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Bilinear decompositions of products of local Hardy and Lipschitz or BMO spaces through wavelets

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Let $p \in (\frac{n}{n+1}, 1]$ and $f \in h^p(\mathbb{R}^n)$ be the local Hardy space in the sense of D. Goldberg. In this paper, the authors establish two bilinear decompositions of the product spaces of $h^p(\mathbb{R}^n)$ and their dual spaces. More precisely, the authors prove that $h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) = L^1(\mathbb{R}^n) + h_*^\Phi(\mathbb{R}^n)$ and, for any $p \in (\frac{n}{n+1}, 1)$, $h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n) = L^1(\mathbb{R}^n) + h^p(\mathbb{R}^n)$, where $\mathrm{bmo}(\mathbb{R}^n)$ denotes the local BMO space, $\Lambda_\alpha(\mathbb{R}^n)$, for any $p \in (\frac{n}{n+1}, 1)$ and $\alpha := n(\frac{1}{p} - 1)$, the inhomogeneous Lipschitz space and $h_*^\Phi(\mathbb{R}^n)$ a variant of the local Orlicz–Hardy space related to the Orlicz function $\Phi(t) := \frac{t}{\log(e+t)}$ for any $t \in [0, \infty)$ which was introduced by Bonami and Feuto. As an application, the authors establish a div-curl lemma at the endpoint case.

Keywords: Local Hardy space; local BMO space; inhomogeneous Lipschitz space; product; paraproduct; renormalization; wavelet; div-curl lemma.

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1. Introduction

The problem of the bilinear decompositions of the products of Hardy spaces and their dual spaces was first raised by Bonami *et al.* [6], motivated by developments in the studies of the weak Jacobian, and has applications in the geometric function theory and nonlinear elasticity [1, 2]. One of the most important results on this direction was made by Bonami *et al.* [5], where Bonami *et al.* proved that the product space $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ of the Hardy space $H^1(\mathbb{R}^n)$ and the BMO space $\mathrm{BMO}(\mathbb{R}^n)$ has a bilinear decomposition of the form

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n).$$

Here $H^{\log}(\mathbb{R}^n)$ denotes the so-called Musielak–Orlicz–Hardy space, introduced in [15], related to the Musielak–Orlicz function

$$\theta(x,\, t) := \frac{t}{\log(e+|x|) + \log(e+t)} \quad \text{for any } (x,\, t) \in \mathbb{R}^n \times [0,\, \infty) \tag{1.1}$$

(see [23] for a complete theory of Musielak–Orlicz–Hardy spaces). This fact was further extended in [7], where, for any $p \in (\frac{n}{n+1},\, 1)$, the authors proved the following bilinear decomposition of the form:

$$H^p(\mathbb{R}^n) \times \Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^p(\mathbb{R}^n)$$

and

$$H^p(\mathbb{R}^n) \times \dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^p_{w_\alpha}(\mathbb{R}^n),$$

where $H^p(\mathbb{R}^n)$ denotes the classical real Hardy space, $\dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ and $\Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ the homogeneous, respectively, the inhomogeneous Lipschitz spaces and, for any $\alpha \in (n(1-p),\, \infty)$, $H^p_{w_\alpha}(\mathbb{R}^n)$ the weighted Hardy space related to the weight function

$$w_\alpha(x) := \frac{1}{(1+|x|)^\alpha} \quad \text{for any } x \in \mathbb{R}^n.$$

Ky [16] also established the bilinear decompositions of the products of Hardy spaces, associated with some Schrödinger operators $-\Delta + V$, and their dual spaces. It was found in [14, 17] that these kinds of bilinear decompositions are very useful in the endpoint estimates for commutators of Calderón–Zygmund operators and $\mathrm{BMO}(\mathbb{R}^n)$ functions.

For the local Hardy space, Bonami *et al.* [3] established some linear decompositions of the products of the local Hardy spaces and their dual spaces. Precisely, they proved that

$$h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + h^\Phi_*(\mathbb{R}^n) \tag{1.2}$$

and, for any $p \in (0,\, 1)$ and $\alpha := n(\frac{1}{p} - 1)$,

$$h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + h^p(\mathbb{R}^n), \tag{1.3}$$

where, for any $p \in (0,\, 1]$, $h^p(\mathbb{R}^n)$ denotes the local Hardy space in the sense of Goldberg [12] (see also (2.14) below for its definition), $\mathrm{bmo}\,(\mathbb{R}^n)$ the local BMO space (see also (2.15) below for its definition), $\Lambda_\alpha(\mathbb{R}^n)$ the inhomogeneous Lipschitz space (see also (2.16) below for its definition) and $h_*^\Phi(\mathbb{R}^n)$ a variant of the local Orlicz–Hardy space related to the Orlicz function

$$\Phi(t) := \frac{t}{\log(e+t)} \quad \text{for any } t \in [0,\,\infty), \tag{1.4}$$

which was introduced in [3] (see also (2.40) below for its definition). Here, we point out that both the decompositions, obtained in (1.2) and (1.3), are linear only in the functions from $\mathrm{bmo}\,(\mathbb{R}^n)$ or from $\Lambda_\alpha(\mathbb{R}^n)$.

Motivated by the aforementioned results in [6, 5, 3], in this paper, we study the bilinear decompositions of the products of local Hardy spaces $h^p(\mathbb{R}^n)$ and their dual spaces in the case when $p<1$ and near to 1. Precisely, let $p \in (\frac{n}{n+1},\,1]$, $\alpha := n(\frac{1}{p}-1)$ and $h_*^\Phi(\mathbb{R}^n)$ be a variant of the local Orlicz–Hardy space related to the Orlicz function Φ as in (1.4). The main result of this paper is to establish the following two bilinear decompositions.

Theorem 1.1. *Let $p \in (\frac{n}{n+1},\,1)$, $\alpha := n(\frac{1}{p}-1)$ and Φ be as in* (1.4). *Then*

(i) *there exist two bounded bilinear operators $S:\ h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)\to L^1(\mathbb{R}^n)$ and $T:\ h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)\to h^p(\mathbb{R}^n)$ such that, for any $(f,g)\in h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)$,*

$$f\times g = S(f,\,g)+T(f,\,g);$$

(ii) *there exist two bounded bilinear operators $S:\ h^1(\mathbb{R}^n)\times\mathrm{bmo}(\mathbb{R}^n)\to L^1(\mathbb{R}^n)$ and $T:\ h^1(\mathbb{R}^n)\times\mathrm{bmo}(\mathbb{R}^n)\to h_*^\Phi(\mathbb{R}^n)$ such that, for any $(f,g)\in h^1(\mathbb{R}^n)\times\mathrm{bmo}(\mathbb{R}^n)$,*

$$f\times g = S(f,\,g)+T(f,\,g),$$

where $h_*^\Phi(\mathbb{R}^n)$ denotes the variant local Orlicz–Hardy space defined as in (2.40) *below with Φ as in* (1.4).

Theorem 1.1 extends the corresponding linear decompositions (1.2) and (1.3) from [3], by showing that the aforementioned two linear decompositions are bilinear when $p \in (\frac{n}{n+1},\,1]$. We point out that the variant local Hardy space $h_*^\Phi(\mathbb{R}^n)$, in Theorem 1.1(ii), can be replaced by the local Musielak–Orlicz–Hardy space $h^{\log}(\mathbb{R}^n)$, which was first introduced in [24] (see Remark 3.1 for more details on this result). Theorem 1.1 is proved in Sec. 3. To show this theorem, we use the idea of the renormalization of the product of two functions (or distributions). This idea, based on the theories of multiresolution analysis and wavelets, originally appeared in Dobyinsky [10] and Coifman *et al.* [8] and was used systematically in [5] to deal with the bilinear decomposition of the product space $H^1(\mathbb{R}^n)\times\mathrm{BMO}(\mathbb{R}^n)$. In this paper, the idea of the renormalization enables us to reduce the multiplication operator to four bilinear operators $\{\Pi_i\}_{i=1}^4$ (see (2.7) through (2.10) below for their definitions). To obtain the boundedness of these bilinear operators, we also need to establish some wavelet coefficient characterizations of the local Hardy spaces $h^p(\mathbb{R}^n)$ and their dual spaces (see Theorems 2.7 and 2.8 below).

As an application of the main result of this paper, we prove a div-curl lemma at the endpoint case $q = \infty$. More precisely, let

$$h^1(\mathbb{R}^n;\,\mathbb{R}^n) := \{\mathbf{F} := (F_1, \ldots, F_n):\ \text{for any } i \in \{1, \ldots, n\},\ F_i \in h^1(\mathbb{R}^n)\}$$

and, for any $\mathbf{F} \in h^1(\mathbb{R}^n;\,\mathbb{R}^n)$, let

$$\|\mathbf{F}\|_{h^1(\mathbb{R}^n;\,\mathbb{R}^n)} := \left[\sum_{i=1}^{n} \|F_i\|^2_{h^1(\mathbb{R}^n)}\right]^{\frac{1}{2}}.$$

The local vector-valued BMO space

$$\mathrm{bmo}\,(\mathbb{R}^n;\,\mathbb{R}^n) := \{\mathbf{G} := (G_1, \ldots, G_n):\ \text{for any } i \in \{1, \ldots, n\},\ G_i \in \mathrm{bmo}\,(\mathbb{R}^n)\}$$

is similarly defined, the details being omitted. We have the following div-curl lemma at the endpoint case.

Theorem 1.2. *Let $\mathbf{F} \in h^1(\mathbb{R}^n;\,\mathbb{R}^n)$ with $\mathrm{curl}\,\mathbf{F} \equiv 0$ in the sense of distributions and $\mathbf{G} \in \mathrm{bmo}\,(\mathbb{R}^n;\,\mathbb{R}^n)$ with $\mathrm{div}\,\mathbf{G} \equiv 0$ in the sense of distributions. Then $\mathbf{F} \cdot \mathbf{G} \in h^{\Phi}_*(\mathbb{R}^n)$, where $h^{\Phi}_*(\mathbb{R}^n)$ denotes the variant local Orlicz–Hardy space defined as in* (2.40) *below with Φ as in* (1.4).

Theorem 1.2 is proved in Sec. 3. This result essentially extends the corresponding div-curl lemmas in [4, 5]. In what follows, let $C_c^\infty(\mathbb{R}^n)$ denote the set of all infinite differentiable functions on $\mathbb{R}^n$ with compact supports. Recall that, for a vector field $\mathbf{F} := (F_1, \ldots, F_n)$ of locally integrable functions on $\mathbb{R}^n$, its *divergence* $\mathrm{div}\,\mathbf{F}$ is defined, in the sense of distributions, by setting, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \mathrm{div}\,\mathbf{F},\, \varphi\rangle := -\int_{\mathbb{R}^n} \mathbf{F}(x) \cdot \nabla\varphi(x)\, dx$$

and its *curl* $\mathrm{curl}\,\mathbf{F}$ is a matrix $\{(\mathrm{curl}\,\mathbf{F})_{i,\,j}\}_{i,\,j\in\{1,\ldots,n\}}$ of distributions, whose entries are defined by setting, for any $i,\, j \in \{1, \ldots, n\}$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle(\mathrm{curl}\,\mathbf{F})_{i,\,j},\, \varphi\rangle := \int_{\mathbb{R}^n} \left[F_j(x)\frac{\partial\varphi}{\partial x_i}(x) - F_i(x)\frac{\partial\varphi}{\partial x_j}(x)\right] dx.$$

This paper is organized as follows. We first, in Sec. 2, collect some basic definitions and facts on the multiresolution analysis, wavelets, local Hardy spaces and their dual spaces. These notions are necessary in the succeeding study of the products of local Hardy spaces and their dual spaces. Then, in the same section, we establish some wavelet coefficient characterizations of the local Hardy spaces $h^p(\mathbb{R}^n)$ and their dual spaces (see Theorems 2.7 and 2.8 below). We also develop some boundedness of the bilinear operators $\{\Pi_i\}_{i=1}^4$, arising from the renormalization of the product, in Sec. 2. Finally, in Sec. 3, we prove the main results, Theorems 1.1 and 1.2, of this paper.

We end this section by making some conventions on the notation. Throughout this paper, we always let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $\alpha \in \mathbb{R}$, let $\lfloor\alpha\rfloor$ be the biggest integer not bigger than α. We use C to denote a *positive* *constant* that is independent of the main parameters involved but whose value may differ from line to line, and $C_{(\alpha,\ldots)}$ to denote a *positive constant* depending on the indicated parameters $\alpha,\ldots$. If $f \le Cg$, we then write $f \lesssim g$. For any set $E \subset \mathbb{R}^n$, χ_E denotes its *characteristic function*. We let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class on $\mathbb{R}^n$, equipped with the well-known classical topology, and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, equipped with the weak-$*$ topology. The notation $f * g$ always denotes the *convolution* of two functions f and g, or a Schwartz function f and a tempered distribution g. Also, for any $x := (x_1,\ldots,x_n) \in \mathbb{R}^n$ and multi-index $\alpha := (\alpha_1,\ldots,\alpha_n) \in (\mathbb{Z}_+)^n$, we let $\partial_x^\alpha := (\frac{\partial}{\partial_{x_1}})^{\alpha_1}\cdots(\frac{\partial}{\partial_{x_n}})^{\alpha_n}$, $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

2. Products of Functions in Local Hardy and Inhomogeneous Lipschitz or Local BMO Spaces

In this section, we first collect some basic definitions and facts that are necessary in the following study of the products of local Hardy spaces and their dual spaces. To this end, we begin with the following definition of multiresolution analysis on $\mathbb{R}$; see, for example, [18, 21, 5] for more details.

Definition 2.1. Let $\{V_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of closed subspaces in $L^2(\mathbb{R})$. Then $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* (MRA) *on* $\mathbb{R}$ if it has the following properties:

(i) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{\theta\}$, where θ denotes the zero function;
(ii) for any $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$;
(iii) for any $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, $f(\cdot) \in V_0$ if and only if $f(\cdot - k) \in V_0$;
(iv) there exists a function $\phi \in L^2(\mathbb{R})$ (called *scaling function* or *father wavelet*) such that $\{\phi_k(\cdot)\}_{k\in\mathbb{Z}} := \{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0.

If $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA, then $\{V_j\}_{j\in\mathbb{Z}_+}$ is called an (*inhomogeneous*) MRA.

Remark 2.2. The (inhomogeneous) MRA first appeared, as far as we know, in [21, Definition 1.49]. As we can see from their definitions, the notions of both the MRA and the (inhomogeneous) MRA are closely related. Indeed, both the MRA and the (inhomogeneous) MRA enjoy the same father wavelet and the same mother wavelet. Since, in this paper, we only use the (inhomogeneous) MRA, to simplify the presentation of this paper, in what follows, we omit (inhomogeneous) when we mention the (inhomogeneous) MRA.

Based on MRA, we now recall some preliminary facts on the wavelet. First, from [9, Sec. 4] (see also [21, Theorem 1.61(ii)]), it follows that, for any $k \in \mathbb{N}$, we can choose the father and the mother wavelets ϕ, $\psi \in C^k(\mathbb{R})$ (the set of all functions with continuous derivatives up to order k) with compact supports such that $\widehat{\phi}(0) = (2\pi)^{-1/2}$ and, for any $l \in \{0,\ldots,k\}$,

$$\int_{\mathbb{R}} x^l \psi(x)\,dx = 0,$$

where $\widehat{\phi}$ denotes the *Fourier transform* of ϕ. Following [5], throughout this paper, we *always assume* that

$$\operatorname{supp}\phi,\ \ \operatorname{supp}\psi \subset \frac{1}{2}+m\left(-\frac{1}{2},\,\frac{1}{2}\right), \tag{2.1}$$

where $1/2+m(-1/2,\,1/2)$ denotes the interval obtained from $(0,\,1)$ via a dilation by m centered at $1/2$, namely, $x\in 1/2+m(-1/2,\,1/2)$ if and only if $|x-1/2|<m/2$. Here m is a positive constant that is independent of the main parameters involved in this paper.

The extension of the above considerations from 1 dimension to n dimension can be established by the standard procedure of tensor products. More precisely, let

$$\vec{\theta}_n := (\overbrace{0,\ldots,0}^{n\text{ times}}) \quad\text{and}\quad E:=\{0,\,1\}^n\backslash\{\vec{\theta}_n\}.$$

Assume that $\mathcal{D}_0$ is the set of all *dyadic cubes* in $\mathbb{R}^n$ with side lengths not greater than 1, namely, for any $I\in\mathcal{D}_0$, there exist $j\in\mathbb{Z}_+$ and $k:=\{k_1,\ldots,k_n\}\in\mathbb{Z}^n$ such that

$$I:=I_{j,\,k}:=\{x\in\mathbb{R}^n:\ k_i\le 2^jx_i<k_i+1 \text{ for any } i\in\{1,\ldots,n\}\}. \tag{2.2}$$

Let mI be the m dilation of I with the same center as I and m as in (2.1). By an argument similar to that used in [5, Sec. 3] or [21, Sec. 1.7], we know that there exist two families $\{\phi_I\}_{I\in\mathcal{D}_0}$ and $\{\psi_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E}$ of functions satisfying

$$\operatorname{supp}\phi_I,\ \ \operatorname{supp}\psi_I^\lambda\subset mI \tag{2.3}$$

and, for any $j\in\mathbb{Z}_+$, $k\in\mathbb{Z}^n$ (with $I=I_{j,\,k}\in\mathcal{D}_0$ as in (2.2)), $\lambda\in E\cup\{\vec{\theta}_n\}$ and $x\in\mathbb{R}^n$,

$$\Psi_I^\lambda(x):=\Psi_{I_{j,\,k}}^\lambda(x):=\begin{cases}\phi_{I_{j,\,k}}(x) & \text{when } j=0,\ k\in\mathbb{Z}^n \text{ and } \lambda=\vec{\theta}_n,\\ \psi^\lambda_{I_{j-1,\,k}}(x) & \text{when } j\in\mathbb{N},\ k\in\mathbb{Z}^n \text{ and } \lambda\in E,\\ 0 & \text{otherwise}\end{cases} \tag{2.4}$$

such that $\{\Psi_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$. Thus, for any $f\in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} f&=\sum_{I\in\mathcal{D}_0}\sum_{\lambda\in E\cup\{\vec{\theta}_n\}}\langle f,\,\Psi_I^\lambda\rangle\Psi_I^\lambda\\ &=\sum_{\substack{I\in\mathcal{D}_0\\ |I|=1}}\langle f,\,\phi_I\rangle\phi_I+\sum_{I\in\mathcal{D}_0}\sum_{\lambda\in E}\langle f,\,\psi_I^\lambda\rangle\psi_I^\lambda \end{aligned} \tag{2.5}$$

in $L^2(\mathbb{R}^n)$, where $\langle\cdot,\,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^n)$. Moreover, for any $j\in\mathbb{Z}_+$, let V_j be the closed subspace of $L^2(\mathbb{R}^n)$ spanned by $\{\phi_I\}_{|I|=2^{-jn}}$. It is known that $\{V_j\}_{j\in\mathbb{Z}_+}$ is an MRA on $\mathbb{R}^n$ (whose definition is similar to that of the MRA on $\mathbb{R}$ in Definition 2.1; see [18, Chap. 2] or [21, Remark 1.52] for more details).

Now, let $f,\, g \in L^2(\mathbb{R}^n)$, we renormalize the pointwise product fg by using the wavelets arising from the MRA. In particular, for any $j \in \mathbb{Z}_+$, let P_j and Q_j be the orthogonal projections of $L^2(\mathbb{R}^n)$ onto V_j, respectively, onto $W_j := V_{j+1} \ominus V_j$ (the *orthogonal complement* of V_j in V_{j+1}). As in [10, p. 313], we write

$$\begin{aligned} fg &= \sum_{j=0}^{\infty} [(P_{j+1}f)\,(P_{j+1}g) - (P_jf)\,(P_jg)] + (P_0f)\,(P_0g) \\ &= \sum_{j=0}^{\infty} (P_jf)\,(Q_jg) + \sum_{j=0}^{\infty} (Q_jf)\,(P_jg) \\ &\quad + \sum_{j=0}^{\infty} (Q_jf)\,(Q_jg) + (P_0f)\,(P_0g) \end{aligned} \tag{2.6}$$

in $L^1(\mathbb{R}^n)$. Now, for any $f,\, g \in L^2(\mathbb{R}^n)$, define

$$\Pi_1(f,\, g) := \sum_{\substack{I,\, I' \in \mathcal{D}_0 \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f,\, \phi_I\rangle \langle g,\, \psi_{I'}^{\lambda}\rangle \phi_I \psi_{I'}^{\lambda}, \tag{2.7}$$

$$\Pi_2(f,\, g) := \sum_{\substack{I,\, I' \in \mathcal{D}_0 \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f,\, \psi_I^{\lambda}\rangle \langle g,\, \phi_{I'}\rangle \psi_I^{\lambda} \phi_{I'}, \tag{2.8}$$

$$\begin{aligned} \Pi_3(f,\, g) := &\sum_{\substack{I,\, I' \in \mathcal{D}_0 \\ |I|=|I'|}} \sum_{\substack{\lambda,\, \lambda' \in E \\ (I,\, \lambda) \neq (I',\, \lambda')}} \langle f,\, \psi_I^{\lambda}\rangle \langle g,\, \psi_{I'}^{\lambda'}\rangle \psi_I^{\lambda} \psi_{I'}^{\lambda'} \\ &+ \sum_{\substack{I,\, I' \in \mathcal{D}_0 \\ |I|=|I'|=1}} \langle f,\, \phi_I\rangle \langle g,\, \phi_{I'}\rangle \phi_I \phi_{I'} \end{aligned} \tag{2.9}$$

and

$$\Pi_4(f,\, g) := \sum_{I \in \mathcal{D}_0} \sum_{\lambda \in E} \langle f,\, \psi_I^{\lambda}\rangle \langle g,\, \psi_I^{\lambda}\rangle (\psi_I^{\lambda})^2. \tag{2.10}$$

We point out that, although the operators $\{\Pi_i\}_{i=1}^4$ are only defined on functions in $L^2(\mathbb{R}^n)$, they can be extended to more general scales of function spaces in the case when f and g are distributions and have wavelet expansions.

From (2.6) and the definitions of V_j and W_j for any $j \in \mathbb{Z}_+$, we deduce that

$$fg = \sum_{i=1}^{4} \Pi_i(f,\, g) \tag{2.11}$$

in $L^1(\mathbb{R}^n)$. Also, as in [10, 5], we let

$$T := \sum_{i=1}^{3} \Pi_i \quad \text{and} \quad S := \Pi_4 \tag{2.12}$$

and call the operator T, which preserves the cancelation property of the product, the *renormalization of the product* fg.

We now recall the definition of the local Hardy spaces $h^p(\mathbb{R}^n)$ for any $p \in (0, \infty)$ from [12]. Recall that, for any $m \in \mathbb{N}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the *local non-tangential grand maximal function* $f^*_{m,\,\mathrm{loc}}$ of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f^*_{m,\,\mathrm{loc}}(x) := \sup_{\varphi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{\substack{|y-x|<t \\ t\in(0,\,1)}} |f * \varphi_t(y)|\,, \tag{2.13}$$

where

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\le m+1}} \left[(1+|x|)^{(m+2)(n+1)} \left|\partial_x^\alpha \varphi(x)\right|\right] \le 1 \right\}.$$

For any $p \in (0, \infty)$, the *local Hardy space* $h^p(\mathbb{R}^n)$ is defined to be the set

$$h^p(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{h^p(\mathbb{R}^n)} := \|f^*_{m,\,\mathrm{loc}}\|_{L^p(\mathbb{R}^n)} < \infty\}. \tag{2.14}$$

It is well known that $h^p(\mathbb{R}^n)$ is independent of the choice of $m \in \mathbb{N} \cap (\lfloor n(\frac{1}{p} - 1)\rfloor, \infty)$. Thus, in what follows, we *always remove* the subscript m in the definition of $\|\cdot\|_{h^p(\mathbb{R}^n)}$ whenever $m \in \mathbb{N} \cap (\lfloor n(\frac{1}{p} - 1)\rfloor, \infty)$. For more properties of the local Hardy spaces $h^p(\mathbb{R}^n)$, we refer the reader to [12].

Recall that a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the *local BMO space* $\mathrm{bmo}\,(\mathbb{R}^n)$ if

$$\|b\|_{\mathrm{bmo}\,(\mathbb{R}^n)} := \sup_{|I|<1} \left\{\frac{1}{|I|}\int_I |b(x) - b_I|\, dx\right\} + \sup_{|I|\ge 1} \left\{\frac{1}{|I|}\int_I |b(x)|\, dx\right\}, \tag{2.15}$$

where $b_I := \frac{1}{|I|}\int_I b(x)\,dx$ denotes the mean of f over the cube I, the first supremum is taken over all cubes I with $|I| < 1$ and the second one over all cubes I with $|I| \ge 1$.

For any $\alpha \in (0, 1]$ and f being continuous, let

$$\|f\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} := \sup_{\substack{x,\,y\in\mathbb{R}^n \\ x\ne y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \quad \text{and} \quad \|f\|_{\Lambda_\alpha(\mathbb{R}^n)} := \|f\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Then the *inhomogeneous Lipschitz space* $\Lambda_\alpha(\mathbb{R}^n)$ is defined by setting

$$\Lambda_\alpha(\mathbb{R}^n) := \{f \text{ is continuous in } \mathbb{R}^n : \|f\|_{\Lambda_\alpha(\mathbb{R}^n)} < \infty\}. \tag{2.16}$$

For the local Hardy space $h^p(\mathbb{R}^n)$, one of its most important properties is its atomic characterization, which was first established in [12].

Definition 2.3. Let $p \in (0, 1]$ and I be a cube in $\mathbb{R}^n$. A function $a \in L^2(\mathbb{R}^n)$ is called an $h^p(\mathbb{R}^n)$-*atom associated with* I if

(i) $\operatorname{supp} a \subset I$;
(ii) $\|a\|_{L^2(\mathbb{R}^n)} \le |I|^{\frac{1}{2}-\frac{1}{p}}$;
(iii) if $|I| < 1$, then, for any multi-index $\alpha \in (\mathbb{Z}_+)^n$ satisfying $|\alpha| \le \lfloor n(\frac{1}{p} - 1)\rfloor$, $\int_{\mathbb{R}^n} x^\alpha a(x)\,dx = 0$.

The following lemma establishes the atomic decomposition of the local Hardy space $h^p(\mathbb{R}^n)$ for any $p \in (0, 1]$.

Lemma 2.4 ([12]). *Let $p \in (0, 1]$ and $f \in h^p(\mathbb{R}^n)$. Then there exist a family $\{a_j\}_{j=1}^\infty$ of $h^p(\mathbb{R}^n)$-atoms and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, there exists a positive constant $\widetilde{C}$, independent of f, $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty$, such that*

$$\left\{\sum_{j=1}^\infty |\lambda_j|^p\right\}^{\frac{1}{p}} \le \widetilde{C}\|f\|_{h^p(\mathbb{R}^n)}.$$

Remark 2.5. It is known that, if the condition $|I| < 1$ in Definition 2.3(iii) is replaced by $|I| < c$ with c being a finite fixed positive constant, depending on the constant m in (2.1), but independent of all the other main parameters involved in this paper, then Lemma 2.4 still holds true with the positive constant $\widetilde{C}$ depending on c.

We now establish the finite atomic decompositions of the elements in $h^p(\mathbb{R}^n)$ having finite wavelet expansions. To this end, we first recall some known facts on the inhomogeneous Triebel–Lizorkin spaces and their sequence spaces; see [11, 20, 21, 26] and their references for more details on these spaces.

Let Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : \ |\xi| \le 2\} \quad \text{and} \quad \operatorname{supp}\widehat{\varphi} \subset \left\{\xi \in \mathbb{R}^n : \ \frac{1}{2} \le |\xi| \le 2\right\}.$$

Assume that there exists a positive constant c such that, for any $\xi, \eta \in \mathbb{R}^n$ satisfying $|\xi| \le \frac{5}{3}$, respectively, $\frac{3}{5} \le |\eta| \le \frac{5}{3}$,

$$|\widehat{\Phi}(\xi)| \ge c > 0 \quad \text{and} \quad |\widehat{\varphi}(\eta)| \ge c > 0,$$

where, for every function f, $\widehat{f}$ denotes its Fourier transform. Then, for any $s \in \mathbb{R}$ and p, $q \in (0, \infty]$, the *inhomogeneous Triebel–Lizorkin space* $F^s_{p,\,q}(\mathbb{R}^n)$ is defined by setting

$$F^s_{p,\,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f\|_{F^s_{p,\,q}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{F^s_{p,\,q}(\mathbb{R}^n)} := \begin{cases} \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\|\left[\displaystyle\sum_{k=1}^\infty \left|2^{ks}\varphi_k * f\right|^q\right]^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)} & \text{if } p \in (0, \infty), \\ \|\Phi * f\|_{L^\infty(\mathbb{R}^n)} & \\ \quad + \displaystyle\sup_{\substack{I \text{ dyadic cube} \\ |I|<1}} \left[\frac{1}{|I|}\int_I \sum_{k=-\log_2 l(I)}^\infty \left|2^{ks}\varphi_k * f(x)\right|^q \, dx\right]^{\frac{1}{q}} & \text{if } p = \infty \end{cases}$$

with the usual modification made when $q = \infty$ and, for any $k \in \mathbb{N}$, $\varphi_k(\cdot) := 2^{kn}\varphi(2^k\cdot)$ and $l(I)$ denotes the *side length* of $I \in \mathcal{D}_0$.

It is well known that inhomogeneous Triebel–Lizorkin spaces include many classical function spaces as special cases. For example, it holds true that (see, for example, [11, Sec. 12] or [20, Sec. 2.3.5])

$$h^p(\mathbb{R}^n) = F^0_{p,\,2}(\mathbb{R}^n), \quad \forall\, p \in (0,\, 1]; \tag{2.17}$$

$$\mathrm{bmo}\,(\mathbb{R}^n) = F^0_{\infty,\,2}(\mathbb{R}^n) \tag{2.18}$$

and

$$\Lambda_\alpha\,(\mathbb{R}^n) = F^\alpha_{\infty,\,\infty}(\mathbb{R}^n), \quad \forall\, \alpha \in (0,\, \infty). \tag{2.19}$$

We also need the following definition of sequence spaces. For any $s \in \mathbb{R}$ and p, $q \in (0, \infty]$, the associated *sequence space* $f^s_{p,\,q}(\mathbb{R}^n)$ is defined by setting

$$f^s_{p,\,q}(\mathbb{R}^n) := \{\{s_I\}_{I\in\mathcal{D}_0} :\ \|\{s_I\}_{I\in\mathcal{D}_0}\|_{f^s_{p,\,q}(\mathbb{R}^n)} < \infty\},$$

where

$$\|\{s_I\}_{I\in\mathcal{D}_0}\|_{f^s_{p,\,q}(\mathbb{R}^n)} := \begin{cases} \left\| \left\{ \displaystyle\sum_{I\in\mathcal{D}_0} [|I|^{-\frac{s}{n}-\frac{1}{2}}|s_I|\chi_I]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} & \text{if } p \in (0,\, \infty), \\ \displaystyle\sup_{P\in\mathcal{D}_0} \left\{ \frac{1}{|P|} \int_P \sum_{I\subset P} [|I|^{-\frac{s}{n}-\frac{1}{2}}|s_I|\chi_I(x)]^q\, dx \right\}^{\frac{1}{q}} & \text{if } p = \infty \end{cases} \tag{2.20}$$

with the usual modification made when $q = \infty$.

It is known that the sequence space has the following dual relationship (see, for example, [11, Sec. 12]): for any $p \in (0, 1)$ and $q \in (1, \infty]$,

$$\left(f^0_{p,\,q}(\mathbb{R}^n)\right)^* = f^{n(\frac{1}{p}-1)}_{\infty,\,\infty}(\mathbb{R}^n) \quad \text{and} \quad \left(f^0_{1,\,2}(\mathbb{R}^n)\right)^* = f^0_{\infty,\,2}(\mathbb{R}^n). \tag{2.21}$$

The following wavelet characterization of the inhomogeneous Triebel–Lizorkin space $F^s_{p,\,q}(\mathbb{R}^n)$ was proved in [21, Theorem 1.64] for any $s \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty]$, and was then extended to the limiting case $p = \infty$ in [26, Corollary 4.2].

Lemma 2.6 ([21, 26]). *Let p, $q \in (0, \infty]$ and $s \in \mathbb{R}$ with the additional restrictions $q \in (0, \infty)$ and $s \in (n \max\{0, \frac{1}{q} - 1\}, \infty)$ when $p = \infty$. Then there exist positive constants C and $\widetilde{C}$, and a family $\{\Psi^\lambda_I\}_{I\in\mathcal{D},\, \lambda\in E\cup\{\vec{\theta}_n\}}$ of wavelets arising from the MRA such that, for any $f \in F^s_{p,\,q}(\mathbb{R}^n)$,*

$$\begin{aligned} C\|f\|_{F^s_{p,\,q}(\mathbb{R}^n)} &\le \|\{\langle f,\, \Psi^\lambda_I\rangle\}_{I\in\mathcal{D}_0,\, \lambda\in E\cup\{\vec{\theta}_n\}}\|_{f^s_{p,\,q}(\mathbb{R}^n)} \\ &\le \widetilde{C}\|f\|_{F^s_{p,\,q}(\mathbb{R}^n)}, \end{aligned}$$

where, for any sequence $\{s_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\subset\mathbb{C}$,

$$\|\{s_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{f^s_{p,\,q}(\mathbb{R}^n)}$$
$$:=\begin{cases}\left\|\left\{\displaystyle\sum_{\substack{I\in\mathcal{D}_0\\ \lambda\in E\cup\{\vec{\theta}_n\}}}[|I|^{-\frac{s}{n}-\frac12}|s_I^\lambda|\chi_I]^q\right\}^{\frac1q}\right\|_{L^p(\mathbb{R}^n)} & \text{if } p\in(0,\,\infty),\\ \displaystyle\sup_{P\in\mathcal{D}_0}\left\{\frac{1}{|P|}\int_P\sum_{\substack{I\subset P\\ \lambda\in E\cup\{\vec{\theta}_n\}}}[|I|^{-\frac{s}{n}-\frac12}|s_I^\lambda|\chi_I(x)]^q\,dx\right\}^{\frac1q} & \text{if } p=\infty\end{cases}$$

is a slight modification of the quasi-norm defined in (2.20). *In particular, if* $p,\,q\in(0,\,\infty)$ *and* $s\in\mathbb{R}$, *the family* $\{\Psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E\cup\{\vec{\theta}_n\}}$ *of wavelets is an unconditional basis in* $F^s_{p,\,q}(\mathbb{R}^n)$.

With the help of the wavelet characterization of the inhomogeneous Triebel–Lizorkin space, we now establish the following finite atomic decompositions of the elements in $h^p(\mathbb{R}^n)$ having finite wavelet expansions.

Theorem 2.7. *Let* $p\in(0,\,1]$ *and* $f\in h^p(\mathbb{R}^n)$ *have a finite wavelet expansion, namely,*

$$f=\sum_{I\in\mathcal{D}_0}\sum_{\lambda\in E\cup\{\vec{\theta}_n\}}\langle f,\,\Psi_I^\lambda\rangle\Psi_I^\lambda \tag{2.22}$$

pointwise, where $\mathcal{D}_0$ *is as in* (2.2), $\{\Psi_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}$ *are as in* (2.4) *and the coefficient* $\langle f,\,\Psi_I^\lambda\rangle\neq0$ *only for a finite number of* $(I,\,\lambda)\in\mathcal{D}_0\times(E\cup\{\vec{\theta}_n\})$. *Then* f *has a finite local atomic decomposition satisfying* $f=\sum_{l=1}^L\mu_la_l$ *and there exists a positive constant* C, *independent of any* $\{\mu_l\}_{l=1}^L,\{a_l\}_{l=1}^L$ *and* f, *such that*

$$\left\{\sum_{l=1}^L|\mu_l|^p\right\}^{\frac1p}\le C\|f\|_{h^p(\mathbb{R}^n)},$$

where $L\in\mathbb{N}$ *and, for any* $l\in\{1,\ldots,L\}$, $\mu_l\in\mathbb{C}$ *and* a_l *is an* $h^p(\mathbb{R}^n)$*-atom associated with the dyadic cube* R_l, *which can be written into the following form:*

$$a_l=\sum_{\substack{I\in\mathcal{D}_0\\ I\subset R_l}}\sum_{\lambda\in E\cup\{\vec{\theta}_n\}}c_{(I,\,\lambda,\,l)}\Psi_I^\lambda \tag{2.23}$$

with $\{c_{(I,\,\lambda,\,l)}\}_{I\subset R_l,\,\lambda\in E\cup\{\vec{\theta}_n\}}$ *being positive constants independent of* $\{a_l\}_{l=1}^L$. *Moreover, for any* $l\in\{1,\ldots,L\}, a_l$ *in* (2.23) *also has a finite wavelet expansion which is extracted from that of* f *in* (2.22).

Proof. We prove this theorem by borrowing some ideas from the proof of [13, Theorem 5.12], where Hernández and Weiss established the wavelet characterization of the Hardy space $H^1(\mathbb{R})$. More precisely, for any f in (2.22) and $x \in \mathbb{R}^n$, let

$$(W_\Psi f)(x) := \left\{ \sum_{I \in \mathcal{D}_0} \sum_{\lambda \in E \cup \{\vec{\theta}_n\}} \left| \langle f, \Psi_I^\lambda \rangle \right|^2 |I|^{-1} \chi_I(x) \right\}^{\frac{1}{2}}$$

and, for any $k \in \mathbb{Z}$,

$$\Omega_k := \{ y \in \mathbb{R}^n : \ (W_\Psi f)(y) > 2^k \}.$$

It is easy to see that $\{|\Omega_k|\}_{k\in\mathbb{Z}}$ is decreasing in k. By a summation by parts argument, (2.17), (2.20) and Lemma 2.6, we find that

$$\sum_{k \in \mathbb{Z}} 2^{kp} |\Omega_k| \lesssim \|W_\Psi f\|_{L^p(\mathbb{R}^n)}^p \sim \|f\|_{h^p(\mathbb{R}^n)}^p. \tag{2.24}$$

Moreover, for any $k \in \mathbb{Z}$, let

$$\mathcal{A}_k := \left\{ I \in \mathcal{D}_0 : \ |I \cap \Omega_k| \geq \frac{1}{2}|I| \right\} \quad \text{and} \quad \mathcal{B}_k := \mathcal{A}_k \backslash \mathcal{A}_{k+1}. \tag{2.25}$$

Also, for any $k \in \mathbb{Z}$, denote by $\widetilde{\mathcal{D}}_k := \{\widetilde{I}_k^i \in \mathcal{B}_k : \ i \in \Lambda_k\}$ the set of all maximal dyadic cubes in $\mathcal{B}_k$, where Λ_k is an index set. From (2.22) and an argument similar to that used in the proof of (5.15) of the proof of [13, Theorem 5.12], it follows that

$$f = \sum_{I \in \mathcal{D}_0} \sum_{\lambda \in E \cup \{\vec{\theta}_n\}} \langle f, \Psi_I^\lambda \rangle \Psi_I^\lambda = \sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} \left[\sum_{\substack{I \subset \widetilde{I}_k^i \\ I \in \mathcal{B}_k}} \sum_{\lambda \in E \cup \{\vec{\theta}_n\}} \langle f, \Psi_I^\lambda \rangle \Psi_I^\lambda \right] \tag{2.26}$$

pointwise.

Now, for any $k \in \mathbb{Z}$ and $i \in \Lambda_k$, let

$$\lambda_{k,i} := |\widetilde{I}_k^i|^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{\substack{I \subset \widetilde{I}_k^i \\ I \in \mathcal{B}_k}} \sum_{\lambda \in E \cup \{\vec{\theta}_n\}} |\langle f, \Psi_I^\lambda \rangle|^2 \right)^{\frac{1}{2}}$$

and

$$a_{k,i} := \begin{cases} \dfrac{1}{\lambda_{k,i}} \displaystyle\sum_{\substack{I \subset \widetilde{I}_k^i \\ I \in \mathcal{B}_k}} \sum_{\lambda \in E \cup \{\vec{\theta}_n\}} \langle f, \Psi_I^\lambda \rangle \Psi_I^\lambda & \text{when } \lambda_{k,i} \neq 0, \\ 0 & \text{when } \lambda_{k,i} = 0, \end{cases} \tag{2.27}$$

where, from the assumption that f has a finite wavelet expansion (2.22), we deduce that the summation in the first formula of (2.27) is of finite terms. Thus, we find that

$$f = \sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} \lambda_{k,i} a_{k,i},$$

where, for each $k \in \mathbb{Z}$ and $i \in \Lambda_k$, we have

$$\|a_{k,i}\|_{L^2(\mathbb{R}^n)} \lesssim |2m\widetilde{I}_k^i|^{\frac{1}{2}-\frac{1}{p}}$$

and $\operatorname{supp} a_{k,\,i} \subset 2m\widetilde{I}_k^i$, where m is as in (2.1). Moreover, if, in (2.27), there exists $(\lambda,\,I) \in (E \cup \{\vec{\theta}_n\}) \times \mathcal{D}_0$ satisfying $\lambda = \vec{\theta}_n$, then, by (2.4), we know that $|I| = 1$ and hence $l_{2m\widetilde{I}_k^i} \geq 2m$, here and hereafter, $l_{2m\widetilde{I}_k^i}$ denotes the side length of the cube $2m\widetilde{I}_k^i$. Otherwise, by (2.4), we know that $l_{2m\widetilde{I}_k^i} < 2m$ and $\int_{\mathbb{R}^n} a_{k,\,i}(x)\,dx = 0$. This, combined with Remark 2.5, immediately shows that, for any $k \in \mathbb{Z}$ and $i \in \Lambda_k$, $a_{k,\,i}$ is a harmless constant multiple of an $h^p(\mathbb{R}^n)$-atom associated with $2m\widetilde{I}_k^i$.

On the other hand, from the definitions of Ω_k, $\mathcal{B}_k$ and $W_\Psi f$, it follows that

$$\begin{aligned}
\sum_{\substack{I\subset \widetilde{I}_k^i\\ I\in\mathcal{B}_k}} \sum_{\lambda\in E\cup\{\vec{\theta}_n\}} \left|\langle f,\,\Psi_I^\lambda\rangle\right|^2 &\lesssim \sum_{\substack{I\subset \widetilde{I}_k^i\\ I\in\mathcal{B}_k}} \sum_{\lambda\in E\cup\{\vec{\theta}_n\}} \left|\langle f,\,\Psi_I^\lambda\rangle\right|^2 \frac{|I\backslash\Omega_{k+1}|}{|I|}\\
&\lesssim \int_{\widetilde{I}_k^i\backslash\Omega_{k+1}} \sum_{I\in\mathcal{D}_0} \sum_{\lambda\in E\cup\{\vec{\theta}_n\}} \left|\langle f,\,\Psi_I^\lambda\rangle\right|^2 \frac{1}{|I|}\chi_I(x)\,dx\\
&\lesssim \int_{\widetilde{I}_k^i\backslash\Omega_{k+1}} [(W_\psi f)(x)]^2\,dx \lesssim 2^{2(k+1)}|\widetilde{I}_k^i|,
\end{aligned}$$

which, together with (2.24), (2.25) and the disjointness of $\{\widetilde{I}_k^i\}_{i\in\Lambda_k}$ for any $k \in \mathbb{Z}$ (which can be deduced from the definition of $\{\widetilde{I}_k^i\}_{i\in\Lambda_k}$; see also the proof of [13, Theorem 5.12] for this fact), implies that

$$\begin{aligned}
\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\Lambda_k} |\lambda_{k,\,i}|^p\right\}^{\frac{1}{p}} &\lesssim \left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\Lambda_k} |\widetilde{I}_k^i| 2^{(k+1)p}\right\}^{\frac{1}{p}}\\
&\lesssim \left\{\sum_{k\in\mathbb{Z}} 2^{kp}|\Omega_k|\right\}^{\frac{1}{p}} \lesssim \|f\|_{h^p(\mathbb{R}^n)}. \qquad (2.28)
\end{aligned}$$

Since f has a finite wavelet expansion, it follows that there exists only a finite number of $(k,\,i) \in \mathbb{Z}\times\mathbb{N}$ such that both $a_{k,\,i}$ and $\lambda_{k,\,i}$ are not zero. By renumbering the subscripts of $\{a_{k,\,i}\}_{k\in\mathbb{Z},\,i\in\Lambda_k}$ and $\{\lambda_{k,\,i}\}_{k\in\mathbb{Z},\,i\in\Lambda_k}$ into $\{a_l\}_{l=1}^L$ and $\{\lambda_l\}_{l=1}^L$, we then complete the proof of Theorem 2.7. $\square$

We also need the following wavelet characterizations of the local BMO space $\mathrm{bmo}(\mathbb{R}^n)$ and the inhomogeneous Lipschitz space $\Lambda_\alpha(\mathbb{R}^n)$.

Theorem 2.8. *Let $\alpha \in (0,\,1]$ and $\mathcal{C}_\alpha(\mathbb{R}^n)$ be the space of all sequences $\{s_I\}_{I\in\mathcal{D}_0}$ of complex numbers satisfying*

$$\|\{s_I\}_{I\in\mathcal{D}_0}\|_{\mathcal{C}_\alpha(\mathbb{R}^n)} := \sup_{J\in\mathcal{D}_0} \left\{\frac{1}{|J|^{\frac{2}{\alpha}-1}} \sum_{\substack{I\in\mathcal{D}_0\\ I\subset J}} |s_I|^2\right\}^{\frac{1}{2}} < \infty. \qquad (2.29)$$

Then

(i) *there exist a positive constant* C *and a family* $\{\Psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E\cup\{\vec{\theta}_n\}}$ *of wavelets arising from the MRA such that, for any* $g\in \mathrm{bmo}\,(\mathbb{R}^n)$,

$$\|\{\langle g,\,\Psi_I^\lambda\rangle\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{\mathcal{C}_1(\mathbb{R}^n)}\le C\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)},$$

where, for any sequence $\{s_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\subset\mathbb{C}$,

$$\|\{s_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{\mathcal{C}_1(\mathbb{R}^n)}:=\sup_{J\in\mathcal{D}_0}\left\{\frac{1}{|J|^{\frac{2}{\alpha}-1}}\sum_{\substack{I\in\mathcal{D}_0,\,I\subset J\\ \lambda\in E\cup\{\vec{\theta}_n\}}}\left|s_I^\lambda\right|^2\right\}^{\frac12}\tag{2.30}$$

is a slight modification of the norm defined as in (2.29);

(ii) *for any* $p\in(0,1)$, *there exist a positive constant* C *and a family* $\{\Psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E\cup\{\vec{\theta}_n\}}$ *of wavelets arising from the MRA such that, for any* $g\in \Lambda_{n(\frac1p-1)}(\mathbb{R}^n)$,

$$\|\{\langle g,\,\Psi_I^\lambda\rangle\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{\mathcal{C}_p(\mathbb{R}^n)}\le C\|g\|_{\Lambda_{n(\frac1p-1)}(\mathbb{R}^n)},$$

where, for any sequence $\{s_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\subset\mathbb{C}$, $\|\{s_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{\mathcal{C}_p(\mathbb{R}^n)}$ *is defined in the same way as that in* (2.30).

Proof. We prove this theorem by transferring the considered problem to the level of inhomogeneous sequence spaces. More precisely, from (2.18) and (2.19), we deduce that $\mathrm{bmo}\,(\mathbb{R}^n)=F^0_{\infty,\,2}(\mathbb{R}^n)$ and $\Lambda_{n(\frac1p-1)}\,(\mathbb{R}^n)=F^{n(\frac1p-1)}_{\infty,\infty}(\mathbb{R}^n)$ for any $p\in(0,\,1)$, which, together with Lemma 2.6, implies that there exist a positive constant C and a family $\{\Psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E\cup\{\vec{\theta}_n\}}$ of wavelets arising from the MRA such that, for any $f\in\mathrm{bmo}\,(\mathbb{R}^n)$,

$$\|\{\langle f,\,\Psi_I^\lambda\rangle\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{f^0_{\infty,\,2}(\mathbb{R}^n)}\le C\|f\|_{\mathrm{bmo}\,(\mathbb{R}^n)}$$

and, for any $f\in\Lambda_{n(\frac1p-1)}\,(\mathbb{R}^n)$,

$$\|\{\langle f,\,\Psi_I^\lambda\rangle\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_n\}}\|_{f^{n(\frac1p-1)}_{\infty,\infty}(\mathbb{R}^n)}\le C\|f\|_{\Lambda_{n(\frac1p-1)}\,(\mathbb{R}^n)},$$

Thus, to finish the proof of Theorem 2.8, it suffices to show that

$$\mathcal{C}_1(\mathbb{R}^n)=f^0_{\infty,\,2}(\mathbb{R}^n)$$

and, for any $p\in(0,\,1)$,

$$\mathcal{C}_p(\mathbb{R}^n)=f^{n(\frac1p-1)}_{\infty,\infty}(\mathbb{R}^n).\tag{2.31}$$

Without loss of generality, we may only prove (2.31). To prove the inclusion $\mathcal{C}_p(\mathbb{R}^n)\subset f^{n(\frac1p-1)}_{\infty,\infty}(\mathbb{R}^n)$, let $\{s_I\}_{I\in\mathcal{D}_0}\in\mathcal{C}_p(\mathbb{R}^n)$. For any $\{c_I\}_{I\in\mathcal{D}_0}\in f^0_{p,\,2}(\mathbb{R}^n)$ and

$k \in \mathbb{Z}$, let

$$\Omega_k := \left\{ x \in \mathbb{R}^n : \left[\sum_{I \in \mathcal{D}_0} |c_I|^2 \, |I|^{-1} \chi_I(x) \right]^{\frac{1}{2}} > 2^k \right\},$$

and $\mathcal{A}_k$, $\mathcal{B}_k$ be as in (2.25). Then, using a calculation similar to that used in the second equality of (2.26) (with the same notation therein), Hölder's inequality and $p \in (0, 1)$, we find that

$$\begin{aligned} \left| \sum_{I \in \mathcal{D}_0} s_I \overline{c}_I \right| &= \left| \sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} \left(\sum_{\substack{I \in \mathcal{B}_k \\ I \subset \widetilde{I}_k^i}} s_I \overline{c}_I \right) \right| \\ &\lesssim \left\{ \sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} \left[\left(\sum_{\substack{I \in \mathcal{B}_k \\ I \subset \widetilde{I}_k^i}} |s_I|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{I \in \mathcal{B}_k \\ I \subset \widetilde{I}_k^i}} |c_I|^2 \right)^{\frac{1}{2}} \right]^p \right\}^{\frac{1}{p}} \\ &\lesssim \sup_{J \in \mathcal{D}_0} \left\{ \frac{1}{|J|^{\frac{2}{p}-1}} \sum_{\substack{I \in \mathcal{D}_0 \\ I \subset J}} |s_I|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} \left[|\widetilde{I}_k^i|^{\frac{1}{p} - \frac{1}{2}} \left(\sum_{\substack{I \in \mathcal{B}_k \\ I \subset \widetilde{I}_k^i}} |c_I|^2 \right)^{\frac{1}{2}} \right]^p \right\}^{\frac{1}{p}}, \end{aligned}$$

which, combined with the definition of $\mathcal{C}_p(\mathbb{R}^n)$ and some calculations similar to those used in (2.28) and (2.24) (replacing $\langle f, \Psi_I^\lambda \rangle$ in the definition of $\lambda_{k,\,i}$ therein by c_I), implies that

$$\left| \sum_{I \in \mathcal{D}_0} s_I \overline{c}_I \right| \lesssim \|\{s_I\}_{I \in \mathcal{D}_0}\|_{\mathcal{C}_p(\mathbb{R}^n)} \|\{c_I\}_{I \in \mathcal{D}_0}\|_{f^0_{p,\,2}(\mathbb{R}^n)}, \tag{2.32}$$

where $\|\{c_I\}_{I \in \mathcal{D}_0}\|_{f^0_{p,\,2}(\mathbb{R}^n)}$ is as in (2.20). This shows that $\{s_I\}_{I \in \mathcal{D}_0} \in \left(f^0_{p,\,2}(\mathbb{R}^n)\right)^*$ and

$$\|\{s_I\}_{I \in \mathcal{D}_0}\|_{(f^0_{p,\,2}(\mathbb{R}^n))^*} \lesssim \|\{s_I\}_{I \in \mathcal{D}_0}\|_{\mathcal{C}_p(\mathbb{R}^n)},$$

which, combined with (2.21), implies $\mathcal{C}_p(\mathbb{R}^n) \subset f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n)$.

To prove $f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n) \subset \mathcal{C}_p(\mathbb{R}^n)$, we borrow some ideas from the proof of [22, Theorem 3.5]. More precisely, fix $J \in \mathcal{D}_0$ and let $S_J := \{I \in \mathcal{D}_0 : \ I \subset J\}$. Define the measure $d\nu$ on S_J by setting, for any $I \in S_J$, $d\nu(I) := |I|/|J|^{\frac{2}{p}-1}$. By an elementary calculation, it is easy to see that

$$\begin{aligned} l^2(S_J,\, d\nu) := \Bigg\{ &\{u_I\}_{I \in S_J} \subset \mathbb{C} : \\ &\|\{u_I\}_{I \in S_J}\|_{l^2(S_J,\, d\nu)} := \left(\frac{1}{|J|^{\frac{2}{p}-1}} \sum_{\substack{I \in \mathcal{D}_0 \\ I \subset J}} |u_I|^2 \, |I| \right)^{\frac{1}{2}} < \infty \Bigg\} \end{aligned}$$

is a Hilbert space equipped with the *inner product* $\langle\cdot,\cdot\rangle_{l^2(S_J,\,d\nu)}$ which is defined by setting, for any $\{u_I\}_{I\in S_J}$, $\{v_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)$,

$$\langle\{u_I\}_{I\in S_J},\,\{v_I\}_{I\in S_J}\rangle_{l^2(S_J,\,d\nu)}:=\frac{1}{|J|^{\frac{2}{p}-1}}\sum_{\substack{I\in\mathcal{D}_0\\ I\subset J}}u_I\overline{v_I}\,|I|.$$

Thus, by the Riesz representation theorem (see, for example, [25, p. 90, Theorem]) on the Hilbert space $l^2(S_J,\,d\nu)$, we know that, for every $\{u_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)$,

$$\begin{aligned}&\|\{u_I\}_{I\in S_J}\|_{l^2(S_J,\,d\nu)}\\ &\qquad=\sup_{\substack{\{v_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)\\ \|\{v_I\}_{I\in S_J}\|_{l^2(S_J,\,d\nu)}\le 1}}\left|\langle\{u_I\}_{I\in S_J},\,\{v_I\}_{I\in S_J}\rangle_{l^2(S_J,\,d\nu)}\right|.\end{aligned}\tag{2.33}$$

We now continue the proof of $f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n)\subset\mathcal{C}_p(\mathbb{R}^n)$. Let $\{c_I\}_{I\in\mathcal{D}_0}\in f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n)$. By (2.21), (2.33), (2.20) and Hölder's inequality, we find that

$$\begin{aligned}&\left(\frac{1}{|J|^{\frac{2}{p}-1}}\sum_{\substack{I\in\mathcal{D}_0\\ I\subset J}}|c_I|^2\right)^{\frac12}\\
&\quad=\left\|\left\{\frac{c_I}{|I|^{\frac12}}\right\}_{I\in S_J}\right\|_{l^2(S_J,\,d\nu)}=\sup_{\substack{\{t_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)\\ \|\{t_I\}_{I\in S_J}\|_{l^2(S_J,\,d\nu)}\le 1}}\left|\sum_{\substack{I\in\mathcal{D}_0\\ I\subset J}}t_I\overline{c}_I\frac{|I|^{\frac12}}{|J|^{\frac{2}{p}-1}}\right|\\
&\quad\lesssim\|\{c_I\}_{I\in\mathcal{D}_0}\|_{(f^0_{p,\,2}(\mathbb{R}^n))^*}\sup_{\substack{\{t_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)\\ \|\{t_I\}_{I\in S_J}\|_{l^2(S_J,\,d\nu)}\le 1}}\left\|\left\{t_I\frac{|I|^{\frac12}}{|J|^{\frac{2}{p}-1}}\right\}_{I\in S_J}\right\|_{f^0_{p,\,2}(\mathbb{R}^n)}\\
&\quad\lesssim\|\{c_I\}_{I\in\mathcal{D}_0}\|_{f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n)}\\
&\qquad\times\sup_{\substack{\{t_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)\\ \|\{t_I\}_{I\in S_J}\|_{l^2(S_J,\,d\nu)}\le 1}}\frac{1}{|J|^{\frac{2}{p}-1}}\left\{\int_J\left[\sum_{\substack{I\in\mathcal{D}_0\\ I\subset J}}|t_I|^2\chi_I(x)\right]^{\frac{p}{2}}dx\right\}^{\frac1p}\\
&\quad\lesssim\|\{c_I\}_{I\in\mathcal{D}_0}\|_{f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n)}\sup_{\substack{\{t_I\}_{I\in S_J}\in l^2(S_J,\,d\nu)\\ \|\{t_I\}_{I\in S_J}\|_{l^2(S_J,\,d\nu)}\le 1}}\left\{\sum_{\substack{I\in\mathcal{D}_0\\ I\subset J}}|t_I|^2\frac{|I|}{|J|^{\frac{2}{p}-1}}\right\}^{\frac12}\\
&\quad\lesssim\|\{c_I\}_{I\in\mathcal{D}_0}\|_{f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n)},\end{aligned}$$

which immediately implies that $\{c_I\}_{I\in\mathcal{D}_0} \in \mathcal{C}_p(\mathbb{R}^n)$. Thus, we conclude that $f^{n(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^n) \subset \mathcal{C}_p(\mathbb{R}^n)$ and hence (2.31) holds true. This finishes the proof of Theorem 2.8. $\square$

We now study the products of the local Hardy and the inhomogeneous Lipschitz or the local BMO spaces. Following [3], let $f \in h^1(\mathbb{R}^n)$ and $g \in \mathrm{bmo}\,(\mathbb{R}^n)$. The *product* $f \times g$ is defined by setting, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle f \times g,\, \phi\rangle := \langle \phi g,\, f\rangle, \tag{2.34}$$

where the last bracket denotes the dual pair between $\mathrm{bmo}\,(\mathbb{R}^n)$ and $h^1(\mathbb{R}^n)$. Recall that Nakai and Yabuta [19, Theorem 3] proved that every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a pointwise multiplier of $\mathrm{bmo}(\mathbb{R}^n)$, namely, for any $g \in \mathrm{bmo}(\mathbb{R}^n)$, $\phi g \in \mathrm{bmo}(\mathbb{R}^n)$. This implies that the definition in (2.34) is meaningful.

If $f \in h^p(\mathbb{R}^n)$ and $g \in \Lambda_\alpha(\mathbb{R}^n)$, where $p \in (0,\, 1)$ and $\alpha = n(\frac{1}{p} - 1)$, then we can define the product $f \times g$ in the same way as in (2.34). This definition is well defined because of the fact that every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is also a pointwise multiplier of the inhomogeneous Lipschitz space $\Lambda_\alpha(\mathbb{R}^n)$ for any $\alpha \in (0,\, 1]$ (see [19, Theorem 3]).

To give the desired bilinear decompositions of the product spaces of the local Hardy spaces and their dual spaces. By using the renormalization (2.11), we are reduced to the study of the boundedness properties of the bilinear operators $\{\Pi_i\}_{i=1}^4$ in (2.7) through (2.10).

The following theorem is an inhomogeneous version of [10, Proposition 1.1]. Recall that the bilinear operators S and T are defined in (2.12).

Theorem 2.9. (i) *The operator S is a bilinear operator bounded from the product space $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

(ii) *The operator T is a bilinear operator bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $h^1(\mathbb{R}^n)$.*

Proof. For any $f,\, g \in L^2(\mathbb{R}^n)$, by the fact that $fg = S(f,\, g)+T(f,\, g)$ and Hölder's inequality, we know that, to finish the proof of this theorem, it suffices to show $T(f,\, g) \in h^1(\mathbb{R}^n)$. Moreover, by (2.7) through (2.9) and [10, Proposition 1.1], we conclude that

$$T(f,\, g) - \sum_{\substack{I,\, I'\in\mathcal{D}_0\\ |I|=|I'|=1}} \langle f,\, \phi_I\rangle\langle g,\, \phi_{I'}\rangle \phi_I\phi_{I'} \in H^1(\mathbb{R}^n).$$

Thus, we only need to show that

$$\sum_{\substack{I,\, I'\in\mathcal{D}_0\\ |I|=|I'|=1}} \langle f,\, \phi_I\rangle\langle g,\, \phi_{I'}\rangle \phi_I\phi_{I'} \in h^1(\mathbb{R}^n).$$

Indeed, since $|I| = |I'| = 1$, it is easy to see that, for each I and I', $\phi_I\phi_{I'}$ is a harmless constant multiple of an h^1-atom associated with mI, where m is as in (2.1).

Thus, by the atomic characterization of $h^1(\mathbb{R}^n)$ (see Lemma 2.4 and Remark 2.5) and Hölder's inequality, we have

$$\begin{aligned}\Bigg\|\sum_{\substack{I,\,I'\in\mathcal{D}_0\\|I|=|I'|=1}}\langle f,\,\phi_I\rangle\langle g,\,\phi_{I'}\rangle\phi_I\phi_{I'}\Bigg\|_{h^1(\mathbb{R}^n)}&\lesssim\sum_{\substack{I,\,I'\in\mathcal{D}_0\\|I|=|I'|=1}}|\langle f,\,\phi_I\rangle|\,|\langle g,\,\phi_{I'}\rangle|\\&\lesssim\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

which immediately implies that

$$\sum_{\substack{I,\,I'\in\mathcal{D}_0\\|I|=|I'|=1}}\langle f,\,\phi_I\rangle\langle g,\,\phi_{I'}\rangle\phi_I\phi_{I'}\in h^1(\mathbb{R}^n)$$

and hence $T(f,\,g)\in h^1(\mathbb{R}^n)$. This finishes the proof of Theorem 2.9. □

We also have the following proposition on the boundedness of the bilinear operators $\{\Pi_i\}_{i=1}^4$.

Proposition 2.10. *Let* $p\in(\frac{n}{n+1},\,1),\alpha:=n(\frac{1}{p}-1)$ *and* $\{\Pi_i\}_{i=1}^4$ *be as in* (2.7) *through* (2.10). *Then*

(i) *the operator* Π_1 *can be extended to a bilinear operator bounded from* $h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)$ *to* $h^p(\mathbb{R}^n)$ *and from* $h^1(\mathbb{R}^n)\times\mathrm{bmo}\,(\mathbb{R}^n)$ *to* $h^1(\mathbb{R}^n)$;
(ii) *the operator* Π_2 *can be extended to a bilinear operator bounded from* $h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)$ *to* $h^p(\mathbb{R}^n)$;
(iii) *the operator* Π_3 *can be extended to a bilinear operator bounded from* $h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)$ *to* $h^p(\mathbb{R}^n)$ *and from* $h^1(\mathbb{R}^n)\times\mathrm{bmo}\,(\mathbb{R}^n)$ *to* $h^1(\mathbb{R}^n)$;
(iv) *the operator* Π_4 *can be extended to a bilinear operator bounded from* $h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$ *and from* $h^1(\mathbb{R}^n)\times\mathrm{bmo}\,(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$.

Proof. We begin the proof of this proposition with proving (i). To this end, we first prove that Π_1 can be extended to a bilinear operator bounded from $h^p(\mathbb{R}^n)\times\Lambda_\alpha(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$. Let $g\in\Lambda_\alpha(\mathbb{R}^n)$ and a be an $h^p(\mathbb{R}^n)$-atom associated with the cube R. By (2.7) and (2.3), we immediately know that $\operatorname{supp}\Pi_1(a,\,g)\subset 2mR$, where m is as in (2.1). Moreover, using the boundedness of Π_1 from $L^2(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ which can be deduced easily from (2.7), we conclude that

$$\|\Pi_1(a,\,g)\|_{L^2(\mathbb{R}^n)}\lesssim\|g\|_{L^\infty(\mathbb{R}^n)}\|a\|_{L^2(\mathbb{R}^n)}\lesssim\|g\|_{\Lambda^\alpha(\mathbb{R}^n)}|R|^{\frac{1}{2}-\frac{1}{p}}. \tag{2.35}$$

This, together with the assumption $p\in(\frac{n}{n+1},\,1)$ and the fact that

$$\int_{\mathbb{R}^n}\Pi_1(a,\,g)(x)\,dx=0,$$

implies that $\Pi_1(a,\,g)$ is a harmless constant multiple of an $h^p(\mathbb{R}^n)$-atom associated with $2mR$, where m is as in (2.1).

To continue the proof of Proposition 2.10, for any $f\in h^p(\mathbb{R}^n)$ having finite wavelet expansion, by Theorem 2.7, we know that $f=\sum_{l=1}^L\mu_l a_l$ has a finite atomic decomposition with the notation same as in Theorem 2.7. This, combined with the bilinearity of the operator Π_1, $p < 1$ and the fact that, for any $l \in \{1, \ldots, L\}$, $\Pi_1(a_l, g)$ is a harmless constant multiple of an $h^p(\mathbb{R}^n)$-atom associated with some cube $2mR_l$, implies that

$$\|\Pi_1(f, g)\|^p_{h^p(\mathbb{R}^n)} \le \sum_{l=1}^{L} |\mu_l|^p \|\Pi_1(a_l, g)\|^p_{h^p(\mathbb{R}^n)} \lesssim \|f\|^p_{h^p(\mathbb{R}^n)} \|g\|^p_{\Lambda^\alpha(\mathbb{R}^n)}. \quad (2.36)$$

We now show that Π_1 can be extended to a bilinear operator bounded from $h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$. For any $f \in h^p(\mathbb{R}^n)$, by Lemma 2.6, we know that there exists a family $\{f_k\}_{k\in\mathbb{N}} \subset h^p(\mathbb{R}^n)$ having finite wavelet expansions such that $\lim_{k\to\infty} f_k = f$ in $h^p(\mathbb{R}^n)$. Thus, we extend the definition of Π_1 to $h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$ by setting, for any $f \in h^p(\mathbb{R}^n)$ and $g \in \Lambda_\alpha(\mathbb{R}^n)$,

$$\Pi_1(f, g) := \lim_{k\to\infty} \Pi_1(f_k, g)$$

in $h^p(\mathbb{R}^n)$. By (2.36), we know that the above definition is well defined. Moreover, from the above definition and (2.36), it follows that

$$\|\Pi_1(f, g)\|_{h^p(\mathbb{R}^n)} = \lim_{k\to\infty} \|\Pi_1(f_k, g)\|_{h^p(\mathbb{R}^n)} \lesssim \|f\|_{h^p(\mathbb{R}^n)} \|g\|_{\Lambda_\alpha(\mathbb{R}^n)}.$$

This immediately shows that Π_1 can be extended to a bilinear operator bounded from $h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$.

We now turn to the proof that Π_1 can be extended to a bounded bilinear operator from $h^1(\mathbb{R}^n) \times \mathrm{bmo}\,(\mathbb{R}^n)$ to $h^1(\mathbb{R}^n)$. Let $f \in h^1(\mathbb{R}^n)$ and $g \in \mathrm{bmo}\,(\mathbb{R}^n)$. Without loss of generality, we may assume that f has a finite wavelet expansion (for the general case $f \in h^1(\mathbb{R}^n)$, we can use the argument same as that used in the proof of the first part of (i)). In this case, by Theorem 2.7, we know that $f = \sum_{l=1}^{L} \mu_l a_l$ has a finite atomic decomposition with the notation same as in Theorem 2.7. Thus, by (2.1) and Theorem 2.9(ii), we find that

$$\begin{aligned} \|\Pi_1(f, g)\|_{h^1(\mathbb{R}^n)} &\lesssim \left[\sum_{l=1}^{L} |\mu_l|\right] \sup_{l\in\{1,\ldots,L\}} \|\Pi_1(a_l, b_l)\|_{h^1(\mathbb{R}^n)} \\ &\lesssim \|f\|_{h^1(\mathbb{R}^n)} \sup_{l\in\{1,\ldots,L\}} \|a_l\|_{L^2(\mathbb{R}^n)} \|b_l\|_{L^2(\mathbb{R}^n)}, \end{aligned} \quad (2.37)$$

where, for any $l \in \{1, \ldots, L\}$,

$$b_l := \sum_{\substack{I\in\mathcal{D}_0 \\ I\subset 2mR_l}} \sum_{\lambda\in E} \langle g, \psi_I^\lambda\rangle \psi_I^\lambda$$

with R_l being the cube associated to a_l. By the orthogonality of the wavelets $\{\psi_I^\lambda\}_{I\in\mathcal{D}_0,\lambda\in E}$ in $L^2(\mathbb{R}^n)$, we know that, for any $l \in \{1, \ldots, L\}$,

$$\|b_l\|_{L^2(\mathbb{R}^n)} \lesssim \sup_{R\in\mathcal{D}_0} \left\{ \sum_{\substack{I\in\mathcal{D}_0 \\ I\subset 2mR}} \sum_{\lambda\in E} \left|\langle g, \psi_I^\lambda\rangle\right|^2 \right\}^{\frac{1}{2}}.$$

By this, combined with (2.37) and Theorem 2.8, we conclude that

$$\begin{aligned}\|\Pi_1(f,\,g)\|_{h^1(\mathbb{R}^n)} &\lesssim \|f\|_{h^1(\mathbb{R}^n)} \sup_{R\in\mathcal{D}_0}\left\{\frac{1}{|R|}\sum_{\substack{I\in\mathcal{D}_0\\ I\subset 2mR}}\sum_{\lambda\in E}\left|\langle g,\,\psi_I^\lambda\rangle\right|^2\right\}^{\frac{1}{2}}\\ &\lesssim \|f\|_{h^1(\mathbb{R}^n)}\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}, \qquad (2.38)\end{aligned}$$

which completes the proof of (i).

The proof of (ii) is similar to that of the first part of (i), the details being omitted.

To prove (iii), for any $f,\,g\in L^2(\mathbb{R}^n)$, by (2.9), we first write

$$\begin{aligned}\Pi_3(f,\,g) &= \sum_{\substack{I,\,I'\in\mathcal{D}_0\\ |I|=|I'|}}\ \sum_{\substack{\lambda,\,\lambda'\in E\\ (I,\,\lambda)\neq(I',\,\lambda')}} \langle f,\,\psi_I^\lambda\rangle\langle g,\,\psi_{I'}^{\lambda'}\rangle\psi_I^\lambda\psi_{I'}^{\lambda'}\\ &\quad+\sum_{\substack{I,\,I'\in\mathcal{D}_0\\ |I|=|I'|=1}}\langle f,\,\phi_I\rangle\langle g,\,\phi_{I'}\rangle\phi_I\phi_{I'}\\ &=: \Pi_{3,\,1}(f,\,g)+\Pi_{3,\,2}(f,\,g).\end{aligned}$$

The bilinear operator $\Pi_{3,\,1}$ can be dealt in a way similar to that used in the proof of (i), the details being omitted.

To prove the boundedness of $\Pi_{3,\,2}$, let $f\in h^p(\mathbb{R}^n)$ and $g\in\Lambda_\alpha(\mathbb{R}^n)$. Without loss of generality, we may also assume that f has a finite wavelet expansion (the general case $f\in h^p(\mathbb{R}^n)$ can be dealt by the argument same as that used in the proof of (i)). Then, by Theorem 2.7 with the notation same as therein, we know that f has a finite atomic decomposition $f=\sum_{l=1}^L \mu_l a_l$, where, for each $l\in\{1,\ldots,L\}$, a_l is an $h^p(\mathbb{R}^n)$-atom associated with the cube R_l and also has a finite wavelet expansion of the following form:

$$a_l=\sum_{\substack{I\in\mathcal{D}_0\\ I\subset R_l}}\ \sum_{\lambda\in E\cup\{\vec{\theta}_n\}} c_{(I,\,\lambda,\,l)}\Psi_I^\lambda. \qquad (2.39)$$

We now consider two cases based on the size of $|R_l|$. If $|R_l|<1$, by (2.39) and (2.4), we immediately know that

$$a_l=\sum_{\substack{I\in\mathcal{D}_0\\ I\subset R_l}}\sum_{\lambda\in E} c_{(I,\,\lambda,\,l)}\psi_I^\lambda,$$

which, together with the orthogonality of the wavelets $\{\phi_I\}_{I\in\mathcal{D}_0,\,|I|=1}$ and $\{\psi_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E}$ (see, for example, [5, Sec. 3]), implies that

$$\Pi_{3,\,2}(a_l,\,g)=\sum_{\substack{I,\,I'\in\mathcal{D}_0\\ |I|=|I'|=1}}\langle a_l,\,\phi_I\rangle\langle g,\,\phi_{I'}\rangle\phi_I\phi_{I'}=0.$$

Thus, such atoms contribute nothing to the desired boundedness of $\Pi_{3,\,2}$.

If $|R_l|\geq 1$, then, by an argument similar to that used in (2.35), we know that

$$\|\Pi_{2,\,3}(a_l,\,g)\|_{L^2(\mathbb{R}^n)}\lesssim\|g\|_{\Lambda^\alpha(\mathbb{R}^n)}|R_l|^{\frac{1}{2}-\frac{1}{p}}.$$

Moreover, by (2.3), we immediately know that $\operatorname{supp} \Pi_{3,\,2}(a,\, g) \subset 2mR_l$, where m is as in (2.1), which, combined with the definition of $h^p(\mathbb{R}^n)$-atoms, shows that $\Pi_{3,\,2}(a_l,\, g)$ is a harmless constant multiple of an $h^p(\mathbb{R}^n)$-atom associated with $2mR_l$. This, together with the bilinearity of the operator $\Pi_{3,\,2}$ and a density argument, immediately shows the extended boundedness of $\Pi_{3,\,2}$ from $h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$.

The proof of the extended boundedness of Π_3 from $h^1(\mathbb{R}^n) \times \mathrm{bmo}\,(\mathbb{R}^n)$ to $h^1(\mathbb{R}^n)$ is similar to the corresponding part of (i) with a slight modification, the details being omitted. This shows (iii).

The proof of (iv) is similar to that of (i), the only difference is that here we need to use Theorem 2.9(i), instead of Theorem 2.9(ii), in the proof of (iv), which justifies the space $L^1(\mathbb{R}^n)$ appearing in (iv). This finishes the proof of Proposition 2.10. $\square$

We now study the boundedness of the bilinear operator Π_2 from the product space $h^1(\mathbb{R}^n) \times \mathrm{bmo}\,(\mathbb{R}^n)$ to the variant local Orlicz–Hardy space $h_*^\Phi(\mathbb{R}^n)$ related to the Orlicz function Φ as in (1.4). Recall that, in [3], Bonami and Feuto introduced the following *variant local Orlicz–Hardy space* $h_*^\Phi(\mathbb{R}^n)$ which is defined to be the set

$$h_*^\Phi(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{h_*^\Phi(\mathbb{R}^n)} := \|f_{\mathrm{loc}}^*\|_{L_*^\Phi(\mathbb{R}^n)} < \infty\}, \tag{2.40}$$

where f_{loc}^* is defined as in (2.13) with some $m \in \mathbb{N} \cap (\lfloor n(1/p-1)\rfloor, \infty)$ and, for any measurable function g,

$$\|g\|_{L_*^\Phi(\mathbb{R}^n)} := \sum_{j\in\mathbb{Z}^n} \|g\|_{L^\Phi(\mathbb{Q}_j)}$$

with $j := (j_1, \ldots, j_n)$, $\mathbb{Q}_j := [j_1,\, j_1+1) \times \cdots \times [j_n,\, j_n+1)$ and

$$\|g\|_{L^\Phi(\mathbb{Q}_j)} := \inf\left\{\lambda \in (0,\,\infty) :\ \int_{\mathbb{Q}_j} \Phi\left(\frac{|g(x)|}{\lambda}\right)\, dx \le 1\right\}.$$

From their definitions, it immediately follows that $h^1(\mathbb{R}^n) \subset h_*^\Phi(\mathbb{R}^n)$. For more properties of the local Orlicz–Hardy spaces, we refer the reader to [3, 24, 23].

Proposition 2.11. *Let* Φ *be as in* (1.4). *Then the operator* Π_2 *can be extended to a bilinear operator bounded from* $h^1(\mathbb{R}^n) \times \mathrm{bmo}\,(\mathbb{R}^n)$ *to* $h_*^\Phi(\mathbb{R}^n)$.

Proof. Let $f \in h^1(\mathbb{R}^n)$ and $g \in \mathrm{bmo}\,(\mathbb{R}^n)$. As in the proof of Proposition 2.10, without loss of generality, we may assume that f has a finite wavelet expansion. By Theorem 2.7 with the notation same as therein, we know that $f = \sum_{l=1}^L \mu_l a_l$ has a finite local atomic decomposition. Thus, it holds true that

$$\Pi_2(f,\, g) = \sum_{l=1}^L \mu_l \Pi_2(a_l,\, g). \tag{2.41}$$

For any $l \in \{1, \ldots, L\}$, since each a_l also has a finite wavelet expansion, by an elementary calculation (see also [7, (3.17)] for a similar calculation), we further obtain

$$\Pi_2(a_l, g) = a_l P_{i_l} g + \Pi_2(a_l, b_l),$$

where, for any $l \in \{1, \ldots, L\}$,

$$b_l := \sum_{\substack{I \in \mathcal{D}_0 \\ I \subset 2mR_l}} \sum_{\lambda \in E \cup \{\vec{\theta}_n\}} \langle g, \Psi_I^\lambda \rangle \Psi_I^\lambda,$$

m is as in (2.1), Ψ_I^λ, for any $I \subset 2mR_l$ and $\lambda \in E \cup \{\vec{\theta}_n\}$, is as in (2.4), R_l denotes the associated dyadic cube of the atom a_l satisfying $|R_l| = 2^{-i_l n}$ for some $i_l \in \mathbb{Z}_+$ and P_{i_l} the orthogonal projection as in (2.6).

We now claim that there exists a positive constant $\widetilde{c}$, depending only on m from (2.1), such that, for any $l \in \{1, \ldots, L\}$, $\Pi_2(a_l, g)$ can be written as

$$\Pi_2(a_l, g) = h_l^{(1)} + \widetilde{c} h_l^{(2)} g_{R_l}, \tag{2.42}$$

where

$$g_{R_l} := \frac{1}{|R_l|} \int_{R_l} g(x)\, dx, \tag{2.43}$$

$h_l^{(1)} \in h^1(\mathbb{R}^n)$ satisfies $\|h_l^{(1)}\|_{h^1(\mathbb{R}^n)} \lesssim \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}$ with the implicit positive constant independent of g and $h_l^{(1)}$, and $h_l^{(2)}$ is an $h^1(\mathbb{R}^n)$-atom satisfying $\|h_l^{(2)}\|_{h^1(\mathbb{R}^n)} \leq 1$.

To show the above claim, for any $l \in \{1, \ldots, L\}$, recalling that R_l is the associated dyadic cube of the atom a_l and $|R_l| = 2^{-i_l n} \leq 1$, we have

$$a_l P_{i_l} g = \sum_{\substack{I \in \mathcal{D}_0 \\ |I| = 2^{-i_l n}}} a_l \langle g, \phi_I \rangle \phi_I. \tag{2.44}$$

Observe that, for any $l \in \{1, \ldots, L\}$, $I \in \mathcal{D}_0$ and $|I| = 2^{-i_l n}$,

$$a_l \phi_I \neq 0 \quad \text{if and only if } I \subset 2mR_l \tag{2.45}$$

and

$$\#\{I \in \mathcal{D}_0 :\ |I| = 2^{-i_l n},\, I \subset 2mR_l\} \leq (2m)^n, \tag{2.46}$$

where $\#E$ for any set E denotes the number of its elements. Thus, by (2.45) and (2.46), we know that the summation in (2.44) is of finite terms with the number not greater than $(2m)^n$. Moreover, for any $l \in \{1, \ldots, L\}$, we write

$$\begin{aligned} a_l \langle g, \phi_I \rangle \phi_I &= a_l \phi_I |R_l|^{\frac{1}{2}} \langle g, |R_l|^{-\frac{1}{2}} \phi_I \rangle \\ &= a_l \phi_I |R_l|^{\frac{1}{2}} \left\langle g, \frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l} \right\rangle + a_l \phi_I |R_l|^{\frac{1}{2}} g_{R_l}. \end{aligned}$$

For any $l \in \{1, \ldots, L\}$, since a_l is an $h^1(\mathbb{R}^n)$-atom associated with R_l, we know that $a_l \phi_I |R_l|^{\frac{1}{2}}$ is a harmless constant multiple of an $h^1(\mathbb{R}^n)$-atom associated with $2mR_l$. Also, for any $l \in \{1, \ldots, L\}$, we know that $\frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l}$ is a harmless constant multiple of an $H^1(\mathbb{R}^n)$-atom associated with $2mR_l$, which, combined with the assumption that $g \in \mathrm{bmo}\,(\mathbb{R}^n) \subset \mathrm{BMO}\,(\mathbb{R}^n)$, shows that

$$\left|\left\langle g,\, \frac{1}{|R_l|^{1/2}}\phi_I - \frac{1}{|R_l|}\chi_{R_l}\right\rangle\right| \lesssim \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}.$$

Thus, we conclude that, for any $l \in \{1, \ldots, L\}$,

$$\left\|a_l\phi_I|R_l|^{\frac{1}{2}}\left\langle g,\, \frac{1}{|R_l|^{1/2}}\phi_I - \frac{1}{|R_l|}\chi_{R_l}\right\rangle\right\|_{h^1(\mathbb{R}^n)} \lesssim \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}. \tag{2.47}$$

Now, for any $l \in \{1, \ldots, L\}$, let

$$h_l^{(1)} := \Pi_2(a_l,\, b_l) + \sum_{\substack{I\in\mathcal{D}_0\\ |I|=2^{-i_l n}}} a_l\phi_I|R_l|^{\frac{1}{2}}\left\langle g,\, \frac{1}{|R_l|^{1/2}}\phi_I - \frac{1}{|R_l|}\chi_{R_l}\right\rangle.$$

By the fact that Π_2 is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $h^1(\mathbb{R}^n)$ (see Theorem 2.9(ii)) and a calculation similar to that used in estimate (2.38), we find that, for any $l \in \{1, \ldots, L\}$,

$$\|\Pi_2(a_l,\, b_l)\|_{h^1(\mathbb{R}^n)} \lesssim \|a_l\|_{L^2(\mathbb{R}^n)}\|b_l\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)},$$

which, together with (2.47), implies that $h_l^{(1)} \in h^1(\mathbb{R}^n)$ and

$$\|h_l^{(1)}\|_{h^1(\mathbb{R}^n)} \lesssim \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)} \tag{2.48}$$

with the implicit constant independent of g and l.

On the other hand, by (2.45) and (2.46), we find a positive constant $\widetilde{c}$, depending only on m from (2.1), such that, for any $l \in \{1, \ldots, L\}$,

$$\left\|\frac{1}{\widetilde{c}}\sum_{\substack{I\in\mathcal{D}_0\\ |I|=2^{-i_l n}}} a_l\phi_l|I|^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)} \le |2mR_l|^{-\frac{1}{2}}. \tag{2.49}$$

Now, for any $l \in \{1, \ldots, L\}$, let

$$h_l^{(2)} := \sum_{\substack{I\in\mathcal{D}_0\\ |I|=2^{-i_l n}}} \frac{1}{\widetilde{c}}a_l\phi_I|I|^{\frac{1}{2}}.$$

For any $l \in \{1, \ldots, L\}$, by (2.49), combined with the fact that a_l is an $h^1(\mathbb{R}^n)$-atom associated with R_l, we know that $h_l^{(2)}$ is also an $h^1(\mathbb{R}^n)$-atom associated with $2mR_l$. This, together with (2.48), shows that the above claim (2.42) holds true.

We now continue the proof of Proposition 2.11 by extending the consideration from atoms to elements in $h^1(\mathbb{R}^n)$ having finite wavelet expansion. From (2.41) and the claim (2.42), we deduce that

$$\begin{aligned}\|\Pi_2(f,\,g)\|_{h_*^\Phi(\mathbb{R}^n)} &= \left\|\sum_{l=1}^L \mu_l \Pi_2(a_l,\,g)\right\|_{h_*^\Phi(\mathbb{R}^n)}\\ &= \left\|\sum_{l=1}^L \mu_l [h_l^{(1)} + \widetilde{c} h_l^{(2)} g_{R_l}]\right\|_{h_*^\Phi(\mathbb{R}^n)}\\ &\lesssim \left\|\sum_{l=1}^L \mu_l h_l^{(1)}\right\|_{h_*^\Phi(\mathbb{R}^n)} + \left\|\sum_{l=1}^L \mu_l h_l^{(2)} g_{R_l}\right\|_{h_*^\Phi(\mathbb{R}^n)} =: \mathrm{A} + \mathrm{D}. \qquad (2.50)\end{aligned}$$

Using the fact that $h^1(\mathbb{R}^n) \subset h_*^\Phi(\mathbb{R}^n)$ and (2.48), we obtain

$$\mathrm{A} \lesssim \left\|\sum_{l=1}^L \mu_l h_l^{(1)}\right\|_{h^1(\mathbb{R}^n)} \lesssim \sum_{l=1}^L |\mu_l| \|h_l^{(1)}\|_{h^1(\mathbb{R}^n)} \lesssim \|f\|_{h^1(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}, \qquad (2.51)$$

which is the desired estimate.

To estimate D, we write

$$\mathrm{D} \lesssim \left\|\sum_{l\in L_1} \mu_l h_l^{(2)} g_{R_l}\right\|_{h_*^\Phi(\mathbb{R}^n)} + \left\|\sum_{l\in L_2} \mu_l h_l^{(2)} g_{R_l}\right\|_{h_*^\Phi(\mathbb{R}^n)} =: \mathrm{D}_1 + \mathrm{D}_2, \qquad (2.52)$$

where $L_1,\, L_2 \subset \mathbb{N}$ satisfy $L_1 \cap L_2 = \emptyset$, $L_1 \cup L_2 = \{1, \ldots, L\}$ and, for each $l \in L_1$, $|R_l| < 1$; for each $l \in L_2$, $|R_l| \geq 1$. Using again the fact that $h^1(\mathbb{R}^n) \subset h_*^\Phi(\mathbb{R}^n)$ and (2.15), we find

$$\begin{aligned}\mathrm{D}_2 &\lesssim \left\|\sum_{l\in L_2} \mu_l h_l^{(2)} g_{R_l}\right\|_{h^1(\mathbb{R}^n)} \lesssim \sum_{l\in L_2} |\mu_l| |g_{R_l}| \|h_l^{(2)}\|_{h^1(\mathbb{R}^n)}\\ &\lesssim \|f\|_{h^1(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}, \qquad (2.53)\end{aligned}$$

which is the desired estimate.

We now turn to the estimate for D_1. In this case, we know that, for each $l \in L_1$, $|R_l| < 1$, which, together with Remark 2.5, implies that, for each $l \in L_1$, $h_l^{(2)}$ is an $H^1(\mathbb{R}^n)$-atom associated with $2mR_l$. Moreover, by (2.40), we write

$$\begin{aligned}\mathrm{D}_1 &\lesssim \left\|\sum_{l\in L_1} \mu_l (h_l^{(2)})^*_{\mathrm{loc}} |g - g_{R_l}|\right\|_{L_*^\Phi(\mathbb{R}^n)} + \left\|\sum_{l\in L_1} \mu_l (h_l^{(2)})^*_{\mathrm{loc}} |g|\right\|_{L_*^\Phi(\mathbb{R}^n)}\\ &=: \mathrm{D}_{1,\,1} + \mathrm{D}_{1,\,2}, \qquad (2.54)\end{aligned}$$

where $(h_l^{(2)})^*_{\mathrm{loc}}$ is defined as in (2.13) with some $m \in \mathbb{N} \cap (\lfloor n(1/p-1)\rfloor, \infty)$.

For $\mathrm{D}_{1,\,1}$, using the facts that $L^1(\mathbb{R}^n) \subset L_*^\Phi(\mathbb{R}^n)$, $h_l^{(2)}$ is an $H^1(\mathbb{R}^n)$-atom in this case and, for any $g \in \mathrm{BMO}\,(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$-atom a associated with some cube R

(see, for example, [5, p. 238]),

$$\int_{\mathbb{R}^n} a^*_{\mathrm{loc}}(x)|g(x) - g_R|\,dx \lesssim \|g\|_{\mathrm{BMO}\,(\mathbb{R}^n)} \lesssim \|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)},$$

we know that

$$\begin{aligned}\mathrm{D}_{1,\,1} &\lesssim \left\|\sum_{l\in L_1} \mu_l (h_l^{(2)})^*_{\mathrm{loc}}\,|g - g_{R_l}|\right\|_{L^1(\mathbb{R}^n)}\\ &\lesssim \sum_{l\in L_1}|\mu_l|\int_{\mathbb{R}^n}(h_l^{(2)})^*_{\mathrm{loc}}(x)|g(x) - g_{R_l}|\,dx\\ &\lesssim \|f\|_{h^1(\mathbb{R}^n)}\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)},\end{aligned} \tag{2.55}$$

which is the desired estimate.

For $\mathrm{D}_{1,\,2}$, by the generalized Hölder inequality (see, for example, [3, Lemma 3.1]), we know that, for any $j \in \mathbb{Z}^n$, $\mathbb{Q}_j$ as in (2.40), $f \in L^1(\mathbb{Q}_j)$ and $g \in L^{\exp}(\mathbb{Q}_j)$, it holds true that

$$\|fg\|_{L^\Phi_*(\mathbb{Q}_j)} \lesssim \|f\|_{L^1(\mathbb{Q}_j)}\|g\|_{L^{\exp}(\mathbb{Q}_j)}, \tag{2.56}$$

where

$$\|g\|_{L^{\exp}(\mathbb{Q}_j)} := \inf\left\{\lambda \in (0,\,\infty) :\ \int_{\mathbb{Q}_j} \exp\left(\frac{|g(x)|}{\lambda}\right)\,dx \le 1\right\}.$$

Thus, by (2.56) and the John–Nirenberg inequality for functions in $\mathrm{bmo}\,(\mathbb{R}^n)$ (see, for example, [3, (22)]), we find that, for any $f \in L^1(\mathbb{R}^n)$ and $g \in \mathrm{bmo}\,(\mathbb{R}^n)$,

$$\begin{aligned}\|fg\|_{L^\Phi_*(\mathbb{R}^n)} &= \sum_{j\in\mathbb{Z}^n}\|fg\|_{L^\Phi(\mathbb{Q}_j)} \lesssim \sum_{j\in\mathbb{Z}^n}\|f\|_{L^1(\mathbb{Q}_j)}\|g\|_{L^{\exp}(\mathbb{Q}_j)}\\ &\lesssim \|f\|_{L^1(\mathbb{R}^n)}\sup_{j\in\mathbb{Z}^n}\|g\|_{L^{\exp}(\mathbb{Q}_j)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}.\end{aligned}$$

By this, we further know that

$$\begin{aligned}\mathrm{D}_{1,\,2} &= \left\|\sum_{l\in L_1}\mu_l(h_l^{(2)})^*_{\mathrm{loc}}\,|g|\right\|_{L^\Phi_*(\mathbb{R}^n)} \lesssim \left\|\sum_{l\in L_1}\mu_l(h_l^{(2)})^*_{\mathrm{loc}}\right\|_{L^1(\mathbb{R}^n)}\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)}\\ &\lesssim \sum_{l\in L_1}|\mu_l|\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)} \lesssim \|f\|_{h^1(\mathbb{R}^n)}\|g\|_{\mathrm{bmo}\,(\mathbb{R}^n)},\end{aligned} \tag{2.57}$$

which is the desired estimate.

Combining (2.50) through (2.57), we conclude that Π_2 can be extended to a bilinear operator bounded from $h^1(\mathbb{R}^n) \times \mathrm{bmo}\,(\mathbb{R}^n)$ to $h^\Phi_*(\mathbb{R}^n)$. This finishes the proof of Proposition 2.11. $\square$

Remark 2.12. Let θ be the Musielak–Orlicz function as in (1.1). Recall that, in [24], the *local Musielak–Orlicz–Hardy space* $h^{\log}(\mathbb{R}^n)$ is defined to be the set

$$h^{\log}(\mathbb{R}^n) := \Bigg\{ f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{h^{\log}(\mathbb{R}^n)} := \|f^*_{\rm loc}\|_{L^{\log}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) :\ \int_{\mathbb{R}^n} \theta\left(x,\, \frac{|f^*_{\rm loc}(x)|}{\lambda}\right) dx \le 1\right\} < \infty \Bigg\}, \tag{2.58}$$

where $f^*_{\rm loc}$ is defined as in (2.13) with some $m \in \mathbb{N} \cap (\lfloor n(1/p-1)\rfloor, \infty)$ (see also [23] for a complete theory of local Musielak–Orlicz–Hardy spaces). From their definitions, we deduce that $h^1(\mathbb{R}^n) \subset h^{\log}(\mathbb{R}^n)$. Moreover, by [5, Proposition 2.1], we know that the following generalized Hölder inequality holds true, namely, for any $f \in L^1(\mathbb{R}^n)$ and $g \in \mathrm{bmo}\,(\mathbb{R}^n)$,

$$\|fg\|_{L^{\log}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}[\|g\|_{\mathrm{BMO}(\mathbb{R}^n)} + |g_{\mathbb{Q}_0}|] \lesssim \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{\mathrm{bmo}(\mathbb{R}^n)}, \tag{2.59}$$

where $\mathbb{Q}_0$ denotes the unit cube and $g_{\mathbb{Q}_0}$ is defined as in (2.43). Thus, if we replace the generalized Hölder inequality (2.56) by (2.59), and follow the same argument used in the remainder of the proof of Proposition 2.11, we then conclude that the operator Π_2 can be extended to a bilinear operator bounded from $h^1(\mathbb{R}^n)\times \mathrm{bmo}\,(\mathbb{R}^n)$ to $h^{\log}(\mathbb{R}^n)$, the details being omitted.

3. Proofs of Theorems 1.1 and 1.2

The main aim of this section is to prove Theorems 1.1 and 1.2. We first prove Theorem 1.1 with the help of Propositions 2.10 and 2.11.

Proof of Theorem 1.1. For any $f,\, g \in L^2(\mathbb{R}^n)$, let $S(f,\, g) := \Pi_4(f,\, g)$ and

$$T(f,\, g) := \sum_{i=1}^{3} \Pi_i(f,\, g).$$

In the case when $f \in h^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $g \in \Lambda_\alpha(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, by Proposition 2.10, we know that $S(f,\, g) \in L^1(\mathbb{R}^n)$ and $T(f,\, g) \in h^p(\mathbb{R}^n)$. In the case when $f \in h^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $g \in \mathrm{bmo}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, by Propositions 2.10 and 2.11, we know that $S(f,\, g) \in L^1(\mathbb{R}^n)$ and $T(f,\, g) \in h^\Phi_*(\mathbb{R}^n)$.

Now, for any $f \in h^p(\mathbb{R}^n)$ and $g \in \Lambda_\alpha(\mathbb{R}^n)$, by Lemma 2.6, we know that there exists a family $\{f_k\}_{k\in\mathbb{N}} \subset h^p(\mathbb{R}^n)$ having finite wavelet expansions such that $\lim_{k\to\infty} f_k = f$ in $h^p(\mathbb{R}^n)$. By the definition of $f \times g$ in (2.34), we know that

$$f \times g = \lim_{k\to\infty} f_k\, g \tag{3.1}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where, for each $k \in \mathbb{N}$, $f_k\, g$ denotes the usual pointwise multiplication of f_k and g. Since f_k has a finite wavelet expansion, it follows that $f_k \in L^2(\mathbb{R}^n)$ and has compact support. Let Q be a cube satisfying $\mathrm{supp}\, f_k \subset Q$, $l(Q) \ge 1$ and let η_k be a cut-off function satisfying $\mathrm{supp}\,\eta_k \subset 8m\sqrt{n}Q$ and $\eta_k \equiv 1$ on $4m\sqrt{n}Q$, where

m is as in (2.1). It is easy to see that $f_k\, g = f_k\,(\eta_k g)$ and $\eta_k g \in L^2(\mathbb{R}^n)$. From (2.6), we deduce that

$$f_k\, g = f_k(\eta_k g) = \sum_{i=1}^{4} \Pi_i(f_k,\, \eta_k g)$$

in $L^1(\mathbb{R}^n)$ and hence in $\mathcal{S}'(\mathbb{R}^n)$.

By (2.7), (2.3) and the definition of η_k, we find that

$$\begin{aligned}\Pi_1(f_k,\, \eta_k g) &= \sum_{\substack{I,\, I' \in \mathcal{D}_0 \\ |I|=|I'|}} \sum_{\lambda\in E} \langle f_k,\, \phi_I\rangle \langle \eta_k g,\, \psi_{I'}^{\lambda}\rangle \phi_I \psi_{I'}^{\lambda} \\ &= \sum_{\substack{I,\, I' \in \mathcal{D}_0 \\ |I|=|I'|}} \sum_{\lambda\in E} \langle f_k,\, \phi_I\rangle \langle g,\, \psi_{I'}^{\lambda}\rangle \phi_I \psi_{I'}^{\lambda} = \Pi_1(f_k,\, g).\end{aligned}$$

Similarly, we have $\Pi_i(f_k,\, \eta_k g) = \Pi_i(f_k,\, g)$ for any $i \in \{2,\, 3,\, 4\}$. Thus, we conclude that $f_k\, g = \sum_{i=1}^{4} \Pi_i(f_k,\, g)$ holds true in $\mathcal{S}'(\mathbb{R}^n)$. Combining (3.1) and Propositions 2.10 and 2.11, we find that

$$f \times g = \sum_{i=1}^{4} \lim_{k\to\infty} \Pi_i(f_k,\, g) = \Pi_4(f,\, g) + \left[\sum_{i=1}^{3} \Pi_i(f,\, g)\right] =: S(f,\, g) + T(f,\, g)$$

in $\mathcal{S}'(\mathbb{R}^n)$, where $S(f,\, g) := \Pi_4(f,\, g) \in L^1(\mathbb{R}^n)$ and $T(f,\, g) := \sum_{i=1}^{3} \Pi_i(f,\, g) \in h^p(\mathbb{R}^n)$. This shows that (i) holds true.

The proof of (ii) follows from an argument similar to that used in the proof of (i), the details being omitted, which completes the proof of Theorem 1.1. □

Remark 3.1. From Remark 2.12, the fact $h^1(\mathbb{R}^n) \subset h^{\log}(\mathbb{R}^n)$ and the proof of Theorem 1.1, we deduce that the operator T can be extended to a bilinear operator bounded from $h^1(\mathbb{R}^n) \times \mathrm{bmo}\,(\mathbb{R}^n)$ to $h^{\log}(\mathbb{R}^n)$, where $h^{\log}(\mathbb{R}^n)$ denotes the local Musielak–Orlicz–Hardy space as in (2.58). This implies that the bilinear decomposition

$$h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + h^{\log}(\mathbb{R}^n)$$

also holds true.

We also point out that the following bilinear decomposition, for any $p \in (\frac{n}{n+1},\, 1)$,

$$h^p(\mathbb{R}^n) \times \Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + h^p(\mathbb{R}^n),$$

obtained in Theorem 1.1, is sharp in view of the following result on pointwise multipliers of $\Lambda_\alpha(\mathbb{R}^n)$, with $\alpha := n(\frac{1}{p} - 1)$, from [19, Theorem 2]: the pointwise multiplier class of $\Lambda_\alpha(\mathbb{R}^n)$ equals to $\Lambda_\alpha = \big(L^1(\mathbb{R}^n) + h^p(\mathbb{R}^n)\big)^*$.

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\mathbf{F} \in h^1(\mathbb{R}^n;\, \mathbb{R}^n)$ with $\operatorname{curl} \mathbf{F} \equiv 0$ in the sense of distributions and $\mathbf{G} \in \mathrm{bmo}(\mathbb{R}^n;\, \mathbb{R}^n)$ with $\operatorname{div} \mathbf{G} \equiv 0$ in the sense of distributions.

By Theorem 1.1, we know that

$$\mathbf{F}\cdot\mathbf{G}=\sum_{i=1}^{n}F_i\times G_i=\sum_{i=1}^{n}S(F_i,\,G_i)+\sum_{i=1}^{n}T(F_i,\,G_i)=:\mathrm{A}(\mathbf{F},\,\mathbf{G})+\mathrm{B}(\mathbf{F},\,\mathbf{G}).$$

From Theorem 1.1(ii), it immediately follows that $\mathrm{B}(\mathbf{F},\,\mathbf{G})\in h_*^{\Phi}(\mathbb{R}^n)$ and

$$\|\mathrm{B}(\mathbf{F},\,\mathbf{G})\|_{h_*^{\Phi}(\mathbb{R}^n)}\lesssim\|\mathbf{F}\|_{h^1(\mathbb{R}^n;\,\mathbb{R}^n)}\|\mathbf{G}\|_{\mathrm{bmo}(\mathbb{R}^n;\,\mathbb{R}^n)},$$

where Φ is as in (1.4). Thus, to finish the proof of Theorem 1.2, it suffices to show $\mathrm{A}(\mathbf{F},\,\mathbf{G})\in h_*^{\Phi}(\mathbb{R}^n)$. To this end, we only need to prove that $\mathrm{A}(\mathbf{F},\,\mathbf{G})\in h^1(\mathbb{R}^n)$ because of the inclusion $h^1(\mathbb{R}^n)\subset h_*^{\Phi}(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n;\,\mathbb{R}^n)$ is dense in $h^1(\mathbb{R}^n;\,\mathbb{R}^n)$, without loss of generality, we may assume that $\mathbf{F}\in h^1(\mathbb{R}^n;\,\mathbb{R}^n)\cap L^2(\mathbb{R}^n;\,\mathbb{R}^n)$ (the general case $\mathbf{F}\in h^1(\mathbb{R}^n;\,\mathbb{R}^n)$ can be proved via a density argument). Using the Helmholtz decomposition (see, for example, [6, Sec. 4] for more details), we find that there exists

$$f:=-\sum_{i=1}^{n}R_i(F_i)\in h^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$$

such that $\mathbf{F}=\nabla(-\Delta)^{-1/2}f$, where, for any $i\in\{1,\ldots,n\}$, R_i denotes the ith *Riesz transform* defined by $R_i:=\frac{\partial}{\partial x_i}(-\Delta)^{-1/2}$. Moreover, since $\operatorname{div}\mathbf{G}\equiv 0$, it follows that $\sum_{i=1}^{n}R_i(G_i)\equiv 0$. Thus, we can write

$$\mathrm{A}(\mathbf{F},\,\mathbf{G})=\sum_{i=1}^{n}S(F_i,\,G_i)=\sum_{i=1}^{n}\left[S(R_i(f),\,G_i)+S(f,\,R_i(G_i))\right].$$

Using (2.10) and the fact that R_i is a Calderón–Zygmund operator with odd kernel, we further find that, for each $i\in\{1,\ldots,n\}$,

$$\begin{aligned}&S(R_i(f),\,G_i)+S(f,\,R_i(G_i))\\&\quad=\sum_{I,\,I'\in\mathcal{D}_0}\sum_{\lambda,\,\lambda'\in E}\langle f,\,\psi_I^{\lambda}\rangle\langle G_i,\,\psi_{I'}^{\lambda'}\rangle\langle R_i\psi_I^{\lambda},\,\psi_{I'}^{\lambda'}\rangle[(\psi_{I'}^{\lambda'})^2-(\psi_I^{\lambda})^2],\end{aligned}$$

which, combined with some similar calculations to those used in the proof of [5, Lemma 6.1], implies that there exists a positive constant $\delta\in(0,\,\frac{1}{2}]$ such that

$$\begin{aligned}&\|S(R_i(f),\,G_i)+S(f,\,R_i(G_i))\|_{h^1(\mathbb{R}^n)}\\&\quad\lesssim\sum_{I,\,I'\in\mathcal{D}_0}\sum_{\lambda,\,\lambda'\in E}\left|\langle f,\,\psi_I^{\lambda}\rangle\right|\left|\langle G_i,\,\psi_{I'}^{\lambda'}\rangle\right|p_\delta(I,\,I'),\end{aligned}$$

where, for any $|I|=2^{-jn}$ and $|I'|=2^{-j'n}$ with centers at x_I, respectively, at $x_{I'}$,

$$p_\delta(I,\,I'):=2^{-|j-j'|(\delta+n/2)}\left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|x_I-x_{I'}|}\right)^{n+\delta}.$$

This shows that the coefficient matrix of A is almost diagonal (see [11, Theorem 3.3, p. 132] for more details), which, combined with a calculation similar to that used in estimate (2.32), the boundedness of the Riesz transform $\nabla(-\Delta)^{-1/2}$ on $h^1(\mathbb{R}^n)$, [11, Theorem 3.3] and Theorem 2.8(i), implies that

$$\begin{aligned}
&\|S(R_i(f),\, G_i) + S(f,\, R_i(G_i))\|_{h^1(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{I'\in\mathcal{D}_0}\sum_{\lambda'\in E}\left[\sum_{I\in\mathcal{D}_0}\sum_{\lambda\in E}\left|\langle f,\, \psi_I^\lambda\rangle\right| p_\delta(I,\, I')\right]|\langle G_i,\, \psi_{I'}^{\lambda'}\rangle| \\
&\quad\lesssim \left\|\left\{\sum_{I\in\mathcal{D}_0}\sum_{\lambda\in E}\left|\langle f,\, \psi_I^\lambda\rangle\right| p_\delta(I,\, I')\right\}_{I'\in\mathcal{D}_0,\, \lambda'\in E}\right\|_{f^0_{1,\,2}(\mathbb{R}^n)} \\
&\qquad\times \|\{\langle G_i,\, \psi_{I'}^{\lambda'}\rangle\}_{I'\in\mathcal{D}_0,\, \lambda'\in E}\|_{\mathcal{C}_1(\mathbb{R}^n)} \\
&\quad\lesssim \|\{\langle f,\, \psi_I^\lambda\rangle\}_{I\in\mathcal{D}_0,\, \lambda\in E}\|_{f^0_{1,\,2}(\mathbb{R}^n)}\|\{\langle G_i,\, \psi_{I'}^{\lambda'}\rangle\}_{I'\in\mathcal{D}_0,\, \lambda'\in E}\|_{\mathcal{C}_1(\mathbb{R}^n)} \\
&\quad\lesssim \|\mathbf{F}\|_{h^1(\mathbb{R}^n;\, \mathbb{R}^n)}\|\mathbf{G}\|_{\mathrm{bmo}(\mathbb{R}^n;\, \mathbb{R}^n)}.
\end{aligned}$$

This shows that $\mathrm{A}(\mathbf{F},\, \mathbf{G}) \in h^1(\mathbb{R}^n)$ and

$$\|\mathrm{A}(\mathbf{F},\, \mathbf{G})\|_{h^1(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{h^1(\mathbb{R}^n;\, \mathbb{R}^n)}\|\mathbf{G}\|_{\mathrm{bmo}(\mathbb{R}^n;\, \mathbb{R}^n)},$$

which completes the proof of Theorem 1.2. □

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