Review: Jean Dieudonné, History of functional analysis

Article *in* Bulletin of the American Mathematical Society · January 1982

Source: OAI

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 2, September 1982

0273-0979/82/0000-0122/$02.50

BOOK REVIEWS

History of functional analysis, by Jean Dieudonné, Notas de Matemática, no. 77, North-Holland Mathematics Studies, no. 49, North-Holland Publishing Company, Amsterdam, 1981, vi + 312 pp., $29.50.

> *The infinite! No other question has ever moved so profoundly the spirit of man.*—David Hilbert (1921)

In the last decade or two there has been renewed interest among mathematicians regarding the history of their subject. Books and papers focusing on the history of various topics within comtemporary mathematics are now commonplace. Many mathematical journals actively solicit and promote high quality survey articles which offer historical perspective. After many years of explosive development, there is now a growing awareness of the importance of interpreting and reflecting on how mathematics arrived at its present state. Although most mathematicians know that many problems have their origins in *classical* problems, few of us (it appears safe to say) have had the time, inclination, interest, patience (or ability?) to unravel the precise sequence of events connecting these problems. The point of view of present-day historians of mathematics is that it is worthwhile to know about these ties with classical problems and the subsequent evolution stimulated by them. The admirable book under review, written by an eminently qualified mathematician, makes a notable contribution to the understanding of the historical process that has shaped what is known today as functional analysis.

What is functional analysis? Perhaps it is surprising that this term is ordinarily not defined even by those who write on the subject. One reason for this appears to stem from the fact that, to date, functional analysis has not completely crystallized as a single discipline but rather suggests a grouping of subjects which, in certain respects, have more in common regarding method then content. It is the essence of functional analysis that concepts and methods of classical analysis and related branches of mathematics be extended to more general objects. Such generalization makes it possible to approach, from a unified point of view, questions which earlier appeared isolated or to have little in common. Furthermore, the very general nature of the techniques of functional analysis often reveal deep insights and new results that otherwise would escape detection.

Let us return to the opening question of the previous paragraph: What is functional analysis? To achieve some degree of focus and simultaneously to allow for sufficient coverage of applications the author adopts the following definition: *Functional analysis* is "the study of topological vector spaces and mappings $u: \Omega \to F$ from a subset Ω of a topological vector space E into a topological vector space F, these mappings being assumed to satisfy various algebraic and topological conditions". This definition is wide enough in scope to include most of the "standard topics" considered to fall within the study of functional analysis. In particular, it includes Hilbert space and the spectral theory of operators (bounded and unbounded), the theory of normed linear spaces, the theory of Banach algebras and operator algebras (C^*-algebras and von Neumann algebras), the general theory of topological vector spaces, generalized functions (distributions), and the theory of partial differential equations.

The fundamental concepts and methods of functional analysis arose gradually (in the early years, at least) from the oldest parts of mathematical analysis: the calculus of variations, integral equations, the theory of orthogonal functions, Chebyshev approximation theory, and the moment problem. Much of the modern day terminology in functional analysis originated within these classical subjects. For example, the concept of 'functional' had its origin in the study of the calculus of variations. On the other hand, the development of set theory, general topology, abstract algebra, and the Lebesgue integral paved the way for a systematic treatment of the new generalized methods in abstract form. Since the most frequently studied mappings between topological vector spaces are linear, it is not surprising that linear algebra should have greatly influenced functional analysis, and this indeed turned out to be the case (however, this influence, especially in the beginning, was not always positive).

The book under review is divided into nine chapters, the first eight of which describe a particular "era" or "chunk" in the history of functional analysis. Each of these chapters is structured around ideas or influential papers which had a major impact on the development of the subject. The final chapter is concerned primarily with applications of functional analysis to the theory of differential equations. The author's writing style is informal (in fact, nearly conversational) which together with his expertise in the subject makes the presentation a delight. His device of combining historical development with a clear (but far from linear) description of the material is particularly effective in giving the reader an unobstructed view of the difficulties encountered by the early writers, as well as their methods of dealing with these difficulties. While protecting the reader from obscurities of language and notation of earlier times except by way of illustration or making a point, Dieudonné's principal concern is the process itself, the actual record of the way functional analysis developed. He skillfully traces who first introduced important ideas, concepts, and methods (the reviewer counted no less than twenty-five statements which began with the phrase "It was first proved by ...") and the corresponding dates. Zipping back and forth through time periods ranging from one to nearly two-hundred years Dieudonné discusses techniques and arguments of the early innovators. He offers praise as well as criticism. The depth and originality of the results are weighed as well as their influence on the historical development of the subject. All of this is accompanied by commentary whose purpose is to place it in proper historical context. Of course, since mathematicians rarely leave behind their inner thoughts concerning how an important theorem came to be formulated and eventually proved, it is inevitable that some speculation is required in a work such as this one. Speculation also arises with questions of priority, i.e., when an important idea or theorem is duplicated (at least in spirit if not in fact) several years after it originated or was published by another mathematician. Dieudonné does not hesitate to point out overlapping results and duplications. However, he usually sidesteps the hard questions of foreknowledge of the results by the later author. It, in fact, appears that Dieudonné generally assumes that all parties are innocent partly on the basis that mathematicians of the caliber he is discussing would not resort to publishing work which is not due to them, and partly on the basis that most of the papers in which there is duplication take a decidedly different direction except for the duplication.

Turning to the contents of the individual chapters one finds a lively description of many important subjects whose development is inextricably connected with the foundations of functional analysis. Chapter I (13 pp.) contains a discussion of the historical relevance of linear differential equations and the Sturm-Liouville problem to the birth of modern spectral theory. There is a discussion of the contributions of Lagrange (who, we are told, introduced the notion of *adjoint* of a linear differential operator), Euler, D'Alembert, D. Bernoulli, J. Fourier, Parseval, Poisson, C. Sturm, J. Liouville, and several lesser known figures. Although this chapter is quite short, it contains an unusually clear summary of the early work on linear differential equations, Fourier analysis, and the Sturm-Liouville problem.

Chapter II (17 pp.) entitled *The crypto-integral equations* is divided into five sections dealing with successive approximations, partial differential equations, the beginnings of potential theory, the Dirichlet principle, and the Beer-Neumann method. The historical scope of the chapter is fairly broad, covering roughly the period 1800 to 1895. Most of the thrust of the chapter centers on attempts to solve the Dirichlet problem. The contributions of H. A. Schwarz, H. Poincaré, C. Beer, and C. Neumann concerning the problem are discussed in detail. While the work of Schwarz and Poincaré described here, based on approximation techniques, did not directly influence the development of functional analysis, the work of Neumann in conjunction with earlier work of Beer was a landmark in functional analysis because it contained the first example of what is now called a "Fredholm integral equation of the second kind."

Chapter III (23 pp.) entitled *The equation of vibrating membranes* is devoted to a paper of H. A. Schwarz written in 1885 and work of H. Poincaré during the period 1887-1895. In his study of minimal surfaces, Schwarz is led to consider the partial differential equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \lambda^2 p v = 0,$$

where p is a strictly positive continuous function defined on a domain D of the xy-plane. So ingenious are the techniques of Schwarz that Dieudonné is led to state "Schwarz's paper is extremely remarkable by the originality of its methods, which do not seem to have been inspired by any previous work." To demonstrate the power of Schwarz's methods Dieudonné proceeds to show that Schwarz's arguments can be translated, almost without change, to the modern theory of selfadjoint compact operators in a separable Hilbert space. Incidently, this 1885 paper contains a double integral version of the famous inequality which is named after Schwarz (the inequality had been considered earlier by Cauchy (1821) and Buniakovskii (1859) in different settings, but Schwarz did not mention this). The remainder of Chapter III is devoted to three long papers of H. Poincaré. Once again we have the opportunity to observe a truly creative mind at work. Concerning Poincaré's attack on the problems at hand Dieudonné states "it is quite remarkable to see Poincaré introducing a whole batch of *completely new ideas*."

The concept of infinite dimension is taken up in Chapter IV (25 pp.). The evolution of linear algebra is summarized to form a backdrop for the ideas which led to functional analysis. Because mathematicians were unwilling to let go of familiar techniques used for solving systems of linear equations, Dieudonné points out that the historical development of linear algebra ended up in exactly the reverse order of what we now consider to be the *logical* order. Infinite determinants are discussed as well as first notions of "function spaces" and operators between them. The passage from "finiteness to infinity" emerged in the first *general* theory of integral equations with the work of Le Roux in 1894 and Volterra in 1896.

Chapter V (24 pp.) is titled *The crucial years and the definition of Hilbert space*. Between 1900 and 1910, there was a sudden crystallization of all the ideas and methods which had been slowly accumulating during the nineteenth century. This was essentially due to the publication of four fundamental papers: (1) a paper of Fredholm on integral equations published in 1900; (2) Lebesgue's 1902 thesis on integration; (3) a paper of Hilbert on spectral theory published in 1906; and (4) Fréchet's 1906 thesis on metric spaces. The central focus of the chapter is on the work of Fredholm and Hilbert although important contributions by E. Schmidt in 1908 are also discussed.

Duality and the definition of normed spaces is the topic of Chapter VI (22 pp.). The remarkable Riesz-Fréchet theorem and the development of the L^p-spaces led to many important concepts in functional analysis. The work of F. Riesz in 1910 is considered by Dieudonné to be second in importance only to Hilbert's 1906 paper on spectral theory. One of the highlights of this chapter is the description of work by the Austrian mathematician E. Helly in two papers, one in 1912 and the other in 1921 (the nine year interval between them is due to World War I, in which Helly was a prisoner of war in Russia). Helly's results were penetrating and deep; furthermore, they applied to *general* "normed sequence spaces" and for the first time did not depend on special features of the space, contrasting with the techniques of E. Schmidt and F. Riesz. Among other things, Helly's work contained the essential *computational* steps necessary in the proof of the Hahn-Banach extension theorem, a fact which had not been widely recognized by the mathematical community until quite recently.

The independent work of H. Hahn and S. Banach on axiomatic normed linear spaces is described in some detail leading up to the publication of Banach's landmark book in 1932. With the publication of this book analysts the world over began to realize the power of the new methods and to apply them to a great variety of problems.

Chapter VII (65 pp.) is titled *Spectral theory after* 1900. This is the longest chapter in the book and contains sections on F. Riesz's theory of compact operators, the spectral theory of Hilbert, the work of Weyl and Carleman, the spectral theory of von Neumann, the theory of Banach algebras, operator algebras, function algebras, harmonic analysis, representations of locally compact groups, and other topics. The discussion of these results is very modern with definitions provided and statements of the most important theorems. The basic viewpoint of the chapter is that the above topics can be absorbed under the general heading of "spectral theory."

Locally convex topological spaces and the theory of distributions is the topic of Chapter VIII (22 pp.). In his thesis Fréchet had already noticed that convergence in a metric space did not always correspond to certain types of "convergence" for functions. Thus there was a need to generalize the concept of metric space but none proved adequate for functional analysis until Hausdorff, in 1914, introduced the concept of neighborhood providing a basis for what is now known as general topology. Even so, it took mathematicians many years to formulate the general notion of topological vector space and they were not the subject of a systematic treatment until 1950. Dieudonné describes the various stages of development, emphasizing the role of boundedness, for the theory of locally convex spaces. Apart from briefly mentioning the important work of A. Grothendieck on the tensor product of two locally convex spaces, he refrains from discussing the vast number of papers on the subject after 1950. A very interesting description of the origin of generalized functions (distributions) and the subsequent efforts of L. Schwartz to weld all the previous ideas into a unified and cohesive theory is given in the second half of Chapter VIII. Dieudonné likens Schwartz's role in distribution theory to that of Newton and Leibniz in the development of calculus.

Finally, as mentioned earlier, Chapter IX (46 pp.) is concerned with applications of functional analysis to differential and partial differential equations. The chapter is divided into five sections dealing with fixed point theorems, Carleman operators, boundary problems, Sobolev spaces and *a priori* inequalities, and pseudo-differential operators. An effort has been made to supply the reader with enough background to make the reading of the sections interesting and intelligible. In addition, references have been given to enable the reader to pursue the applications further if he desires. Several books, at varying levels of sophistication, whose purpose is to apply the techniques of functional analysis have recently appeared (see [**1**, **5**, **6**, **11**]).

The author has justifiably chosen to stay within certain conventional boundaries in writing this history. There are, for example, no excursions into the historical origins of nonlinear functional analysis [**3**], partially ordered topological vector spaces [**7**, **9**], non-Archimedean functional analysis [**10**], general topological algebras [**2**], or Banach modules and categorical considerations in functional analysis [**4**]. The reasons for this are quite obvious since most of these topics are quite recent or have developed along lines distinct from the ideas of the present volume.

It should be mentioned that the book under review is not the first attempt to trace the history of functional analysis. Indeed, in 1973 A. F. Monna [**8**] published a book with many of the same objectives, and raised several questions (see [**8**, p. 55 and p. 133]) concerning the contributions of the Italian school of mathematicians to the foundations of functional analysis. His primary concern appears to be that their pioneering work before 1900 concerning the definition of normed linear spaces has not been properly acknowledged. This book by Monna is not listed among the references in Dieudonné's bibliography and no reference is made to it in the text. The reviewer would have appreciated seeing Dieudonné's response to some of Monna's questions.

Another reference missing from Dieudonné's bibliography is Alaoglu's 1938 Bulletin *announcement* [Bull. Amer. Math. Soc. **44** (1938), 196, 459] concerning the weak*-compactness of the closed unit ball in the dual of a normed space, and related results. This reference clearly establishes priority of Alaoglu over Bourbaki concerning these results, contrary to the account given on page 212. The absence of Alaoglu's announcement from the bibliography puzzles the reviewer. Indeed, Dieudonné referred to it in a 1949 review of his own (Mathematical Reviews **10** (1949), 611), where he acknowledges Alaoglu's priority concerning the above!

Putting aside such things, Dieudonné has given us a very readable and exciting account of how functional analysis has evolved. He has tried to communicate his message in several ways. For example, at the end of the introduction (p. 8) a complicated diagram has been constructed that depicts graphically (in some detail) the successive stages of this history and its interaction with other parts of mathematics. There are 235 entries in the bibliography (many of them collected works), an author index, and a subject index. A small number of misprints (< 20) have been detected but none of them should cause problems for the reader.

After the dust has settled, one is impressed that the remarkable feature of this history is that, after a slow beginning, spectral theory, in the span of a few years, reached complete maturity, giving rise in the process to the important concept of linear duality. A lot of information is packed into 312 pages! This is essential reading for functional analysts who wish to know how their subject came into existence.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 2, September 1982

0273-0979/82/0000-0121/$01.50

Operator inequalities, by Johann Schröder, Mathematics in Science and Engineering, vol. 147, Academic Press, New York, 1980, xvi + 367 pp., $39.50.

There exists an extensive literature on the theory of differential inequalities relative to initial value problems in finite and infinite dimensional spaces, including random differential inequalities [**2**, **3**, **5**, **6–8**, **9**]. This theory is also known as the theory of comparison principle. The corresponding theory of differential inequalities related to boundary value problems of ordinary and partial differential equations has also developed substantially [**1**, **4**, **5**, **9**]. The treatment of this general theory of differential inequalities is not for its own sake. The essential unity is achieved by the wealth of its applications to various qualitative and quantitative problems of a variety of dynamical systems. This theory can be applied employing as a candidate a suitable norm or more generally a Lyapunov-like function, to provide an effective mechanism for investigating various problems. It is therefore natural to expect the development of an abstract theory so as to bring out the unifying theme of various theories of inequalities. The present book is an attempt in this direction.

As the title suggests, this book is concerned with inequalities that are described by operators which may be matrices, differential operators, or integral operators. As an example the inverse-positive linear operators M, may be described by the property that $Mu \geqslant 0$ implies $u \geqslant 0$. These are operators M which have a positive inverse M^{-1}. For an inverse-positive operator M one can derive estimates of $M^{-1}r$ from properties of r without knowing the inverse M^{-1} explicitly. This property can be used to derive a priori estimates for solutions of equations $Mu = r$. There are important applications as well. For example, if M is inverse-positive, an equation $Mu = Nu$ with a nonlinear operator N may be transformed into a fixed-point equation $u = M^{-1}Nu$, to which then a monotone iteration method or other methods may be applied, if N has suitable properties. Moreover, for inverse-positive M the eigenvalue