|^2\,dx\right]dt\right. \\
&\quad \left.+\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|\nabla^{m-1}u(x,\,t)|^2\,dx\right]dt\right\}. \qquad (6.7)
\end{aligned}$$

By symmetry, we know that the estimate of II_2 is similar to that of II_1. Also, using an argument similar to that used in the estimate for (6.7), we obtain

$$\begin{aligned}
\mathrm{II}_3 &\lesssim \int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|u(x,\,t)|^2\,dx\right]dt \\
&\quad + \int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|\nabla^{m-1}u(x,\,t)|^2\,dx\right]dt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathrm{II} &\lesssim \epsilon\mathcal{A}+C_{(\epsilon)}\left\{\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|u(x,\,t)|^2\,dx\right]dt\right. \\
&\quad \left.+\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}|\nabla^{m-1}u(x,\,t)|^2\,dx\right]dt\right\}. \qquad (6.8)
\end{aligned}$$

For I, we first write

$$\begin{aligned}\mathrm{I} \lesssim & \left| \Re e \left\{ \int_{t_0-2r}^{t_0+2r} \sum_{0\le|\alpha|,\,|\beta|\le m} \left[\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t) \right.\right.\right.\\ & \left.\left.\left. \times \left(\overline{\partial^\alpha u(x,\,t)}\,[\eta(x)]^{2m} - \overline{\partial^\alpha (u\eta^{2m})(x,\,t)} \right) dx \right] \gamma(t)\,dt \right\} \right| \\ & + \left| \Re e \left\{ \int_{t_0-2r}^{t_0+2r} \sum_{0\le|\alpha|,\,|\beta|\le m} \left[\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t)\overline{\partial^\alpha (u\eta^{2m})(x,\,t)}\,dx \right] \right.\right.\\ & \left.\left. \times\,\gamma(t)\,dt \right\} \right| =: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

The estimate of I_2 is similar to that of (3.6) in the proof of [15, Proposition 3.1] (using the fact that $\partial_t e^{-tL} = -Le^{-tL}$, integration by parts, and the definition of the cut-off function γ) and we have

$$\mathrm{I}_2 \lesssim \frac{1}{r^{2m}} \int_{t_0-2r}^{t_0+2r} \int_{B(x_0,\,2r)} |u(x,\,t)|^2\,dxdt. \tag{6.9}$$

We now bound I_1. Using the definition of η, Leibniz's rule, Cauchy's inequality with ϵ, the interpolation inequality (2.4) and the fact that $\alpha,\,\beta \in \{0,\ldots,\,m\}$, we conclude that, for all $\epsilon \in (0,\,\infty)$, there exist positive constants $C_{(\epsilon)}$ and $C_{(\alpha,\,\widetilde{\alpha})}$ such that

$$\begin{aligned}\mathrm{I}_1 \lesssim & \left| \Re e \left\{ \int_{t_0-2r}^{t_0+2r} \sum_{0\le|\alpha|,\,|\beta|\le m} \left[\int_{B(x_0,\,2r)} a_{\alpha,\,\beta}(x)\partial^\beta u(x,\,t) \right.\right.\right.\\ & \left.\left.\left. \times \left\{ \sum_{\theta\le\widetilde{\alpha}<\alpha} C_{(\alpha,\,\widetilde{\alpha})}\partial^{\widetilde{\alpha}} u(x,\,t)\partial^{\alpha-\widetilde{\alpha}}(\eta^{2m})(x) \right\} dx \right] \gamma(t)\,dt \right\} \right| \\ \lesssim & \,\epsilon\mathcal{A} + C_{(\epsilon)} \left\{ \int_{t_0-2r}^{t_0+2r} \left[\int_{B(x_0,\,2r)} |u(x,\,t)|^2\,dx \right] dt \right.\\ & \left. + \int_{t_0-2r}^{t_0+2r} \left[\int_{B(x_0,\,2r)} |\nabla^{m-1} u(x,\,t)|^2\,dx \right] dt \right\} \\ & + \sum_{k=0}^{m}\sum_{l=0}^{k-1} \frac{C_{(\epsilon)}}{r^{2(k-l)}} \int_{t_0-2r}^{t_0+2r} \int_{B(x_0,\,2r)} |\nabla^l u(x,\,t)|^2\,dxdt,\end{aligned}$$

where $\theta := (0,\ldots,0) \in (\mathbb{Z}_+)^n$, $\theta \le \widetilde{\alpha}$ means that each component of $\widetilde{\alpha}$ is larger than 0, and $\widetilde{\alpha} < \alpha$ means that each component of $\widetilde{\alpha}$ is not larger than the corresponding component of α and $|\widetilde{\alpha}| < |\alpha|$. This, combined with (6.4) through (6.9), implies that (6.2) holds true.

The inequality (6.3) follows immediately from (6.2), the interpolation inequality (2.4) and Young's inequality with ϵ, the details being omitted. This finishes the proof of Lemma 6.1. $\square$

The following lemma improves the parabolic Caccioppoli's inequality (6.2) in Lemma 6.1, by removing all the gradient terms on the right-hand side, based on an idea of induction from Barton [10], where the author established some interesting elliptic Caccioppoli's inequalities for higher order divergence form elliptic systems.

Lemma 6.2. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\widetilde{\mathcal{E}_1}), t \in (0, \infty), f \in L^2(\mathbb{R}^n)$ *and* $u(x, t) := e^{-t^{2m}L}(f)(x)$ *for all* $x \in \mathbb{R}^n$. *Then there exists a positive constant C, independent of f, such that, for all* $x_0 \in \mathbb{R}^n, r \in (0, \infty)$ *and* $t_0 \in (3r, \infty)$,

$$\int_{t_0-r}^{t_0+r} \int_{B(x_0, r)} |\nabla^m u(x, t)|^2 \, dxdt \leq \sum_{k=0}^{m} \frac{C}{r^{2k}} \int_{t_0-2r}^{t_0+2r} \int_{B(x_0, 2r)} |u(x, t)|^2 \, dxdt. \tag{6.10}$$

Proof. To prove Lemma 6.2, we borrow some ideas from the proof of [10, Theorem 3.10]. First, we make the following *claim* that, to finish the proof of Lemma 6.2, it suffices to show that there exists a positive constant C, independent of f, such that, for all $j \in \{1, \ldots, m\}$ and $0 < \zeta < \xi \leq 2r$,

$$\begin{aligned}\int_{t_0-\zeta}^{t_0+\zeta} \int_{B(x_0, \zeta)} |\nabla^j u(x, t)|^2 \, dxdt &\leq C \left\{ \int_{t_0-\xi}^{t_0+\xi} \int_{B(x_0, \xi)} |u(x, t)|^2 \, dxdt + \int_{t_0-\xi}^{t_0+\xi} \int_{B(x_0, \xi)} |\nabla^{j-1} u(x, t)|^2 \, dxdt \right\} \\ &\quad + \sum_{k=0}^{j} \sum_{l=0}^{k-1} \frac{C}{(\xi-\zeta)^{2(k-l)}} \int_{t_0-\xi}^{t_0+\xi} \int_{B(x_0, \xi)} |\nabla^l u(x, t)|^2 \, dxdt.\end{aligned} \tag{6.11}$$

Indeed, if (6.11) holds true for all $j \in \{1, \ldots, m\}$ and $0 < \zeta < \xi \leq 2r$, then let

$$r = r_0 < r_1 < r_2 < \cdots < r_m = 2r$$

be an average decomposition of $(r, 2r)$ and

$$A_{s, j} := \int_{t_0-s}^{t_0+s} \int_{B(x_0, s)} |\nabla^j u(x, t)|^2 \, dxdt \tag{6.12}$$

with $j \in \{0, \ldots, m\}$ and $s \in (0, \infty)$. By repetitively using (6.11) with $j = m, m-1, \ldots, 1$, we know that

$$\int_{t_0-r}^{t_0+r} \int_{B(x_0, r)} |\nabla^m u(x, t)|^2 \, dxdt = A_{r_0, m} \lesssim A_{r_1, 0} + A_{r_1, m-1} + \sum_{k=0}^{m} \sum_{l=0}^{k-1} \frac{1}{r^{2(k-l)}} A_{r_1, l}$$

$$\sim A_{r_1,\,0}+\left(1+\frac{1}{r^2}\right)A_{r_1,\,m-1}+\sum_{l=0}^{m-2}\frac{1}{r^{2(m-l)}}A_{r_1,\,l}$$

$$+\sum_{k=0}^{m-1}\sum_{l=0}^{k-1}\frac{1}{r^{2(k-l)}}A_{r_1,\,l}\lesssim\sum_{k=0}^{m}\frac{1}{r^{2k}}A_{2r,\,0},$$

which immediately implies that (6.10) holds true.

We now turn to the proof of (6.11). Recall that, if $j = m$, then (6.11) can be obtained by following the same line of the proof of the parabolic Caccioppoli's inequality (6.2) with r and $2r$, respectively, by ζ and ξ. Thus, by induction, to finish the proof of (6.11), it remains to show that, if (6.11) holds true for some $j+1$, then (6.11) also holds true for j.

Now, for all $i\in\mathbb{Z}_+$, let $\{\rho_i\}_{i\in\mathbb{Z}_+}$ be a sequence of increasing numbers satisfying

$$\zeta=\rho_0<\rho_1<\cdots<\xi,$$

$\delta_i:=\rho_{i+1}-\rho_i$ and $\widetilde{\rho}_i:=\rho_i+\frac{\delta_i}{2}$, where the exact value of ρ_i will be determined later. Let $\varphi_i\in C_c^\infty(B(x_0,\,\widetilde{\rho}_i))$ satisfy $\operatorname{supp}\varphi_i\subset B(x_0,\,\widetilde{\rho}_i)$, $\varphi_i\equiv 1$ on $B(x_0,\,\rho_i)$, $\|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^n)}\lesssim\frac{1}{\delta_i}$ and $\|\nabla^2\varphi_i\|_{L^\infty(\mathbb{R}^n)}\lesssim\frac{1}{\delta_i^2}$. By properties of φ_i and the Fourier transform, and Hölder's inequality, we know that, for all $i\in\mathbb{Z}_+$,

$$\begin{aligned}
&\int_{t_0-\rho_i}^{t_0+\rho_i}\int_{B(x_0,\,\rho_i)}|\nabla^j u(x,\,t)|^2\,dxdt\\
&\quad\le\int_{t_0-\widetilde{\rho}_i}^{t_0+\widetilde{\rho}_i}\int_{B(x_0,\,\widetilde{\rho}_i)}|\nabla(\varphi_i\nabla^{j-1}u)(x,\,t)|^2\,dxdt\\
&\quad\lesssim\left\{\int_{t_0-\widetilde{\rho}_i}^{t_0+\widetilde{\rho}_i}\int_{B(x_0,\,\widetilde{\rho}_i)}|\nabla^{j-1}u(x,\,t)|^2\,dxdt\right\}^{\frac12}\\
&\qquad\times\left\{\int_{t_0-\widetilde{\rho}_i}^{t_0+\widetilde{\rho}_i}\int_{B(x_0,\,\widetilde{\rho}_i)}|\nabla^2(\varphi_i\nabla^{j-1}u)(x,\,t)|^2\,dxdt\right\}^{\frac12}\\
&\quad\lesssim\left\{\int_{t_0-\widetilde{\rho}_i}^{t_0+\widetilde{\rho}_i}\int_{B(x_0,\,\widetilde{\rho}_i)}|\nabla^{j-1}u(x,\,t)|^2\,dxdt\right\}^{\frac12}\\
&\qquad\times\left\{\int_{t_0-\widetilde{\rho}_i}^{t_0+\widetilde{\rho}_i}\int_{B(x_0,\,\widetilde{\rho}_i)}\left|\left(\nabla^{j+1}u+\frac{1}{\delta_i}\nabla^j u+\frac{1}{\delta_i^2}\nabla^{j-1}u\right)(x,\,t)\right|^2\,dxdt\right\}^{\frac12}\\
&\quad\lesssim[A_{\widetilde{\rho}_i,\,j-1}]^{1/2}\left[A_{\widetilde{\rho}_i,\,j+1}+\frac{1}{\delta_i^2}A_{\widetilde{\rho}_i,\,j}+\frac{1}{\delta_i^4}A_{\widetilde{\rho}_i,\,j-1}\right]^{1/2},
\end{aligned}$$

where $A_{\widetilde{\rho}_i,\,j+1}$, $A_{\widetilde{\rho}_i,\,j}$ and $A_{\widetilde{\rho}_i,\,j-1}$ are defined as in (6.12).

From the assumption that (6.11) holds true for $j+1$ and Cauchy's inequality with ϵ, we further conclude that, for any $\epsilon\in(0,\,\infty)$, there exists a positive constant

$C_{(\epsilon)}$ such that

$$\begin{aligned}
&\int_{t_0-\rho_i}^{t_0+\rho_i}\int_{B(x_0,\,\rho_i)} |\nabla^j u(x,\,t)|^2\,dxdt \\
&\quad\lesssim [A_{\rho_{i+1},\,j-1}]^{\frac{1}{2}}\Bigg[A_{\rho_{i+1},\,0}+A_{\rho_{i+1},\,j}+\sum_{k=0}^{j+1}\sum_{l=0}^{k-1}\frac{1}{\delta_i^{2(k-l)}}A_{\rho_{i+1},\,l} \\
&\qquad+\frac{1}{\delta_i^2}A_{\rho_{i+1},\,j}+\frac{1}{\delta_i^4}A_{\rho_{i+1},\,j-1}\Bigg]^{\frac{1}{2}} \\
&\quad\sim\ [A_{\rho_{i+1},\,j-1}]^{\frac{1}{2}}\ \Bigg[A_{\rho_{i+1},\,0}+A_{\rho_{i+1},\,j}+\sum_{k=0}^{j}\sum_{l=0}^{k-1}\frac{1}{\delta_i^{2(k-l)}}A_{\rho_{i+1},\,l} \\
&\qquad+\sum_{l=0}^{j}\frac{1}{\delta_i^{2(j+1-l)}}A_{\rho_{i+1},\,l}+\frac{1}{\delta_i^2}A_{\rho_{i+1},\,j}+\frac{1}{\delta_i^4}A_{\rho_{i+1},\,j-1}\Bigg]^{\frac{1}{2}} \\
&\quad\lesssim\ C_{(\epsilon)}A_{\rho_{i+1},\,j-1}+\epsilon A_{\rho_{i+1},\,0}+\epsilon A_{\rho_{i+1},\,j}+\epsilon\sum_{k=0}^{j}\sum_{l=0}^{k-1}\frac{1}{\delta_i^{2(k-l)}}A_{\rho_{i+1},\,l} \\
&\qquad+\frac{C_{(\epsilon)}}{\delta_i^2}A_{\rho_{i+1},\,j-1}+\epsilon\sum_{l=0}^{j}\frac{1}{\delta_i^{2(j-l)}}A_{\rho_{i+1},\,l}+\epsilon A_{\rho_{i+1},\,j}+\frac{\epsilon}{\delta_i^2}A_{\rho_{i+1},\,j-1} \\
&\quad\lesssim C_{(\epsilon)}A_{\rho_{i+1},\,j-1}+\left[\frac{C_{(\epsilon)}}{\delta_i^2}+\epsilon\right]A_{\rho_{i+1},\,j-1}+\epsilon A_{\rho_{i+1},\,0} \\
&\qquad+\epsilon\sum_{k=0}^{j}\sum_{l=0}^{k-1}\frac{1}{\delta_i^{2(k-l)}}A_{\rho_{i+1},\,l}+3\epsilon A_{\rho_{i+1},\,j}.
\end{aligned}$$

By letting ϵ be small enough, we find that there exists a positive constant $\widetilde{C}$ such that, for all $i\in\mathbb{Z}_+$,

$$\begin{aligned}
&\int_{t_0-\rho_i}^{t_0+\rho_i}\int_{B(x_0,\,\rho_i)} |\nabla^j u(x,\,t)|^2\,dxdt \\
&\quad\le\widetilde{C}\left[A_{\rho_{i+1},\,j-1}+A_{\rho_{i+1},\,0}+\sum_{k=0}^{j}\sum_{l=0}^{k-1}\frac{1}{\delta_i^{2(k-l)}}A_{\rho_{i+1},\,l}\right]+\frac{1}{2}A_{\rho_{i+1},\,j} \\
&\quad=:\widetilde{C}B_{\rho_{i+1},\,j}+\frac{1}{2}A_{\rho_{i+1},\,j},
\end{aligned}\tag{6.13}$$

which immediately implies that

$$\begin{aligned}
A_{\rho_0,\,j}&\le\widetilde{C}B_{\rho_1,\,j}+\frac{1}{2}A_{\rho_{1,\,j}}\le\widetilde{C}B_{\rho_1,\,j}+\frac{1}{2}\left[\widetilde{C}B_{\rho_2,\,j}+\frac{1}{2}A_{\rho_{2,\,j}}\right] \\
&\le\widetilde{C}\sum_{i=1}^{\infty}2^{1-i}B_{\rho_i,\,j}.
\end{aligned}\tag{6.14}$$

By the fact $\zeta = \rho_0 < \rho_1 < \cdots < \xi$, we see that

$$\sum_{i=1}^{\infty} 2^{1-i} \widetilde{C} A_{\rho_{i+1},\, j-1} \lesssim A_{\xi,\, j-1} \sum_{i=1}^{\infty} 2^{1-i} \lesssim A_{\xi,\, j-1}. \tag{6.15}$$

Similarly, we have

$$\sum_{i=1}^{\infty} 2^{1-i} \widetilde{C} A_{\rho_{i+1},\, 0} \lesssim A_{\xi,\, 0} \sum_{i=1}^{\infty} 2^{1-i} \lesssim A_{\xi,\, 0}. \tag{6.16}$$

Moreover, for all $i \in \mathbb{Z}_+$, take $\tau \in (2^{-1/(2m)}, 1)$, $\rho_i := \zeta + (\xi - \zeta)(1-\tau) \sum_{s=1}^{i} \tau^s$ and hence $\delta_i = (\xi - \zeta)(1-\tau)\tau^{i+1}$, we then have

$$\begin{aligned}
&\sum_{i=1}^{\infty} 2^{1-i} \widetilde{C} \sum_{k=0}^{j} \sum_{l=0}^{k-1} \frac{1}{\delta_i^{2(k-l)}} A_{\rho_{i+1},\, l} \\
&\quad \lesssim \sum_{i=1}^{\infty} 2^{1-i} \widetilde{C} \sum_{k=0}^{j} \sum_{l=0}^{k-1} \frac{1}{[(\xi-\zeta)(1-\tau)\tau^i]^{2(k-l)}} A_{\rho_{i+1},\, l} \\
&\quad \lesssim \sum_{k=0}^{j} \sum_{l=0}^{k-1} \sum_{i=1}^{\infty} \frac{1}{[2\tau^{2(k-l)}]^i} A_{\rho_{i+1},\, l} \\
&\quad \lesssim \sum_{k=0}^{j} \sum_{l=0}^{k-1} \frac{1}{(\xi-\zeta)^{2(k-l)}} A_{\xi,\, l},
\end{aligned} \tag{6.17}$$

where the implicit positive constants depend on m and τ.

Combining the estimates (6.14) through (6.17) and the definition of $B_{i,\, j}$ in (6.13), we conclude that

$$\begin{aligned}
\int_{t_0-\zeta}^{t_0+\zeta} \int_{B(x_0,\, \zeta)} |\nabla^j u(x,\, t)|^2 \, dx dt = A_{\rho_0,\, j} &\lesssim \sum_{i=1}^{\infty} 2^{1-i} B_{i,\, j} \\
&\lesssim A_{\xi,\, j-1} + A_{\xi,\, 0} + \sum_{k=0}^{j} \sum_{l=0}^{k-1} \frac{1}{(\xi-\zeta)^{2(k-l)}} A_{\xi,\, l},
\end{aligned}$$

which immediately implies that (6.11) holds true for j. Thus, by induction, we know that (6.11) holds true for $j \in \{1, \ldots, m\}$, which completes the proof of Lemma 6.2. $\square$

Now, let δ be as in Definition 4.1 and $\widetilde{\delta} \in [\delta, \infty)$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $S_{h,\, L+\widetilde{\delta}}(f)$ be the *Lusin-area function*, associated with the heat semigroup, of f, defined by setting

$$S_{h,\, L+\widetilde{\delta}}(f)(x) := \left\{ \iint_{\Gamma(x)} \sum_{k=0}^{m} |t^m \nabla^k (e^{-t^{2m}[L+\widetilde{\delta}]}(f))(y)|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2}, \tag{6.18}$$

where $\Gamma(x)$ for all $x \in \mathbb{R}^n$ is as in (1.2). Here, to distinguish with square functions of the form (1.5), we use the terminology *Lusin-area function* to denote square functions with gradients as in (6.18).

Proposition 6.3. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\widetilde{\mathcal{E}}_1)$, δ *be as in Definition* 4.1 *and* $p \in (0, 1]$. *Then, for all $\delta \le \delta_1 < \delta_2 < \infty$, there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\|S_{L+\delta_2}(f)\|_{L^p(\mathbb{R}^n)} \le C\|S_{h,\,L+\delta_1}(f)\|_{L^p(\mathbb{R}^n)}.$$

Proof. To prove Proposition 6.3, we use the same notation as in the proof of [15, Proposition 3.3] (see also [42] for similar notation). For all $0 < \epsilon \ll R < \infty$, $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^n$, the *truncated cone* $\Gamma^{\epsilon,\,R,\,\lambda}(x)$ is defined by setting

$$\Gamma^{\epsilon,\,R,\,\lambda}(x) := \{(y,\,t) \in \mathbb{R}^n \times (\epsilon,\,R) : |x-y| < \lambda t\}. \tag{6.19}$$

Let $\eta \in C_0^\infty(\Gamma^{\epsilon/2,\,2R,\,3/2}(x))$ satisfy $\eta \equiv 1$ on $\Gamma^{\epsilon,\,R,\,1}(x)$, $0 \le \eta \le 1$ and, for all $k \in \mathbb{N}$ with $k \le m$ and $(y,\,t) \in \Gamma^{\epsilon/2,\,2R,\,3/2}(x)$,

$$|\nabla^k \eta(y,\,t)| \lesssim \frac{1}{t^k}.$$

Let $L + \delta_2 =: \sum_{0\le|\alpha|,\,|\beta|\le m}(-1)^{|\alpha|}\partial^\alpha(a_{\alpha,\,\beta,\,\delta_2}\partial^\beta)$. By the definition of $L+\delta_2$, Leibniz's rule and Hölder's inequality, we first have

$$\begin{aligned}
&\left\{\iint_{\Gamma^{\epsilon,\,R,\,1}(x)} |t^{2m}(L+\delta_2)e^{-t^{2m}(L+\delta_2)}(f)(y)|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&\quad\lesssim \left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |t^{2m}(L+\delta_2)e^{-t^{2m}(L+\delta_2)}(f)(y)|^2\eta(y,\,t)\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&\quad\sim \sum_{0\le|\alpha|,\,|\beta|\le m}\left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} t^{2m}a_{\alpha,\,\beta,\,\delta_2}(y)\partial^\beta(e^{-t^{2m}[L+\delta_2]}(f))(y)\right.\\
&\qquad\left.\times\,\overline{\partial^\alpha(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f)\eta)(y,\,t)}\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&\quad\lesssim \sum_{0\le|\alpha|,\,|\beta|\le m}\left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} t^{2m}a_{\alpha,\,\beta,\,\delta_2}(y)\partial^\beta(e^{-t^{2m}[L+\delta_2]}(f))(y)\right.\\
&\qquad\left.\times\,\overline{\left[\sum_{\theta\le\widetilde{\alpha}\le\alpha}C_{(\alpha,\,\widetilde{\alpha})}\partial^{\widetilde{\alpha}}(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(x)\partial^{\alpha-\widetilde{\alpha}}\eta(y,\,t)\right]}\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&\quad\lesssim \sum_{k=0}^m\sum_{\widetilde{k}=0}^m\sum_{l=0}^{\widetilde{k}}\left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |t^m\nabla^k(e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/4}
\end{aligned}$$

$$\times \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |t^{m-\widetilde{k}+l}\nabla^l (t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2 \,\frac{dy\,dt}{t^{n+1}} \right\}^{1/4}$$

$$=: \sum_{k=0}^{m}\sum_{\widetilde{k}=0}^{m}\sum_{l=0}^{\widetilde{k}} \mathrm{I}_k \times \mathrm{J}_{\widetilde{k},\,l}, \tag{6.20}$$

where $\theta := (0, \ldots, 0) \in \mathbb{N}^n$ and, for all multi-indices α and $\widetilde{\alpha}$, $C_{(\alpha,\,\widetilde{\alpha})}$ is a positive constant depending only on α and $\widetilde{\alpha}$.

We first bound I_k. By the fact that $e^{-t^{2m}(\delta_2-\delta_1)} \le 1$, it is easy to see that, for all $k \in \{0, \ldots, m\}$,

$$\mathrm{I}_k \lesssim \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} \sum_{k=0}^{m} |t^m \nabla^k (e^{-t^{2m}(L+\delta_1)}(f))(y)|^2 \,\frac{dy\,dt}{t^{n+1}} \right\}^{1/4}. \tag{6.21}$$

To bound $\mathrm{J}_{\widetilde{k},\,l}$, for $l = 0$, by the fact that, for all $j \in \mathbb{Z}_+$,

$$t^j e^{-t^{2m}(\delta_2-\delta_1)} \lesssim 1, \tag{6.22}$$

we conclude that

$$\begin{aligned}
\mathrm{J}_{\widetilde{k},\,0} &\lesssim \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |t^{m-\widetilde{k}}(t^{2m}[L+\delta_1]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2 \,\frac{dy\,dt}{t^{n+1}} \right\}^{1/4} \\
&\quad + \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |t^{m-\widetilde{k}}(t^{2m}[\delta_2-\delta_1]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2 \,\frac{dy\,dt}{t^{n+1}} \right\}^{1/4} \\
&\lesssim \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |t^{2m}(L+\delta_1)e^{-t^{2m}(L_2+\delta_1)}(f)(y)|^2 \,\frac{dy\,dt}{t^{n+1}} \right\}^{1/4} \\
&\quad + \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,\frac{3}{2}}(x)} |e^{-t^{2m}(L+\delta_1)}(f)(y)|^2 \,\frac{dy\,dt}{t^{n+1}} \right\}^{1/4}.
\end{aligned} \tag{6.23}$$

For $\widetilde{k} = m = l$, let $Q(z, 2r)$ be the cube with center z and sidelength $2r$ in $\mathbb{R}^{n+1}_+$. Write $z := (z^*, t)$ with $z^* \in \mathbb{R}^n$ and $t \in (0, \infty)$. Assume that $\{Q(z_j, 2r_j)\}_j$ is a covering of $\Gamma^{\epsilon/2,\,2R,\,3/2}(x)$ satisfying

$$\Gamma^{\epsilon/2,\,2R,\,3/2}(x) \subset \bigcup_{j\in\mathbb{N}} Q(z_j, 2r_j) \subset \bigcup_{j\in\mathbb{N}} Q(z_j, 4\sqrt{n}r_j) \subset \Gamma^{\epsilon/4,\,3R,\,2}(x),$$

$$d(z_j, (\Gamma^{\epsilon/4,\,3R,\,2}(x))^{\complement}) \sim r_j \sim d(z_j, \{t=0\})$$

and the collection $\{B(z_j^*, \sqrt{n}r_j) \times (t_j - \sqrt{n}r_j, t_j + \sqrt{n}r_j)\}_j$ has a bounded overlap, where, for all $j \in \mathbb{N}$, $z_j := (z_j^*, t_j)$.

Moreover, we know that

$$\begin{aligned}\Gamma^{\epsilon/2,\,2R,\,3/2}(x) &\subset \bigcup_{j\in\mathbb{N}} Q(z_j,\,2r_j)\\ &\subset \bigcup_{j\in\mathbb{N}} B(z_j^*,\,\sqrt{n}r_j)\times(t_j-\sqrt{n}r_j,\,t_j+\sqrt{n}r_j)\\ &\subset \bigcup_{j\in\mathbb{N}} B(z_j^*,\,2\sqrt{n}r_j)\times(t_j-2\sqrt{n}r_j,\,t_j+2\sqrt{n}r_j)\\ &\subset \bigcup_{j\in\mathbb{N}} Q(z_j,\,4\sqrt{n}r_j)\\ &\subset \Gamma^{\epsilon/4,\,3R,\,2}(x).\end{aligned}$$

Thus, by the parabolic Caccioppoli's inequality (6.3) and (6.22), we obtain

$$\begin{aligned}(\mathrm{J}_{m,\,m})^4 &\lesssim \sum_j \int_{t_j-\sqrt{n}r_j}^{t_j+\sqrt{n}r_j}\int_{B(z_j^*,\,\sqrt{n}r_j)} |t^m\nabla^m\\ &\quad\times(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\lesssim \epsilon\sum_j \int_{t_j-2\sqrt{n}r_j}^{t_j+2\sqrt{n}r_j}\int_{B(z_j^*,\,2\sqrt{n}r_j)} |t^m\nabla^m\\ &\quad\times(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\quad+C_{(\epsilon)}\sum_j\sum_{k=0}^m \frac{1}{r_j^{2k}}\int_{t_j-2\sqrt{n}r_j}^{t_j+2\sqrt{n}r_j}\int_{B(z_j^*,\,2\sqrt{n}r_j)}\\ &\quad\times|t^{3m}(L+\delta_2)e^{-t^{2m}[L+\delta_2]}(f)(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\lesssim \epsilon\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)} |t^m\nabla^m(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\quad+C_{(\epsilon)}\sum_{k=0}^m\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)} |t^{m-k}t^{2m}(L+\delta_2)e^{-t^{2m}(L+\delta_2)}(f)(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\lesssim \epsilon\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)} |t^m\nabla^m(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\quad+C_{(\epsilon)}\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)} |t^{2m}(L+\delta_1)e^{-t^{2m}(L+\delta_1)}(f)(y)|^2\frac{dy\,dt}{t^{n+1}}\\ &\quad+C_{(\epsilon)}\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)} |e^{-t^{2m}(L+\delta_1)}(f)(y)|^2\frac{dy\,dt}{t^{n+1}}.\end{aligned}\tag{6.24}$$

For general $\widetilde{k}$ and l, by the interpolation inequality (2.4), we see that

$$\begin{aligned}(\mathrm{J}_{\widetilde{k},\,l})^4 &\lesssim \int_{\epsilon/2}^{2R}\int_{B(x,\,\frac{3}{2}t)} |t^{m-\widetilde{k}+l}\nabla^l(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\,\frac{dy\,dt}{t^{n+1}}\\ &\lesssim \int_{\epsilon/2}^{2R}\int_{B(x,\,\frac{3}{2}t)} |t^m\nabla^m(t^{m-\widetilde{k}}t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\,\frac{dy\,dt}{t^{n+1}}\\ &\quad+\int_{\epsilon/2}^{2R}\int_{B(x,\,\frac{3}{2}t)} |t^{m-\widetilde{k}}(t^{2m}[L+\delta_2]e^{-t^{2m}[L+\delta_2]}(f))(y)|^2\,\frac{dy\,dt}{t^{n+1}},\end{aligned}$$

which, together with (6.20) through (6.24) and an argument similar to the corresponding part in the proof of [15, Proposition 3.3], completes the proof of Proposition 6.3. □

Now, we control the $L^p(\mathbb{R}^n)$ quasi-norm of $S_{h,\,L+\delta_1}(f)$ by that of the associated nontangential maximal function.

Proposition 6.4. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\widetilde{\mathcal{E}}_1)$, $p\in(0,\,1]$, $\delta\in(0,\,\infty)$ *be as in Definition* 4.1. *Then there exist* $\delta_1\in(\delta,\,\infty)$, *and positive constants γ and C such that, for all $f\in L^2(\mathbb{R}^n)$,*

$$\|S_{h,\,L+\delta_1}(f)\|_{L^p(\mathbb{R}^n)}\le C\|\mathcal{N}^{\gamma}_{h,\,L+\delta}(f)\|_{L^p(\mathbb{R}^n)},$$

where, for all $\gamma\in(0,\,\infty)$, $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, the non-tangential maximal function $\mathcal{N}^{\gamma}_{h,\,L+\delta}(f)$ is defined by setting

$$\mathcal{N}^{\gamma}_{h,\,L+\delta}(f)(x):=\sup_{(y,t)\in\Gamma_\gamma(x)}\left\{\frac{1}{(\gamma t)^n}\int_{B(y,\,\gamma t)}|e^{-t^{2m}(L+\delta)}(f)(z)|^2\,dz\right\}^{\frac12}\tag{6.25}$$

with

$$\Gamma_\gamma(x):=\{(y,\,t)\in\mathbb{R}^{n+1}_+:|y-x|<\gamma t\}$$

for all $x\in\mathbb{R}^n$.

Proof. To prove Proposition 6.4, we use the same notation as in the proof of [15, Proposition 3.7]. Let $\sigma\in(0,\,\infty)$, $\delta\in(0,\,\infty)$ be as in Definition 4.1 and $f\in L^2(\mathbb{R}^n)$. Assume that $\gamma\in(0,\,\infty)$, whose exact value will be determined later. Let

$$E:=\{x\in\mathbb{R}^n:\mathcal{N}^{\gamma}_{h,\,L+\delta}(f)(x)\le\sigma\}.$$

Its *subset E^* of global* 1/2 *density* is defined by

$$E^*:=\left\{x\in\mathbb{R}^n:\ \text{for all balls } B(x,\,r) \text{ in } \mathbb{R}^n,\quad \frac{|E\cap B(x,\,r)|}{|B(x,\,r)|}\ge\frac12\right\}.$$

For all $0<\epsilon\ll R<\infty$, let $\mathcal{R}^{\epsilon,\,R,\,\gamma}(E^*):=\bigcup_{x\in E^*}\Gamma^{\epsilon,\,R,\,\gamma}(x)$ be the *sawtooth region based on E^**. Moreover, for all $y\in\mathbb{R}^n$ and $t\in(0,\,\infty)$, let $u(y,\,t):=e^{-t^{2m}(L+\delta_1)}(f)(y)$, where $\delta_1\in(\delta,\,\infty)$. Using the interpolation inequality (2.4) and

Fubini's theorem, we conclude that

$$\begin{aligned}
&\int_{E^*} [S^{2\epsilon,\,R,\,1/2}_{h,\,L+\delta_1}(f)(x)]^2\,dx \\
&\quad \lesssim \int_{E^*} \left\{ \iint_{\Gamma^{2\epsilon,\,R,\,1/2}(x)} [|t^m e^{-t^{2m}(L+\delta_1)}(f)(y)|^2 \right. \\
&\qquad \left. + |t^m \nabla^m (e^{-t^{2m}[L+\delta_1]}(f))(y)|^2] \frac{dy\,dt}{t^{n+1}} \right\} dx \\
&\quad \sim \iint_{\mathcal{R}^{2\epsilon,\,R,\,1/2}(E^*)} t^{2m}[|u(y,\,t)|^2 + |\nabla^m u(y,\,t)|^2]\,\frac{dy\,dt}{t} =: \mathcal{A},
\end{aligned}$$

where $S^{2\epsilon,\,R,\,1/2}_{h,\,L+\delta_1}(f)$ is defined as in (6.18) with $\Gamma(x)$ replaced by $\Gamma^{2\epsilon,\,R,\,\frac{1}{2}}(x)$ (see (6.19)).

Before controlling $\mathcal{A}$, we first let $\sum_{0\le|\alpha|,\,|\beta|\le m}(-1)^{|\alpha|}\partial^\alpha(a_{\alpha,\,\beta,\,\delta_1}\partial^\beta) := L+\delta_1$ and write

$$\begin{aligned}
&\Re e\left\{ \iint_{\mathbb{R}^{n+1}_+} \left[t^{2m} \sum_{0\le|\alpha|,\,|\beta|\le m} a_{\alpha,\,\beta,\,\delta_1}(y)\partial^\beta u(y,\,t)\overline{\partial^\alpha u(y,\,t)}\eta(y,\,t) \right] \frac{dy\,dt}{t} \right\} \\
&\quad \ge \Re e\left\{ \iint_{\mathbb{R}^{n+1}_+} t^{2m} \left[\delta_1 u(y,\,t)\overline{u(y,\,t)} + \sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta}\partial^\beta u(y,\,t)\overline{\partial^\alpha u(y,\,t)} \right] \right. \\
&\quad \left. \times\, \eta(y,\,t)\frac{dy\,dt}{t} \right\} - \sum_{0\le|\alpha|,\,|\beta|\le m,\,|\alpha|+|\beta|<2m} \left\{ \iint_{\mathbb{R}^{n+1}_+} |t^{2m} a_{\alpha,\,\beta,\,\delta_1} \right. \\
&\quad \left. \times\,(y)\partial^\beta u(y,\,t)\overline{\partial^\alpha u(y,\,t)}\eta(y,\,t)|\frac{dy\,dt}{t} \right\} \\
&\quad =: \mathrm{K}_1 - \sum_{0\le|\alpha|,\,|\beta|\le m,\,|\alpha|+|\beta|<2m} \mathrm{K}_{\alpha,\,\beta}.
\end{aligned} \tag{6.26}$$

To control $\mathrm{K}_{\alpha,\,\beta}$, without loss of generality, we may assume that $|\alpha| = m$ and $|\beta| < m$, since other cases can be estimated in a similar way. By Cauchy's inequality with ϵ, we see that, for all $\epsilon \in (0,\,\infty)$, there exists a positive constant $C_{(\epsilon)}$ such that

$$\begin{aligned}
\mathrm{K}_{\alpha,\,\beta} &\lesssim \left\{ \iint_{\mathbb{R}^{n+1}_+} t^{2m}|\partial^\beta u(y,\,t)|^2\eta(y,\,t)\,\frac{dy\,dt}{t} \right\}^{1/2} \\
&\quad \times \left\{ \iint_{\mathbb{R}^{n+1}_+} t^{2m}|\partial^\alpha u(y,\,t)|^2\eta(y,\,t)\,\frac{dy\,dt}{t} \right\}^{1/2} \\
&\lesssim \epsilon \iint_{\mathbb{R}^{n+1}_+} t^{2m}|\partial^\alpha u(y,\,t)|^2\eta(y,\,t)\,\frac{dy\,dt}{t}
\end{aligned}$$

$$
\begin{aligned}
&+ C_{(\epsilon)} \iint_{\mathbb{R}^{n+1}_+} t^{2m} |\partial^\beta u(y,\,t)|^2 \eta(y,\,t)\, \frac{dy\,dt}{t} \\
&=: \epsilon \widetilde{\mathrm{K}}_\alpha + C_{(\epsilon)} \widetilde{\mathrm{K}}_\beta. \qquad (6.27)
\end{aligned}
$$

For term $\widetilde{\mathrm{K}}_\alpha$, by the Ellipticity condition $(\widetilde{\mathcal{E}}_1)$, we immediately obtain

$$
\widetilde{\mathrm{K}}_\alpha \lesssim \Re e \left\{ \iint_{\mathbb{R}^{n+1}_+} t^{2m} \sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta}(y) \partial^\beta u(y,\,t) \overline{\partial^\alpha u(y,\,t)} \eta(y,\,t) \frac{dy\,dt}{t} \right\}. \qquad (6.28)
$$

For term $\widetilde{\mathrm{K}}_\beta$, using the Ellipticity condition $(\widetilde{\mathcal{E}}_1)$, the weighted interpolation inequality (2.5) and Cauchy's inequality with ϵ, we know that, for all $\epsilon \in (0,\,\infty)$, there exists a positive constant $C_{(\epsilon)}$ such that

$$
\begin{aligned}
\widetilde{\mathrm{K}}_\beta &\lesssim \left\{ \iint_{\mathbb{R}^{n+1}_+} t^m |\nabla^m u(y,\,t)|^2 \eta(y,\,t)\, \frac{dy\,dt}{t} \right\}^{\frac{|\beta|}{m}} \\
&\quad \times \left\{ \iint_{\mathbb{R}^{n+1}_+} t^m |u(y,\,t)|^2 \eta(y,\,t)\, \frac{dy\,dt}{t} \right\}^{\frac{m-|\beta|}{m}} \\
&\lesssim \epsilon \iint_{\mathbb{R}^{n+1}_+} t^m |\nabla^m u(y,\,t)|^2 \eta(y,\,t)\, \frac{dy\,dt}{t} + C_{(\epsilon)} \iint_{\mathbb{R}^{n+1}_+} t^m |u(y,\,t)|^2 \eta(y,\,t)\, \frac{dy\,dt}{t} \\
&\lesssim \epsilon \Re e \left\{ \iint_{\mathbb{R}^{n+1}_+} t^{2m} \sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta}(y) \partial^\beta u(y,\,t) \overline{\partial^\alpha u(y,\,t)} \eta(y,\,t) \frac{dy\,dt}{t} \right\} \\
&\quad + C_{(\epsilon)} \iint_{\mathbb{R}^{n+1}_+} t^m |u(y,\,t)|^2\, \frac{dy\,dt}{t}. \qquad (6.29)
\end{aligned}
$$

By letting δ_1 sufficiently large and ϵ in (6.27) and (6.29) sufficiently small, together with (6.28), we see that

$$
\sum_{0\le|\alpha|,\,|\beta|\le m,\,|\alpha|+|\beta|<2m} \mathrm{K}_{\alpha,\,\beta} \le \frac{1}{2} \mathrm{K}_1,
$$

which, combined with (6.26), implies that

$$
\begin{aligned}
&\Re e \left\{ \iint_{\mathbb{R}^{n+1}_+} \left[t^{2m} \sum_{0\le|\alpha|,\,|\beta|\le m} a_{\alpha,\,\beta,\,\delta_1}(y) \partial^\beta u(y,\,t) \overline{\partial^\alpha u(y,\,t)} \eta(y,\,t) \right] \frac{dy\,dt}{t} \right\} \\
&\quad \ge \frac{1}{2} \Re e \left\{ \iint_{\mathbb{R}^{n+1}_+} t^{2m} \left[\delta_1 u(y,\,t) \overline{u(y,\,t)} + \sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta} \partial^\beta u(y,\,t) \overline{\partial^\alpha u(y,\,t)} \right] \right. \\
&\qquad \left. \times\, \eta(y,\,t) \frac{dy\,dt}{t} \right\}.
\end{aligned}
$$

From this, together with $\delta_1 \in (\delta, \infty)$, the Ellipticity condition $(\widetilde{\mathcal{E}}_1)$, integration by parts and Leibniz's rule, we deduce that

$$\begin{aligned}
\mathcal{A} &\sim \iint_{\mathcal{R}^{2\epsilon,\, R,\, 1/2}(E^*)} t^{2m}[|u(y,\, t)|^2 + |\nabla^m u(y,\, t)|^2]\,\frac{dy\,dt}{t} \\
&\lesssim \Re e\left\{\iint_{\mathbb{R}^{n+1}_+} t^{2m}\left[\delta_1|u(y,\, t)|^2 + \sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta}(y)\partial^\beta u(y,\, t)\overline{\partial^\alpha u(y,\, t)}\eta(y,\, t)\right]\right. \\
&\qquad \left.\times\frac{dy\,dt}{t}\right\} \\
&\lesssim \Re e\left\{\iint_{\mathbb{R}^{n+1}_+}\left[t^{2m}\sum_{0\le|\alpha|,\,|\beta|\le m} a_{\alpha,\,\beta,\,\delta_1}(y)\partial^\beta u(y,\, t)\overline{\partial^\alpha u(y,\, t)}\eta(y,\, t)\right]\frac{dy\,dt}{t}\right\} \\
&\lesssim \left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+}\left[t^{2m}\sum_{0\le|\alpha|,\,|\beta|\le m}\eta(y,\, t)(-1)^{|\alpha|}\partial^\alpha(a_{\alpha,\,\beta,\,\delta_1}\partial^\beta u)(y,\, t)\overline{u(y,\, t)}\right]\right.\right. \\
&\qquad \left.\left.\times\frac{dy\,dt}{t}\right\}\right| + \left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+}\left[t^{2m}\sum_{0\le|\alpha|,\,|\beta|\le m}\sum_{\theta<\widetilde{\alpha}\le\alpha}\partial^{\widetilde{\alpha}}\eta(y,\, t)\partial^{\alpha-\widetilde{\alpha}}\right.\right.\right. \\
&\qquad \left.\left.\left.\times(a_{\alpha,\,\beta,\,\delta_1}\partial^\beta u)(y,\, t)\overline{u(y,\, t)}\right]\frac{dy\,dt}{t}\right\}\right| =: \mathrm{I} + \mathrm{J},
\end{aligned}$$

where $\theta := (0, \ldots, 0) \in (\mathbb{Z}_+)^n$.

The estimate of I is similar to that of J_0 in the proof of [15, Proposition 3.7] and hence we conclude that there exists a positive constant γ such that

$$\begin{aligned}
\mathrm{I} &\lesssim \left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+} t^{2m}\eta(y,\, t)(L+\delta_1)(u)(y,\, t)\overline{u(y,\, t)}\,\frac{dy\,dt}{t}\right\}\right| \\
&\lesssim \int_E |\mathcal{N}^3_{h,\, L+\delta}(f)(z)|^2\,dz + |B^*|\left[\sup_{z\in E}\mathcal{N}^\gamma_{h,\, L+\delta}(f)(z)\right]^2.
\end{aligned}$$

To bound J, via integral by parts, Hölder's inequality and the definition of η, we see that

$$\begin{aligned}
\mathrm{J} &\lesssim \sum_{k=0}^m\sum_{\widetilde{k}=0}^m\sum_{|\beta|=k}\sum_{|\alpha|=\widetilde{k}}\sum_{l=1}^{\widetilde{k}}\sum_{|\widetilde{\alpha}|=l,\,\widetilde{\alpha}\le\alpha} \\
&\qquad \times\left\{\iint_{\mathcal{R}^{\epsilon/2,\, 2R,\, 3/2}(E^*)\setminus\mathcal{R}^{\epsilon,\, R,\, 1/2}(E^*)} |t^m\partial^\beta u(y,\, t)|^2\,\frac{dy\,dt}{t}\right\}^{\frac12}
\end{aligned}$$

$$\times \left\{ \iint_{\mathcal{R}^{\epsilon/2,\,2R,\,3/2}(E^*)\setminus\mathcal{R}^{\epsilon,\,R,\,1/2}(E^*)} |t^m \partial^{\alpha-\widetilde{\alpha}}((\partial^{\widetilde{\alpha}}\eta)u)(y,\,t)|^2 \,\frac{dy\,dt}{t} \right\}^{\frac{1}{2}}$$

$$=: \sum_{k=0}^{m} \sum_{\widetilde{k}=0}^{m} \sum_{|\beta|=k} \sum_{|\alpha|=\widetilde{k}} \sum_{l=1}^{\widetilde{k}} \sum_{|\widetilde{\alpha}|=l,\,\widetilde{\alpha}\le\alpha} \mathrm{J}_\beta \times \widetilde{\mathrm{J}}_{\alpha,\,\widetilde{\alpha}}.$$

The remainder of the proof is similar to the corresponding part of the proof of [15, Proposition 3.6], by using the parabolic Caccioppoli's inequality (6.10), the interpolation inequality (2.4) and (6.22), the details being omitted. This finishes the proof of Proposition 6.4. □

Propositions 6.3, 6.4 and Corollary 4.10 imply the following maximal function characterization of $H^p_{L+\delta}(\mathbb{R}^n)$, the details being omitted.

Definition 6.5. Let L be as in (1.1) and satisfy the Weak ellipticity condition $(\widetilde{\mathcal{E}})$. For all $p \in (0,\,\infty)$, the *Hardy space* $H^p_{\mathcal{N}_{h,\,L+\delta}}(\mathbb{R}^n)$ is defined as the completion of

$$\{f \in L^2(\mathbb{R}^n) : \mathcal{N}_{h,\,L+\delta}(f) \in L^p(\mathbb{R}^n)\}$$

with respect to the *quasi-norm* $\|f\|_{H^p_{\mathcal{N}_{h,\,L+\delta}}(\mathbb{R}^n)} := \|\mathcal{N}_{h,\,L+\delta}(f)\|_{L^p(\mathbb{R}^n)}$, where $\mathcal{N}_{h,\,L+\delta}(f)$ is as in (6.25).

Theorem 6.6. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\widetilde{\mathcal{E}}_1)$, $p \in (0,\,1]$ *and* $\delta \in (0,\,\infty)$ *be as in Definition* 4.1. *Then, for all* $p \in (0,\,1]$, $H^p_{L+\delta}(\mathbb{R}^n) = H^p_{\mathcal{N}_{h,\,L+\delta}}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

Proof. The proof of the inclusion that $H^p_{\mathcal{N}_{h,\,L+\delta}}(\mathbb{R}^n) \subset H^p_{L+\delta}(\mathbb{R}^n)$ follows immediately from Propositions 6.3 and 6.4, and Corollary 4.10. For the converse inclusion, by Theorem 4.5, we only need to show that, for all $p \in (0,\,1]$, $\epsilon \in (0,\,\infty)$, $M \in \mathbb{N}$ such that $M > \frac{n}{2m}(\frac{1}{p} - \frac{1}{2})$ and $(p,\,2,\,M,\,\epsilon)_{L+\delta}$-molecules α, $\mathcal{N}_{h,\,L+\delta}(\alpha)$ is uniformly bounded in $L^p(\mathbb{R}^n)$. This can be proved by using the off-diagonal estimates of $L+\delta$ in Proposition 2.10 (see also the proof of [42, Theorem 6.3] for more details), the details being omitted. This finishes the proof of Theorem 6.6. □

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