Multiplication between Hardy spaces and their dual spaces

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Abstract

For any $p \in (0,1)$ and $\alpha = 1/p - 1$, let $H^p(\mathbb{R}^n)$ and $\mathfrak{C}_\alpha(\mathbb{R}^n)$ be the Hardy and the Campanato spaces on the n-dimensional Euclidean space $\mathbb{R}^n$, respectively. In this article, the authors find suitable Musielak–Orlicz functions Φ_p, defined by setting, for any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$,

$$\Phi_p(x, t) := \begin{cases} \dfrac{t}{1 + [t(1+|x|)^n]^{1-p}} & \text{when } n(1/p - 1) \notin \mathbb{N}, \\ \dfrac{t}{1 + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p - 1) \in \mathbb{N}, \end{cases}$$

and then establish a bilinear decomposition theorem for multiplications of functions in $H^p(\mathbb{R}^n)$ and its dual space $\mathfrak{C}_\alpha(\mathbb{R}^n)$. To be precise, for any $f \in$

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$H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, the authors prove that the product of f and g, viewed as a distribution, can be decomposed into $S(f,g) + T(f,g)$, where S is a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and T a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ associated with the above Musielak–Orlicz function Φ_p. Such a bilinear decomposition is sharp when $n\alpha \notin \mathbb{N}$, in the sense that any other vector space $\mathcal{Y} \subset H^{\Phi_p}(\mathbb{R}^n)$ adapted to the above bilinear decomposition should satisfy $(L^1(\mathbb{R}^n) + \mathcal{Y})^* = (L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n))^*$. To obtain the sharpness, the authors establish a characterization of the class of pointwise multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ by means of the dual space of $H^{\Phi_p}(\mathbb{R}^n)$, which is of independent interest. As an application, an estimate of the div-curl product involving the space $H^{\Phi_p}(\mathbb{R}^n)$ is discussed. This article naturally extends the known sharp bilinear decomposition of $H^1(\mathbb{R}^n) \times \mathrm{BMO}\,(\mathbb{R}^n)$.

Résumé

Etant donnés $p \in (0,1)$ et $\alpha = 1/p - 1$, nous désignons par $H^p(\mathbb{R}^n)$ et $\mathfrak{C}_\alpha(\mathbb{R}^n)$ respectivement les espaces de Hardy et de Campanato sur $\mathbb{R}^n$. Nous définissons dans cet article une famille de fonctions de type Musielak–Orlicz appelées Φ_p en posant, pour tout $x \in \mathbb{R}^n$ et $t \in [0, \infty)$,

$$\Phi_p(x,\, t) := \begin{cases} \dfrac{t}{1 + [t(1+|x|)^n]^{1-p}} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ \dfrac{t}{1 + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N}. \end{cases}$$

Nous établissons un théorème de décomposition bilinéaire pour les multiplications de fonctions qui sont respectivement dans $H^p(\mathbb{R}^n)$ et dans son dual $\mathfrak{C}_\alpha(\mathbb{R}^n)$. Plus précisément, quelles que soient les fonctions $f \in H^p(\mathbb{R}^n)$ et $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, le produit de f et g au sens des distributions se décompose en $S(f,g)+T(f,g)$, où S est un opérateur bilinéaire continu de $H^p(\mathbb{R}^n)\times\mathfrak{C}_\alpha(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$ and T un opérateur bilinéaire continu de $H^p(\mathbb{R}^n)\times\mathfrak{C}_\alpha(\mathbb{R}^n)$ dans l'espace de Hardy de type Musielak–Orlicz $H^{\Phi_p}(\mathbb{R}^n)$ associé à la fonction Φ_p. Une telle decomposition bilinéaire est critique lorsque $n\alpha \notin \mathbb{N}$, en ce sens que tout autre espace $\mathcal{Y} \subset H^{\Phi_p}(\mathbb{R}^n)$ pour lequel une telle décomposition serait possible serait tel que $(L^1(\mathbb{R}^n) + \mathcal{Y})^* = (L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n))^*$. Pour conclure cet argument, nous montrons que l'espace des multiplicateurs de $\mathfrak{C}_\alpha(\mathbb{R}^n)$ s'identifie à l'intersection de $L^\infty(\mathbb{R}^n)$ avec le dual de $H^{\Phi_p}(\mathbb{R}^n)$, ce qui peut présenter un intérêt séparé. Comme application nous donnons un lemme div-curl généralisé dans lequel l'estimation est en termes de l'espace

space $H^{\Phi_p}(\mathbb{R}^n)$. Cet article est une extension naturelle des décompositions des produits $H^1(\mathbb{R}^n) \times \mathrm{BMO}\,(\mathbb{R}^n)$.

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1. Introduction and main results

Let $p \in (0, 1]$ and $\alpha = 1/p - 1$. Denote by $H^p(\mathbb{R}^n)$ the Hardy space on the n-dimensional Euclidean space $\mathbb{R}^n$. Denote by $\mathfrak{C}_\alpha(\mathbb{R}^n)$ the Campanato space on $\mathbb{R}^n$, which is just the dual space of $H^p(\mathbb{R}^n)$. Certainly, when $\alpha = 0$, the space $\mathfrak{C}_0(\mathbb{R}^n)$ turns out to be the space $\mathrm{BMO}(\mathbb{R}^n)$ of bounded mean oscillations. The main purpose of this article is to study the following problem on the multiplication between $H^p(\mathbb{R}^n)$ and its dual space $\mathfrak{C}_\alpha(\mathbb{R}^n)$:

> *For any $p \in (0, 1]$ and $\alpha = 1/p - 1$, find the 'smallest' linear vector space $\mathcal{Y}$ so that $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ has the following bilinear decomposition of the form:*
>
> $$H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + \mathcal{Y}. \tag{1.1}$$

The precise interpretation of (1.1) is as follows: the product of $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ can be written as $S(f, g) + T(f, g)$, where

$$S : \ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$T : \ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to \mathcal{Y}$$

are bounded bilinear mappings.

In (1.1), elements in the product space $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ are understood as Schwartz distributions (see [8, 6]). Let us be more precise. Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, equipped with the well-known topology determined by a countable family of seminorms. Denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak-$*$ topology. For any $f \in H^p(\mathbb{R}^n)$ with $p \in (0,\, 1)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ with $\alpha = 1/p - 1$, the *product* $f \times g$ is defined to be a Schwartz distribution in $\mathcal{S}'(\mathbb{R}^n)$, whose action on a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is as follows:

$$\langle f \times g,\, \phi\rangle := \langle \phi g,\, f\rangle, \tag{1.2}$$

where the last bracket denotes the dual pair between $\mathfrak{C}_\alpha(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$. Equality (1.2) is well defined because every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a pointwise multiplier on $\mathfrak{C}_\alpha(\mathbb{R}^n)$ (see [3, p. 59] or Corollary 3.2 below), that is, $\phi g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ for any $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$. This fact also implies the product $f \times g$ in (1.2) can be viewed as a distribution on the class of pointwise multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$.

It should be emphasized that, while for duality the spaces $\mathfrak{C}_\alpha(\mathbb{R}^n)$ have to be interpreted as quotient spaces modulo polynomials of degree $\lfloor n\alpha \rfloor$ (basically H^p distributions may be thought as orthogonal to such polynomials), we speak here of functions in the Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n)$. Here and hereafter, the *symbol* $\lfloor s \rfloor$ for any $s \in \mathbb{R}$ denotes the largest integer not greater than s.

In this sense, if the distribution f in $H^p(\mathbb{R}^n)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, then the product (1.2) coincides with the pointwise product of f and the Campanato function g, as the dual pair in (1.2) equals to

$$\langle \phi g,\, f\rangle = \int_{\mathbb{R}^n} (\phi g)(x) f(x)\, dx.$$

The study of the decomposition problems like (1.1) was initiated by Bonami et al. [8] (in the case $p = 1$), motivated by developments in the geometric function theory and nonlinear elasticity [1, 2, 39, 40]. A good understanding of the structure of this product can help us to improve the boundedness of many nonlinear qualities such as div-curl products and weak Jacobians (see [16, 6, 4]) as well as the endpoint boundedness of commutators (see [31, 36]), which are fundamental in various research areas of mathematics such as the compensated compactness theory in nonlinear partial differential equations and the study of the existence and regularity for solutions to partial differential equations where the uniform ellipticity condition is lost (see [41, 44, 34, 8, 28, 30] and their references).

Let us give a brief review of the progress on the study of the decomposition problem (1.1). The following

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^\Phi_w(\mathbb{R}^n) \tag{1.3}$$

linear decomposition was proved in [8, Theorem 1.6], where $H^\Phi_w(\mathbb{R}^n)$ denotes the weighted Orlicz–Hardy space associated to the weight function $w(x) := 1/\log(e+|x|)$ for any $x \in \mathbb{R}^n$ and to the Orlicz function $\Phi(t) := t/\log(e+t)$ for any $t \in [0,\infty)$. Being more precise, for any given $f \in H^1(\mathbb{R}^n)$, there exist two bounded linear operators

$$S_f : \mathrm{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$T_f : \mathrm{BMO}(\mathbb{R}^n) \to H^\Phi_w(\mathbb{R}^n)$$

such that, for any $g \in \mathrm{BMO}(\mathbb{R}^n)$,

$$f \times g = S_f g + T_f g.$$

Moreover, it was conjectured in [8] that one can find two bounded bilinear operators S and T such that the aforementioned decomposition is also linear in f.

Via wavelet multiresolution analysis, the above conjecture of [8] was solved by Bonami et al. [6] who proved the following *bilinear* decomposition

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n),$$

where the space $H^{\log}(\mathbb{R}^n)$ (see also [32]) denotes the Hardy space of Musielak–Orlicz type associated to the Musielak–Orlicz function

$$\theta(x,t) := \frac{t}{\log(e+t) + \log(e+|x|)}, \qquad \forall\, x \in \mathbb{R}^n,\ \forall\, t \in [0,\infty). \tag{1.4}$$

Such a Musielak–Orlicz Hardy space $H^{\log}(\mathbb{R}^n)$ is smaller than $H^\Phi_w(\mathbb{R}^n)$ in (1.3). By proving that the dual space of $H^{\log}(\mathbb{R}^n)$ is the generalized BMO space that had been introduced by Nakai and Yabuta [42] to characterize multipliers of $\mathrm{BMO}(\mathbb{R}^n)$, Bonami et al. in [6] deduced that $H^{\log}(\mathbb{R}^n)$ is in some sense sharp. They came back to this sharpness in further work to prove that this is indeed the smallest space in dimension one [7, 5] and gave a partial result in higher dimension by proving that every atom of $H^{\log}(\mathbb{R}^n)$ can be written as a finite combination of products, with the required norm estimates. In this way, problem (1.1) for the case $p = 1$ may be considered as solved in [6] with $\mathcal{Y} = H^{\log}(\mathbb{R}^n)$.

The two articles [6, 8] were a source of inspiration for succeeding work on the (bi)linear decomposition of the product functions in Hardy spaces and their dual spaces. Partial results for $H^p(\mathbb{R}^n)$ were obtained in [3, 21]. Cao et al. [12] obtained the bilinear decomposition of product functions in the local Hardy space $h^p(\mathbb{R}^n)$ and its dual space for p close to 1.

Motivated by the aforementioned articles, it is a very natural question to seek suitable Hardy-type spaces $\mathcal{Y}$ that can give a linear or bilinear decomposition of (1.1) when $p \in (0,1)$. This is the aim of this article. Our first result concerns linear decompositions. It involves the weighted Hardy space $H^p_{w_p}(\mathbb{R}^n)$, which consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{H^p_{w_p}(\mathbb{R}^n)} := \|f^*\|_{L^p_{w_p}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} [f^*(x)]^p w_p(x)\,dx\right\}^{1/p} < \infty.$$

Here f^* denotes the grand maximal function defined in Definition 2.1 below and w_p is an $A_1(\mathbb{R}^n)$-weight (see Lemma 2.13) defined by setting, for any $x \in \mathbb{R}^n$,

$$w_p(x) := \begin{cases} \dfrac{1}{(1+|x|)^{n(1-p)}} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ \dfrac{1}{(1+|x|)^{n(1-p)}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N}. \end{cases} \tag{1.5}$$

We have the following results.

Theorem 1.1. *Let $p \in (0,1)$ and $\alpha = 1/p - 1$. Then, for any given $f \in H^p(\mathbb{R}^n)$, one can find two bounded linear operators*

$$S_f : \mathfrak{C}_\alpha(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$T_f : \mathfrak{C}_\alpha(\mathbb{R}^n) \to H^p_{w_p}(\mathbb{R}^n)$$

such that, for any $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$,

$$f \times g = S_f(g) + T_f(g).$$

Moreover, there exists a positive constant C, independent of f, such that both operators have norms bounded by $C\|f\|_{H^p(\mathbb{R}^n)}$.

We have already mentioned that what is involved here in the product is a function of the Campanato space and not its equivalence class modulo polynomials. So, for any $\alpha \in (0, \infty)$, we need to equip $\mathfrak{C}_\alpha(\mathbb{R}^n)$ with a norm and, for any $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, we choose here to define

$$\|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} := \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} + \frac{1}{|B(\vec{0}_n, 1)|} \int_{B(\vec{0}_n,1)} |g(x)|\, dx.$$

Here and hereafter, we use $\vec{0}_n$ to denote the *origin* of $\mathbb{R}^n$ and

$$B(\vec{0}_n, 1) := \{x \in \mathbb{R}^n : \ |x| < 1\}.$$

Theorem 1.1 can be interpreted as the fact that the product $f \times g$ can be written as the sum of an integrable part and a part that keeps some of the oscillation properties of $H^p(\mathbb{R}^n)$. When considering the duality $\langle g, f\rangle$, only the first part gives a non zero quantity. When considering other operators, it is the second part that plays the main role. This is why it is natural to ask whether or not it is possible to cut bilinearly the product into two parts.

This is the aim of our main result, but the decomposition that we prove is a little different. Instead of the weighted Hardy space $H^p_{w_p}(\mathbb{R}^n)$, we consider the *Musielak–Orlicz Hardy space* $H^{\Phi_p}(\mathbb{R}^n)$ associated with the Musielak–Orlicz function

$$\Phi_p(x, t) := \begin{cases} \dfrac{t}{1 + [t(1+|x|)^n]^{1-p}} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ \dfrac{t}{1 + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N}, \\ \dfrac{t}{\log(e+t) + \log(e+|x|)} & \text{when } p = 1, \end{cases} \tag{1.6}$$

where $x \in \mathbb{R}^n$ and $t \in [0, \infty)$; see Section 2.3 below. We mention that the case $p = 1$, which can be treated in a unified way in some of our results, is a source of inspiration for this article. But we concentrate on the cases $p \in (0, 1)$, which correspond to the new results obtained here. We show in Section 2.2 below that Φ_p in (1.6) are Musielak–Orlicz functions satisfying the growth conditions used in Ky [32], so that the corresponding Musielak–Orlicz Hardy spaces $H^{\Phi_p}(\mathbb{R}^n)$ fall into the scope of Musielak–Orlicz Hardy spaces studied in [32, 35, 47].

When studying these particular Musielak–Orlicz Hardy spaces, simplifications are due to the fact that the growth function Φ_p is equivalent to the minimum of two growth functions. Namely, for any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$,

$$\Phi_p(x,\,t) \sim \begin{cases} \min\left\{t,\ \dfrac{t^p}{(1+|x|)^{n(1-p)}}\right\} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ \min\left\{t,\ \dfrac{t^p}{(1+|x|)^{n(1-p)}[\log(e+|x|)]^p}\right\} & \text{when } n(1/p-1) \in \mathbb{N}, \end{cases} \tag{1.7}$$

with positive equivalence constants independent of x and t. The Musielak–Orlicz Hardy spaces that correspond to these two functions are respectively $H^1(\mathbb{R}^n)$ and the weighted Orlicz–Hardy spaces $H^p_{w_p}(\mathbb{R}^n)$ that we have already encountered. Moreover, it was proved in the recent work [13] that $H^{\Phi_p}(\mathbb{R}^n)$ coincides with the sum of quasi-Banach spaces $H^1(\mathbb{R}^n) + H^p_{w_p}(\mathbb{R}^n)$. So, obviously, we could as well replace $H^p_{w_p}(\mathbb{R}^n)$ by $H^{\Phi_p}(\mathbb{R}^n)$ in Theorem 1.1.

The main result of this article is as follows.

Theorem 1.2. *Let $p \in (0,1)$, $\alpha = 1/p - 1$ and Φ_p be as in* (1.6)*. Then there exist two bounded bilinear operators*

$$S:\ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$T:\ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to H^{\Phi_p}(\mathbb{R}^n)$$

such that, for any $(f, g) \in H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$,

$$f \times g = S(f,\,g) + T(f,\,g) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, there exists a positive constant C such that, for any $(f, g) \in H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$,

$$\|S(f,g)\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{H^p(\mathbb{R}^n)}\|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}$$

and

$$\|T(f,g)\|_{H^{\Phi_p}(\mathbb{R}^n)} \le C\|f\|_{H^p(\mathbb{R}^n)}\|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}.$$

Again, we mention here that the corresponding conclusion of Theorem 1.2 for the case $p = 1$ was proved in [6]. Theorem 1.2 is proved in Section 4. As in [6], the main strategy we used to prove Theorem 1.2 is based on a technique of renormalization of products of functions (or distributions) via wavelets introduced by Coifman et al. [14, 18], which enables us to write

$$f \times g = \sum_{i=1}^{4} \Pi_i(f, g);$$

see (4.7) through (4.10) below for the precise definitions of bilinear operators $\{\Pi_i\}_{i=1}^4$. Then the problem can be reduced to the study of each $\Pi_i(f, g)$, with f being an atom a which has a finite wavelet expansion. The main obstacle is the treatment of Π_2. Recall that the study of Π_2 in [6] utilized the following fact: if a is an $H^1(\mathbb{R}^n)$-atom supported on some dyadic cube I and a has a finite wavelet expansion, then $a\phi_I$ satisfies only a zero order moment condition by the orthogonality of wavelet basis. This is enough to make $a\phi_I$ a harmlessly constant multiple of an $H^p(\mathbb{R}^n)$-atom when $p \in (n/(n+1), 1]$, but insufficient when $p \in (0, n/(n+1)]$. To overcome this obstacle, we borrow some ideas from [22]. We reduce the estimation of the bilinear operator $\Pi_2(a, g)$ to that of $aP_{B,s}g$, where $P_{B,s}g$ denotes the minimizing polynomial of g on the ball B with degree $\leq \lfloor n(1/p-1) \rfloor$. As $P_{B,s}g$ is a polynomial, the term $aP_{B,s}g$ can still enjoy the higher order moment condition by requiring the wavelets ψ_I^λ to have the sufficiently higher order moment conditions. Proving that $aP_{B,s}g \in H^{\Phi_p}(\mathbb{R}^n)$ is done in Proposition 2.24, by using some growth estimates of $P_{B,s}g$ established in Proposition 2.22. Let us mention that all these estimates involving $P_{B,s}g$ are delicate.

Another contribution of this article is the following characterization of the pointwise multipliers on $\mathfrak{C}_\alpha(\mathbb{R}^n)$ for a general $\alpha \in (0, \infty)$ satisfying $n\alpha \notin \mathbb{N}$, by means of the dual space of $H^{\Phi_p}(\mathbb{R}^n)$ (see Section 2.3 below), where $\alpha = 1/p - 1$. Recall that, for any quasi-Banach space X equipped with a quasi-norm $\|\cdot\|_X$, a function g defined on $\mathbb{R}^n$ is called a *pointwise multiplier* on X if there exists a positive constant C such that $\|gf\|_X \leq C\|f\|_X$ for any $f \in X$.

Theorem 1.3. *Let $p \in (0, 1)$, $\alpha = 1/p - 1$ satisfy $n\alpha \notin \mathbb{N}$ and Φ_p be as in* (1.6)*. Denote by $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ the dual space of $H^{\Phi_p}(\mathbb{R}^n)$. For any function g on $\mathbb{R}^n$, the following assertions are equivalent:*

(i) *$g \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$;*

(ii) *g is a pointwise multiplier of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ and, for any $f \in \mathfrak{C}_\alpha(\mathbb{R}^n)$,*

$$\|gf\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \leq C\|f\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)}[\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)}],$$

where C is a positive constant independent of f and g.

When $p = 1$, the corresponding conclusion of Theorem 1.3 was already known in [42, 32, 6]. We mention here that the class of pointwise multiplies of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ for any $n\alpha \in (0, 1)$ was characterized by [42]. However, the results of [42] were not connected with the dual of $H^{\Phi_p}(\mathbb{R}^n)$.

The proof of Theorem 1.3 needs some intrinsic properties of Campanato spaces. Proposition 2.22 proves that a function $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ (so does $P_{B,\lfloor n\alpha\rfloor} g$) on the ball B has a polynomial growth of order $n\alpha$ when $n\alpha \notin \mathbb{N}$, but can have an extra logarithm growth factor when $n\alpha \in \mathbb{N}$. In contrast to the $\mathfrak{C}_\alpha(\mathbb{R}^n)$-functions constructed in Propositions 3.3 and 3.7, we know that the estimates in Proposition 2.22 are best possible. By Proposition 2.22, the proof that (ii) implies (i) of Theorem 1.3 is given in Theorem 3.1 below, and the proof that (i) implies (ii) of Theorem 1.3 is given in Theorem 3.6 below.

Theorem 1.3 for the case $n\alpha \in \mathbb{N}$ is still unsolved. Indeed, when $n\alpha \in \mathbb{N}$, the proof of the non-integer case shows that (i) of Theorem 1.3 implies (ii) of Theorem 1.3 and that a pointwise multiplier on $\mathfrak{C}_\alpha(\mathbb{R}^n)$ is also bounded; the difficulty lies in proving that a pointwise multiplier on $\mathfrak{C}_\alpha(\mathbb{R}^n)$ belongs to the space $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. See Section 3.3 for more on this problem.

Remark 1.4. Theorem 1.3 implies that the bilinear decomposition in Theorem 1.2 is in some sense sharp when $n\alpha \notin \mathbb{N}$. Being more precise, if Theorem 1.2 holds true with $H^{\Phi_p}(\mathbb{R}^n)$ therein replaced by any other linear vector space $\mathcal{Y} \subset H^{\Phi_p}(\mathbb{R}^n)$, then $(L^1(\mathbb{R}^n) + \mathcal{Y})^* = (L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n))^*$; see Remark 4.12 below. In analogy, the sharpness of Theorem 1.2 for the case $n\alpha \in \mathbb{N}$ follows directly if one could show Theorem 1.3 for the case $n\alpha \in \mathbb{N}$.

Notice that there exists no contradiction between Theorems 1.1 and 1.2: it is known that $H^{\Phi_p}(\mathbb{R}^n)$ and $H^p_{w_p}(\mathbb{R}^n)$ have the same dual (see, for instance, [13, Remark 3.1]). Contrarily to what happens in Orlicz–Hardy spaces, Musielak–Orlicz Hardy spaces may have the same dual, but they may actually differ. However, inspired by the fact that $H^{\Phi_p}(\mathbb{R}^n) = H^1(\mathbb{R}^n) + H^p_{w_p}(\mathbb{R}^n)$ in [13], one may ask whether or not the Musielak–Orlicz Hardy space in Theorem 1.2 can be replaced by $H^p_{w_p}(\mathbb{R}^n)$, which is still unknown.

Theorem 1.2 can be applied to study the div-curl product. Denote by $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. For a vector field $\mathbf{F} := (F_1, \ldots, F_n)$ of locally integrable functions on $\mathbb{R}^n$, we define its *divergence* $\operatorname{div} \mathbf{F}$ as a distribution, whose action on

$\varphi \in C_c^\infty(\mathbb{R}^n)$ is defined by setting

$$\langle \operatorname{div} \mathbf{F},\, \varphi \rangle := -\int_{\mathbb{R}^n} \mathbf{F}(x) \cdot \nabla \varphi(x)\, dx,$$

and define its *curl* $\operatorname{curl} \mathbf{F}$ as a matrix $\{(\operatorname{curl} \mathbf{F})_{i,\,j}\}_{i,\,j \in \{1,\,\dots,\,n\}}$ of distributions, with the action of each entry $(\operatorname{curl} \mathbf{F})_{i,\,j}$ on $\varphi \in C_c^\infty(\mathbb{R}^n)$ being defined by setting

$$\langle (\operatorname{curl} \mathbf{F})_{i,\,j},\, \varphi \rangle := \int_{\mathbb{R}^n} \left[F_j(x) \frac{\partial \varphi}{\partial x_i}(x) - F_i(x) \frac{\partial \varphi}{\partial x_j}(x) \right] dx;$$

see, for instance, [15, p. 507]. Notice that div-curl estimates have been investigated in [4, 6, 16].

For any $p \in (0,\, 1)$, let

$$H^p(\mathbb{R}^n;\, \mathbb{R}^n) := \{\mathbf{F} := (F_1,\, \dots,\, F_n) : \text{ for any } i \in \{1,\, \dots,\, n\},\ F_i \in H^p(\mathbb{R}^n)\} \tag{1.8}$$

equipped with the quasi-norm

$$\|\mathbf{F}\|_{H^p(\mathbb{R}^n;\, \mathbb{R}^n)} := \left[\sum_{i=1}^n \|F_i\|_{H^p(\mathbb{R}^n)}^2 \right]^{\frac{1}{2}}.$$

In a similar way, we define $L^2(\mathbb{R}^n;\, \mathbb{R}^n)$ and the vector-valued Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n;\, \mathbb{R}^n)$ as well as the norms $\|\cdot\|_{L^2(\mathbb{R}^n;\, \mathbb{R}^n)}$, $\|\cdot\|_{\mathfrak{C}_\alpha(\mathbb{R}^n;\, \mathbb{R}^n)}$ and $\|\cdot\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n;\, \mathbb{R}^n)}$, where $\alpha \in (0, \infty)$.

Applying Theorem 1.2, we are able to prove the following a priori estimate of the div-curl product involving the space $H^{\Phi_p}(\mathbb{R}^n)$.

Theorem 1.5. *Let* $p \in (0,\, 1)$, $\alpha = 1/p - 1$, Φ_p *be as in* (1.6) *and* $\mathbf{F} \in L^2(\mathbb{R}^n;\, \mathbb{R}^n)$. *Assume further that* $\mathbf{F} \in H^p(\mathbb{R}^n;\, \mathbb{R}^n)$ *with* $\operatorname{curl} \mathbf{F} \equiv 0$ *and* $\mathbf{G} \in \mathfrak{C}_\alpha(\mathbb{R}^n;\, \mathbb{R}^n)$ *with* $\operatorname{div} \mathbf{G} \equiv 0$ (*both of the equalities hold true in the sense of distributions*). *Then the inner product* $\mathbf{F} \cdot \mathbf{G} \in H^{\Phi_p}(\mathbb{R}^n)$ *and*

$$\|\mathbf{F} \cdot \mathbf{G}\|_{H^{\Phi_p}(\mathbb{R}^n)} \leq C \|\mathbf{F}\|_{H^p(\mathbb{R}^n;\, \mathbb{R}^n)} \|\mathbf{G}\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n;\, \mathbb{R}^n)},$$

where C *is a positive constant independent of* $\mathbf{F}$ *and* $\mathbf{G}$.

Theorem 1.5 extends the result of [6, Theorem 1.2], while the latter proved that $\mathbf{F} \cdot \mathbf{G} \in H^{\log}(\mathbb{R}^n)$ whenever $\mathbf{F} \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ satisfies $\operatorname{curl} \mathbf{F} \equiv 0$ and $\mathbf{G} \in \mathrm{BMO}(\mathbb{R}^n; \mathbb{R}^n)$ satisfies $\operatorname{div} \mathbf{G} \equiv 0$.

This article is organized as follows.

Section 2 concerns some basic properties of the function spaces involved in this article. In Section 2.1, we recall the definitions of the Hardy space $H^p(\mathbb{R}^n)$ and the Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n)$. Section 2.2 shows that Φ_p defined in (1.6) is a Musielak–Orlicz function satisfying some growth conditions as in [32], so that it makes sense for us to introduce the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ and its dual space $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ as in [32, 35, 47] (see Section 2.3). Moreover, an equivalent characterization of $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ is given in Section 2.3 (see Proposition 2.18 below). In Section 2.4, we establish the pointwise growth estimates for functions in the Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n)$ (see Proposition 2.22 below), which ensures that $aP_{B,s}g$, with a being an $H^p(\mathbb{R}^n)$-atom and $P_{B,s}g$ the minimizing polynomial of g on B with degree $\leq s = \lfloor n\alpha \rfloor$, is an element of $H^{\Phi_p}(\mathbb{R}^n)$ (see Proposition 2.24 below).

It should be mentioned that estimates in Section 2.4 play an important role in the proofs of Theorems 1.1, 1.2 and 1.3.

Section 3 is devoted to the proof of Theorem 1.3. We prove the necessary part (including also the case $n\alpha \in \mathbb{N}$) in Section 3.1, and the sufficient part in Section 3.2, with also a comment on the sufficient part when $n\alpha \in \mathbb{N}$ given in Section 3.3. As a consequence of Theorem 1.3, we show that functions in $\mathcal{S}(\mathbb{R}^n)$ belong to the class of pointwise multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$. Applying this and Proposition 2.24, we give the proof of Theorem 1.1 in Section 3.4.

The aim of Section 4 is to establish Theorem 1.2. We begin with some basic definitions of the multiresolution analysis (for short, MRA) in Section 4.1. In Section 4.2, we then recall the renormalization of the products in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ from [14, 18]. Later, in Section 4.3, we give three auxiliary lemmas on the atomic decomposition of the Hardy space $H^p(\mathbb{R}^n)$ and the wavelet characterization of $H^p(\mathbb{R}^n)$ and its dual $\mathfrak{C}_\alpha(\mathbb{R}^n)$. In Section 4.4, we prove Theorem 1.2 based on the boundedness results of the four bilinear operators introduced in Section 4.2.

Theorem 1.5 is proved in Section 5 as an application of Theorem 1.2.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, denote by $B(x, r)$ the ball with center x and radius r, that is,

$$B(x, r) := \{y \in \mathbb{R}^n : \ |x - y| < r\}.$$

For any ball $B \subset \mathbb{R}^n$, we always denote by c_B its center and r_B its radius. We use $\vec{0}_n$ to denote the origin of $\mathbb{R}^n$. For any $\lambda \in (0, \infty)$ and any ball B, denote by λB the ball with center c_B and radius λr_B. For any set $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes its *characteristic function* and

$$\fint_E := \frac{1}{|E|} \int_E .$$

We use C to denote a *positive constant* that is independent of the main parameters involved, whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. If $f \le Cg$, we also write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. For any $s \in \mathbb{R}$, let $\lfloor s \rfloor$ (resp., $\lceil s \rceil$) be the largest integer not greater than s (resp., the smallest integer not smaller than s). For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, define $D^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ with $\partial_{x_j} := \frac{\partial}{\partial x_j}$ for any $j \in \{1, \dots, n\}$.

2. Hardy-type spaces and their dual spaces

This section concerns some basic properties of the Hardy space $H^p(\mathbb{R}^n)$, the Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n)$, the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ and its dual space $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. The main results of this section are Propositions 2.22 and 2.24, which play key roles in the proofs of Theorems 1.1 and 1.2.

2.1. Hardy and Campanato spaces

In this section, we recall the notions of Hardy and Campanato spaces.

Definition 2.1. Let $p \in (0, \infty)$ and $m \in \mathbb{Z}_+$ satisfy $m \ge \lfloor n(1/p - 1) \rfloor$.

(i) For any $f \in \mathcal{S}'(\mathbb{R}^n)$, its *non-tangential grand maximal function* f_m^* is defined by setting, for any $x \in \mathbb{R}^n$

$$f_m^*(x) := \sup_{\varphi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t,\, t\in(0,\infty)} |f * \varphi_t(y)|, \tag{2.1}$$

where $\varphi_t(z) := t^{-n}\varphi(t^{-1}z)$ for any $t \in (0, \infty)$ and $z \in \mathbb{R}^n$, and

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \le m+1} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{(m+2)(n+1)} |D^\alpha \varphi(x)| \le 1 \right\}.$$

(ii) If $m = \lfloor n(1/p-1) \rfloor$, then we write f_m^* simply as f^*. The *Hardy space* $H^p(\mathbb{R}^n)$ is defined to be the collection of all Schwartz distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p(\mathbb{R}^n)} := \|f^*\|_{L^p(\mathbb{R}^n)} < \infty.$$

Note that, if $p \in (0,\infty)$ and $m \geq \lfloor n(1/p-1) \rfloor$, then $H^p(\mathbb{R}^n)$ can be equivalently defined by using the (quasi-)norm $\|f_m^*\|_{L^p(\mathbb{R}^n)}$ (see, for instance, [37, Chapter 1] or [47, Chapter 1]). Moreover, when $p \in (1,\infty)$, the Hardy space $H^p(\mathbb{R}^n)$ coincides to the Lebesgue space $L^p(\mathbb{R}^n)$ with equivalent norms. We refer the reader to [20, 25, 43, 37] for more properties on $H^p(\mathbb{R}^n)$.

The dual of the Hardy space turns out to be the Campanato space, which was first introduced by Campanato in [10, 11]. For any $s \in \mathbb{Z}_+$, denote by $\mathcal{P}_s(\mathbb{R}^n)$ the space of all polynomials on $\mathbb{R}^n$ with degree $\leq s$.

Definition 2.2. Let $\alpha \in [0,\infty)$, $q \in [1,\infty]$ and $s \in \mathbb{Z}_+$ be such that $s \geq \lfloor n\alpha \rfloor$. The *Campanato space* $\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)$ is defined to be the collection of all locally integrable functions g such that

$$\|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} := \begin{cases} \sup\limits_{B\subset\mathbb{R}^n} \dfrac{1}{|B|^\alpha} \left\{ \fint_B |g(x) - P_{B,s}g(x)|^q \, dx \right\}^{1/q} \\ \qquad\qquad\qquad\qquad \text{when } q \in [1,\infty), \\ \sup\limits_{B\subset\mathbb{R}^n} \operatorname*{ess\,sup}\limits_{x\in B} \dfrac{|g(x) - P_{B,s}g(x)|}{|B|^\alpha} \\ \qquad\qquad\qquad\qquad \text{when } q = \infty \text{ and } \alpha \neq 0 \end{cases}$$

is finite, where the suprema are taken over all balls B of $\mathbb{R}^n$. Here and hereafter, $P_{B,s}g$ denotes the minimizing polynomial of g on B with degree $\leq s$, that is, $P_{B,s}g$ is the unique polynomial with degree $\leq s$ such that, for any polynomial $Q \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\int_B [g(x) - P_{B,s}g(x)]\, Q(x)\, dx = 0. \tag{2.2}$$

In particular, when $s = \lfloor n\alpha \rfloor$ and $q = 1$, we simply write $\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)$ as $\mathfrak{C}_\alpha(\mathbb{R}^n)$.

With all the notation as in Definition 2.2, then a function g satisfies $\|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} = 0$ if and only if g coincides almost everywhere with a polynomial in $\mathcal{P}_s(\mathbb{R}^n)$. Moreover, for any function $g \in \mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)$, we introduce the following inhomogeneous norm

$$\|g\|_{\mathfrak{C}^{+}_{\alpha,q,s}(\mathbb{R}^n)} := \|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} + \fint_{B(\vec{0}_n,1)} |g(x)|\,dx. \tag{2.3}$$

Also, when $q \in [1,\infty)$, an equivalent definition of the Campanato norm is as follows (see, for instance, [25, p. 292]):

$$\|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} \sim \sup_{B\subset\mathbb{R}^n} \inf_{P\in\mathcal{P}_s(\mathbb{R}^n)} \frac{1}{|B|^\alpha} \left\{ \fint_B |g(x) - P(x)|^q\,dx \right\}^{1/q}, \tag{2.4}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and the positive equivalence constants are independent of g.

We give several remarks on the relations between Campanato spaces and some other related function spaces.

Remark 2.3. Let α, q and s be as in Definition 2.2.

(i) When $p \in (0,\,1]$ is such that $\alpha = 1/p - 1$, we deduce from [25, Theorem 5.30] or [37, p. 55, Theorem 4.1] that

$$(H^p(\mathbb{R}^n))^* = \mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)/\mathcal{P}_s(\mathbb{R}^n).$$

This implies that the quotient spaces

$$\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)/\mathcal{P}_s(\mathbb{R}^n) \quad \text{and} \quad \mathfrak{C}_{\alpha}(\mathbb{R}^n)/\mathcal{P}_{\lfloor n\alpha \rfloor}(\mathbb{R}^n)$$

are consistent, and

$$\|\cdot\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} \sim \|\cdot\|_{\mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)}.$$

(ii) If $\alpha = s = 0$ and $q \in [1,\,\infty)$, then $\mathfrak{C}_0(\mathbb{R}^n) = \mathfrak{C}_{0,q,0}(\mathbb{R}^n)$ is just the space $\mathrm{BMO}(\mathbb{R}^n)$ (see [25, p. 292]), where $\mathrm{BMO}(\mathbb{R}^n)$ denotes the space of all locally integrable functions g on $\mathbb{R}^n$ such that

$$\|g\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \fint_B |g(x) - g_B|\,dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and

$$g_B := \fint_B g(y)\,dy.$$

The additional term $⨍_{B(\vec{0}_n,1)} |g(x)|\,dx$ in the expression of the norm $\|\cdot\|_{\mathfrak{C}^+_{\alpha,q,s}(\mathbb{R}^n)}$ in (2.3) has certain degree of freedom and can be replaced by many quantities adapted to the ball $B(\vec{0}_n, 1)$. The following lemma expresses this possibility.

Lemma 2.4. *Let $\alpha \in (0, \infty)$, $q \in [1, \infty]$ and $s \in \mathbb{Z}_+$ be such that $s \geq \lfloor n\alpha \rfloor$. Then, for any $g \in \mathfrak{C}^+_{\alpha,q,s}(\mathbb{R}^n)$,*

$$\begin{aligned}\|g\|_{\mathfrak{C}^+_{\alpha,q,s}(\mathbb{R}^n)} &\sim \|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} + \int_{B(\vec{0}_n,1)} \left|P_{B(\vec{0}_n,1),s}\, g(x)\right| dx \\ &\sim \|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} + \sum_{|\gamma|\leq s} \left| \int_{B(\vec{0}_n,1)} x^\gamma g(x)\,dx \right| \\ &\sim \|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} + \sup_{x\in B(\vec{0}_n,1)} |g(x)|,\end{aligned}$$

where the positive equivalence constants are independent of g.

Proof. The first estimate of this lemma follows immediately from the fact that the integral average of $g - P_{B(\vec{0}_n,1),s}\, g$ over the ball $B(\vec{0}_n, 1)$ is bounded by $\|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)}$. But all norms are equivalent on the finite dimensional vector space $\mathcal{P}_s(\mathbb{R}^n)$, so that we can as well replace $\int_{B(\vec{0}_n,1)} |P_{B(\vec{0}_n,1),s}\, g(x)|\,dx$ in the first estimate by $\sup_{x\in B(\vec{0}_n,1)} |P_{B(\vec{0}_n,1),s}\, g(x)|$ or by

$$\sum_{|\gamma|\leq s} \left| \int_{B(\vec{0}_n,1)} x^\gamma P_{B(\vec{0}_n,1),s}\, g(x)\,dx \right|.$$

From this and (2.2), the second estimate follows. Noticing that

$$\sup_{x\in B(0,1)} \left| g(x) - P_{B(\vec{0}_n,1),s}\, g(x) \right| \lesssim \|g\|_{\mathfrak{C}_{\alpha,\infty,s}(\mathbb{R}^n)} \sim \|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)},$$

we obtain the equivalence of the third estimate, which concludes the proof. □

Finally it is classical that, when $\alpha \in (0, \infty)$, the space $\mathfrak{C}_\alpha(\mathbb{R}^n)$ reduces to the homogeneous Lipschitz space $\dot{\Lambda}_{n\alpha}(\mathbb{R}^n)$ with equivalent norms (see [25, 26, 27]). Let us recall its definition.

Definition 2.5. (i) Let $\sigma \in (0,1)$. The *homogeneous Lipschitz space* $\dot{\Lambda}_\sigma(\mathbb{R}^n)$ is defined to be the collection of (equivalent classes of) continuous functions g such that

$$\|g\|_{\dot{\Lambda}_\sigma(\mathbb{R}^n)} := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\sigma} < \infty.$$

(ii) Let $\sigma = 1$. The *homogeneous Lipschitz space* $\dot{\Lambda}_\sigma(\mathbb{R}^n)$, which is also called the *homogeneous Zygmund space*, is defined to be the collection of (equivalent classes of) continuous functions g such that

$$\|g\|_{\dot{\Lambda}_1(\mathbb{R}^n)} := \sup_{x,t\in\mathbb{R}^n, t\neq 0} \frac{|g(x+t)+g(x-t)-2g(x)|}{|t|} < \infty.$$

(iii) Let $\sigma \in (1,\infty)$. The *homogeneous Lipschitz space* $\dot{\Lambda}_\sigma(\mathbb{R}^n)$ is defined to be the collection of $C^{\sigma_0}(\mathbb{R}^n)$ such that all its derivatives of order σ_0 belong to $\dot{\Lambda}_{\sigma-\sigma_0}(\mathbb{R}^n)$, where σ_0 denotes the largest integer strictly less than σ (hence $\sigma_0 < \sigma$). Moreover, let

$$\|g\|_{\dot{\Lambda}_\sigma(\mathbb{R}^n)} := \sum_{|\beta|=\sigma_0} \left\|D^\beta g\right\|_{\dot{\Lambda}_{\sigma-\sigma_0}}(\mathbb{R}^n).$$

Observe that the semi-norm $\|\cdot\|_{\dot{\Lambda}_\sigma(\mathbb{R}^n)}$ vanishes precisely over the space $\mathcal{P}_{\lfloor\sigma\rfloor}(\mathbb{R}^n)$.

Notice that, when $\sigma \in \mathbb{N}$, a function in $\dot{\Lambda}_\sigma(\mathbb{R}^n)$ may not be in $C^\sigma(\mathbb{R}^n)$. We will see an example later on.

We state the identification of Campanato and Lipschitz spaces in the next lemma; see [25, pp. 301-302].

Lemma 2.6. *Let $\alpha \in (0,\infty)$ and $q \in [1,\infty]$. Then any function $g \in \dot{\Lambda}_{n\alpha}(\mathbb{R}^n)$ if and only if $g \in \mathfrak{C}_{\alpha,q,\lfloor n\alpha\rfloor}(\mathbb{R}^n)$ after modifying g, if necessary, on a set of measure 0. Moreover,*

$$\|g\|_{\dot{\Lambda}_{n\alpha}(\mathbb{R}^n)} \sim \|g\|_{\mathfrak{C}_{\alpha,q,\lfloor n\alpha\rfloor}(\mathbb{R}^n)},$$

with positive equivalence constants independent of g.

2.2. The Musielak–Orlicz growth function Φ_p

In this section, we show that the function Φ_p as in (1.6) is a Musielak–Orlicz function as in [32]. To this end, we first recall some notions from Ky [32] (see also [47]).

Definition 2.7. A nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if $\phi(0) = 0$, $\phi(t) > 0$ for any $t \in (0,\infty)$ and $\lim_{t\to\infty} \phi(t) = \infty$. For any given $p \in (0,\infty)$, an Orlicz function ϕ is said to be of *positive lower* (resp., *upper*) *type* p if there exists a positive constant C such that, for any $t \in [0,\infty)$ and $s \in (0,1]$ (resp., $s \in [1,\infty)$), $\phi(st) \le Cs^p\phi(t)$.

Definition 2.8. Let $\phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be such that $\phi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for any $x \in \mathbb{R}^n$. For any given $p \in (0,\infty)$, the function ϕ is said to be of *positive uniformly lower* (resp., *upper*) *type* p if there exists a positive constant C such that, for any $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in (0,1]$ (resp., $s \in [1,\infty)$),

$$\phi(x, st) \le Cs^p\phi(x,t).$$

Let

$$i(\phi) := \sup\{p \in (0,\infty) : \phi \text{ is of positive uniformly lower type } p\}$$

and

$$I(\phi) := \inf\{p \in (0,\infty) : \phi \text{ is of positive uniformly upper type } p\}.$$

Definition 2.9. Let $\phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfy that $\phi(\cdot,t) : \mathbb{R}^n \to [0,\infty)$ is a measurable function for any $t \in [0,\infty)$. For any given $q \in [1,\infty)$, the function ϕ is said to satisfy the *uniformly Muckenhoupt* $\mathbb{A}_q(\mathbb{R}^n)$ *condition*, denoted by $\phi \in \mathbb{A}_q(\mathbb{R}^n)$, if

$$[\phi]_{\mathbb{A}_q(\mathbb{R}^n)} := \begin{cases} \displaystyle\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left[\frac{1}{|B|}\int_B \phi(z,t)\,dz\right] & \\ \displaystyle\qquad \times \left[\frac{1}{|B|}\int_B \{\phi(z,t)\}^{-\frac{1}{q-1}}\,dz\right]^{q-1} & \text{when } q \in (1,\infty), \\ \displaystyle\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \sup_{x\in B} \left[\frac{1}{|B|}\int_B \phi(z,t)\,dz\right][\phi(x,t)]^{-1} & \text{when } q = 1 \end{cases}$$

is finite, where the second suprema are taken over all balls B of $\mathbb{R}^n$. Let

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

Equivalently, $\phi \in \mathbb{A}_\infty(\mathbb{R}^n)$ if and only if there exist $0 < \delta, \gamma < 1$ such that

$$|E| \geq \gamma|B| \ \text{ implies } \ \phi(E,t) \geq \delta\phi(B,t),$$

for any $t \in (0,\infty)$, ball $B \subset \mathbb{R}^n$ and $E \subset B$, where $\phi(F,t) := \int_F \phi(x,\, t)\,dx$ for any measurable set $F \subset \mathbb{R}^n$. Define the *critical weight index* $q(\phi)$ of $\phi \in \mathbb{A}_\infty(\mathbb{R}^n)$ by setting

$$q(\phi) := \inf\left\{q \in [1,\infty) :\ \phi \in \mathbb{A}_q(\mathbb{R}^n)\right\}.$$

Definition 2.10. A function $\phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *Musielak–Orlicz function* if the function $\phi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for any $x \in \mathbb{R}^n$, and the function $\phi(\cdot,t)$ is a measurable function for any $t \in [0,\infty)$.

Definition 2.11. A Musielak–Orlicz function $\phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if $\phi \in \mathbb{A}_\infty(\mathbb{R}^n)$, ϕ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

The following proposition shows that the function Φ_p in (1.6) is a growth function.

Proposition 2.12. *Let $p \in (0,1]$. Then Φ_p in* (1.6) *is a Musielak–Orlicz function satisfying that*

(i) *Φ_p is of uniformly lower type p and uniformly upper type* 1*;*

(ii) $\Phi_p \in \mathbb{A}_1(\mathbb{R}^n)$.

In particular, Φ_p is a growth function as in Definition 2.11.

The main argument for the proof of Proposition 2.12 is contained in the following lemma which will also be useful in the remainder of this article; see also [19, Lemma 2.3(iv)] for another proof of (2.5).

Lemma 2.13. *Let $\gamma \in [0, 1)$ and $\beta \in [0, \infty)$. Then, for any $R \in (0, \infty)$,*

$$\fint_{B(0,R)} (1 + |x|)^{-n\gamma} [\log(e + |x|)]^{-\beta} \, dx \sim (1 + R)^{-n\gamma} [\log(e + R)]^{-\beta}. \tag{2.5}$$

Moreover, the function $(1 + |x|)^{-n\gamma} [\log(e + |x|)]^{-\beta}$ is in the classical Muckenhoupt class $A_1(\mathbb{R}^n)$, that is, for any ball $B \subset \mathbb{R}^n$,

$$\fint_{B} (1 + |x|)^{-n\gamma} [\log(e + |x|)]^{-\beta} \, dx \lesssim \inf_{z \in B} \left[(1 + |z|)^{-n\gamma} [\log(e + |z|)]^{-\beta} \right]. \tag{2.6}$$

In particular, for any ball $B \subset \mathbb{R}^n$,

$$\fint_{B} (1 + |x|)^{-n\gamma} [\log(e + |x|)]^{-\beta} \, dx \sim (1 + |c_B| + r_B)^{-n\gamma} [\log(e + |c_B| + r_B)]^{-\beta}. \tag{2.7}$$

Here, in (2.5) through (2.7), the positive equivalence constants are independent of R and B.

Proof. Let us first show (2.5). For any $R \in (0, \infty)$, the bound below comes directly from the integral in $\{x \in \mathbb{R}^n : \ R/2 < |x| < R\}$. So let us concentrate on the bound above. Taking radial coordinates and making a change of variables, to show (2.5), we only need to prove that

$$\int_0^1 \frac{t^{n(1-\gamma)}}{[\log(e + Rt)]^\beta} \frac{dt}{t} \lesssim [\log(e + R)]^{-\beta}.$$

This inequality is straightforward for $R \leq 4$. So let us assume that $R > 4$. When we integrate on the interval $0 < Rt < 4$, the integral is bounded by a power of R^{-1}, which is smaller than the right hand side. Finally, we observe that, for $Rt \geq 4$, we have the inequality

$$\frac{\log R}{\log(Rt)} \leq 1 + \log(t^{-1}),$$

which, together with the assumptions $R > 4$ and $\gamma \in [0,\, 1)$, implies that

$$\begin{aligned}
\int_{4/R}^1 \frac{t^{n(1-\gamma)}}{[\log(e + Rt)]^\beta} \frac{dt}{t} &\leq \int_{4/R}^1 \frac{t^{n(1-\gamma)}}{[\log(Rt)]^\beta} \frac{dt}{t} \\
&\leq (\log R)^{-\beta} \int_0^1 t^{n(1-\gamma)} \, (1 - \log t)^\beta \, \frac{dt}{t} \\
&\sim [\log(e + R)]^{-\beta}.
\end{aligned}$$

This allows us to conclude the proof of (2.5).

In order to show $(1+|x|)^{-n\gamma}[\log(e+|x|)]^{-\beta} \in A_1(\mathbb{R}^n)$, let us come to the proof of (2.6). To this end, we fix a ball B with center $c_B \in \mathbb{R}^n$ and radius $r_B \in (0,\infty)$. If $|c_B| \geq 2r_B$, then the distance from any point in B to the origin is at least r_B, which implies that $|x| \sim |z|$ for any $x, z \in B$, and hence (2.6) holds true. If $|c_B| < 2r_B$, we directly obtain (2.6) by using (2.5) for the ball centered at $\vec{0}_n$ with radius $3r_B$, which contains B. This finishes the proof of (2.6).

Observe that the fact $(1+|x|)^{-n\gamma}[\log(e+|x|)]^{-\beta}$ for any $x \in \mathbb{R}^n$ is in $A_1(\mathbb{R}^n)$ implies directly (2.7). We have completed the proof of this lemma. $\square$

Proof of Proposition 2.12. Notice that this proposition was known when $p = 1$ (see [32]). It remains to consider the case $p \in (0,1)$. By (1.6) and Definition 2.10, it is easy to see that Φ_p is a Musielak–Orlicz function. Next, we observe that the function Φ_p is, for any given $p \in (0,1)$, equivalent to the minimum of two functions that are Orlicz functions with weights. Indeed, as we said in the introduction, for any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$,

$$\Phi_p(x,\, t) \sim \begin{cases} \min\left\{t, \dfrac{t^p}{(1+|x|)^{n(1-p)}}\right\} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ \min\left\{t, \dfrac{t^p}{(1+|x|)^{n(1-p)}[\log(e+|x|)]^p}\right\} & \\ \qquad\qquad\qquad\qquad\qquad \text{when } n(1/p-1) \in \mathbb{N} \cup \{0\}. & \end{cases} \tag{2.8}$$

It is easy to see that all Orlicz functions involved in these expressions are of lower type p and upper type 1. From this, we directly deduce that the minimum is also of uniformly lower type p and uniformly upper type 1. Also, it is easily seen that the minimum of two growth functions in $\mathbb{A}_1(\mathbb{R}^n)$ is still a growth function in $\mathbb{A}_1(\mathbb{R}^n)$ (see, for instance, [19, Lemma 2.3 (i)]). So, it suffices to prove that the functions $(1+|x|)^{-n(1-p)}$ and $(1+|x|)^{-n(1-p)}[\log(e+|x|)]^{-p}$ are in $A_1(\mathbb{R}^n)$. But these are already proved in Lemma 2.13. Altogether, we have completed the proof of Proposition 2.12. $\square$

2.3. Musielak–Orlicz Hardy spaces $H^{\Phi_p}(\mathbb{R}^n)$ and their dual spaces

Given any Musielak–Orlicz function ϕ that satisfies the growth condition in Definition 2.11, it was built in [32, 35] a real-variable theory of Musielak–Orlicz Hardy and Musielak–Orlicz Campanato spaces associated with ϕ.

Definition 2.14. Let ϕ be a growth function and $m(\phi) := \lfloor n[q(\phi)/i(\phi)-1]\rfloor$. The *Musielak–Orlicz–Lebesgue space* $L^\phi(\mathbb{R}^n)$ is defined to be the collection of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^\phi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \phi(x, |f(x)|/\lambda)\, dx \le 1\right\} < \infty. \qquad (2.9)$$

The *Musielak–Orlicz Hardy space* $H^\phi(\mathbb{R}^n)$ is defined to be the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^*_{m(\phi)}$ belongs to $L^\phi(\mathbb{R}^n)$, where $f^*_{m(\phi)}$ is as in (2.1) with m replaced by $m(\phi)$. For any $f \in H^\phi(\mathbb{R}^n)$, its *quasi-norm* $\|f\|_{H^\phi(\mathbb{R}^n)}$ is defined by setting

$$\|f\|_{H^\phi(\mathbb{R}^n)} := \left\|f^*_{m(\phi)}\right\|_{L^\phi(\mathbb{R}^n)}.$$

Definition 2.15. Let ϕ be a growth function and $s \in \mathbb{Z}_+$. The *Musielak–Orlicz Campanato space* $\mathfrak{C}_{\phi,1,s}(\mathbb{R}^n)$ is defined to be the collection of all locally integrable functions g on $\mathbb{R}^n$ such that

$$\|g\|_{\mathfrak{C}_{\phi,1,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\mathbf{1}_B\|_{L^\phi(\mathbb{R}^n)}} \int_B |g(x) - P_{B,s}g(x)|\, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. The semi-norm $\|\cdot\|_{\mathfrak{C}_{\phi,1,s}(\mathbb{R}^n)}$ vanishes precisely over the space $\mathcal{P}_s(\mathbb{R}^n)$.

The following duality result was established by Liang and Yang [35, Theorem 3.5] (see also [47, Theorem 5.2.1]), whose special case when $nq(\phi) < (n+1)i(\phi)$, that is, $\lfloor n[q(\phi)/i(\phi)-])\rfloor = 0$, was obtained by Ky [32, Theorem 3.2].

Lemma 2.16. *Let ϕ be a growth function and $s \in \mathbb{Z}_+$ such that $s \ge \lfloor n[q(\phi)/i(\phi)-1]\rfloor$. Then $(H^\phi(\mathbb{R}^n))^* = \mathfrak{C}_{\phi,1,s}(\mathbb{R}^n)/\mathcal{P}_s(\mathbb{R}^n)$.*

Remark 2.17. Let $p \in (0,1]$ and $s \in \mathbb{Z}_+$ be such that $s \ge \lfloor n(1/p-1)\rfloor$.

(i) According to Proposition 2.12, every Φ_p in (1.6) is a growth function with indices $q(\Phi_p) = 1$ and $i(\Phi_p) = p$ (both are not attainable), and hence the index $m(\Phi_p)$ is equal to $\lfloor n(1/p-1)\rfloor$. This indicates that the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ and the Musielak–Orlicz Campanato space $\mathfrak{C}_{\Phi_p,1,s}(\mathbb{R}^n)$ with $s \ge \lfloor n(1/p-1)\rfloor$ are well defined, with the function ϕ in Definitions 2.14 and 2.15 therein replaced by Φ_p. Further, we deduce from Lemma 2.16 that the dual space of $H^{\Phi_p}(\mathbb{R}^n)$ is $\mathfrak{C}_{\Phi_p,1,s}(\mathbb{R}^n)/\mathcal{P}_s(\mathbb{R}^n)$.

(ii) When $s = \lfloor n(1/p-1) \rfloor$, we simply write $\mathfrak{C}_{\Phi_p,1,s}(\mathbb{R}^n)$ as $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. Moreover,

$$\|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \inf_{P\in\mathcal{P}_s(\mathbb{R}^n)} \frac{1}{\|\mathbf{1}_B\|_{L^{\Phi_p}(\mathbb{R}^n)}} \int_B |g(x) - P(x)|\, dx < \infty.$$

(iii) Based on the equivalent expression of Φ_p in (1.7), we easily observe that $H^1(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n)$ and $H^p_{w_p}(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n)$. Moreover,

$$\|\cdot\|_{H^{\Phi_p}(\mathbb{R}^n)} \lesssim \min\left\{\|\cdot\|_{H^1(\mathbb{R}^n)},\, \|\cdot\|_{H^p_{w_p}(\mathbb{R}^n)}\right\}. \tag{2.10}$$

Next, we give an equivalent characterization of the Musielak–Orlicz Campanato space $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$, where $p \in (0,1)$.

Proposition 2.18. *Let $p \in (0,\,1)$, $\alpha = 1/p - 1$ and Φ_p be as in (1.6). For any ball $B \subset \mathbb{R}^n$, let*

$$\Psi_\alpha(B) := \begin{cases} \dfrac{|B|^\alpha}{(1+|c_B|+r_B)^{n\alpha}} & \text{when } n\alpha \notin \mathbb{N}, \\ \dfrac{|B|^\alpha}{(1+|c_B|+r_B)^{n\alpha}\log(e+|c_B|+r_B)} & \text{when } n\alpha \in \mathbb{N}. \end{cases} \tag{2.11}$$

Then

$$\|\mathbf{1}_B\|_{L^{\Phi_p}(\mathbb{R}^n)} \sim \|\mathbf{1}_B\|_{L^p_{w_p}(\mathbb{R}^n)} \sim \Psi_\alpha(B)|B|. \tag{2.12}$$

Consequently, for any locally integrable function g on $\mathbb{R}^n$,

$$\|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)} \sim \sup_{B\subset\mathbb{R}^n} \frac{1}{\Psi_\alpha(B)} \fint_B |g(x) - P_{B,\lfloor n\alpha\rfloor} g(x)|\, dx \tag{2.13}$$

whenever either side of (2.13) *is finite. Here, the positive equivalence constants in* (2.12) *and* (2.13) *are independent of g and B.*

The proof of Proposition 2.18 is a consequence of the following proposition, which is of independent interest.

Proposition 2.19. *Assume that the growth function ϕ may be written as* $\min\{\phi_1, \phi_2\}$*, where ϕ_1 and ϕ_2 are two growth functions. Then*

$$\|\mathbf{1}_B\|_{L^{\phi}(\mathbb{R}^n)} \sim \min\left\{\|\mathbf{1}_B\|_{L^{\phi_1}(\mathbb{R}^n)}, \|\mathbf{1}_B\|_{L^{\phi_2}(\mathbb{R}^n)}\right\}, \tag{2.14}$$

where the positive equivalence constants are independent of $B \subset \mathbb{R}^n$.

Proof. The fact that

$$\|\mathbf{1}_B\|_{L^{\phi}(\mathbb{R}^n)} \lesssim \min\left\{\|\mathbf{1}_B\|_{L^{\phi_1}(\mathbb{R}^n)}, \|\mathbf{1}_B\|_{L^{\phi_2}(\mathbb{R}^n)}\right\}$$

comes from the definition of the norm. Let us prove the converse inequality, that is,

$$\min\left\{\phi_1(B, \lambda^{-1}), \phi_2(B, \lambda^{-1})\right\} \leq C$$

for some uniform constant C that is dependent of B. Here $\lambda = \|\mathbf{1}_B\|_{L^{\phi}(\mathbb{R}^n)}$ and, for any $i \in \{1, 2\}$,

$$\phi_i(B, \lambda^{-1}) := \int_B \phi_i(x, \lambda^{-1})\, dx.$$

Without loss of generality, we may assume that the set $E := \{x \in B : \phi_1(x, \lambda^{-1}) \leq \phi_2(x, \lambda^{-1})\}$ has Lebesgue measure larger than $|B|/2$ (otherwise, we may consider instead the set $F := \{x \in B : \phi_1(x, \lambda^{-1}) \geq \phi_2(x, \lambda^{-1})\}$). By assumption, we have

$$\phi_1(E, \lambda^{-1}) = \int_E \phi_1(x,\, \mathbf{1}_B(x)/\lambda)\, dx \leq 1.$$

Because of the facts that ϕ_1 is in $\mathbb{A}_\infty(\mathbb{R}^n)$ and $|B| \leq 2|E|$, we conclude from the definition of $\mathbb{A}_\infty(\mathbb{R}^n)$ that $\phi_1(B, \lambda^{-1}) \leq C$, which proves Proposition 2.19. $\square$

Proof of Proposition 2.18. Notice that (2.13) follows from (2.12) and the definition of $\|\cdot\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)}$. Thus, to finish the proof of Proposition 2.18, it suffices to prove (2.12).

Apply Proposition 2.19 with $\phi_1(x, t) := t$ and $\phi_2(x, t) := t^p w_p(x)$ for any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, where w_p is as in (1.5). Notice that $\|\mathbf{1}_B\|_{L^{\phi_1}(\mathbb{R}^n)} = |B|$. Also, the estimate (2.7) implies that

$$\|\mathbf{1}_B\|_{L^{\phi_2}(\mathbb{R}^n)} = \|\mathbf{1}_B\|_{L^p_{w_p}(\mathbb{R}^n)} \sim |B|\Psi_\alpha(B),$$

where $\Psi_\alpha(B)$ is as in (2.11). Then, invoking the fact that $\min\{1, \Psi_\alpha(B)\} \sim \Psi_\alpha(B)$ and Proposition 2.19, we obtain (2.12). This concludes the proof of Proposition 2.18. $\square$

Remark 2.20. Let all the notation be as in Proposition 2.18. Notice that

$$\Psi_\alpha(B) \sim \begin{cases} \min\left\{1, \left(\dfrac{r_B}{1+|c_B|}\right)^{n\alpha}\right\} & \text{when } n\alpha \notin \mathbb{N}, \\ \min\left\{1, \left(\dfrac{r_B}{1+|c_B|}\right)^{n\alpha}\right\} \dfrac{1}{\log(e+|c_B|+r_B)} & \text{when } n\alpha \in \mathbb{N}, \end{cases}$$

with the positive equivalence constants independent of B, where $\Psi_\alpha(B)$ is as in (2.11). As a consequence of the fact that Ψ_α is given by the minimum of two quantities, it follows that $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ is the intersection of two spaces. For instance, when $\alpha n \notin \mathbb{N}$, it is the intersection of the space $\mathrm{BMO}(\mathbb{R}^n)$ with the space of all functions g such that

$$\sup_B \frac{(1+|c_B|)^{n\alpha}}{|B|^\alpha} \fint_B |g(x) - P_{B,\lfloor n\alpha\rfloor} g(x)|dx < \infty.$$

In particular, g belongs to $\mathfrak{C}_\alpha(\mathbb{R}^n)$. So g is in particular of class $C^{\lceil n\alpha\rceil -1}(\mathbb{R}^n)$ in view of Definition 2.5. The same inclusion is valid for $n\alpha \in \mathbb{N}$.

A first example of functions in $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ is given by the Schwartz functions. We state it as a lemma.

Lemma 2.21. *If $p \in (0, 1)$, then $\mathcal{S}(\mathbb{R}^n)$ embeds continuously into $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$.*

Proof. Let $p \in (0, 1)$, $\alpha = 1/p - 1$ and $g \in \mathcal{S}(\mathbb{R}^n)$. We need to prove that, for any ball $B = B(c_B, r_B) \subset \mathbb{R}^n$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, there exists a polynomial $P \in \mathcal{P}_s(\mathbb{R}^n)$ with $s = \lfloor n\alpha \rfloor$ such that

$$\frac{1}{\Psi_\alpha(B)} \fint_B |g(x) - P(x)|\, dx \lesssim 1,$$

where $\Psi_\alpha(B)$ is as in (2.11). As before, we can reduce to two cases: either $r_B \leq |c_B|/2$ or $r_B > |c_B|/2$. In the first case, we take P to be the Taylor polynomial of g at the point c_B with degree s. Then, for any integer N larger than $s+1$, we have

$$\sup_{x\in B} |g(x) - P(x)| \lesssim \frac{r_B^{s+1}}{(1+|c_B|)^N},$$

which is uniformly bounded when divided by $\Psi_\alpha(B)$.

Let us consider the case $r_B > |c_B|/2$. When $r_B \le 1$, again we take P to be the Taylor polynomial of g at the point c_B with degree s, so that

$$|g(x) - P(x)| \lesssim |x - c_B|^{s+1} \lesssim r_B^{s+1} \lesssim r_B^{n\alpha} \sim \Psi_\alpha(B)$$

inside the ball B and we conclude again directly. When $r_B > 1$, by taking $P = 0$ and using the fact that $g \in L^1(\mathbb{R}^n)$, we find that

$$⨍_B |g(x) - P(x)|\, dx \lesssim |B|^{-1} \lesssim \Psi_\alpha(B).$$

When tracking constants it is easy to see that the embedding is continuous, which completes the proof of Lemma 2.21. $\square$

2.4. The growth of Campanato functions

The first result of this section is the following pointwise estimate, which indicates that a function $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ has polynomial growth of order $n\alpha$ when $n\alpha \notin \mathbb{N}$, but can have an extra logarithm growth factor when $n\alpha \in \mathbb{N}$.

Proposition 2.22. *Let $\alpha \in (0, \infty)$ and $s = \lfloor n\alpha \rfloor$. Then there exists a positive constant C such that, for any $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ and any ball $B \subset \mathbb{R}^n$,*

$$\sup_{x\in B} |g(x)| \le \begin{cases} C(1 + |c_B| + r_B)^{n\alpha} \|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} & \text{when } n\alpha \notin \mathbb{N}, \\ C(1 + |c_B| + r_B)^{n\alpha} \log(e + |c_B| + r_B) \|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} & \\ & \text{when } n\alpha \in \mathbb{N}. \end{cases} \tag{2.15}$$

Moreover, the same estimates hold true for $\sup_{x\in B} |P_{B,s} g(x)|$.

Proof. Recall that $\mathfrak{C}_\alpha(\mathbb{R}^n) = \mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)$ with $s = \lfloor n\alpha \rfloor$. Once we have proved (2.15), then the projection $P_{B,s} g$ satisfies the same estimates as those of g, because Lemma 2.6 implies that

$$\sup_{x\in B} |g(x) - P_{B,s} g(x)| \le \|g\|_{\mathfrak{C}_{\alpha,\infty,s}(\mathbb{R}^n)} \sim \|g\|_{\mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)}.$$

So we only need to prove (2.15). To this end, it suffices to prove that, for any $x \in \mathbb{R}^n$,

$$|g(x)| \lesssim \begin{cases} (1 + |x|)^{n\alpha} \|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} & \text{when } n\alpha \notin \mathbb{N}, \\ (1 + |x|)^{n\alpha} \log(e + |x|) \|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} & \text{when } n\alpha \in \mathbb{N} \end{cases} \tag{2.16}$$

uniformly in g and x. The proof of this inequality is standard even if one has to be careful with the norm $\|\cdot\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}$. One can restrict to the case when $n\alpha \leq 1$ by using an induction argument. Under this restriction, the proof is classical.

To be precise, we first prove that it suffices to show this inequality for $n\alpha \leq 1$. Indeed, assume that $n\alpha > 1$. By Definition 2.5(iii) and Lemma 2.6, we know that g is in $\mathfrak{C}_\alpha(\mathbb{R}^n)$ if and only if $\partial_{x_j} g \in \mathfrak{C}_{\alpha-\frac{1}{n}}(\mathbb{R}^n)$ for any $j \in \{1, \ldots, n\}$, after modifying g on a set of measure zero if necessary. Moreover, we have

$$\|\partial_{x_j} g\|_{\mathfrak{C}_{\alpha-\frac{1}{n}}(\mathbb{R}^n)} \sim \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}.$$

Let us show that we have as well

$$\|\partial_{x_j} g\|_{\mathfrak{C}^+_{\alpha-1/n}(\mathbb{R}^n)} \lesssim \|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}.$$

We use Lemma 2.4 for this. It follows from integration by parts that

$$\begin{aligned}\sum_{|\gamma|\leq s-1} \left| \int_{B(\vec{0}_n,1)} x^\gamma \partial_{x_j} g(x)dx \right| &\lesssim \sum_{|\gamma|\leq s-1} \left| \int_{x\in B(\vec{0}_n,1)} x^\gamma g(x)dx \right| + \sup_{x\in B(\vec{0}_n,1)} |g(x)| \\ &\lesssim \|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)},\end{aligned}$$

where we have used Lemma 2.4 to prove the last inequality.

So, assuming that (2.16) holds true for $\alpha n \leq k$ and wanting to prove it for $k < n\alpha \leq k+1$, we have the required inequality with $\alpha - \frac{1}{n}$ in place of α and $\partial_{x_j} g$ in place of g. The inequality for g is obtained by integration.

It remains to prove (2.16) for $n\alpha \leq 1$. This is straightforward for $n\alpha < 1$. Indeed, we deduce from Lemma 2.6 that, for any x, $y \in \mathbb{R}^n$,

$$|g(x) - g(y)| \lesssim \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} |x-y|^{n\alpha},$$

and integrate in y inside the ball $B(\vec{0}_n, 1)$ to obtain the required estimate.

Assume now that $\alpha n = 1$. It suffices to prove that, for any $x \in \mathbb{R}^n$,

$$|g(x)| \lesssim \left[\sup_{x\in B(\vec{0}_n,1)} |g(y)| + \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \right] (1+|x|) \log(e+|x|). \tag{2.17}$$

Indeed, if $|x| < 1$, then (2.17) follows directly from

$$|g(x)| \leq \sup_{x\in B(\vec{0}_n,\,1)} |g(x)|.$$

If $|x| \geq 1$, then Definition 2.5(ii) implies that

$$
\begin{aligned}
|g(x)| &\leq |g(x) - 2g(2^{-1}x) + g(\vec{0}_n)| + 2|g(2^{-1}x)| + |g(\vec{0}_n)| \\
&\leq 2^{-1}|x|\|g\|_{\dot{\Lambda}_1(\mathbb{R}^n)} + 2|g(2^{-1}x)| + |g(\vec{0}_n)|
\end{aligned}
$$

and an inductive argument further gives that

$$
|g(x)| \leq k2^{-1}|x|\|g\|_{\dot{\Lambda}_1(\mathbb{R}^n)} + 2^k|g(2^{-k}x)| + |g(\vec{0}_n)| \sum_{j=1}^{k} 2^{j-1}
$$

whenever $k \in \mathbb{N}$. In particular, choose $k = n_0$ satisfying $2^{n_0-1} \leq |x| < 2^{n_0}$. Then, using $2^{-n_0}x \in B(\vec{0}_n, 1)$ and $n_0 \sim \log(e + |x|)$, we conclude that $|g(x)|$ has the desired estimate as in the right hand side of (2.17). This finishes the proof of Proposition 2.22. □

2.5. A first decomposition

We apply these bounds above to find a first decomposition of products, which is an analog for any given $p \in (0, 1)$ of the one obtained in [8]. Let us first recall the definition of atoms (see, for instance, [20, 25, 37] for more details).

Definition 2.23. Let $p \in (0, 1)$ and $l \in \mathbb{Z}_+$. A function $a \in L^2(\mathbb{R}^n)$ is called a (p, l)-*atom* if

- (i) there exists a ball B such that $\operatorname{supp} a \subset B$;
- (ii) $\|a\|_{L^2(\mathbb{R}^n)} \leq |B|^{1/2-1/p}$;
- (iii) $\int_{\mathbb{R}^n} x^\alpha a(x)\, dx = 0$ for any multi-index $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ satisfying $|\alpha| := \sum_{i=1}^{n} \alpha_i \leq l$.

Observe that, if the ball B in Definition 2.23 is replaced by a cube $Q \subset \mathbb{R}^n$, we obtain an alternative equivalent definition of (p, l)-atoms supported on cubes. By an abuse of terminology, we still call the latter case a (p, l)-atom (see Sections 4.3 and 4.4 below).

We recall that, as soon as $p \in (0, 1)$ and $l \geq s$ with $s = \lfloor n(1/p - 1) \rfloor$, these (p, l)-atoms have uniformly bounded $H^p(\mathbb{R}^n)$-norms. Moreover, one has an atomic decomposition, that is, distributions in $H^p(\mathbb{R}^n)$ may be obtained as limits of finite linear combinations of atoms. We will go back to this later on but at this point we want to have a first estimate on the product ag, where

a is an atom related to a ball B and g is in $\mathfrak{C}_\alpha(\mathbb{R}^n)$ with $\alpha = 1/p - 1$. If we write

$$ag = a(g - P_{B,s}g) + aP_{B,s}g,$$

we know at once that the first term is in $L^1(\mathbb{R}^n)$. The next proposition states that the product $aP_{B,s}g$ lies in the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ under the condition that the order l is large enough. This result plays a crucial role in the proof of Theorems 1.1 and 1.2.

Proposition 2.24. *Let $p \in (0, 1)$, $\alpha = 1/p - 1$, $s = \lfloor n(1/p - 1)\rfloor$ and $l \in \mathbb{Z}_+ \cap [2s, \infty)$. Assume that $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ and a is a (p, l)-atom supported in a ball $B \subset \mathbb{R}^n$. Then*

$$\|aP_{B,s}g\|_{H^{\Phi_p}(\mathbb{R}^n)} \le C\|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)},$$

where C is a positive constant independent of a and g.

Proof. Without loss of generality, we may assume that $\|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} = 1$. Let $h := aP_{B,s}g$ and h^* be the non-tangential maximal function of h as in (2.1). Using (2.10), we only need to show that

$$\|h\|^p_{H^p_{w_p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} [h^*(x)]^p w_p(x)\, dx \lesssim 1, \tag{2.18}$$

where the weight w_p is given in (1.5). Since the weight w_p is radial, we let $\overline{w}_p$ be the function on $(0,\infty)$ such that $\overline{w}_p(|x|) = w_p(x)$ for any $x \in \mathbb{R}^n$.

Because a has vanishing moments up to order $l \ge 2s$ and $P_{B,s}g$ is a polynomial of order s, we have $\int_{\mathbb{R}^n} x^\beta h(x)\, dx = 0$ for any multi-index β satisfying $|\beta| \le s$. Moreover, from Proposition 2.22, we deduce the estimate that, for any $x \in B = B(c_B, r_B)$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,

$$[\overline{w}_p(|c_B| + r_B)]^{1/p}\, |P_{B,s}g(x)| \lesssim 1.$$

Thus, in particular, $\widetilde{a} := [\overline{w}_p(|c_B| + r_B)]^{1/p} h$ is, up to a uniform constant, a (p, s)-atom. Now, the proof of (2.18) falls into the following estimate:

$$\mathcal{J} := \int_{\mathbb{R}^n} [(\widetilde{a})^*(x)]^p w_p(x)\, dx \lesssim \overline{w}_p(|c_B| + r_B).$$

Applying first the Hölder inequality and then (2.7) to the function

$$[w_p]^{\frac{1}{1-p/2}} = \begin{cases} (1+|x|)^{-n(1-p)/(1-p/2)} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ (1+|x|)^{-n(1-p)/(1-p/2)}[\log(e+|x|)]^{-p/(1-p/2)} & \\ & \text{when } n(1/p-1) \in \mathbb{N}, \end{cases}$$

we obtain

$$\int_{4B}[(\widetilde{a})^*(x)]^p w_p(x)\,dx \le \|(\widetilde{a})^*\|_{L^2(\mathbb{R}^n)}^p \left[\int_{4B}[w_p(x)]^{\frac{1}{1-p/2}}\,dx\right]^{1-p/2}$$
$$\lesssim \|a\|_{L^2(\mathbb{R}^n)}^p |B|^{1-p/2}\overline{w}_p(|c_B|+r_B) \lesssim \overline{w}_p(|c_B|+r_B).$$

According to [43, p. 106], we have

$$(\widetilde{a})^*(x) \lesssim \frac{1}{|B|^{1/p}}\left(\frac{r_B}{|x-c_B|}\right)^{n+s+1}, \qquad \forall\, x\in\mathbb{R}^n\setminus 4B$$

and hence, by (2.7) and the fact $s+1>n(1/p-1)$, we further conclude that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus(4B)}[(\widetilde{a})^*(x)]^p w_p(x)\,dx &= \sum_{j=1}^\infty \int_{(4^{j+1}B)\setminus(4^jB)}[(\widetilde{a})^*(x)]^p w_p(x)\,dx\\ &\lesssim \sum_{j=1}^\infty 2^{-j[(n+s+1)p-n]} ⨍_{4^{j+1}B} w_p(x)\,dx\\ &\lesssim \overline{w}_p(|c_B|+r_B).\end{aligned}$$

Thus, we obtain the desired estimate for $\mathcal{J}$.

This finishes the proof of Proposition 2.24. □

3. Pointwise multipliers of Campanato spaces

The main aim of this section is to prove Theorems 1.1 and 1.3. We begin with the proof of Theorem 1.3 by dividing it into two steps: the necessary part and the sufficient part.

3.1. Necessary part of Theorem 1.3

Theorem 3.1. *Let $p\in(0,1)$, $\alpha=1/p-1$ and Φ_p be as in (1.6). Then there exists a positive constant C such that, for any $g\in L^\infty(\mathbb{R}^n)\cap\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ and $f\in\mathfrak{C}_\alpha(\mathbb{R}^n)$,*

$$\|gf\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} \le C\|f\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)}[\|g\|_{L^\infty(\mathbb{R}^n)}+\|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)}].$$

Proof. Let $g \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. Recall that $\mathfrak{C}_\alpha(\mathbb{R}^n) = \mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)$ with $s = \lfloor n\alpha \rfloor$. Then, for any $f \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, we have

$$\fint_{B(\vec{0}_n,1)} |g(x)f(x)|\,dx \le \|g\|_{L^\infty(\mathbb{R}^n)} \fint_{B(\vec{0}_n,1)} |f(x)|\,dx \le \|g\|_{L^\infty(\mathbb{R}^n)} \|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}.$$

For any ball $B \subset \mathbb{R}^n$ and $x \in B$, we write

$$\begin{aligned}|g(x)f(x) - P_{B,s}f(x)P_{B,s}g(x)| &\le |f(x) - P_{B,s}f(x)|\;|g(x)| \\ &\quad + |P_{B,s}f(x)|\,|g(x) - P_{B,s}g(x)|\,.\end{aligned}$$

From Proposition 2.22 and (2.11), it follows that

$$\sup_{x\in B} |P_{B,s}f(x)| \lesssim \frac{|B|^\alpha}{\Psi_\alpha(B)} \|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)},$$

here and hereafter, $\Psi_\alpha(B)$ is as in (2.11). By this and (2.13), we conclude that

$$\begin{aligned}&\fint_B |g(x)f(x) - P_{B,s}f(x)P_{B,s}g(x)|\;dx \\ &\quad\lesssim \fint_B |f(x) - P_{B,s}f(x)|\;dx\,\|g\|_{L^\infty(\mathbb{R}^n)} \\ &\qquad + \|f\|_{\mathfrak{C}^+_{\alpha,1,s}(\mathbb{R}^n)} \frac{|B|^\alpha}{\Psi_\alpha(B)} \fint_B |g(x) - P_{B,s}g(x)|\,dx \\ &\quad\lesssim |B|^\alpha \left[\|f\|_{\mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{\mathfrak{C}^+_{\alpha,1,s}(\mathbb{R}^n)} \|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)}\right] \\ &\quad\lesssim |B|^\alpha \|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} [\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)}].\end{aligned}$$

As $P_{B,s}f\,P_{B,s}g \in \mathcal{P}_{2s}(\mathbb{R}^n)$, then we utilize (2.4) to obtain $gf \in \mathfrak{C}_{\alpha,1,2s}(\mathbb{R}^n)$ with

$$\|gf\|_{\mathfrak{C}_{\alpha,1,2s}(\mathbb{R}^n)} \lesssim \|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} [\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)}].$$

Further, by the equivalence $\mathfrak{C}_{\alpha,1,2s}(\mathbb{R}^n)/\mathcal{P}_{2s}(\mathbb{R}^n) = \mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)/\mathcal{P}_s(\mathbb{R}^n)$ in Remark 2.3(i), we know that there exists $Q \in \mathcal{P}_{2s}(\mathbb{R}^n)$ such that $gf - Q \in \mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)$. Due to (2.16), the function $gf - Q$ has at most polynomial growth of order $n\alpha$ (with an extra logarithm growth factor for the integer case) at infinity, so does gf because $g \in L^\infty(\mathbb{R}^n)$ and $f \in \mathfrak{C}_\alpha(\mathbb{R}^n)$. This forces Q to be a polynomial of order no more than $n\alpha$. In other words, we have

$Q \in \mathcal{P}_s(\mathbb{R}^n)$. Then the previous fact $gf - Q \in \mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)$ implies that gf itself is in $\mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)$. Moreover, noticing that

$$\|gf\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} = \|gf\|_{\mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)} = \|gf - Q\|_{\mathfrak{C}_{\alpha,1,s}(\mathbb{R}^n)} = \|gf\|_{\mathfrak{C}_{\alpha,1,2s}(\mathbb{R}^n)},$$

we conclude the proof of Theorem 3.1. □

As an application of Theorem 3.1 and Lemma 2.21, we prove in the following corollary that any Schwartz function is a pointwise multiplier of the Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n)$, with $\alpha \in (0, \infty)$, which hence justifies the definition of the product in (1.2). Recall that this fact has also been pointed out in [3, p. 59].

Corollary 3.2. *Let $\alpha \in (0, \infty)$. Then, for any $g \in \mathcal{S}(\mathbb{R}^n)$, g is a pointwise multiplier of $\mathfrak{C}_\alpha(\mathbb{R}^n)$.*

Proof. By Theorem 3.1, it suffices to prove that $g \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$, where the number p satisfies that $\alpha = 1/p - 1 > 0$. It is obvious that Schwartz functions are bounded. The fact that they are in $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ has been proved in Lemma 2.21, which completes the proof of Corollary 3.2. □

3.2. Sufficient part of Theorem 1.3 for the non-integer case

In this section, for the non-integer case $n\alpha \notin \mathbb{N}$, we discuss the sufficient part of Theorem 1.3 by constructing two examples of functions in $\mathfrak{C}_\alpha(\mathbb{R}^n)$.

Proposition 3.3. *Let $\alpha \in (0, \infty)$ and $n\alpha \notin \mathbb{N}$. Let f be a continuous function on $\mathbb{R}^n$, which is homogeneous of degree $n\alpha$ and has continuous derivatives on $\mathbb{R}^n \setminus \{\vec{0}_n\}$ up to order $1 + \lfloor n\alpha \rfloor$. Then f is in the space $\mathfrak{C}_\alpha(\mathbb{R}^n)$. In particular, $|x|^{n\alpha} \in \mathfrak{C}_\alpha(\mathbb{R}^n)$.*

Proof. Let f be as in the proposition. Then, if $n\alpha > 1$, all its derivatives satisfy the same assumptions as f, except that they are now homogeneous of degree $n\alpha - 1$. So an easy induction shows that it suffices to prove the proposition for $0 < n\alpha < 1$. By the homogeneity assumption, we have $|f(x)| \le C|x|^{n\alpha}$ for any $x \in \mathbb{R}^n$, as well as $|\nabla f(x)| \le C|x|^{n\alpha-1}$ for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$.

We want to prove that $|f(x) - f(y)| \lesssim |x - y|^{n\alpha}$ for any $x, y \in \mathbb{R}^n$. By symmetry, we consider only the case $|x| \le |y|$. When $|y - x| \le \frac{|y|}{2}$, we conclude that the whole segment joining x to y lies in the complement of the ball centered at $\vec{0}_n$ and of radius $|y|/2$. By Taylor's theorem, we obtain

$$|f(x) - f(y)| \lesssim |x - y|\, |y|^{n\alpha-1},$$

from which we deduce the desired conclusion. Assume now that $|y-x| \geq \frac{|y|}{2}$. We then conclude the desired conclusion by the estimate $|f(x) - f(y)| \lesssim (|x| + |y|)^{n\alpha}$.

This finishes the proof of Proposition 3.3. $\square$

Proposition 3.4. *Let $\alpha \in (0, \infty)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi \leq 1$, $\operatorname{supp} \varphi \subset B(\vec{0}_n, 2)$ and $\varphi \equiv 1$ on $B(\vec{0}_n, 1)$. Given a ball $B = B(c_B, r_B) \subset \mathbb{R}^n$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ satisfying $|c_B| > \max\{2, 4r_B\}$, define*

$$f^{(B)}(x) := |c_B|^{n\alpha} \varphi\left(\frac{4(x - c_B)}{|c_B|}\right), \qquad \forall\, x \in \mathbb{R}^n. \tag{3.1}$$

Then $f^{(B)}$ has the following properties:

(i) *$f^{(B)} \equiv |c_B|^{n\alpha}$ on B, and $f^{(B)} \equiv 0$ on $B(\vec{0}_n, 1)$*

(ii) *$f^{(B)} \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ and $\|f^{(B)}\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} = \|f^{(B)}\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \leq C$, where C is a positive constant independent of B.*

Proof. Notice that (i) follows directly from the assumptions of φ and $|c_B| > \max\{2, 4r_B\}$. From (i), the equality $\|f^{(B)}\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} = \|f^{(B)}\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}$ in (ii) follows immediately.

Now we show that $\|f^{(B)}\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \lesssim 1$ uniformly in B. Since the semi-norm in (2.4) is invariant by rotation and translation, and is homogeneous of order $n\alpha$, we deduce immediately from the fact that φ is in $\mathfrak{C}_\alpha(\mathbb{R}^n)$ that

$$\left\|\varphi\left(\frac{4(\cdot - c_B)}{|c_B|}\right)\right\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \sim |c_B|^{-n\alpha} \|\varphi\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \sim |c_B|^{-n\alpha}.$$

The result $\|f^{(B)}\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \lesssim 1$ follows directly.

This finishes the proof of Proposition 3.4. $\square$

Also, we need the following lemma, which we can find in [11] and [37, p. 54, Lemma 4.1].

Lemma 3.5. *Let $s \in \mathbb{Z}_+$. Then there exists a positive constant C such that, for any locally integrable function g and for any ball $B \subset \mathbb{R}^n$,*

$$\sup_{x\in B} |P_{B,s}g(x)| \le C \fint_B |g(y)|\, dy.$$

Theorem 3.6. *Let $p \in (0, 1)$, $\alpha = 1/p - 1$ and assume that $n\alpha \notin \mathbb{N}$. If a function g is a pointwise multiplier of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ with operator norm*

$$|||g||| := \sup_{f\neq 0} \frac{\|gf\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}}{\|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}},$$

where the supremum is taken over all $f \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ satisfying $f \neq 0$, then $g \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of g, such that

$$\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)} \le C|||g|||. \tag{3.2}$$

Proof. Let g be a pointwise multiplier of $\mathfrak{C}_\alpha(\mathbb{R}^n)$, which we can assume of norm 1. Then, for any $f \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, we have

$$\|gf\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} \le \|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}. \tag{3.3}$$

Let us first point out that, by testing on the function 1, we find that g itself is in $\mathfrak{C}_\alpha(\mathbb{R}^n)$ and $\|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} \le 1$. We now prove that g is bounded. As a consequence of Proposition 3.3, the function

$$f(x) := 1 + |x|^{n\alpha}, \qquad \forall\, x \in \mathbb{R}^n$$

belongs to $\mathfrak{C}_\alpha(\mathbb{R}^n)$. So $\|gf\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} \le \|f\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} \lesssim 1$. Applying (2.16), we obtain, for any $x \in \mathbb{R}^n$,

$$|g(x)f(x)| \lesssim (1 + |x|)^{n\alpha},$$

which further implies that $g \in L^\infty(\mathbb{R}^n)$.

It remains to prove $\|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)} \lesssim 1$. Let $s = \lfloor n\alpha \rfloor$. We need to prove that, for any ball $B \subset \mathbb{R}^n$,

$$\fint_B |g(x) - P_{B,s}g(x)|\, dx \lesssim \Psi_\alpha(B) \sim \left(\frac{r_B}{1 + |c_B| + r_B}\right)^{n\alpha}, \tag{3.4}$$

where $\Psi_\alpha(B)$ is as in (2.11). We already know that the left hand side is uniformly bounded in terms of $r_B^{n\alpha}$ because $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$. So it suffices to consider balls $B = B(c_B, r_B)$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ satisfying $|c_B| + r_B > 4$.

For the case $|c_B| \le 4r_B$, we apply $g \in L^\infty(\mathbb{R}^n)$ and Lemma 3.5 to derive that

$$\fint_B |g(x) - P_{B,s}g(x)|\, dx \lesssim \fint_B |g(x)|\, dx \lesssim \|g\|_{L^\infty(\mathbb{R}^n)} \lesssim 1 \sim \Psi_\alpha(B).$$

For the case $|c_B| > 4r_B$, we have $|c_B| > \max\{2, 4r_B\}$. Let $f^{(B)}$ be as in Proposition 3.4. Notice that, for any $x \in B$,

$$\left|g(x)f^{(B)}(x) - P_{B,s}(f^{(B)}g)(x)\right| = |c_B|^{n\alpha}|g(x) - P_B g(x)|.$$

From this and (3.3), it follows that

$$\begin{aligned}\fint_B |g(x) - P_{B,s}g(x)|\, dx &= |c_B|^{-n\alpha} \fint_B \left|g(x)f^{(B)}(x) - P_{B,s}(f^{(B)}g)(x)\right|\, dx \\ &\lesssim |c_B|^{-n\alpha}|B|^\alpha \left\|gf^{(B)}\right\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \lesssim \left[\frac{r_B}{|c_B|}\right]^{n\alpha} \left\|f^{(B)}\right\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} \\ &\lesssim \left[\frac{r_B}{1 + |c_B| + r_B}\right]^{n\alpha}.\end{aligned}$$

Combining all the estimates, we obtain (3.4), which completes the proof of Proposition 3.6. □

3.3. Comments on the sufficient part of Theorem 1.3 for the integer case

We will give here partial results on the sufficient part of Theorem 1.3 when $n\alpha = k$ is an integer. Let us first give examples of functions in $\mathfrak{C}_\alpha(\mathbb{R}^n)$ under this assumption.

Proposition 3.7. *Let $\alpha \in (0, \infty)$ be such that $n\alpha \in \mathbb{N}$ and let $k = n\alpha$. For any $j \in \{1, \dots, n\}$ and $x \in \mathbb{R}^n$, define $f_j(x) := x_j^k \log|x_j|$. Then the function f_j is in $\mathfrak{C}_\alpha(\mathbb{R}^n)$.*

Proof. Let $j \in \{1, \dots, n\}$. It is well known that $\log|x_j|$ belongs to the space $\mathrm{BMO}(\mathbb{R}^n)$. All derivatives of order k of f are 0, except for the derivative

$$\frac{\partial^k f_j}{\partial x_j^k}(x) = k! \log|x_j| + \sum_{i=0}^{k-1} \binom{k}{i} k!(-1)^{k-i}, \qquad \forall\, x \in \mathbb{R}^n,$$

where $\binom{k}{i}$ denotes the binomial coefficient. Thus, all the k-th order derivatives of f_j belong to $\mathrm{BMO}(\mathbb{R}^n)$.

For any $\theta \in \mathbb{R}$ and $p,\ q \in (0,\infty]$, denote by $\dot{F}^{\theta}_{p,q}(\mathbb{R}^n)$ the homogeneous Triebel-Lizorkin space (see Frazier and Jawerth [23] or Triebel [45] for its precise definition). We know by [45, p. 244, Theorem] the following continuous embedding

$$\mathrm{BMO}(\mathbb{R}^n) = \dot{\mathrm{F}}^0_{\infty,2}(\mathbb{R}^n) \subset \dot{\mathrm{F}}^0_{\infty,\infty}(\mathbb{R}^n).$$

Also, for any $k \in \mathbb{N}$, it follows from [46, Theorem 1.5] and [48, Proposition 3.1(viii)] that

$$\sum_{|\nu|=k} \|D^\nu f\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}^k_{\infty,\infty}(\mathbb{R}^n)}$$

whenever either side is finite. This implies that $f_j \in \dot{F}^k_{\infty,\infty}(\mathbb{R}^n)$.

With the number p taken to satisfy $\alpha = 1/p - 1$, we know that $H^p(\mathbb{R}^n)$ and $\dot{F}^0_{p,2}(\mathbb{R}^n)$ coincide with equivalent (quasi)-norms (see [45, p. 244, Theorem]), while the dual spaces of these two spaces are $\mathfrak{C}_\alpha(\mathbb{R}^n)/\mathcal{P}_{\lfloor n\alpha\rfloor}(\mathbb{R}^n)$ and $\dot{F}^k_{\infty,\infty}(\mathbb{R}^n)$ (see [23, p.79, (5.14)]), respectively. Therefore, we conclude that $f_j \in \mathfrak{C}_\alpha(\mathbb{R}^n)$. This finishes the proof of Proposition 3.7. $\square$

Assume that $k = n\alpha \in \mathbb{N}$ and g is a pointwise multiplier of $\mathfrak{C}_\alpha(\mathbb{R}^n)$, which we still assume of norm 1. We test the multiplication by g the functions $h_j = 1 + f_j$, with each f_j as in Proposition 3.7 and $j \in \{1,\ldots,n\}$. Using inequality (2.16), we find that

$$\sup_{j\in\{1,\ldots,n\}} |g(x)h_j(x)| \le C(1+|x|)^k \log(e+|x|), \qquad \forall\, x \in \mathbb{R}^n,$$

where C is a positive constant independent of x, which implies that $g \in L^\infty(\mathbb{R}^n)$.

The unsolved part is the proof of $g \in \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ with p satisfying $\alpha = 1/p - 1$. At this point, for the case $n\alpha = k \in \mathbb{N}$, using the same proof as that for the case $n\alpha \notin \mathbb{N}$, we can prove that a pointwise multiplier g of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ is bounded and satisfies the condition: for any ball $B = B(c_B, r_B)$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,

$$\fint_B |g(x) - P_{B,k}g(x)|\, dx \le C \frac{r_B^k}{(1+|c_B|+r_B)^k}, \tag{3.5}$$

where C is a positive constant independent of B. This is not critical, as seen from below. One may conjecture that one has the following necessary condition: for any ball $B = B(c_B, r_B)$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,

$$\fint_B |g(x) - P_{B,k}g(x)|\,dx \le C\frac{r_B^k}{(1 + |c_B| + r_B)^k \log(e + |c_B| + r_B)}, \tag{3.6}$$

where C is a positive constant independent of B. We can not prove this but show that the condition (3.5) is not sufficient. We do this when $n = 1$ and $k = 1$. Observe that, whenever g is a multiplier, the same holds true for the function $g(-x)$, so that we can assume that g is odd or even. If we assume that g is odd, testing the corresponding multiplier of the function $x \log |x|$ and taking the second difference at 0, we find that $|xg(x)| \log |x| \le C|x|$ for any $x \in \mathbb{R} \setminus \{0\}$, where C is a positive constant independent of x. We conclude that there exists a positive constant C such that, for any $x \in \mathbb{R}$,

$$|g(x)| \le \frac{C}{\log(e + |x|)}.$$

But the function $\frac{x}{(1+x^2)^{1/2}}$ for any $x \in \mathbb{R}$ does not satisfy this last property while it satisfies (3.5). To show (3.6), it seems that one needs to find more intrinsic properties of pointwise multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ when $n\alpha \in \mathbb{N}$.

3.4. The linear decomposition: proof of Theorem 1.1

Notice that Corollary 3.2 shows that the definition (1.2) makes sense. Then, with the help of Proposition 2.24, we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, where $\alpha = 1/p - 1 \in (0, \infty)$. From the atomic characterization of $H^p(\mathbb{R}^n)$ (see, for instance, [25, 37]), it follows that there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_j\}_{j\in\mathbb{N}}$ of (p, l)-atoms with $l \in \mathbb{Z}_+ \cap [2s, \infty)$ and $s = \lfloor n\alpha \rfloor$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in } H^p(\mathbb{R}^n) \quad \text{and} \quad \sum_{j\in\mathbb{N}} |\lambda_j|^p \sim \|f\|^p_{H^p(\mathbb{R}^n)}.$$

By (1.2) and the duality theory between $H^p(\mathbb{R}^n)$ and $\mathfrak{C}_\alpha(\mathbb{R}^n)$, we write, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\langle f \times g, \phi\rangle &= \langle \phi g, f\rangle = \sum_{j\in\mathbb{N}} \lambda_j \langle \phi g, a_j\rangle = \sum_{j\in\mathbb{N}} \lambda_j \langle a_j g, \phi\rangle \\ &= \sum_{j\in\mathbb{N}} \lambda_j \left[\langle a_j(g - P_{B_j,s}g), \phi\rangle + \langle a_j P_{B_j,s}g, \phi\rangle\right],\end{aligned}$$

where B_j denotes the ball of $\mathbb{R}^n$ such that $\operatorname{supp} a_j \subset B_j$ for any $j \in \mathbb{N}$.

Now let $S_f(g) := \sum_{j\in\mathbb{N}} \lambda_j a_j(g - P_{B_j,s}g)$ and $T_f(g) := \sum_{j\in\mathbb{N}} \lambda_j a_j P_{B_j,s}g$. By Definition 2.2, we have

$$\begin{aligned}
\|S_f(g)\|_{L^1(\mathbb{R}^n)} &\lesssim \sum_{j\in\mathbb{N}} |\lambda_j| \left\|a_j(g - P_{B_j,s}g)\right\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \sum_{j\in\mathbb{N}} |\lambda_j| \|a_j\|_{L^2(\mathbb{R}^n)} \left\|g - P_{B_j,s}g\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{j\in\mathbb{N}} |\lambda_j| \|g\|_{\mathfrak{C}_{\alpha,2,s}(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)} \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)},
\end{aligned}$$

which further implies that $S_f(g)$ converges in $L^1(\mathbb{R}^n)$ and hence in $\mathcal{S}'(\mathbb{R}^n)$. By (2.18) in the proof of Proposition 2.24, we know that

$$\begin{aligned}
\|T_f(g)\|^p_{H^p_{w_p}(\mathbb{R}^n)} &\le \sum_{j\in\mathbb{N}} |\lambda_j|^p \left\|a_j P_{B_j,s}g\right\|^p_{H^p_{w_p}(\mathbb{R}^n)} \lesssim \sum_{j\in\mathbb{N}} |\lambda_j|^p \|g\|^p_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)} \\
&\lesssim \|f\|^p_{H^p(\mathbb{R}^n)} \|g\|^p_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)},
\end{aligned}$$

which implies that $T_f(g)$ converges in $H^p_{w_p}(\mathbb{R}^n)$ and hence in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, we know that S_f and T_f are well defined linear operators on the space $\mathfrak{C}_\alpha(\mathbb{R}^n)$. In particular, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, both $\sum_{j\in\mathbb{N}} \lambda_j \langle a_j(g - P_{B_j,s}g), \phi\rangle$ and $\sum_{j\in\mathbb{N}} \lambda_j \langle a_j P_{B_j,s}g, \phi\rangle$ converge. Altogether, we conclude the desired linear decomposition

$$\langle f \times g, \phi\rangle = \langle S_f(g) + T_f(g), \phi\rangle, \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^n).$$

This finishes the proof of Theorem 1.1. $\square$

4. Bilinear decomposition for $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$

In this section, we show Theorem 1.2 by using the renormalization technique based on wavelet multiresolution analysis (for short, MRA).

4.1. A few prerequisites on the MRA

Let us begin with the following definition of multiresolution analysis (for short, MRA) of $L^2(\mathbb{R}^n)$ (see, for instance, [38, p. 21]).

Definition 4.1. Let $\{V_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of closed linear subspaces in $L^2(\mathbb{R}^n)$. Then $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* (for short, MRA) *of* $L^2(\mathbb{R}^n)$ if it has the following properties:

(a) $\bigcap_{j\in\mathbb{Z}} V_j = \{\mathbf{0}\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$, where $\mathbf{0}$ denotes the zero element of $L^2(\mathbb{R}^n)$;

(b) for any $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^n)$, $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$;

(c) for any $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{Z}^n$, $f(\cdot) \in V_0$ if and only if $f(\cdot - k) \in V_0$;

(d) there exists a function $\phi \in L^2(\mathbb{R}^n)$ (called a *scaling function* or *father wavelet*) such that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^n}$ is a *Riesz basis* of V_0, that is, for every sequence $\{\alpha_k\}_{k\in\mathbb{Z}^n}$ of scalars,

$$\left\|\sum_{k\in\mathbb{Z}^n} \alpha_k \phi(\cdot - k)\right\|_{L^2(\mathbb{R}^n)} \sim \left(\sum_{k\in\mathbb{Z}^n} |\alpha_k|^2\right)^{1/2},$$

where the positive equivalence constants are independent of $\{\alpha_k\}_{k\in\mathbb{Z}^n}$.

In the literature, the definition of MRA is usually restricted to the one-dimensional case. However, the extension from one dimension to higher dimension is classical via the tensor product method (see [17, p. 921] or [38, Section 3.9]). As was pointed out in [38, Section 2.3], we can construct an orthonormal basis of V_0 based on the Riesz basis in Definition 4.1(d).

For any $j \in \mathbb{Z}$, let $\{V_j\}_{j\in\mathbb{Z}}$ be as in Definition 4.1 and W_j the *orthogonal complement* of V_j in V_{j+1}. It is easy to see that

$$V_{j+1} = \bigoplus_{i=-\infty}^{j} W_i \quad \text{and} \quad L^2(\mathbb{R}^n) = \bigoplus_{i=-\infty}^{\infty} W_i, \tag{4.1}$$

where $\bigoplus$ denotes the *orthogonal direct sum* in $L^2(\mathbb{R}^n)$. Let $\mathcal{D}$ be the class of all dyadic cubes $I := \{x \in \mathbb{R}^n : 2^j x - k \in [0,1)^n\}$ with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ in $\mathbb{R}^n$, and

$$E := \{0, 1\}^n \setminus \{(\overbrace{0, \ldots, 0}^{n \text{ times}})\}.$$

Fix $r \in \mathbb{N}$. According to [38, Sections 3.8 and 3.9], there exist families of *father wavelets* $\{\phi_I\}_{I\in\mathcal{D}}$ and *mother wavelets* $\{\psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E}$ having the following properties:

(P1) for any $j \in \mathbb{Z}$, the family $\{\phi_I\}_{|I|=2^{-jn}}$ forms an orthonormal basis of V_j and the family $\{\psi_I^\lambda\}_{|I|=2^{-jn},\lambda\in E}$ an orthonormal basis of W_j. In particular, the family $\{\psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$;

(P2) there exists a large positive constant m, independent of the main parameters included in the whole article, such that, for any $I \in \mathcal{D}$ and $\lambda \in E$,

$$\operatorname{supp} \phi_I \subset mI \quad \text{and} \quad \operatorname{supp} \psi_I^\lambda \subset mI,$$

where mI denotes the m dilation of I with the same center as I;

(P3) for any multi-index α of order $|\alpha| \le r$, there exists a positive constant C such that, for any $I \in \mathcal{D}$, $\lambda \in E$ and $x \in \mathbb{R}^n$, it holds true that

$$|D^\alpha \phi_I(x)| + \left|D^\alpha \psi_I^\lambda(x)\right| \le C\ell_I^{-n/2-|\alpha|},$$

where ℓ_I denotes the side length of I;

(P4) for any $I \in \mathcal{D}$, $\lambda \in E$ and any multi-index ν of order $|\nu| \le r$, it holds true that

$$\int_{\mathbb{R}^n} x^\nu \psi_I^\lambda(x)\, dx = 0$$

and, for any $I \in \mathcal{D}$,

$$\int_{\mathbb{R}^n} \phi_I(x)\, dx \ne 0;$$

(P5) For any I, $I' \in \mathcal{D}$ satisfying $|I| \le |I'|$ and $\lambda \in E$,

$$\int_{\mathbb{R}^n} \psi_I^\lambda(x)\phi_{I'}(x)\, dx = 0. \tag{4.2}$$

Indeed, let W_j and $V_{j'}$ be the linear subspaces of $L^2(\mathbb{R}^n)$ defined as in (4.1) with $|I| = 2^{-jn}$ and $|I'| = 2^{-j'n}$. Since $|I| \le |I'|$, we deduce $j' \le j$, which, combined with (4.1), shows that $W_j \perp V_{j'}$. By this and the above property (P1), we conclude the validity of (4.2).

Let us point out that the constants m and C in the above properties (P2) and (P3) depend on the *regularity constant* r (see [17] or [38, p.96]). Note that, even in the one-dimensional case, there does not exist a wavelet basis in $L^2(\mathbb{R})$ whose elements are both infinitely differentiable and have compact supports (see, for instance, [29, Theorem 3.8]).

As the family $\{\psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$, we know that any function $f \in L^2(\mathbb{R}^n)$ has the following *wavelet expansion*

$$f = \sum_{I\in\mathcal{D}} \sum_{\lambda\in E} \langle f, \psi_I^\lambda\rangle \psi_I^\lambda, \tag{4.3}$$

where the equality holds true in $L^2(\mathbb{R}^n)$ and $\langle\cdot,\,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^n)$. A function f in $L^2(\mathbb{R}^n)$ is said to have a *finite wavelet expansion* if the coefficients $\{\langle f,\,\psi_I^\lambda\rangle\}_{I\in\mathcal{D},\lambda\in E}$ in (4.3) have only finite non-zero terms.

4.2. Renormalization of functions in the product $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$

Applying the wavelet theory, Coifman et al. [14] and Dobyinsky [18] studied the renormalization of functions in the product $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Below we briefly recall the main results of Dobyinsky [18], which was also nicely summarized in [6, Section 4].

For any $j\in\mathbb{Z}$, let P_j and Q_j be the orthogonal projectors of $L^2(\mathbb{R}^n)$ onto V_j and W_j, respectively. In other words, for any $j\in\mathbb{Z}$ and $f\in L^2(\mathbb{R}^n)$, we have

$$P_jf=\sum_{\substack{I\in\mathcal{D}\\|I|=2^{-jn}}}\langle f,\phi_I\rangle\phi_I \tag{4.4}$$

and

$$Q_jf=\sum_{\substack{I\in\mathcal{D}\\|I|=2^{-jn}}}\sum_{\lambda\in E}\langle f,\psi_I^\lambda\rangle\psi_I^\lambda. \tag{4.5}$$

Assume that $f,\,g\in L^2(\mathbb{R}^n)$ have finite wavelet expansions. Then Dobyinsky [18] proved that

$$fg=\sum_{j\in\mathbb{Z}}(P_jf)(Q_jg)+\sum_{j\in\mathbb{Z}}(Q_jf)(P_jg)+\sum_{j\in\mathbb{Z}}(Q_jf)(Q_jg)\qquad\text{in } L^1(\mathbb{R}^n).$$

Further, using the properties (P1) through (P5) of $\{\phi_I\}_{I\in\mathcal{D}}$ and $\{\psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E}$ stated in the previous section, as well as (4.4) and (4.5), we write

$$fg=\sum_{i=1}^4\Pi_i(f,\,g)\qquad\text{in } L^1(\mathbb{R}^n), \tag{4.6}$$

where

$$\Pi_1(f,\,g):=\sum_{\substack{I,\,I'\in\mathcal{D}\\|I|=|I'|}}\sum_{\lambda\in E}\langle f,\,\phi_I\rangle\langle g,\,\psi_{I'}^\lambda\rangle\phi_I\psi_{I'}^\lambda, \tag{4.7}$$

$$\Pi_2(f,\, g) := \sum_{\substack{I,\, I'\in\mathcal{D}\\ |I|=|I'|}} \sum_{\lambda\in E} \langle f,\, \psi_I^\lambda\rangle \langle g,\, \phi_{I'}\rangle \psi_I^\lambda \phi_{I'}, \tag{4.8}$$

$$\Pi_3(f,\, g) := \sum_{\substack{I,\, I'\in\mathcal{D}\\ |I|=|I'|}} \sum_{\substack{\lambda,\,\lambda'\in E\\ (I,\,\lambda)\neq(I',\,\lambda')}} \langle f,\, \psi_I^\lambda\rangle \langle g,\, \psi_{I'}^{\lambda'}\rangle \psi_I^\lambda \psi_{I'}^{\lambda'} \tag{4.9}$$

and

$$\Pi_4(f,\, g) := \sum_{I\in\mathcal{D}} \sum_{\lambda\in E} \langle f,\, \psi_I^\lambda\rangle \langle g,\, \psi_I^\lambda\rangle \left(\psi_I^\lambda\right)^2. \tag{4.10}$$

From their definitions in (4.7) through (4.10), it follows easily that the four operators $\{\Pi_i\}_{i=1}^4$ are bilinear operators for any $f,\, g \in L^2(\mathbb{R}^n)$ having finite wavelet expansions. Moreover, by [6, Lemmas 4.1 and 4.2] (see also [18, Proposition 1.1]), we have the following lemma.

Lemma 4.2. *Let* $\{\Pi_i\}_{i=1}^4$ *be as in* (4.7) *through* (4.10), *which are well defined whenever* f *and* g *have finite wavelet expansions. Then* $\{\Pi_i\}_{i=1}^3$ *can be extended to bounded bilinear operators from* $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *to* $H^1(\mathbb{R}^n)$ *and* Π_4 *to a bounded bilinear operator from* $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$.

Proof. It was proved in [6, Lemma 4.2] that Π_1 and Π_2 can both be extended to bounded bilinear operators from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. Moreover, using [18, Proposition 1.1] and (4.6) through (4.10), we conclude that $\sum_{i=1}^3 \Pi_i$ can be extended to a bounded bilinear operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$, which further implies that Π_3 can also be extended to a bounded bilinear operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$.

Similarly to the proof of [6, Lemma 4.1], by the Hölder inequality and $\|(\psi_I^\lambda)^2\|_{L^1(\mathbb{R}^n)} = 1$, we know that

$$\begin{aligned}\|\Pi_4(f,g)\|_{L^1(\mathbb{R}^n)} &\le \sum_{I\in\mathcal{D}} \sum_{\lambda\in E} |\langle f,\, \psi_I^\lambda\rangle|\, |\langle g,\, \psi_I^\lambda\rangle| \|(\psi_I^\lambda)^2\|_{L^1(\mathbb{R}^n)}\\ &\le \left(\sum_{I\in\mathcal{D}} \sum_{\lambda\in E} |\langle f,\, \psi_I^\lambda\rangle|^2\right)^{1/2} \left(\sum_{I\in\mathcal{D}} \sum_{\lambda\in E} |\langle g,\, \psi_I^\lambda\rangle|^2\right)^{1/2}\\ &\lesssim \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

This implies that Π_4 can be extended to a bounded bilinear operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, which completes the proof of Lemma 4.2. $\square$

Remark 4.3. If we assume that only f has a finite wavelet expansion and g is a general $L^2(\mathbb{R}^n)$ function, then (4.6) remains true by passing to the limit in its both sides. Consequently, the equations $\{\Pi_i\}_{i=1}^4$ in (4.7) through (4.10) are well defined whenever $g \in L^2(\mathbb{R}^n)$ and f has a finite wavelet expansion.

In what follows, we use the *symbol* $L^2_{\rm loc}(\mathbb{R}^n)$ to denote the collection of all measurable functions which are locally in $L^2(\mathbb{R}^n)$.

Remark 4.4. Assume that f has a finite wavelet expansion as in (4.3) and $g \in L^2_{\rm loc}(\mathbb{R}^n)$. Then we may as well assume that f is supported on a cube R large enough such that, for any I as in (4.3), $I \subset R$. Take η to be a smooth cut-off function such that $\operatorname{supp}\eta \subset 9mR$ and $\eta \equiv 1$ on $5mR$, where m is as in property (P2) in Section 4.1. Though g may not belong to $L^2(\mathbb{R}^n)$, it makes sense to understand the formal expression of each $\Pi_i(f, g)$ as

$$\Pi_i(f, g) = \Pi_i(f, \eta g), \qquad i \in \{1, 2, 3, 4\}. \tag{4.11}$$

Let us take $i = 1$ for example to illustrate (4.11). Since f has a finite wavelet expansion and $\eta g \in L^2(\mathbb{R}^n)$, it follows from Remark 4.3 that

$$\Pi_1(f,\, \eta g) = \sum_{\substack{I,\, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f,\, \phi_I\rangle \langle \eta g,\, \psi^\lambda_{I'}\rangle \phi_I \psi^\lambda_{I'} \qquad \text{in } L^1(\mathbb{R}^n).$$

Based on the above properties (P2) and (P5) in Section 4.1, the factor $\langle f,\, \phi_I\rangle \phi_I \psi^\lambda_{I'}$ in the above summation is non-zero only when $(mI) \cap (mI') \neq \emptyset$, $(mI) \cap R \neq \emptyset$ and $|I| \le |R|$, which automatically gives that $mI' \subset 5mR$ so that $\eta(x) \equiv 1$ on $\operatorname{supp}\psi^\lambda_{I'}$. Consequently, we can remove the function η in the pairing $\langle \eta g,\, \psi^\lambda_{I'}\rangle$ and hence obtain

$$\Pi_1(f,\, \eta g) = \sum_{\substack{I,\, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f,\, \phi_I\rangle \langle g,\, \psi^\lambda_{I'}\rangle \phi_I \psi^\lambda_{I'} \qquad \text{in } L^1(\mathbb{R}^n),$$

while the right hand side of the above equality is just the formal expression of $\Pi_1(f, g)$. Thus, (4.11) makes sense when $i = 1$.

Since $f \in L^2(\mathbb{R}^n)$ has a finite wavelet expansion and $\eta g \in L^2(\mathbb{R}^n)$, it follows from Lemma 4.2 and Remark 4.3 that every $\Pi_i(f, g)$ is well defined in $L^1(\mathbb{R}^n)$. In particular, (4.11) holds true in $L^1(\mathbb{R}^n)$ and also in $\mathcal{S}'(\mathbb{R}^n)$.

4.3. Three auxiliary lemmas

From now on, we assume that the regularity parameter r appearing in (P3) and (P4) of Section 4.1 satisfies that

$$r > \lfloor n\alpha \rfloor = \lfloor n(1/p - 1) \rfloor, \tag{4.12}$$

whenever the Hardy space $H^p(\mathbb{R}^n)$ or the Campanato space $\mathfrak{C}_\alpha(\mathbb{R}^n)$ with $\alpha = 1/p - 1$ is involved.

The next first lemma gives a finite atomic decomposition of elements in Hardy spaces that have finite wavelet expansions. Indeed, Lemma 4.5 below for the case $p = 1 = n$ was essentially proved in [29, Theorem 5.12 of Section 6.5], while the case $0 < p < 1 = n$ was discussed in item 7 of [29, Section 6.8]. For any $p \in (0, 1]$ and general dimension $n \in \mathbb{N}$, we easily derive Lemma 4.5 by following the proof of Theorem 5.12 in [29, Section 6.5], with the details being omitted here.

Lemma 4.5. *Let $p \in (0,\, 1]$ and $s \in \mathbb{Z}_+$ with $s \geq \lfloor n(1/p-1) \rfloor$. Assume that $f \in H^p(\mathbb{R}^n)$ has a finite wavelet expansion, namely,*

$$f = \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \langle f,\, \psi_I^\lambda \rangle \psi_I^\lambda, \tag{4.13}$$

where the coefficient $\langle f,\, \psi_I^\lambda \rangle \neq 0$ only for a finite number of $(I,\, \lambda) \in \mathcal{D} \times E$. Then f has a finite atomic decomposition satisfying $f = \sum_{l=1}^L \mu_l a_l$, where $L \in \mathbb{N}$ and the following properties hold true:

(i) *there exists a positive constant C, independent of $\{\mu_l\}_{l=1}^L$, $\{a_l\}_{l=1}^L$ and f, such that*

$$\left\{ \sum_{l=1}^L |\mu_l|^p \right\}^{\frac{1}{p}} \leq C \|f\|_{H^p(\mathbb{R}^n)};$$

(ii) *for any $l \in \{1,\, \ldots,\, L\}$, a_l is a (p, s)-atom supported on some dyadic cube R_l, which can be written into the following form:*

$$a_l = \sum_{I \in \mathcal{D},\, I \subset R_l} \sum_{\lambda \in E} c_{(I,\, \lambda,\, l)} \psi_I^\lambda \tag{4.14}$$

with $\{c_{(I,\, \lambda,\, l)}\}_{I \subset R_l,\, \lambda \in E,\, l \in \{1,\, \ldots,\, L\}}$ being positive constants independent of $\{a_l\}_{l=1}^L$;

(iii) *for each* $l \in \{1, \ldots, L\}$, a_l *in* (4.14) *has a finite wavelet expansion, whose non-zero terms are extracted from the finite wavelet expansion of* f *in* (4.13).

The next lemma concerns the wavelet characterizations of Hardy spaces on $\mathbb{R}^n$. Its proof when $n = 1$ was given in [24, Theorem 4.2] and a similar discussion also works for any $n \in \mathbb{N}$. We omit the details here.

Lemma 4.6. *Let* $p \in (0, 1]$. *Then* $f \in H^p(\mathbb{R}^n)$ *if and only if*

$$\|\mathcal{W}_\psi f\|_{L^p(\mathbb{R}^n)} := \left\| \left\{ \sum_{I\in\mathcal{D}} \sum_{\lambda\in E} \left|\langle f,\, \psi_I^\lambda\rangle\right|^2 |I|^{-1} \mathbf{1}_I \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Moreover, it holds true that $\|f\|_{H^p(\mathbb{R}^n)} \sim \|\mathcal{W}_\psi f\|_{L^p(\mathbb{R}^n)}$ *with positive equivalence constants independent of* f.

The following lemma is on the wavelet characterization of Campanato spaces on $\mathbb{R}^n$. We refer the reader to [33, Corollary 2] for the case $n = 1$, while the proof for any $n \in \mathbb{N}$ is similar and the details are omitted.

Lemma 4.7. *Let* $\alpha \in [0, \infty)$. *Then* $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ *if and only if its wavelet coefficients* $\{s_{I,\lambda}\}_{I\in\mathcal{D},\,\lambda\in\mathrm{E}} := \{\langle g,\, \psi_I^\lambda\rangle\}_{I\in\mathcal{D},\,\lambda\in\mathrm{E}}$ *satisfy that*

$$\|\{s_{I,\lambda}\}_{I\in\mathcal{D},\,\lambda\in E}\|_{\mathcal{C}_\alpha(\mathbb{R}^n)} := \sup_{I\in\mathcal{D}} \left\{ \frac{1}{|I|^{2\alpha+1}} \sum_{\substack{J\in\mathcal{D}\\ J\subset I}} \sum_{\lambda\in E} |s_{J,\lambda}|^2 \right\}^{\frac{1}{2}} < \infty.$$

Moreover, $\|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \sim \|\{s_{I,\lambda}\}_{I\in\mathcal{D},\,\lambda\in E}\|_{\mathcal{C}_\alpha(\mathbb{R}^n)}$ *with positive equivalence constants independent of* g.

4.4. Proof of Theorem 1.2

In this section, we still assume (4.12). Applying Lemmas 4.5, 4.6 and 4.7, we now prove the following four propositions.

Proposition 4.8. *Let* $p \in (0,1)$ *and* $\alpha = 1/p-1$. *Then the bilinear operator* Π_1, *defined as in* (4.7), *can be extended to a bilinear operator bounded from* $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ *to* $H^1(\mathbb{R}^n)$.

Proof. Assume that $f \in H^p(\mathbb{R}^n)$ has a finite wavelet expansion and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$. Note that $g \in L^2_{\rm loc}(\mathbb{R}^n)$. Let $s \in \mathbb{Z}_+$ be such that $s \geq \lfloor n(1/p-1) \rfloor$. In this case, by Lemma 4.5, we know that $f = \sum_{l=1}^{L} \mu_l a_l$ has a finite atomic decomposition with the same notation as therein. Assume that every (p, s)-atom a_l is supported on a dyadic cube R_l. For each $l \in \{1, \ldots, L\}$, define

$$b_l := \sum_{\substack{I\in\mathcal{D}\\ I\subset 5mR_l}} \sum_{\lambda\in E} \langle g,\, \psi_I^\lambda\rangle \psi_I^\lambda.$$

Applying property (P1) in Section 4.1 and Lemma 4.7, one easily has

$$\|b_l\|_{L^2(\mathbb{R}^n)} \lesssim \left(\sum_{\substack{I\in\mathcal{D}\\ I\subset 5mR_l}} \sum_{\lambda\in E} \left|\langle g,\, \psi_I^\lambda\rangle\right|^2 \right)^{\frac{1}{2}} \lesssim |R_l|^{\alpha+1/2} \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}.$$

Moreover, according to Lemma 4.5(iii), the wavelet expansion of a_l has only finite terms. By this, $b_l \in L^2(\mathbb{R}^n)$, Lemma 4.2 and Remark 4.3, we know that $\Pi_1(a_l, b_l)$ is well defined and

$$\|\Pi_1(a_l,\, b_l)\|_{H^1(\mathbb{R}^n)} \lesssim \|a_l\|_{L^2(\mathbb{R}^n)} \|b_l\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}. \tag{4.15}$$

Observe that

$$\begin{aligned} \Pi_1(a_l,\, b_l) &= \sum_{\substack{I,\, I'\in\mathcal{D}\\ |I|=|I'|}} \sum_{\lambda\in E} \langle a_l,\, \phi_I\rangle \langle b_l,\, \psi_{I'}^\lambda\rangle \phi_I \psi_{I'}^\lambda \\ &= \sum_{\substack{I,\, I'\in\mathcal{D}\\ |I|=|I'|,\, I'\subset 5mR_l}} \sum_{\lambda\in E} \langle a_l,\, \phi_I\rangle \langle g,\, \psi_{I'}^\lambda\rangle \phi_I \psi_{I'}^\lambda. \end{aligned} \tag{4.16}$$

By properties (P2) and (P5) in Section 4.1, together with the expression of a_l in (4.14), we know that $\langle a_l, \phi_I\rangle \neq 0$ only for these I satisfying $|I| \leq |R_l|$ and $R_l \cap (mI) \neq \emptyset$. Again, property (P2) in Section 4.1 implies that $\phi_I \psi_{I'}^\lambda$ is a non-zero function only if $(mI) \cap (mI') \neq \emptyset$. From this, one easily deduces that $I' \subset 5mR_l$. Therefore, the restriction in the last term of (4.16) can be removed and hence we then have

$$\Pi_1(a_l,\, b_l) = \sum_{\substack{I,\, I'\in\mathcal{D}\\ |I|=|I'|}} \sum_{\lambda\in E} \langle a_l,\, \phi_I\rangle \langle g,\, \psi_{I'}^\lambda\rangle \phi_I \psi_{I'}^\lambda = \Pi_1(a_l, g)$$

pointwisely. Consequently, by the fact that Π_1 is bilinear, we obtain

$$\Pi_1(f,g) = \sum_{l=1}^{L} \mu_l \, \Pi_1(a_l,\, g) = \sum_{l=1}^{L} \mu_l \, \Pi_1(a_l,\, b_l),$$

where all the equalities hold true pointwisely. Moreover, by (4.15), we have $\Pi_1(f,\, g) \in H^1(\mathbb{R}^n)$ and

$$\begin{aligned}
\|\Pi_1(f,\, g)\|_{H^1(\mathbb{R}^n)} &= \left\| \sum_{l=1}^{L} \mu_l \Pi_1(a_l,\, b_l) \right\|_{H^1(\mathbb{R}^n)} \\
&\le \sum_{l=1}^{L} |\mu_l| \|\Pi_1(a_l,\, b_l)\|_{H^1(\mathbb{R}^n)} \lesssim \left(\sum_{l=1}^{L} |\mu_l|^p \right)^{1/p} \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} \\
&\lesssim \|f\|_{H^p(\mathbb{R}^n)} \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}. \qquad (4.17)
\end{aligned}$$

For a general $f \in H^p(\mathbb{R}^n)$, under (4.12), since the family $\{\psi_I^\lambda\}_{I\in\mathcal{D},\,\lambda\in E}$ of wavelets is an unconditional basis of $H^p(\mathbb{R}^n)$ (see, for instance, [9, Theorem 5.8]), it follows that there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset H^p(\mathbb{R}^n)$ having finite wavelet expansions such that $\lim_{k\to\infty} f_k = f$ in $H^p(\mathbb{R}^n)$. Thus, we extend the definition of Π_1 by setting, for any $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$,

$$\Pi_1(f,\, g) := \lim_{k\to\infty} \Pi_1(f_k,\, g) \qquad \text{in } H^1(\mathbb{R}^n).$$

Estimate (4.17) ensures that the above definition is independent of the choice of the sequence $\{f_k\}_{k\in\mathbb{N}}$ and hence is well defined. Based on this extension, we derive from (4.17) that

$$\|\Pi_1(f,\, g)\|_{H^1(\mathbb{R}^n)} = \lim_{k\to\infty} \|\Pi_1(f_k,\, g)\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)} \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}.$$

This implies that Π_1 can be extended to a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. This finishes the proof of Proposition 4.8. □

Proposition 4.9. *Let $p \in (0,1)$ and $\alpha = 1/p - 1$. Then the bilinear operator Π_2, defined as in (4.8), can be extended to a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $H^{\Phi_p}(\mathbb{R}^n)$.*

Proof. Let $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ and $s = \lfloor n\alpha \rfloor$. Assume that a is a $(p, 2s)$-atom supported on a dyadic cube R and a has a finite wavelet expansion. Denote by

B the smallest ball in $\mathbb{R}^n$ containing R and $P_{B,s}g$ the minimizing polynomial of g on B with degree $\leq s$ as in (2.2).

Let η be a smooth cut-off function such that $\operatorname{supp}\eta \subset 9mR$ and $\eta \equiv 1$ on $5mR$, where m is as in property (P2) in Section 4.1. Applying Remark 4.4 and property (P4) in Section 4.1 with r therein satisfying (4.12), together with the expressions (4.7) through (4.10), we know that

$$aP_{B,s}g = a(\eta P_{B,s}g) = \sum_{i=1}^{4} \Pi_i(a,\, \eta P_{B,s}g) = \Pi_2(a,\, \eta P_{B,s}g) = \Pi_2(a,\, P_{B,s}g).$$

Again, by Remark 4.4, we write

$$\begin{aligned}\Pi_2(a,\, g) &= \Pi_2(a,\, \eta g) = \Pi_2(a,\, \eta[g - P_{B,s}g]) + \Pi_2(a, \eta P_{B,s}g)\\ &= \Pi_2(a,\, \eta[g - P_{B,s}g]) + aP_{B,s}g.\end{aligned}$$

Notice that the expression of the function Φ_p easily implies that $H^1(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n)$. Moreover, by the fact that Π_2 is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$ (see Lemma 4.2), a is a $(p, 2s)$-atom, $\operatorname{supp}\eta \subset 9mR$ and $\mathfrak{C}_\alpha(\mathbb{R}^n) = \mathfrak{C}_{\alpha,2,s}(\mathbb{R}^n)$, we conclude that

$$\begin{aligned}\|\Pi_2(a,\, \eta[g - P_{B,s}g])\|_{H^{\Phi_p}(\mathbb{R}^n)} &\lesssim \|\Pi_2(a,\, \eta[g - P_{B,s}g])\|_{H^1(\mathbb{R}^n)}\\ &\lesssim \|a\|_{L^2(\mathbb{R}^n)}\|\eta(g - P_{B,s}g)\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)}.\end{aligned}$$

From this and Proposition 2.24, we deduce that

$$\begin{aligned}\|\Pi_2(a,\, g)\|_{H^{\Phi_p}(\mathbb{R}^n)} &\lesssim \|\Pi_2(a,\, g - P_{B,s}g)\|_{H^{\Phi_p}(\mathbb{R}^n)} + \|aP_{B,s}g\|_{H^{\Phi_p}(\mathbb{R}^n)}\\ &\lesssim \|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}.\end{aligned} \tag{4.18}$$

We now extend the above boundedness from an atom a to a general $f \in H^p(\mathbb{R}^n)$ with a finite wavelet expansion. Such f has a finite atomic decomposition $f = \sum_{l=1}^{L} \mu_l a_l$, with the same notation as in Lemma 4.5. By the definition of $\|\cdot\|_{H^{\Phi_p}(\mathbb{R}^n)}$, it suffices to show that there exists a positive constant C such that

$$\int_{\mathbb{R}^n} \Phi_p\left(x,\, \frac{(\Pi_2(f, g))^*(x)}{C\|f\|_{H^p(\mathbb{R}^n)}\|g\|_{\mathfrak{C}^+_\alpha(\mathbb{R}^n)}}\right) dx \leq 1, \tag{4.19}$$

where $(\Pi_2(f, g))^*$ denotes the non-tangential maximal function of $\Pi_2(f, g)$ as in (2.1) with $m = \lfloor n(1/p - 1)\rfloor$. Without loss of generality, we may assume that $\|f\|_{H^p(\mathbb{R}^n)} = 1$ and $\|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} = 1$. Otherwise, we use $\widetilde{f} := f/\|f\|_{H^p(\mathbb{R}^n)}$ and $\widetilde{g} := g/\|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)}$ in the argument below.

Now, we prove (4.19). Lemma 4.5 implies that

$$\left(\sum_{l=1}^{L} |\mu_l|^p\right)^{1/p} \le \widetilde{C}\|f\|_{H^p(\mathbb{R}^n)} = \widetilde{C}$$

for some positive constant $\widetilde{C}$. Without loss of generality, we may as well assume that $\widetilde{C} \ge 1$. By the expression of Φ_p in (1.6), it is easy to see that $\Phi_p(\cdot,\, t)$ is strictly increasing in t. Observe that

$$(\Pi_2(f, g))^* \le \sum_{l=1}^{L} |\mu_l| \left(\Pi_2(a_l, g)\right)^* .$$

Notice that Lemma 4.5 implies that every a_l has a finite wavelet expansion, so that (4.18) holds true with the atom a therein replaced by a_l. By this, $\|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} = 1$ and the definition of $\|\cdot\|_{H^{\Phi_p}(\mathbb{R}^n)}$, we conclude that there exists a positive constant C_1, independent of a_l and g, such that

$$\int_{\mathbb{R}^n} \Phi_p\left(x,\, \frac{\left(\Pi_2(a_l,\, g)\right)^*(x)}{C_1}\right) dx \le 1, \tag{4.20}$$

where $(\Pi_2(a_l,\, g))^*$ denotes the non-tangential maximal function of $\Pi_2(a_l,\, g)$ as in (2.1) with $m = \lfloor n(1/p-1)\rfloor$. As was proved in Proposition 2.12(i) that Φ_p is of uniformly upper type 1, we conclude that, for any $x \in \mathbb{R}^n$ and any sequence $\{t_j\}_{j\in\mathbb{N}} \subset [0,\, \infty)$,

$$\Phi_p\left(x, \sum_{j\in\mathbb{N}} t_j\right) = \sum_{i\in\mathbb{N}} \left[\frac{t_i}{\sum_{j\in\mathbb{N}} t_j} \Phi_p\left(x, \sum_{j\in\mathbb{N}} t_j\right)\right] \le \sum_{i\in\mathbb{N}} \Phi_p(x, t_i).$$

Let $M := 2^{1/p}\widetilde{C}$. Then $M \ge 1$ and

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi_p\left(x,\, \frac{(\Pi_2(f, g))^*(x)}{MC_1}\right) dx &\le \int_{\mathbb{R}^n} \Phi_p\left(x,\, \frac{\sum_{l=1}^{L} |\mu_l| \left(\Pi_2(a_l, g)\right)^*(x)}{MC_1}\right) dx \\ &\le \sum_{l=1}^{L} \int_{\mathbb{R}^n} \Phi_p\left(x,\, \frac{|\mu_l| \left(\Pi_2(a_l, g)\right)^*(x)}{MC_1}\right) dx \\ &=: \sum_{l=1}^{L} \mathrm{D}_l.\end{aligned}$$

If $|\mu_l| \le M$, then (4.20) and the fact that Φ_p is of uniformly lower type p [see Proposition 2.12(i)] imply that

$$\mathrm{D}_l \le \frac{|\mu_l|^p}{M^p} \int_{\mathbb{R}^n} \Phi_p\left(x, \frac{(\Pi_2(a_l, g))^*(x)}{C_1}\right) dx \le \frac{|\mu_l|^p}{M^p}.$$

If $|\mu_l| > M$, then (4.20) and the fact that Φ_p is of uniformly upper type 1 [see Proposition 2.12(i) again] imply that

$$\mathrm{D}_l \le \frac{|\mu_l|}{M} \int_{\mathbb{R}^n} \Phi_p\left(x, \frac{(\Pi_2(a_l, g))^*(x)}{C_1}\right) dx \le \frac{|\mu_l|}{M}.$$

Combining the last three formulae, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi_p\left(x, \frac{(\Pi_2(f, g))^*(x)}{MC_1}\right) dx \\
&\quad \le \frac{1}{M^p} \sum_{\{1\le l\le L:\, |\mu_l|\le M\}} |\mu_l|^p + \frac{1}{M} \sum_{\{1\le l\le L:\, |\mu_l|>M\}} |\mu_l| \\
&\quad \le \frac{\widetilde{C}^p}{M^p} + \frac{\widetilde{C}}{M} < 1,
\end{aligned}$$

which implies that (4.19) holds true with the constant C therein taken as MC_1. Thus, we arrive at the conclusion that

$$\|\Pi_2(f, g)\|_{H^{\Phi_p}(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)} \|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)}$$

whenever $f \in H^p(\mathbb{R}^n)$ has a finite wavelet expansion.

As in Proposition 4.8, from the fact that $\{\psi_I^\lambda\}_{I\in\mathcal{D},\, \lambda\in E}$ is an unconditional basis of $H^p(\mathbb{R}^n)$ and a standard argument, we can deduce that the definition of $\Pi_2(f, g)$ can be extended to general $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ with the desired boundedness estimate, the details being omitted. This finishes the proof of Proposition 4.9. $\square$

Similarly to the proof of Proposition 4.8, we deduce the following results on the boundedness of the bilinear operators Π_3 and Π_4.

Proposition 4.10. *Let $p \in (0, 1)$ and $\alpha = 1/p-1$. Then the bilinear operator Π_3, defined as in (4.9), can be extended to a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$.*

Proof. According to Lemma 4.2 and Remark 4.3, the operator $\Pi_3(f, g)$ is well defined whenever $g \in L^2(\mathbb{R}^n)$ and f has a finite wavelet expansion, and can be extended to a bounded bilinear operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. With this, we follow the same lines as in the proof of Proposition 4.8 and can obtain the desired conclusion, the details being omitted. $\square$

Proposition 4.11. *Let $p \in (0, 1)$ and $\alpha = 1/p-1$. Then the bilinear operator Π_4, defined as in* (4.10)*, can be extended to a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Proof. The proof of this proposition is similar to that of Proposition 4.8, but now we use the boundedness of Π_4 from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ (see Lemma 4.2). The details are omitted. $\square$

Proof of Theorem 1.2. Let $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$. By (4.12), we know that the wavelet system $\{\psi_I^\lambda\}_{I\in\mathcal{D},\, \lambda\in E}$ is an unconditional basis of $H^p(\mathbb{R}^n)$ (see, for instance, [9, Theorem 5.8]), so there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset H^p(\mathbb{R}^n)$, with finite wavelet expansions, satisfying $\lim_{k\to\infty} f_k = f$ in $H^p(\mathbb{R}^n)$. By the definition of $f \times g$ in (1.2) and Corollary 3.2, we conclude that

$$f \times g = \lim_{k\to\infty} f_k\, g \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $f_k\, g$ denotes the usual pointwise product of f_k and g. Since f_k has a finite wavelet expansion, it follows that $f_k \in L^2(\mathbb{R}^n)$ and f_k is supported on a ball $B(\vec{0}_n, R_k)$ for some $R_k \in (0, \infty)$. Let η_k be a cut-off function satisfying $\operatorname{supp} \eta_k \subset B(\vec{0}_n, 9mR_k)$ and $\eta_k \equiv 1$ on $B(\vec{0}_n, 5mR_k)$, where m is as in property (P2) in Section 4.1. By Remark 4.4, we find that, for any $k \in \mathbb{N}$ and $i \in \{1,\, 2,\, 3,\, 4\}$,

$$\Pi_i(f_k,\, \eta_k g) = \Pi_i(f_k,\, g) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

so that

$$f_k g = f_k(\eta_k g) = \sum_{i=1}^{4} \Pi_i(f_k, \eta_k g) = \sum_{i=1}^{4} \Pi_i(f_k, g) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Notice that the convergence of a sequence in $H^p(\mathbb{R}^n)$ or $H^{\Phi_p}(\mathbb{R}^n)$ implies its convergence in $\mathcal{S}'(\mathbb{R}^n)$; see [26, Proposition 6.4.10] and [32, Proposition 5.1].

By this, $\lim_{k\to\infty} f_k = f$ in $H^p(\mathbb{R}^n)$ and Propositions 4.8 through 4.11, we know that, for any $i \in \{1, 2, 3, 4\}$,

$$\lim_{k\to\infty} \Pi_i(f_k,\, g) = \Pi_i(f,\, g) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Therefore, we have

$$f \times g = \lim_{k\to\infty} f_k\, g = \sum_{i=1}^{4} \lim_{k\to\infty} \Pi_i(f_k,\, g) = \sum_{i=1}^{4} \Pi_i(f,\, g) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Thus, if we define

$$S(f,\, g) := \Pi_4(f,\, g) \in L^1(\mathbb{R}^n) \tag{4.21}$$

and

$$T(f,\, g) := \sum_{i=1}^{3} \Pi_i(f,\, g) \in H^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n), \tag{4.22}$$

then, applying Propositions 4.8 through 4.11, we obtain the desired conclusion of Theorem 1.2. $\square$

Remark 4.12. Let $p \in (0, 1)$, $\alpha = 1/p - 1$ and $n\alpha \notin \mathbb{N}$. Assume that $(f, g) \in H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$. Since $L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ characterizes the class of pointwise multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$, it follows that the largest range of φ that makes

$$\langle f \times g,\, \varphi\rangle = \langle g\varphi, f\rangle$$

meaningful is $\varphi \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. For any $i \in \{1, 2, 3, 4\}$, by Propositions 4.8 through 4.11 and $(H^{\Phi_p}(\mathbb{R}^n))^* = \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$, we know that $\langle \Pi_i(f, g), \varphi\rangle$ makes sense whenever $\varphi \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. Then the proof of Theorem 1.2 implies that the bilinear decomposition holds true in the following sense:

$$\langle f \times g,\, \varphi\rangle = \langle S(f,\, g) + T(f,\, g),\, \varphi\rangle, \qquad \forall\, \varphi \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n).$$

Now we consider the sharpness of Theorem 1.2 when $n\alpha \notin \mathbb{N}$. Suppose that Theorem 1.2 holds true with $H^{\Phi_p}(\mathbb{R}^n)$ therein replaced by a smaller vector space $\mathcal{Y}$. Then, for any $\varphi \in (L^1(\mathbb{R}^n) + \mathcal{Y})^*$, the pairing $\langle f \times g,\, \varphi\rangle$ is meaningful whenever $(f, g) \in H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$, so that φ is a pointwise multiplier of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ and hence $\varphi \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. We therefore obtain $(L^1(\mathbb{R}^n) + \mathcal{Y})^* \subset L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$. From this and $\mathcal{Y} \subset H^{\Phi_p}(\mathbb{R}^n)$, we deduce that

$$(L^1(\mathbb{R}^n) + \mathcal{Y})^* = L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n) = (L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n))^*.$$

In this sense, Theorem 1.2 when $n\alpha \notin \mathbb{N}$ is sharp.

5. The div-curl estimate

In this section, we utilize Theorem 1.2 to prove Theorem 1.5. We first recall the following Helmholtz decomposition for vector fields in $L^2(\mathbb{R}^n;\mathbb{R}^n)$ (see, for instance, [8, p. 1421]).

Lemma 5.1. *Let* $\mathbf{F} := (F_1, \ldots, F_n) \in L^2(\mathbb{R}^n; \mathbb{R}^n)$. *Then there exists a unique decomposition*

$$\mathbf{F} = \mathbf{H} + \mathbf{K}$$

in $L^2(\mathbb{R}^n;\mathbb{R}^n)$ *with* $\mathbf{H} := (H_1, \ldots, H_n) \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $\mathbf{K} := (K_1, \ldots, K_n) \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, curl $\mathbf{H} \equiv 0$ *and* div $\mathbf{K} \equiv 0$. *Moreover, for any* $j \in \{1, \ldots, n\}$,

$$H_j = -\sum_{i=1}^n R_j R_i(F_i) \qquad \text{and} \qquad K_j = F_j + \sum_{i=1}^n R_j R_i(F_i),$$

where $R_j := \partial_{x_j}(-\Delta)^{-1/2}$, *with* $\Delta := -\sum_{j=1}^n \partial^2_{x_j}$, *denotes the* j*-th Riesz transform.*

Let $p \in (0,1]$ and $\alpha \in [0,\infty)$. For any sequence $\{a_I\}_{I\in\mathcal{D}}$ of complex numbers that is indexed by the set $\mathcal{D}$ of dyadic cubes, we define

$$\|\{a_I\}_{I\in\mathcal{D}}\|_{\mathcal{C}_\alpha(\mathbb{R}^n)} := \sup_{I\in\mathcal{D}} \left\{ \frac{1}{|I|^{2\alpha+1}} \sum_{J\in\mathcal{D},\, J\subset I} |a_J|^2 \right\}^{\frac{1}{2}}$$

and

$$\|\{a_I\}_{I\in\mathcal{D}}\|_{\dot{f}^0_{p,\,2}(\mathbb{R}^n)} := \left\| \left[\sum_{I\in\mathcal{D}} \left(|a_I|\, |I|^{-\frac{1}{2}} \mathbf{1}_I \right)^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)},$$

which induce suitable norms for the Carleson sequence space $\mathcal{C}_\alpha(\mathbb{R}^n)$ and the homogeneous Triebel-Lizorkin sequence space $\dot{f}^0_{p,\,2}(\mathbb{R}^n)$ (see [23]), respectively. We also need the following duality conclusion for sequence spaces, whose proof is similar to that of [12, (2.32)] and hence the details are omitted.

Lemma 5.2. *Let* $p \in (0,\, 1]$. *Then there exists a positive constant* C *such that, for any sequences of complex numbers* $\{a_I\}_{I\in\mathcal{D}}$ *and* $\{b_I\}_{I\in\mathcal{D}}$,

$$\left| \sum_{I\in\mathcal{D}} a_I b_I \right| \le C\, \|\{a_I\}_{I\in\mathcal{D}}\|_{\dot{f}^0_{p,\,2}(\mathbb{R}^n)}\, \|\{b_I\}_{I\in\mathcal{D}}\|_{\mathcal{C}_{1/p-1}(\mathbb{R}^n)}$$

whenever the right hand side of the above inequality is finite.

Using Theorem 1.2 and Lemmas 5.1 and 5.2, we now turn to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\mathbf{F} \in H^p(\mathbb{R}^n;\, \mathbb{R}^n) \cap L^2(\mathbb{R}^n;\, \mathbb{R}^n)$ with $\operatorname{curl} \mathbf{F} \equiv 0$ and $\mathbf{G} \in \mathfrak{C}_\alpha(\mathbb{R}^n;\, \mathbb{R}^n)$ with $\operatorname{div} \mathbf{G} \equiv 0$. With bilinear operators S and T defined as in (4.21) and (4.22), we write

$$\begin{aligned}\mathbf{F} \cdot \mathbf{G} &= \sum_{i=1}^n F_i \times G_i = \sum_{i=1}^n S(F_i,\, G_i) + \sum_{i=1}^n T(F_i,\, G_i)\\ &=: \mathrm{A}(\mathbf{F},\, \mathbf{G}) + \mathrm{B}(\mathbf{F},\, \mathbf{G}).\end{aligned}$$

By Theorem 1.2, we know that $\mathrm{B}(\mathbf{F},\, \mathbf{G}) \in H^{\Phi_p}(\mathbb{R}^n)$ and

$$\|\mathrm{B}(\mathbf{F},\, \mathbf{G})\|_{H^{\Phi_p}(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n;\, \mathbb{R}^n)} \|\mathbf{G}\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n;\, \mathbb{R}^n)}.$$

To estimate $\mathrm{A}(\mathbf{F},\, \mathbf{G})$, by the assumption $\mathbf{F} \in H^p(\mathbb{R}^n;\, \mathbb{R}^n) \cap L^2(\mathbb{R}^n;\, \mathbb{R}^n)$ and Lemma 5.1, we find that there exists

$$f := -\sum_{i=1}^n R_i(F_i) \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

such that $\mathbf{F} = \nabla(-\Delta)^{-1/2} f$, where R_i, for any $i \in \{1,\, \ldots,\, n\}$, denotes the i-th Riesz transform. Since $\operatorname{div} \mathbf{G} \equiv 0$, it follows that $\sum_{i=1}^n R_i(G_i) \equiv 0$. Thus, we can write

$$\mathrm{A}(\mathbf{F},\, \mathbf{G}) = \sum_{i=1}^n S(F_i,\, G_i) = \sum_{i=1}^n \left[S(R_i(f),\, G_i) + S(f,\, R_i(G_i))\right].$$

Using (4.10) and the fact that R_i is a Calderón-Zygmund operator with odd kernel, we further find that, for each $i \in \{1,\, \ldots,\, n\}$,

$$\begin{aligned}&S(R_i(f),\, G_i) + S(f,\, R_i(G_i))\\ &\quad= \sum_{I,\, I' \in \mathcal{D}} \sum_{\lambda,\, \lambda' \in E} \langle f,\, \psi_I^\lambda\rangle \langle G_i,\, \psi_{I'}^{\lambda'}\rangle \langle R_i \psi_I^\lambda,\, \psi_{I'}^{\lambda'}\rangle \left[\left(\psi_{I'}^{\lambda'}\right)^2 - \left(\psi_I^\lambda\right)^2\right].\end{aligned}$$

By a similar calculation to that used in the proof of [6, Lemma 6.1], we obtain

$$\begin{aligned}&\|S(R_i(f),\, G_i) + S(f,\, R_i(G_i))\|_{H^1(\mathbb{R}^n)}\\ &\quad\lesssim \sum_{I,\, I' \in \mathcal{D}} \sum_{\lambda,\, \lambda' \in E} \left|\langle f,\, \psi_I^\lambda\rangle\right| \left|\langle G_i,\, \psi_{I'}^{\lambda'}\rangle\right| p_\delta(I,\, I'),\end{aligned}$$

where, for any $\delta \in (0, \frac{1}{2}]$, $|I| = 2^{-jn}$ with center at x_I, and $|I'| = 2^{-j'n}$ with center $x_{I'}$,

$$p_\delta(I, I') := 2^{-|j-j'|(\delta+n/2)} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |x_I - x_{I'}|} \right)^{n+\delta}.$$

This shows that the coefficient matrix $\{p_\delta(I, I')\}_{I,I' \in \mathcal{D}}$ is *almost diagonal* (see [23, p. 53] for the precise definition). Furthermore, from Lemma 5.2 and [23, Theorem 3.3], together with Lemmas 4.6 and 4.7, we deduce that

$$\begin{aligned}
&\|S(R_i(f), G_i) + S(f, R_i(G_i))\|_{H^1(\mathbb{R}^n)} \\
&\lesssim \sum_{I' \in \mathcal{D}} \sum_{\lambda' \in E} \left[\sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \left| \langle f, \psi_I^\lambda \rangle \right| p_\delta(I, I') \right] \left| \langle G_i, \psi_{I'}^{\lambda'} \rangle \right| \\
&\lesssim \left\| \left\{ \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \left| \langle f, \psi_I^\lambda \rangle \right| p_\delta(I, I') \right\}_{I' \in \mathcal{D}, \lambda' \in E} \right\|_{\dot{f}^0_{p,2}(\mathbb{R}^n)} \\
&\quad \times \left\| \left\{ \langle G_i, \psi_{I'}^{\lambda'} \rangle \right\}_{I' \in \mathcal{D}, \lambda' \in E} \right\|_{\mathcal{C}_{1/p-1}(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \langle f, \psi_I^\lambda \rangle \right\}_{I \in \mathcal{D}, \lambda \in E} \right\|_{\dot{f}^0_{p,2}(\mathbb{R}^n)} \left\| \left\{ \langle G_i, \psi_{I'}^{\lambda'} \rangle \right\}_{I' \in \mathcal{D}, \lambda' \in E} \right\|_{\mathcal{C}_{1/p-1}(\mathbb{R}^n)} \\
&\lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} \|\mathbf{G}\|_{\mathfrak{C}_{1/p-1}(\mathbb{R}^n; \mathbb{R}^n)},
\end{aligned}$$

which implies $\mathrm{A}(\mathbf{F}, \mathbf{G}) \in H^1(\mathbb{R}^n)$ and

$$\|\mathrm{A}(\mathbf{F}, \mathbf{G})\|_{H^1(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} \|\mathbf{G}\|_{\mathfrak{C}_{1/p-1}(\mathbb{R}^n; \mathbb{R}^n)}.$$

This, combined with the fact $H^1(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n)$, shows that $\mathrm{A}(\mathbf{F}, \mathbf{G}) \in H^{\Phi_p}(\mathbb{R}^n)$ and hence finishes the proof of Theorem 1.5. $\square$

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