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Real interpolation of weighted tent spaces

Jun Cao[a], Der-Chen Chang[b,c], Zunwei Fu[d] and Dachun Yang[e]

[a]Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, P.R. China; [b]Department of Mathematics & Statistics, Georgetown University, Washington, DC 20057, USA; [c]Department of Mathematics, Fu Jen Catholic University, Taipei 242, Taiwan, China; [d]Department of Mathematics, Linyi University, Linyi 276005, P.R. China; [e]School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China

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ABSTRACT

Let $p\in(0,\infty)$ and $w\in A_\infty(\mathbb{R}^n)$ be a Muckenhoupt weight. In this article, the authors study the real interpolation of the weighted tent space $T^p_w(\mathbb{R}^{n+1}_+)$. For all $w\in A_\infty(\mathbb{R}^n)$, $\theta\in(0,1)$, $0<p_0<p_1<\infty$ and $q\in(0,\infty]$, the authors show that $(T^{p_0}_w(\mathbb{R}^{n+1}_+), T^{p_1}_w(\mathbb{R}^{n+1}_+))_{\theta,q}=T^{p,q}_w(\mathbb{R}^{n+1}_+)$, where $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $T^{p,q}_w(\mathbb{R}^{n+1}_+)$ denotes the weighted Lorentz-tent space, which is introduced in this article. As an application, the authors prove a real interpolation result on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for all $p\in(0,1]$ and $w\in A_\infty(\mathbb{R}^n)$, which, when $w\equiv 1$, seals a gap existing in the original proof of a corresponding result of Fefferman et al. [Trans. Amer. Math. Soc. 1974;191:75–81].

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1. Introduction

The tent spaces $T^p(\mathbb{R}^{n+1}_+)$ on the upper half space $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,\infty)$, for all $p\in(0,\infty]$, were first introduced by Coifman et al. [1,2]. It is well known that these spaces lead to unifications and simplifications of many basic techniques in harmonic analysis. For example, they provide natural settings for the study of such things as maximal functions, square functions and Carleson measures. Also, as the retracts of many important function spaces, the method of tent spaces can be used as a simple and united way to deduce various useful properties of the associated function spaces; see, for example [2–4].

Among all the literatures on the tent spaces, the study of the real interpolations of tent spaces has always attracted a lot of attentions (see, e.g. [2,5–9]). In particular, in their seminal paper, Coifman et al. [2] first established the real interpolation of the tent spaces $T^p(\mathbb{R}^{n+1}_+)$, for all $p\in[1,\infty]$, by showing that, for all $\theta\in(0,1)$, $1\le p_0<p<p_1\le\infty$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$,

$$\left(T^{p_0}(\mathbb{R}^{n+1}_+),\,T^{p_1}(\mathbb{R}^{n+1}_+)\right)_{\theta,p}=T^p(\mathbb{R}^{n+1}_+) \tag{1.1}$$

(see Definitions 1.1 and 2.1 for the precise definitions of tent spaces $T^p(\mathbb{R}^{n+1}_+)$ and their real interpolations). Based on this result, Coifman et al. [2] provided a very simple proof of the real interpolations of Hardy spaces, $H^p(\mathbb{R}^n)=(H^{p_0}(\mathbb{R}^n),H^{p_1}(\mathbb{R}^n))_{\theta,p}$, where the exponents θ, p, p_0 and p_1 have the same ranges as those in (1.1).

CONTACT Dachun Yang dcyang@bnu.edu.cn
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Later, in the setting of product domains, by establishing the equivalent relations between the associated K-functionals, Long [5] improved the aforementioned result in [2] via extending the exponents to the full range. More precisely, in the case of Euclidean spaces, Long [5] proved that, for all $\theta \in (0, 1)$, $0 < p_0 < p < p_1 \le \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $q \in (0, \infty]$,

$$\left(T^{p_0}(\mathbb{R}^{n+1}_+),\, T^{p_1}(\mathbb{R}^{n+1}_+)\right)_{\theta,q} = T^{p,q}(\mathbb{R}^{n+1}_+), \tag{1.2}$$

where $T^{p,q}(\mathbb{R}^{n+1}_+)$ denotes the so-called Lorentz-tent space (see Definition 1.4). Recall that the Lorentz-tent space $T^{p,q}(\mathbb{R}^{n+1}_+)$ was also studied by Bonami et al. [6] with emphasis on two particular cases $T^{p,1}(\mathbb{R}^{n+1}_+)$ and $T^{p,\infty}(\mathbb{R}^{n+1}_+)$, due to the observation that the Lorentz space $L^{p,1}(\mathbb{R}^n)$ has a good analogue of the atomic decomposition as the Hardy space $H^p(\mathbb{R}^n)$ (see [10]). Also, in [8], Alvarez et al. considered the real interpolation of some extreme tent spaces and the associated dual spaces of Carleson measures. As was pointed out in [2], these extreme tent spaces play an important role in the study of maximal operators.

On the other hand, various versions of weighted tent spaces naturally arose due to further developments of the theory of tent spaces and their applications. For example, in [9], to study the real interpolation of tent spaces with a function parameter, Soria introduced one kind of weighted tent spaces with the weight function w being a non-negative function on $R^+ :=(0, \infty)$. Russ [11] studied the tent spaces on spaces of homogeneous type, which include a large number of weighted tent spaces as special cases; see also [12] for a more recent development of this kind of tent spaces (in particular, for the problem of the complex interpolation in this setting). We point out that the study of the weighted tent spaces also has motivations from the theory of the associated weighted function spaces, in view of the retract relations between the weighted tent spaces and those weighted function spaces. On many occasions, the associated weighted tent spaces may beyond the scope of tent spaces on spaces of homogeneous type studied in [11]. In these cases, many useful tools which work well in the settings of Euclidean spaces and spaces of homogeneous type may lose their powers (see the discussion after Theorem 1.6 for more details). Thus, to study this kind of weighted tent spaces, some new techniques and methods are needed.

In this article, motivated by the aforementioned results in [2,5–8,12], we study the real interpolation of some weighted tent spaces $T^p_w(\mathbb{R}^n)$ for all $p \in (0, \infty)$ with w belonging to $A_\infty(\mathbb{R}^n)$ (the class of Muckenhoupt weights) (see Theorem 1.6). This result extends the corresponding unweighted results (1.1) and (1.2) from [2,5]. As an application of Theorem 1.6, we prove a result on the real interpolation of weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ (see Theorem 1.7), which, even in the case $w \equiv 1$, seals the gap existing in the original proof of Fefferman et al. [13]. More precisely, recall that the key idea, used in the proof of the main result in [13], depends on a decomposition of Schwartz functions into “good” and “bad” parts and the density of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in the considered function spaces. However, as was pointed out, in [14], this density result may not hold true in the case of weak Hardy spaces (see the discussion after Theorem 1.7 for more details). We also point out that, in [15], Bui considered the real interpolation of the weighted Besov and Triebel–Lizorkin spaces by using a very similar idea via establishing some retract relationships between these spaces and the corresponding weighted vector-valued Hardy spaces. Unfortunately, the Lorentz-type spaces were not considered in [15].

To state our main result, we first introduce some notation. Recall that, for all $q \in [1,\infty)$, a non-negative locally integrable function w on $\mathbb{R}^n$ is said to belong to the *Muckenhoupt class* $A_q(\mathbb{R}^n)$, denoted by $w \in A_q(\mathbb{R}^n)$, if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$, when $q \in (1,\infty)$,

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$$\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\left\{\frac{1}{|B|}\int_B [w(x)]^{-1/(q-1)}\,\mathrm{d}x\right\}^{q-1}\le C$$

and, when $q=1$,

$$\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\le C\operatorname*{ess\,inf}_{x\in B} w(x).$$

Moreover, let $A_\infty(\mathbb{R}^n):=\cup_{q\in[1,\infty)}A_q(\mathbb{R}^n)$. For the notational convenience, in what follows, we let, for any measurable set $E\subset\mathbb{R}^n$,

$$w(E):=\int_E w(x)\,\mathrm{d}x \tag{1.3}$$

and, for any $p\in(0,\infty]$ and any measurable function f on $(\mathbb{R}^n,\ w(x)\,\mathrm{d}x)$ (as a space of homogeneous type),

$$\|f\|_{L^p_w(\mathbb{R}^n)}:=\left\{\int_{\mathbb{R}^n}|f(x)|^p\,w(x)\,\mathrm{d}x\right\}^{\frac{1}{p}} \tag{1.4}$$

with the usual modification made when $p=\infty$.

Let F be a measurable function on $\mathbb{R}^{n+1}_+$. For all $p\in(0,\infty)$, $\alpha\in(0,\infty)$ and $x\in\mathbb{R}^n$, the *$\mathcal{A}$-functional of aperture α* of F, $\mathcal{A}^{(\alpha)}(F)$, is defined by setting

$$\mathcal{A}^{(\alpha)}(F)(x):=\left\{\iint_{\Gamma_\alpha(x)}|F(y,t)|^2\,\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\right\}^{\frac{1}{2}}, \tag{1.5}$$

where

$$\Gamma_\alpha(x):=\left\{(y,t)\in\mathbb{R}^{n+1}_+:\ |y-x|<\alpha t\right\}. \tag{1.6}$$

In what follows, to simplify the notation, we always write $\mathcal{A}(F)$ instead of $\mathcal{A}^{(1)}(F)$ whenever $\alpha=1$.

Definition 1.1: Let $p\in(0,\infty)$, $w\in A_\infty(\mathbb{R}^n)$ and $\alpha\in(0,\infty)$. A function F on $\mathbb{R}^{n+1}_+$ is said to be in the *weighted tent space $T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)$ of aperture α* if $\mathcal{A}^{(\alpha)}(F)\in L^p_w(\mathbb{R}^n)$. For all $F\in T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)$, its quasi-norm is defined by setting

$$\|F\|_{T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)}:=\left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)}, \tag{1.7}$$

where $\|\cdot\|_{L^p_w(\mathbb{R}^n)}$ is as in (1.4).

For $p=\infty$, a function F on $\mathbb{R}^{n+1}_+$ is said to be in $T^\infty_{w,\alpha}(\mathbb{R}^{n+1}_+)$ if its *$\mathcal{C}_{w,\alpha}$-functional*

$$\mathcal{C}_{w,\alpha}(F)(x):=\sup_{B\ni x}\left\{\frac{1}{w(B)}\iint_{\widehat{B}_\alpha}|F(y,t)|^2\,\frac{t^n}{w(B(y,t))}\,\frac{\mathrm{d}y\,\mathrm{d}t}{t}\right\}^{\frac{1}{2}}\in L^\infty(\mathbb{R}^n),$$

where $x\in\mathbb{R}^n$, the supremum is taken over all balls B of $\mathbb{R}^n$ containing x and

$$\widehat{B}_\alpha:=\{(y,t)\in\mathbb{R}^{n+1}_+:\ B(y,\alpha t)\subset B\}.$$

For any $F\in T^\infty_{w,\alpha}(\mathbb{R}^{n+1}_+)$, its quasi-norm is defined by setting

$$\|F\|_{T^\infty_{w,\alpha}(\mathbb{R}^{n+1}_+)}:=\|\mathcal{C}_{w,\alpha}(F)\|_{L^\infty(\mathbb{R}^n)}.$$

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Here and hereafter, we do not indicate the parameter α whenever $\alpha = 1$.

Before moving on, we give some remarks on the definition of the weighted tent space $T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)$.

Remark 1.2:

(i) If $w \equiv 1$, the weighted tent space $T^p_{1,\alpha}(\mathbb{R}^{n+1}_+)$ in Definition 1.1 is just the tent space $T^p(\mathbb{R}^{n+1}_+)$ introduced in [2]. Recall that the space $T^p(\mathbb{R}^{n+1}_+)$ is invariant under the change of aperture α, which is one of the key facts that play important roles in the real interpolation of tent spaces. However, on the weighted tent space $T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)$, this invariance may depend on the range of the weight w (see Lemma 2.6).

(ii) The definition of the $\mathcal{C}_{w,\alpha}$*-functional* in Definition 1.1 is from [16], where Harboure et al. used this $\mathcal{C}_{w,\alpha}$-functional to give out a Carleson measure characterization of some weighted BMO spaces on $\mathbb{R}^n$.

We now recall the definition of weighted Lorentz spaces. To this end, we need the following notion of weighted decreasing rearrangements. Let $w \in A_\infty(\mathbb{R}^n)$ and f be measurable on $(\mathbb{R}^n, w(x)\, \mathrm{d}x)$. For all $\alpha \in [0, \infty)$, the *weighted distribution function* $\lambda_{w,f}(\alpha)$ of f is defined by setting

$$\lambda_{w,f}(\alpha) := w\left(\left\{x \in \mathbb{R}^n : |f(x)| > \alpha\right\}\right), \tag{1.8}$$

where, for any measurable set E, $w(E)$ is defined as in (1.3). For all $t \in (0, \infty)$, the *weighted decreasing rearrangement* f^*_w of f is defined by setting

$$f^*_w(t) := \inf\left\{\alpha \in [0, \infty) : \lambda_{w,f}(\alpha) \le t\right\}. \tag{1.9}$$

Definition 1.3: Let $w \in A_\infty(\mathbb{R}^n)$, $p \in (0, \infty]$ and $q \in (0, \infty]$. For any measurable function f on $(\mathbb{R}^n, w(x)\, \mathrm{d}x)$, define

$$\|f\|_{L^{p,q}_w(\mathbb{R}^n)} := \begin{cases} \left\{\displaystyle\int_0^\infty \left[t^{\frac{1}{p}} f^*_w(t)\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}, & q \in (0, \infty), \\ \displaystyle\sup_{t \in (0,\infty)} \left\{t^{\frac{1}{p}} f^*_w(t)\right\}, & q = \infty, \end{cases}$$

where, we let $\frac{1}{\infty} := 0$. Then the *weighted Lorentz space* $L^{p,q}_w(\mathbb{R}^n)$ is defined to be the set

$$L^{p,q}_w(\mathbb{R}^n) := \left\{f \text{ is measurable on } (\mathbb{R}^n, w(x)\, \mathrm{d}x) : \|f\|_{L^{p,q}_w(\mathbb{R}^n)} < \infty\right\}.$$

Combining the definitions of weighted tent spaces and weighted Lorentz spaces, we introduce the following notion of weighted Lorentz-tent spaces. For more properties of these spaces with no weight, we refer to [6,7,17].

Definition 1.4: Let $w \in A_\infty(\mathbb{R}^n)$. A measurable function F on $\mathbb{R}^{n+1}_+$ is said to be in the *weighted Lorentz-tent space* $T^{p,q}_w(\mathbb{R}^{n+1}_+)$ for all $p \in (0, \infty)$ and $q \in (0, \infty]$ if $\mathcal{A}(F) \in L^{p,q}_w(\mathbb{R}^n)$, where $\mathcal{A}(F)$ denotes the $\mathcal{A}$-functional of F defined as in (1.5) with $\alpha = 1$. For $p = \infty$ and $q \in (0, \infty]$, the weighted Lorentz-tent space $T^{\infty,q}_w(\mathbb{R}^{n+1}_+)$ is defined in the same way via replacing $\mathcal{A}(F)$ by the $\mathcal{C}_{w,\alpha}$-functional $\mathcal{C}_{w,\alpha}(F)$ in Definition 1.1 with $\alpha = 1$.

Remark 1.5: We point out that, if $w \equiv 1$, $p \in (0, 1]$ and $q = \infty$, then the weighted Lorentz-tent space $T^{p,\infty}(\mathbb{R}^{n+1}_+) := T^{p,\infty}_1(\mathbb{R}^{n+1}_+)$ is also called the *weak tent space*, which has been studied in [18]. Moreover, for all $p \in (0, 1]$, $w \in A_\infty(\mathbb{R}^n)$ and $F \in T^{p,\infty}_w(\mathbb{R}^{n+1}_+)$, F has a weak atomic decomposition, which plays an important role in the proofs of our results (see Lemma 2.4 for more details). Recall

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also that, in [6], Bonami et al. established another kind of atomic decompositions for the unweighted Lorentz-tent space, where an atom supports in an open set.

Now, we state the main result of the present article.

Theorem 1.6: Let $w \in A_\infty(\mathbb{R}_+^{n+1})$, $\theta \in (0,1)$, $q \in (0,\infty]$, $0 < p_0 < p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then

$$\left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q} = T_w^{p,q}(\mathbb{R}_+^{n+1}),$$

where $T_w^{p_0}(\mathbb{R}_+^{n+1})$ and $T_w^{p_1}(\mathbb{R}_+^{n+1})$ are the weighted tent spaces defined as in Definition 1.1 with $\alpha = 1$, and $T_w^{p,q}(\mathbb{R}_+^{n+1})$ is the weighted Lorentz-tent space defined as in Definition 1.4 (see also Definition 2.1 for the precise definition of the real interpolation).

Theorem 1.6 is proved in Section 3. As in the unweighted case, the main difficulty to prove Theorem 1.6 is to show the inclusion $T_w^{p,q}(\mathbb{R}_+^{n+1}) \subset (T_w^{p_0}(\mathbb{R}_+^{n+1}), T_w^{p_1}(\mathbb{R}_+^{n+1}))_{\theta,q}$. To this end, one usually needs to find a suitable decomposition for every $F \in T_w^{p,q}(\mathbb{R}_+^{n+1})$. In the present article, we use the decomposition $F = F_0 + F_1$ such that $F_0 \in T_{w,\frac{1}{3}}^{p_0}(\mathbb{R}_+^{n+1})$ and $F_1 \in T_{w,\frac{1}{3}}^{p_1}(\mathbb{R}_+^{n+1})$ (see Lemma 2.11). Thus, we need the fact that the weighted tent spaces are equivalent under the change of apertures (see Lemma 2.6). Unlike in the unweighted case, since our weighted tent spaces cannot be viewed as tent spaces on spaces of homogeneous type, this fact makes many useful tricks which are available in the unweighted case (see, e.g. the trick of the dual norm representation in the proof of [2, Proposition 4]) lose their effectiveness. To overcome this difficulty, we use the vector-valued approach of tent spaces from [19]. This approach enables us to reduce the above problem to the boundedness of the Hardy-Littlewood maximal function on some weighted vector-valued Lebesgue spaces. Thus, to obtain this boundedness, we need to restrict our weights to some $A_p(\mathbb{R}^n)$ class. This restriction is finally removed by using the reiteration theorem of the real interpolation (see Lemma 2.13).

As an application of Theorem 1.6, we consider the real interpolation of the weighted Hardy space $H_w^p(\mathbb{R}^n)$. To this end, we first recall the definition of $H_w^p(\mathbb{R}^n)$ (see [20–23] for more related properties of the weighted Hardy spaces).

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function in the Schwartz class having the following three properties:

(i) $\operatorname{supp}\phi \subset B(0,1)$, where $B(0,1)$ denotes the ball of $\mathbb{R}^n$ centered at 0 with radius 1;
(ii) for all multi-indices γ satisfying $|\gamma| \le N := \lfloor n(\frac{1}{p}-1)\rfloor$, $\int_{\mathbb{R}^n}\phi(x)x^\gamma\,dx = 0$, here and hereafter, $\lfloor s\rfloor$ for any $s \in \mathbb{R}$ denotes the largest integer smaller than or equal to s;
(iii) for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\int_0^\infty |\widehat{\phi}(t\xi)|^2\,\frac{dt}{t} = 1$, where $\widehat{\phi}$ denotes the *Fourier transform* of ϕ.

Let $w \in A_\infty(\mathbb{R}^n)$. For all $p \in (0,\infty)$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to be in the *weighted Hardy space* $H_w^p(\mathbb{R}^n)$ if

$$\mathcal{M}_\phi(f)(\cdot) := \sup_{t\in(0,\infty)}\left|(\phi_t * f)(\cdot)\right| = \sup_{t\in(0,\infty)}\left|\int_{\mathbb{R}^n}\frac{1}{t^n}\phi\left(\frac{\cdot - y}{t}\right)f(y)\,dy\right| \in L_w^p(\mathbb{R}^n) \quad (1.10)$$

and, for any $f \in H_w^p(\mathbb{R}^n)$, let $\|f\|_{H_w^p(\mathbb{R}^n)} := \|\mathcal{M}_\phi(f)\|_{L_w^p(\mathbb{R}^n)}$. Recall that, if $w \equiv 1$, then $H_1^p(\mathbb{R}^n)$ is just the Hardy space $H^p(\mathbb{R}^n)$ introduced in [20,24].

Moreover, let $w \in A_\infty(\mathbb{R}^n)$. For all $p \in (0,\infty)$ and $q \in (0,\infty]$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to be in the *weighted Lorentz-Hardy space* $H_w^{p,q}(\mathbb{R}^n)$ if

$$\mathcal{M}_\phi(f) \in L_w^{p,q}(\mathbb{R}^n) \quad (1.11)$$

and, for any $f \in H_w^{p,q}(\mathbb{R}^n)$, let $\|f\|_{H_w^{p,q}(\mathbb{R}^n)} := \|\mathcal{M}_\phi(f)\|_{L_w^{p,q}(\mathbb{R}^n)}$, where $\mathcal{M}_\phi(f)$ is as in (1.10) and $L_w^{p,q}(\mathbb{R}^n)$ denotes the weighted Lorentz space defined as in Definition 1.3 (see [13,25] for more

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properties on the Lorentz-Hardy spaces). In particular, when $q = \infty$, $H_w^{p,\infty}(\mathbb{R}^n)$ is the weighted weak Hardy space which has already appeared in [26–28].

Our second result of the present article is as follows.

Theorem 1.7: Let $w \in A_\infty(\mathbb{R}^n)$. Then, for all $0 < p_0 < p_1 \le 1$ and $\theta \in (0, 1)$,

$$\left(H_w^{p_0}(\mathbb{R}^n), H_w^{p_1}(\mathbb{R}^n)\right)_{\theta,\infty} = H_w^{p,\infty}(\mathbb{R}^n),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Theorem 1.7 is proved in Section 3. The main idea to prove Theorem 1.7 is to establish a retract relation between the weighted Hardy spaces and the weighted tent spaces, which enables us to use Theorem 1.6 to deduce the desired conclusion. To establish this desired retract relation, the weak atomic decompositions of weighted weak tent spaces established in [18] (see also Lemma 2.4), play an important role.

Recall that Fefferman et al. [13, Theorem 1] established the following real interpolation of the Hardy space $H^p(\mathbb{R}^n)$.

Theorem 1.8: [13] For all $p_0 \in (0, 1)$, $\theta \in (0, 1)$ and $q \in (0, \infty]$, it holds true that

$$\left(H^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{\theta,q} = H^{p,q}(\mathbb{R}^n),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $H^{p,q}(\mathbb{R}^n)$ denotes the Lorentz–Hardy space defined as in (1.11) with $w = 1$.

Fefferman et al. proved [13, Theorem 1] by using the strategy that they first restricted the considered Hardy and Lorentz–Hardy spaces to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and then decomposed functions in $\mathcal{S}(\mathbb{R}^n)$ into "good" and "bad" parts, which finally leads to the desired estimates for the associated K-functionals. Thus, the validity of the proof in [13] depends heavily on the density of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in those Hardy and Lorentz–Hardy spaces. It is well known that this density always holds true for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, \infty)$ and the Lorentz–Hardy spaces $H^{p,q}(\mathbb{R}^n)$ with $p \in (0, \infty)$ and $q \in (0, \infty)$. However, as was pointed out in [14], if $q = \infty$, then $\mathcal{S}(\mathbb{R}^n)$ may not be dense in the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$ for all $p \in (0, \infty)$ (see [14,26] for counterexamples). Indeed, in [14], He constructed a tempered distribution $f := \delta_1 - \delta_{-1} \in H^{1,\infty}(\mathbb{R}^n)$, with δ_1 and δ_{-1} being the Dirac delta functions, and proved that, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\|f - \phi\|_{H^{1,\infty}(\mathbb{R}^n)} \ge \frac{1}{20}$. Thus, there exists a gap in the proof of [13, Theorem 1] when $q = \infty$. Notice that the original proof for [13, Theorem 1] is still valid for $q \in (0, \infty)$.

To explain how Theorem 1.7 fills the gap in the proof of [13, Theorem 1] in the case $q = \infty$, we observe that, for any $p_1 \in (0, \infty)$, $\theta \in (0, 1)$ and $q \in (0, \infty]$,

$$\left(L^{p_1}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{\theta,q} = L^{p,q}(\mathbb{R}^n),$$

where $\frac{1}{p} = \frac{1-\theta}{p_1}$ and $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz space as in Definition 1.3 with $w = 1$. This, together with the fact that, for any $p \in (1, \infty)$, the Hardy $H^p(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$, implies that, for any $p_1 \in (1, \infty)$ and $\theta \in (0, 1)$,

$$\left(H^{p_1}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{\theta,\infty} = L^{p,\infty}(\mathbb{R}^n),$$

where $\frac{1}{p} = \frac{1-\theta}{p_1}$. Moreover, by Theorem 1.7, we know that, for all $0 < p_0 < p_1 \le 1$ and $\theta \in (0, 1)$,

$$\left(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)\right)_{\theta,\infty} = H^{p,\infty}(\mathbb{R}^n),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Thus, by Wolff's reiteration theorem of the real interpolation (see, e.g. [29]), we see that Theorem 1.8 holds true in the case $q = \infty$.

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Finally, we point out that the real interpolation of weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ in Theorem 1.7 further implies that $(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_{\theta,\infty}$ has the interpolation property, which is useful when considering the boundedness of many important operators on the associated weighted Hardy and Lorentz–Hardy spaces (see Remark 2.2 for more details on the interpolation property). Recall that this interpolation property has also been proved in [14, Theorem 7] by using a different method.

We end this section by making some conventions on the notation. Throughout the whole article, we always let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We use C to denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line, and $C_{(\alpha,\ldots)}$ to denote a *positive constant* depending on the parameters $\alpha, \ldots$. *Constants with subscripts*, such as C_1, do not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $\lambda \in (0,\infty)$, $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\lambda B := B(x, \lambda r)$. Also, for any set $E \subset \mathbb{R}^n$, χ_E denotes its *characteristic function*.

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2. Preliminaries

In this section, to give the proof of Theorem 1.6, we provide some preliminary results. To this end, we first recall some basic facts on the real interpolation on quasi-Banach spaces (see, e.g. [15,30–33] for more details). Let A_0 and A_1 be two quasi-Banach spaces. The pair (A_0, A_1) is said to be *compatible* if there exists a Hausdorff space $\mathbb{X}$ such that $A_0 \subset \mathbb{X}$ and $A_1 \subset \mathbb{X}$. For any compatible pair of quasi-Banach spaces (A_0, A_1), the *sum space* $A_0 + A_1$ is defined by setting

$$A_0 + A_1 := \{a \in \mathbb{X} : \exists\, a_0 \in A_0 \text{ and } a_1 \in A_1 \text{ such that } a = a_0 + a_1\}$$

endowed with the quasi-norm

$$\|a\|_{A_0+A_1} := \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0 \text{ and } a_1 \in A_1\right\},$$

where the infimum is taken over all the above decompositions of a. It is easy to see that $A_0 + A_1$ is also a quasi-Banach space (see, e.g. [30, Lemma 2.3.1] for a similar proof in the setting of Banach space).

For any $a \in A_0 + A_1$ and $t \in (0, \infty)$, Peetre's *K-functional* $K(t, a; A_0, A_1)$ is defined by setting

$$K(t, a; A_0, A_1) := \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1 \text{ with } a_0 \in A_0 \text{ and } a_1 \in A_1\right\},$$

where the infimum is taken over all the above decompositions of a.

Definition 2.1: Let (A_0, A_1) be a compatible pair of quasi-Banach spaces. For all $\theta \in (0, 1)$ and $q \in (0, \infty]$, the *real interpolation space* $(A_0, A_1)_{\theta,q}$ is defined by setting

$$(A_0, A_1)_{\theta,q} := \left\{a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} := \left\{\int_0^\infty \left[t^{-\theta}K(t, a; A_0, A_1)\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}} < \infty\right\}$$

with the usual modification made when $q = \infty$.

Remark 2.2: It is easy to see that, for all $\theta \in (0, 1)$ and $q \in (0, \infty]$, $(A_0, A_1)_{\theta,q}$ is also a quasi-Banach space. Moreover, the real method of interpolation always has the following *interpolation property*: if (A_0, A_1) and (B_0, B_1) are two couples of quasi-Banach spaces, and T is a linear operator from $A_0 + A_1$ to $B_0 + B_1$ that is bounded from A_0 to B_0 and from A_1 to B_1, then T is also bounded from $(A_0, A_1)_{\theta,q}$ to $(B_0, B_1)_{\theta,q}$ (see, e.g. [32,34]). Moreover, if (B_0, B_1) is further a couple of quasi-Banach lattices of functions on a measure space, then the condition that T is linear can be weakened to that T is quasi-linear (see [35,36] for more details).

The following lemma collects some basic facts on the weighted Lorentz space introduced in Definition 1.3, which are needed in the proof of Theorem 1.6.

Lemma 2.3: [37] Let $w \in A_\infty(\mathbb{R}^n)$.

(i) If $p \in (0, \infty]$, then $L_w^{p,p}(\mathbb{R}^n) = L_w^p(\mathbb{R}^n)$ with

$$\|f\|_{L_w^p(\mathbb{R}^n)} = \left\{\int_0^\infty \left[f_w^*(t)\right]^p \,\mathrm{d}t\right\}^{\frac{1}{p}}$$

(here the usual modification being made when $p = \infty$) holding true, where $L_w^p(\mathbb{R}^n)$ is the *weighted Lebesgue space* with the quasi-norm $\|\cdot\|_{L_w^p(\mathbb{R}^n)}$ defined as in (1.4).

(ii) For all $p \in (0, \infty]$ and $0 < q_1 < q_2 \leq \infty$, $L_w^{p,q_1}(\mathbb{R}^n) \subset L_w^{p,q_2}(\mathbb{R}^n)$.

(iii) For $p \in (0, \infty)$,

$$\|f\|_{L_w^{p,\infty}(\mathbb{R}^n)} := \left\{\sup_{\lambda\in(0,\infty)} \left[\lambda^p w\left(\left\{x \in \mathbb{R}^n : |f(x)| > \lambda\right\}\right)\right]\right\}^{\frac{1}{p}} = \sup_{s\in(0,\infty)} s\left[\lambda_{w,f}(s)\right]^{\frac{1}{p}},$$

where $\lambda_{w,f}(s)$ is the weighted distribution function of f defined as in (1.8) ($L_w^{p,\infty}(\mathbb{R}^n)$ is also called the *weighted weak Lebesgue space*).

We now introduce some technical lemmas on the weighted tent space. The first result is on the weak atomic decomposition of weighted weak tent spaces. Recall, in [38], that, for all $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$, a measurable function A on $\mathbb{R}_+^{n+1}$ is called a *weighted tent atom* if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\operatorname{supp} A \subset \widehat{B}$, here and hereafter, $\widehat{B} := \{(y, t) \in \mathbb{R}_+^{n+1} : B(y, t) \subset B\}$;

(ii) for any $q \in (1, \infty)$,

$$\|A\|_{T^q(\mathbb{R}_+^{n+1})} \leq |B|^{\frac{1}{q}}[w(B)]^{-\frac{1}{p}},$$

where $\|\cdot\|_{T^q(\mathbb{R}_+^{n+1})}$ denotes the quasi-norm of the tent space $T^q(\mathbb{R}_+^{n+1})$ defined as in Definition 1.1 with $w = 1$ and $\alpha = 1$.

We point out that, if $w \equiv 1$, then the weighted tent atom is just the tent atom introduced in [2].

The following lemma gives the weak atomic decompositions of weak weighted tent spaces, which play an important role in the proofs of both Theorems 1.6 and 1.7.

Lemma 2.4: [18] Let $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$. For all $F \in T^{p,\infty}(\mathbb{R}_+^{n+1})$, there exist a sequence $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}} \subset \mathbb{C}$ satisfying

$$\sup_{i\in\mathbb{Z}} \left\{\sum_{j\in\mathbb{N}} |\lambda_{i,j}|^p\right\}^{\frac{1}{p}} \lesssim \|F\|_{T^{p,\infty}(\mathbb{R}_+^{n+1})}$$

and a sequence $\{A_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ of weighted tent atoms such that

$$F = \sum_{i\in\mathbb{Z},j\in\mathbb{N}} \lambda_{i,j} A_{i,j}$$

almost everywhere in $\mathbb{R}_+^{n+1}$.

Proof: The proof of this lemma is essentially similar to that of [18, Theorem 2.6], where the authors considered the case $w \equiv 1$, the details being omitted. □

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The following lemma establishes the dual relationship between weighted tent spaces $T^1_w(\mathbb{R}^{n+1}_+)$ and $T^\infty_w(\mathbb{R}^{n+1}_+)$.

Lemma 2.5: Let $w \in A_\infty(\mathbb{R}^n)$. The pairing

$$(F, G) := \iint_{\mathbb{R}^{n+1}_+} F(y, t)\overline{G(y, t)}\,\frac{\mathrm{d}y\,\mathrm{d}t}{t}$$

realizes $T^\infty_w(\mathbb{R}^{n+1}_+)$ as equivalent to the Banach space dual of $T^1_w(\mathbb{R}^{n+1}_+)$.

Proof: We first show that $T^\infty_w(\mathbb{R}^{n+1}_+) \subset [T^1_w(\mathbb{R}^{n+1}_+)]^*$. Let $G \in T^\infty_w(\mathbb{R}^{n+1}_+)$. For all $F \in T^1_w(\mathbb{R}^{n+1}_+)$, by [16, Theorem 3], we know that F has a weighted atomic decomposition $F = \sum_j \lambda_j A_j$ with $\{\lambda_j\}_j \subset \mathbb{C}$ satisfying $\sum_j |\lambda_j| \lesssim \|F\|_{T^1_w(\mathbb{R}^{n+1}_+)}$, where A_j for j is a weighted $T^1_w(\mathbb{R}^{n+1}_+)$ tent atom, namely, A_j supports in some tent $\widehat{B}_j$ over the ball B_j and satisfies the size condition

$$\left[\iint_{\mathbb{R}^{n+1}_+} \left|A_j(y, t)\right|^2 \frac{w(B(y, t))}{t^n}\frac{\mathrm{d}y\,\mathrm{d}t}{t}\right]^{\frac{1}{2}} \le [w(B_j)]^{-1/2}.$$

Thus, by Hölder's inequality and the definition of weighted $T^1_w(\mathbb{R}^{n+1}_+)$ tent atoms, we see that

$$\begin{aligned}
|(F, G)| &\le \iint_{\mathbb{R}^{n+1}_+} \left|F(y, t)\right|\left|G(y, t)\right|\,\frac{\mathrm{d}y\,\mathrm{d}t}{t}\\
&\lesssim \sum_j |\lambda_j| \iint_{\widehat{B}_j} \left|A_j(y, t)\right|\left|G(y, t)\right|\,\frac{\mathrm{d}y\,\mathrm{d}t}{t}\\
&\lesssim \sum_j |\lambda_j| \left\{\iint_{\widehat{B}_j} \left|A_j(y, t)\right|^2 \frac{w(B(y, t))}{t^n}\,\frac{\mathrm{d}y\,\mathrm{d}t}{t}\right\}^{\frac{1}{2}}\\
&\quad \times \left\{\iint_{\widehat{B}_j} \left|G(y, t)\right|^2 \frac{t^n}{w(B(y, t))}\,\frac{\mathrm{d}y\,\mathrm{d}t}{t}\right\}^{\frac{1}{2}}\\
&\sim \sum_j |\lambda_j| \|\mathcal{C}_w(G)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|F\|_{T^1_w(\mathbb{R}^{n+1}_+)}\|G\|_{T^\infty_w(\mathbb{R}^{n+1}_+)},
\end{aligned}$$

which immediately implies that G can be extended into a bounded linear functional on $T^1_w(\mathbb{R}^{n+1}_+)$. Thus, $T^\infty_w(\mathbb{R}^{n+1}_+) \subset [T^1_w(\mathbb{R}^{n+1}_+)]^*$.

We now turn to the proof of the converse inclusion. Let $l \in [T^1_w(\mathbb{R}^{n+1}_+)]^*$. For any compact set $K \subset \mathbb{R}^{n+1}_+$ and $F_K \in T^2_w(\mathbb{R}^{n+1}_+)$ supported in K, by Hölder's inequality, we know that

$$\|F_K\|_{T^1_w(\mathbb{R}^{n+1}_+)} \lesssim \|F_K\|_{T^2_w(K)},$$

where the implicit positive constant depends on K, which immediately implies that

$$|l(F_K)| \le \|l\|_{[T^1_w(\mathbb{R}^{n+1}_+)]^*}\|F_K\|_{T^1_w(\mathbb{R}^{n+1}_+)} \lesssim \|l\|_{[T^1_w(\mathbb{R}^{n+1}_+)]^*}\|F_K\|_{T^2_w(K)}.$$

Thus, l induces a bounded linear functional on $T^2_w(K)$.

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Moreover, from Fubini's theorem, it follows that

$$\|F\|_{T^2_w(K)} = \left\{\iint_K |F(y,t)|^2 \frac{w(B(y,t))}{t^n}\frac{dy\,dt}{t}\right\}^{\frac12}$$
$$=: \left\{\iint_K |F(y,t)|^2\, W(y,t)\,\frac{dy\,dt}{t}\right\}^{\frac12} =: \|F\|_{L^2_W(K)},$$

where, for all $(y,t)\in\mathbb{R}^{n+1}_+$, $W(y,t):=\frac{w(B(y,t))}{t^n}$ is a weight function on $\mathbb{R}^{n+1}_+$. This, together with the fact that $[L^2_W(K)]^* = L^2_{W^{-1}}(K)$ endowed with the norm that

$$\|F\|_{L^2_{W^{-1}}(K)} := \left\{\iint_K |F(y,t)|^2 \frac{t^n}{w(B(y,t))}\frac{dy\,dt}{t}\right\}^{\frac12}$$

(this fact can be proved in a way similar to the proof of the classical fact $[L^2(\mathbb{R}^n)]^* = L^2(\mathbb{R}^n)$; see, e.g. [39, Theorem 6.15], the details being omitted), shows that there exists a function $G_K\in L^2_{W^{-1}}(K)$ such that, for all K,

$$l(F_K) = \iint_{\mathbb{R}^{n+1}_+} F_K(y,t)\,\overline{G_K(y,t)}\,\frac{dy\,dt}{t}. \tag{2.1}$$

Taking an increasing family of such sets $\{K\}$ which exhaust $\mathbb{R}^{n+1}_+$, we obtain a function G in $\mathbb{R}^{n+1}_+$ which is locally in $L^2_{W^{-1}}(\mathbb{R}^{n+1}_+)$ and, for all compact sets K, $G|_K = G_K$. Moreover, for all $F\in T^2_w(\mathbb{R}^{n+1}_+)$ with compact supports, we have

$$l(F) = \iint_{\mathbb{R}^{n+1}_+} F(y,t)\,\overline{G(y,t)}\,\frac{dy\,dt}{t}. \tag{2.2}$$

We now prove that $\|\mathcal{C}_w(G)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|l\|_{[T^1_w(\mathbb{R}^{n+1}_+)]^*}$. Indeed, observe that, for all $F\in L^2_W(\mathbb{R}^{n+1}_+)$ with $\|F\|_{L^2_W(\mathbb{R}^{n+1}_+)}\le 1$ and supported in the tent $\widehat{B}$ over the ball B, from its definition, it follows that $\frac{f}{[w(B)]^{1/2}}$ is a weighted $T^1_w(\mathbb{R}^{n+1}_+)$ tent atom associated with the ball B. Thus, for all balls B, by the dual norm representation of $L^2_{W^{-1}}(\mathbb{R}^{n+1}_+)$ and (2.2), we know that

$$\begin{aligned}
&\left\{\frac{1}{w(B)}\iint_{\widehat{B}} |G(y,t)|^2 \frac{t^n}{w(B(y,t))}\frac{dy\,dt}{t}\right\}^{\frac12}\\
&\quad = \left\|\frac{1}{[w(B)]^{1/2}} G\chi_{\widehat{B}}\right\|_{L^2_{W^{-1}}(\mathbb{R}^{n+1}_+)}\\
&\quad = \sup_{\|F\|_{L^2_W(\mathbb{R}^{n+1}_+)}\le 1}\left|\iint_{\mathbb{R}^{n+1}_+}\frac{F(y,t)}{[w(B)]^{1/2}}\overline{G(y,t)}\,\frac{dy\,dt}{t}\right| \le \|l\|_{[T^1_w(\mathbb{R}^{n+1}_+)]^*},
\end{aligned} \tag{2.3}$$

which implies that $\|\mathcal{C}_w(G)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|l\|_{[T^1_w(\mathbb{R}^{n+1}_+)]^*}$.

Now, let

$$\left(T^2_{w,c}(\mathbb{R}^{n+1}_+)\cap T^1_w(\mathbb{R}^{n+1}_+),\ \|\cdot\|_{T^1_w(\mathbb{R}^{n+1}_+)}\right)$$

be the set of all functions in $T^2_w(\mathbb{R}^{n+1}_+)\cap T^1_w(\mathbb{R}^{n+1}_+)$ with compact supports endowed with the norm of $T^1_w(\mathbb{R}^{n+1}_+)$. By (2.1) and (2.3), we conclude that l induces a bounded linear functional on

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$$(T^2_{w,c}(\mathbb{R}^{n+1}_+) \cap T^1_w(\mathbb{R}^{n+1}_+), \|\cdot\|_{T^1_w(\mathbb{R}^{n+1}_+)}).$$

Moreover, from the weighted atomic decomposition of $T^1_w(\mathbb{R}^{n+1}_+)$, it follows that

$$(T^2_{w,c}(\mathbb{R}^{n+1}_+) \cap T^1_w(\mathbb{R}^{n+1}_+), \|\cdot\|_{T^1_w(\mathbb{R}^{n+1}_+)})$$

is dense in $T^1_w(\mathbb{R}^{n+1}_+)$. Thus, for any $F \in T^1_w(\mathbb{R}^{n+1}_+)$, let

$$\{F_m\}_{m\in\mathbb{N}} \subset \left[T^2_{w,c}(\mathbb{R}^{n+1}_+) \cap T^1_w(\mathbb{R}^{n+1}_+)\right]$$

such that $\lim_{m\to\infty} F_m = F$ in $T^1_w(\mathbb{R}^{n+1}_+)$. Then, by Lebesgue's dominate convergence theorem, we have

$$l(F) = \lim_{m\to\infty} l(F_m) = \lim_{m\to\infty} \iint_{\mathbb{R}^{n+1}_+} F_m(y,t)\overline{G(y,t)}\,\frac{dy\,dt}{t} = \iint_{\mathbb{R}^{n+1}_+} F(y,t)\overline{G(y,t)}\,\frac{dy\,dt}{t}.$$

This, combined with (2.3), shows the inclusion $[T^1_w(\mathbb{R}^{n+1}_+)]^* \subset T^\infty_w(\mathbb{R}^{n+1}_+)$ and hence finishes the proof of Lemma 2.5. □

The following lemma shows that the weighted tent spaces $T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)$ are equivalent under the change of apertures.

Lemma 2.6: Let $\alpha, \beta \in (0,\infty)$ and $w \in A_\infty(\mathbb{R}^n)$. Then,

(i) for all $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$, there exist positive constants C_1 and C_2 such that, for all measurable functions F on $\mathbb{R}^{n+1}_+$,

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)} \le C_1 \left\|\mathcal{A}^{(\beta)}(F)\right\|_{L^p_w(\mathbb{R}^n)} \le C_2 \left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)};$$

(ii) for all $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, there exist positive constants C_1 and C_2 such that, for all measurable functions F on $\mathbb{R}^{n+1}_+$,

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)} \le C_1 \left\|\mathcal{A}^{(\beta)}(F)\right\|_{L^p_w(\mathbb{R}^n)} \le C_2 \left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)}.$$

Proof: Without loss of generality, we may assume that $\alpha \in (1,\infty)$, $\beta = 1$ and only prove

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)} \lesssim \|\mathcal{A}(F)\|_{L^p_w(\mathbb{R}^n)}. \tag{2.4}$$

We first prove (2.4) in the case $p \in (0,1]$. Without loss of generality, we may further assume that $\|\mathcal{A}(F)\|_{L^p_w(\mathbb{R}^n)} < \infty$. This implies that $F \in T^p_w(\mathbb{R}^{n+1}_+)$. Using the weighted atomic decomposition of $T^p_w(\mathbb{R}^{n+1}_+)$ from [38], we know that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence of weighted tent atoms, $\{A_j\}_j$, satisfying $F = \sum_j \lambda_j A_j$ almost everywhere on $\mathbb{R}^{n+1}_+$ and

$$\left\{\sum_j |\lambda_j|^p\right\}^{\frac{1}{p}} \lesssim \|F\|_{T^p_w(\mathbb{R}^{n+1}_+)}. \tag{2.5}$$

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Moreover, A_j is supported in some tent $\widehat{B}_j$ over the ball B_j and satisfies the size condition that, for all $q \in (1, \infty)$,

$$\|A_j\|_{T^q(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(A_j)\|_{L^q(\mathbb{R}^n)} \le |B|^{\frac{1}{q}} [w(B)]^{-\frac{1}{p}}.$$

Since $w \in A_\infty(\mathbb{R}^n)$, by the property of Muckenhoupt weights, we know that there exists $q \in (1, \infty)$ such that w is in the *reverse Hölder class* $RH_{q'}(\mathbb{R}^n)$ with $pq > 1$, namely,

$$\left\{\frac{1}{|B|}\int_B [w(x)]^{q'}\,\mathrm{d}x\right\}^{1/q'} \lesssim \frac{1}{|B|}\int_B w(x)\,\mathrm{d}x,$$

here and hereafter, $\frac{1}{q} + \frac{1}{q'} = 1$. Then, by Hölder's inequality, we see that, for all $\alpha \in (1, \infty)$ and j,

$$\begin{aligned}
\left\|\mathcal{A}^{(\alpha)}(A_j)\right\|_{L^p_w(\mathbb{R}^n)} &= \left\{\int_{\alpha B_j}\left[\iint_{\widehat{B}_j\cap\Gamma_\alpha(x)} |A_j(y,t)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\right]^{p/2} w(x)\,\mathrm{d}x\right\}^{1/p}\\
&\le \left\{\int_{\alpha B_j}\left[\iint_{\widehat{B}_j} |A_j(y,t)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\right]^{\frac{pq}{2}} \mathrm{d}x\right\}^{\frac{1}{pq}} \left\{\int_{\alpha B_j} [w(x)]^{q'}\right\}^{\frac{1}{pq'}}\\
&\lesssim \|A_j\|_{T^{pq}(\mathbb{R}^{n+1}_+)} |B_j|^{\frac{1}{pq'}-\frac{1}{p}} [w(B_j)]^{1/p}\\
&\lesssim |B_j|^{\frac{1}{pq}} [w(B_j)]^{-1/p} |B_j|^{\frac{1}{pq'}-\frac{1}{p}} [w(B_j)]^{1/p} \lesssim 1.
\end{aligned}$$

This, combined with (2.5), implies that

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)} \lesssim \|F\|_{T^p_w(\mathbb{R}^{n+1}_+)} \lesssim \|\mathcal{A}(F)\|_{L^p_w(\mathbb{R}^n)}.$$

Thus, (2.4) holds true in the case $p \in (0, 1]$.

To prove (2.4) under the case $p \in (1, \infty)$, we need to use the vector-valued approach from [19]. For all $p \in (1, \infty)$, let $L^p_{2,w}$ be the weighted vector-valued Lebesgue space on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$ endowed with the norm

$$\|f\|_{L^p_{2,w}} := \left\{\int_{\mathbb{R}^n}\left[\iint_{\mathbb{R}^{n+1}_+} |f(x,y,t)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\right]^{\frac{p}{2}} w(x)\,\mathrm{d}x\right\}^{\frac{1}{p}}.$$

For all $\alpha \in (1, \infty)$, $F \in T^p_w(\mathbb{R}^{n+1}_+)$ and $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$, let i_α be the *operator* defined by setting

$$i_\alpha(F)(x,y,t) := \chi_{\Gamma_\alpha(x)}(y,t) F(y,t),$$

where $\Gamma_\alpha(x)$ is as in (1.6). From its definition, it follows that

$$\|i_\alpha(F)\|_{L^p_{2,w}} = \left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)}. \tag{2.6}$$

Moreover, for all $f \in L^p_{2,w}$ and $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$, let

$$\mathcal{N}_\alpha(f)(x, y, t) := \chi_{\Gamma_\alpha(x)}(y, t) \frac{1}{(\alpha t)^n} \int_{\{z\in\mathbb{R}^n:\ |z-x|<\alpha t\}} f(z, y, t)\, dz$$

and

$$\mathcal{M}_\alpha(f)(x, y, t) := \sup_{s\in(0,\infty)} \frac{1}{(\alpha s)^n} \int_{\{z\in\mathbb{R}^n:\ |z-x|<\alpha s\}} \left|f(z, y, t)\right| dz.$$

By the weighted vector-valued maximal operator theory from [40], we know that, for every $w \in A_p(\mathbb{R}^n)$ and $f \in L^p_{2,w}$,

$$\|\mathcal{N}_\alpha(f)\|_{L^p_{2,w}} \le \|\mathcal{M}_\alpha(f)\|_{L^p_{2,w}} \lesssim \|f\|_{L^p_{2,w}}. \tag{2.7}$$

Moreover, from an argument same as that used in the proof of [12, Proposition 3.21], we deduce that, for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$,

$$\left|i_\alpha(F)(x, y, t)\right| \lesssim \mathcal{N}_\alpha\left(\chi_{\Gamma(x)}(y, t)\,|F(y, t)|\right),$$

which, together with (2.6) and (2.7), implies that

$$\begin{aligned}\left\|\mathcal{A}^{(\alpha)}(F)\right\|_{L^p_w(\mathbb{R}^n)} &= \|i_\alpha(F)\|_{L^p_{2,w}} \lesssim \left\|\mathcal{N}_\alpha\left(\chi_{\Gamma(\cdot)}|F|\right)\right\|_{L^p_{2,w}}\\ &\lesssim \left\|\mathcal{M}_\alpha\left(\chi_{\Gamma(\cdot)}|F|\right)\right\|_{L^p_{2,w}} \lesssim \|\mathcal{A}(F)\|_{L^p_{2,w}}.\end{aligned}$$

This shows that (2.4) holds true in the case $p \in (1, \infty)$, which completes the proof of Lemma 2.6. □

Remark 2.7: Lemmas 2.5 and 2.6 immediately imply that the weighted tent spaces $T^p_{w,\alpha}(\mathbb{R}^{n+1}_+)$ for all $p \in (0, \infty]$ are equivalent under the change of the aperture α.

We also need the following technical lemmas on the weighted decreasing rearrangement.

Lemma 2.8: [37]

(i) Let $N \in \mathbb{N}$, $w \in A_\infty(\mathbb{R}^n)$ and $f = \sum_{i=1}^N a_i \chi_{E_i}$ be a simple function with $w(E_i) < \infty$ and $0 < a_N < \cdots < a_1 < \infty$. For all $j \in \{1, \ldots, N\}$, let

$$B_j := \sum_{i=1}^{j} w(E_i) \tag{2.8}$$

and $B_0 = 0$. Then, for all $t \in (0, \infty)$,

$$f^*_w(t) = \sum_{j=1}^{N} a_j \chi_{[B_{j-1}, B_j)}(t),$$

where f^*_w is the weighted decreasing rearrangement defined as in (1.9).

(ii) Let $\{f_j\}_j$ be a sequence of measurable functions, which increases in j satisfying $\lim_{j\to\infty} f_j = f$. Then

$$\lim_{j\to\infty} (f_j)^*_w = f^*_w.$$

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Lemma 2.9: Let $w \in A_\infty(\mathbb{R}^n)$, f be a non-negative measurable function on $\mathbb{R}^n$, $t_0 \in (0, \infty)$ and $O := \{x \in \mathbb{R}^n : |f(x)| > f_w^*(t_0)\}$. Then

(i)

$$(f\chi_O)_w^*(s) \le \begin{cases} f_w^*(s), & s \in (0, t_0), \\ 0, & s \in [t_0, \infty); \end{cases}$$

(ii)

$$(f\chi_{O^\complement})_w^*(s) \le \begin{cases} f_w^*(t_0), & s \in (0, t_0), \\ f_w^*(s), & s \in [t_0, \infty). \end{cases}$$

Proof: We first prove (i). Assume that $f = \sum_{i=1}^N a_i\chi_{E_i}$ is a non-negative simple function with $N \in \mathbb{N}$, $w(E_i) < \infty$ and $0 < a_N < \cdots < a_1 < \infty$. Let $a_{N+1} := 0$. By Lemma 2.8, we know that, for all $s \in [0, \infty)$,

$$f_w^*(s) = \sum_{j=1}^N a_j\chi_{[B_{j-1}, B_j)}(s), \tag{2.9}$$

where the set B_j is defined as in (2.8). Thus, $f_w^*(t_0) \in \{0, a_1, \dots, a_N\}$. We now consider three cases based on the size of $f_w^*(t_0)$.

Case (i) $f_w^*(t_0) = 0$. In this case, by the definition of O, we know that

$$f\chi_O = f. \tag{2.10}$$

Moreover, from (2.9), it follows that $t_0 \ge B_N$. Observe that $f_w^*(s) = 0$ when $s \ge B_N$. This, combined with (2.9) and (2.10), implies that

$$(f\chi_O)_w^*(s) = f_w^*(s) = \begin{cases} f_w^*(s), & s \in (0, t_0), \\ 0, & s \in [t_0, \infty), \end{cases}$$

which shows that (i) holds true in Case (i).

Case (ii) $f_w^*(t_0) = a_1$. In this case, since $O = \{x \in \mathbb{R}^n : |f(x)| > a_1\} = \emptyset$. we know that $f\chi_O \equiv 0$. Thus, (i) also holds true.

Case (iii) There exists $i_0 \in \{2, \dots, N\}$ such that $f_w^*(t_0) = a_{i_0}$. In this case, by (2.9), we see that $t_0 \in [B_{i_0-1}, B_{i_0})$ and hence $f\chi_O := \sum_{i=1}^{i_0-1} a_i\chi_{E_i}$. Thus,

$$(f\chi_O)_w^*(s) = \sum_{j=1}^{i_0-1} a_j\chi_{[B_{j-1}, B_j)}(s),$$

which immediately implies that

$$(f\chi_O)_w^*(s) \le \begin{cases} f_w^*(s), & s \in (0, B_{i_0-1}) \subset (0, t_0), \\ 0, & s \in [B_{i_0-1}, \infty) \supset [t_0, \infty) \end{cases} \le \begin{cases} f_w^*(s), & s \in (0, t_0), \\ 0, & s \in [t_0, \infty). \end{cases}$$

This, together with Cases (i) and (ii), implies that (i) of Lemma 2.9 holds true for all simple functions.

For any general non-negative measurable function f, let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of positive simple functions increasing in j such that $\lim_{j\to\infty} f_j = f$ pointwisely. This implies that $\lim_{j\to\infty} f_j\chi_O = f\chi_O$.

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Thus, by Lemma 2.3(ii), we obtain

$$\left(f\chi_O\right)_w^*(s) = \lim_{j\to\infty}\left(f_j\chi_O\right)_w^*(s) \le \begin{cases} \lim_{j\to\infty}(f_j)_w^*(s) = f_w^*(s), & s\in(0,\ t_0),\\ 0, & s\in[t_0,\ \infty),\end{cases}$$

which immediately implies that (i) of Lemma 2.9.

We now turn to the proof of (ii). As in the case of (i), we first assume that f is a simple function. Let $f = \sum_{i=1}^N a_i\chi_{E_i}$ with the same notation as in the proof of (i). Thus, we also have

$$f_w^*(t_0) \in \{0,\ a_1,\ \dots,\ a_N\}.$$

We now consider three cases based on the size of $f_w^*(t_0)$.

Case (i) $f_w^*(t_0) = 0$. In this case, $O^{\complement} = \{x \in \mathbb{R}^n :\ |f(x)| \le 0\} = \{x \in \mathbb{R}^n :\ |f(x)| = 0\}$. Thus, $f\chi_{O^{\complement}} \equiv 0$, which implies that (ii) of Lemma 2.9 holds true.

Case (ii) $f_w^*(t_0) = a_1$. In this case, $O^{\complement} = \{x \in \mathbb{R}^n :\ |f(x)| \le a_1\} = \mathbb{R}^n$. Thus, $f\chi_{O^{\complement}} = f$. This, together with (2.9), implies that $t_0 \in [0,\ B_1)$. Hence,

$$\left(f\chi_{O^{\complement}}\right)_w^*(s) = f_w^*(s) = \begin{cases} f_w^*(t_0), & s\in(0,\ t_0),\\ f_w^*(s), & s\in[t_0,\ \infty).\end{cases}$$

Case (iii) There exists $i_0 \in \{2,\ \dots,\ N\}$ such that $f_w^*(t_0) = a_{i_0}$. In this case, using (2.9), we know that $t_0 \in [B_{i_0-1},\ B_{i_0})$. Thus,

$$f\chi_{O^{\complement}} = \sum_{i=i_0}^N a_i\chi_{E_i}.$$

Moreover, by Lemma 2.8(i), we know that, for all $t\in(0,\ \infty)$,

$$\left(f\chi_{O^{\complement}}\right)_w^*(t) = \sum_{j=i_0}^N a_j\chi_{[\widetilde{B}_{j-1},\ \widetilde{B}_j)}(t),$$

where, for all $j \in \{i_0-1,\ \dots,\ N\}$, $\widetilde{B}_j := B_j - \sum_{i=1}^{i_0-1} w(E_i)$ and B_j is as in (2.8). Using an elementary calculation, we find that, for all $s\in(0,\ t_0)\subset(0,\ B_{i_0})$,

$$(f\chi_{O^{\complement}})_w^*(s) \le a_{i_0} = f_w^*(t_0).$$

Moreover, by the decreasing property of f_w^*, we know that, for all $s \in [t_0,\ \infty)$, $(f\chi_{O^{\complement}})(s) \le f_w^*(s)$. This, combined with Cases (i) and (ii), shows that (ii) of Lemma 2.9 holds true for all simple functions.

The general case that f is measurable follows from an argument similar to that used in the proof of (i) of Lemma 2.9, the details being omitted, which completes the proof of Lemma 2.9(ii) and hence Lemma 2.9. □

Now, let $w \in A_\infty(\mathbb{R}^n)$. For all $\beta\in(0,\ 1]$, $p,\ q\in(0,\ \infty)$ and $F \in T_w^{p,q}(\mathbb{R}_+^{n+1})$, let $f := \mathcal{A}^{(3\beta)}(F)$, f_w^* be the weighted decreasing rearrangement of f and $\lambda := f_w^*(t^\alpha)$ where $t\in(0,\ \infty)$ and $\alpha\in(0,\ 1)$ satisfying $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$. Let

$$O := \left\{x \in \mathbb{R}^n :\ \mathcal{A}^{(3\beta)}(F)(x) > \lambda\right\} \tag{2.11}$$

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and $\widehat{O}_\beta := \mathbb{R}^{n+1}_+ \backslash (\cup_{x\in O^\complement} \Gamma_\beta(x))$ be the *tent* over O. Moreover, let

$$F_0 := F\chi_{\widehat{O}_\beta} \tag{2.12}$$

and

$$F_1 := F\chi_{\mathbb{R}^{n+1}_+ \backslash \widehat{O}_\beta}. \tag{2.13}$$

Then it is easy to see that $F = F_0 + F_1$.

The following lemma gives a suitable decomposition for every $F \in T^{p,q}_w(\mathbb{R}^{n+1}_+)$.

Lemma 2.10: Let $\beta \in (0, 1]$, $w \in A_\infty(\mathbb{R}^n)$, $p, q \in (0, \infty)$, $F \in T^{p,q}_w(\mathbb{R}^{n+1}_+)$, and F_0 and F_1 be, respectively, as in (2.12) and (2.13). Then,

(i) $\operatorname{supp} \mathcal{A}^{(\beta)}(F_0) \subset O$;

(ii)

$$\mathcal{A}^{(\beta)}(F_1)(x) \le \begin{cases} \mathcal{A}^{(\beta)}(F)(x), & x \in O^\complement, \\ \\ f^*_w(t^\alpha), & x \in O. \end{cases}$$

Proof: We first prove (i). Recall that $\widehat{O}_\beta = \mathbb{R}^{n+1}_+ \backslash (\cup_{x\in O^\complement} \Gamma_\beta(x))$. If $x \in O^\complement$, then, for all $(y, t) \in \Gamma_\beta(x)$, we have $(y, t) \in \cup_{x\in O^\complement} \Gamma_\beta(x)$. Thus,

$$\mathcal{A}^{(\beta)}(F_0)(x) = \left\{ \iint_{\Gamma_\beta(x)} \left| \left(F\chi_{\widehat{O}_\beta}\right)(y, t)\right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}} = 0$$

and hence $\operatorname{supp} \mathcal{A}^{(\beta)}(F_0) \subset O$.

We now turn to the proof of (ii). For all $x \in O^\complement$, by an elementary argument, we know that

$$\mathcal{A}^{(\beta)}(F_1)(x) \le \mathcal{A}^{(\beta)}(F)(x).$$

For any $x \in O$, let $x_1 \in O^\complement$ satisfy

$$|x - x_1| = \inf\left\{|x - z| : z \in O^\complement\right\}. \tag{2.14}$$

Since $O^\complement$ is closed, such x_1 exists.

Moreover, by the fact that $\widehat{O}_\beta = \{(y, t) \in \mathbb{R}^{n+1}_+ : B(y, \beta t) \subset O\}$, we know that

$$\mathbb{R}^{n+1}_+ \setminus \widehat{O}_\beta = \left\{(y, t) \in \mathbb{R}^{n+1}_+ : B(y, \beta t) \cap O^\complement \ne \emptyset\right\}.$$

Thus, for all $(y, t) \in \Gamma_\beta(x) \cap (\mathbb{R}^{n+1}_+ \backslash \widehat{O}_\beta)$, there exists $z_0 \in O^\complement$ such that $|y-x| < \beta t$ and $|y-z_0| < \beta t$, which, together with (2.14), then implies that

$$|y - x_1| \le |y - x| + |x - x_1| < \beta t + |x - z_0| < \beta t + |x - y| + |y - z_0| < 3\beta t.$$

This shows that $\Gamma_\beta(x) \cap (\mathbb{R}^{n+1}_+ \setminus \widehat{O}_\beta) \subset \Gamma_{3\beta}(x_1)$. Moreover, since

$$x_1 \in O^\complement = \{\widetilde{x} \in \mathbb{R}^n : \mathcal{A}^{(3\beta)}(F)(\widetilde{x}) \le (\mathcal{A}^{(3\beta)}(F))^*_w(t^\alpha)\},$$

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it follows that

$$\mathcal{A}^{(\beta)}(F_1)(x)=\left\{\iint_{\Gamma_\beta(x)}\left|F(y,t)\,\chi_{(\widehat{O})^{\complement}}\right|^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac12}$$
$$\le\left\{\iint_{\Gamma_{3\beta}(x_1)}|F(y,t)|^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac12}\le\left(\mathcal{A}^{(3\beta)}(F)\right)_w^*(t^\alpha),$$

which shows that (ii) of Lemma 2.10 holds true and hence completes the proof of Lemma 2.10. □

Lemma 2.11: Let $\beta\in(0,\frac13]$, $w\in A_\infty(\mathbb{R}^n)$, $p,p_0,p_1,q\in(0,\infty)$ and $\theta\in(0,1)$ satisfy $\frac1p=\frac{\theta}{p_0}+\frac{1-\theta}{p_1}$, $F\in T_w^{p,q}(\mathbb{R}_+^{n+1})$, and F_0 and F_1 be, respectively, as in (2.12) and (2.13). Then $F_0\in T^{p_0}_{w,\beta}(\mathbb{R}_+^{n+1})$ and $F_1\in T^{p_1}_{w,\beta}(\mathbb{R}_+^{n+1})$.

Proof: We first prove $F_0\in T^{p_0}_{w,\beta}(\mathbb{R}_+^{n+1})$. Indeed, let $\widetilde{q}\in(0,q)$. By (i) and (ii) of Lemma 2.3, Lemmas 2.10(i) and 2.9(i), Hölder's inequality, $\beta\in(0,\frac13]$ and the decreasing property of weighted decreasing rearrangements, we see that

$$\|F_0\|_{T^{p_0}_{w,\beta}(\mathbb{R}_+^{n+1})}\lesssim\|F_0\|_{T^{p_0,\widetilde{q}}_{w,\beta}(\mathbb{R}_+^{n+1})}\sim\left\{\int_0^\infty\left[s^{\frac{1}{p_0}}\left(\mathcal{A}^{(\beta)}(F_0)\right)_w^*(s)\right]^{\widetilde{q}}\frac{ds}{s}\right\}^{\frac{1}{\widetilde{q}}}$$
$$\lesssim\left\{\int_0^{t^\alpha}\left[s^{\frac1p}\left(\mathcal{A}^{(\beta)}(F_0)\right)_w^*(s)\right]^{\widetilde{q}}s^{(\frac{1}{p_0}-\frac1p)\widetilde{q}}\frac{ds}{s}\right\}^{\frac{1}{\widetilde{q}}}$$
$$\lesssim\left\{\int_0^\infty\left[s^{\frac1p}\left(\mathcal{A}^{(\beta)}(F)\right)_w^*(s)\right]^{q}\frac{ds}{s}\right\}^{\frac1q}\left\{\int_0^{t^\alpha}s^{(\frac{1}{p_0}-\frac1p)\widetilde{q}(q/\widetilde{q})'}\frac{ds}{s}\right\}^{\frac{1}{\widetilde{q}(q/\widetilde{q})'}}$$
$$\lesssim\|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}\,t^{\alpha(\frac{1}{p_0}-\frac1p)}<\infty,$$

which immediately implies that $F_0\in T^{p_0}_{w,\beta}(\mathbb{R}_+^{n+1})$.

To show $F_1\in T^{p_1}_{w,\beta}(\mathbb{R}_+^{n+1})$, we first write

$$\|F_1\|_{T^{p_1}_{w,\beta}(\mathbb{R}_+^{n+1})}\sim\left\{\int_0^\infty\left[\left(\mathcal{A}^{(\beta)}(F_1)\right)_w^*(s)\right]^{p_1}ds\right\}^{\frac{1}{p_1}}$$
$$\lesssim\left\{\int_0^{t^\alpha}\left[\left(\mathcal{A}^{(\beta)}(F_1)\right)_w^*(s)\right]^{p_1}ds\right\}^{\frac{1}{p_1}}+\left\{\int_{t^\alpha}^\infty\dots ds\right\}^{\frac{1}{p_1}}=:\mathrm{I}+\mathrm{II}.$$

We first estimate I. Let O be the set defined as in (2.11). From Lemmas 2.10(ii) and 2.9(ii), and the assumption $\beta\in(0,\frac13]$, we deduce that

$$\mathrm{I}\lesssim\left\{\int_0^{t^\alpha}\left[\left(\mathcal{A}^{(\beta)}(F_1)\chi_{O^\complement}\right)_w^*(s)\right]^{p_1}ds\right\}^{\frac{1}{p_1}}+\left\{\int_0^{t^\alpha}\left[\left(\mathcal{A}^{(\beta)}(F_1)\chi_O\right)_w^*(s)\right]^{p_1}ds\right\}^{\frac{1}{p_1}}$$
$$\lesssim\left\{\int_0^{t^\alpha}\left[\left(\mathcal{A}^{(3\beta)}(F)\right)_w^*(t^\alpha)\right]^{p_1}ds\right\}^{\frac{1}{p_1}}\lesssim(\mathcal{A}(F))_w^*(t^\alpha)\,t^{\frac{\alpha}{p_1}}<\infty.$$

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To estimate II, we consider three cases based on the sizes of p_1, p and q. If $p_1 \geq q \geq p$, then, by the decreasing property of the weighted decreasing rearrangement and $\beta \in (0, \frac{1}{3}]$, we know that

$$\begin{aligned} \mathrm{II} &\leq \left[\left(\mathcal{A}^{(\beta)}(F)\right)_w^*(t^\alpha)\right]^{\frac{p_1-q}{p_1}} \left\{\int_{t^\alpha}^\infty \left[\left(\mathcal{A}^{(\beta)}(F)\right)_w^*(s)\, s^{\frac{1}{p}}\right]^q s^{1-\frac{q}{p}} \frac{ds}{s}\right\}^{\frac{1}{p_1}} \\ &\leq t^{(1-\frac{q}{p})\frac{\alpha}{p_1}} \left[\left(\mathcal{A}^{(\beta)}(F)\right)_w^*(t^\alpha)\right]^{\frac{p_1-q}{p_1}} \left\{\int_{t^\alpha}^\infty \left[s^{\frac{1}{p}} \left(\mathcal{A}^{(\beta)}(F)\right)_w^*(s)\right]^q \frac{ds}{s}\right\}^{\frac{1}{q}\frac{q}{p_1}} \\ &\lesssim t^{(1-\frac{q}{p})\frac{\alpha}{p_1}} \left[\left(\mathcal{A}(F)\right)_w^*(t^\alpha)\right]^{\frac{p_1-q}{p_1}} \|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}^{\frac{q}{p_1}} < \infty. \end{aligned} \tag{2.15}$$

If $p_1 > p > q$, then, by Lemma 2.3(iii) and a calculation similar to that in (2.15), we find that

$$\mathrm{II} \lesssim \left[\left(\mathcal{A}(F)\right)_w^*(t^\alpha)\right]^{\frac{p_1-p}{p_1}} \|F\|_{T^{p,p}(\mathbb{R}_+^{n+1})}^{\frac{p}{p_1}} \lesssim \left[\left(\mathcal{A}(F)\right)_w^*(t^\alpha)\right]^{\frac{p_1-p}{p_1}} \|F\|_{T^{p,q}(\mathbb{R}_+^{n+1})}^{\frac{p}{p_1}} < \infty. \tag{2.16}$$

If $q > p_1$, by Hölder's inequality with $\frac{p_1}{q} + \frac{1}{(\frac{q}{p_1})'} = 1$, we then also have

$$\begin{aligned} \mathrm{II} &\leq \left\{\int_{t^\alpha}^\infty \left[\left(\mathcal{A}^{(\beta)}(F)\right)_w^*(s)\, s^{\frac{1}{p}}\right]^{p_1} s^{1-\frac{p_1}{p}} \frac{ds}{s}\right\}^{\frac{1}{p_1}} \\ &\leq \left\{\int_{t^\alpha}^\infty \left[s^{\frac{1}{p}} \left(\mathcal{A}^{(\beta)}(F)\right)_w^*(s)\right]^q \frac{ds}{s}\right\}^{\frac{1}{p_1}\frac{p_1}{q}} \left\{\int_{t^\alpha}^\infty s^{(1-\frac{p_1}{p})(\frac{q}{p_1})'} \frac{ds}{s}\right\}^{\frac{1}{(\frac{q}{p_1})'}\frac{1}{p_1}} \\ &\lesssim t^{\alpha(\frac{1}{p_1}-\frac{1}{p})} \|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})} < \infty, \end{aligned}$$

which, combined with (2.15) through (2.16), implies that $F_1 \in T_{w,\beta}^{p_1}(\mathbb{R}_+^{n+1})$ and hence completes the proof of Lemma 2.11. □

We also need the following Hardy's inequality (see, e.g. [41, p.196, Lemma 3.14]).

Lemma 2.12: Let $q \in [1, \infty)$, $r \in (0, \infty)$ and g be a non-negative measurable function defined on $(0, \infty)$. Then

(i)

$$\left\{\int_0^\infty \left[\int_0^t g(u)\, du\right]^q t^{-r-1}\, dt\right\}^{\frac{1}{q}} \leq \frac{q}{r} \left\{\int_0^\infty [ug(u)]^q u^{-r-1}\, du\right\}^{\frac{1}{q}};$$

(ii)

$$\left\{\int_0^\infty \left[\int_t^\infty g(u)\, du\right]^q t^{r-1}\, dt\right\}^{\frac{1}{q}} \leq \frac{q}{r} \left\{\int_0^\infty [ug(u)]^q u^{r-1}\, du\right\}^{\frac{1}{q}}.$$

We end this section by recalling the following reiteration theorem of the real interpolation. For more details on the reiteration theorem, we refer to [30,42,43].

Lemma 2.13: Let (A_0, A_3) be a compatible pair of quasi-Banach spaces, $\lambda, \theta_0, \theta_1 \in (0, 1)$, $q_0, q_1, q \in (0, \infty]$ and $\theta := (1-\lambda)\theta_0 + \lambda\theta_1$. Assume that $(A_0, A_3)_{\theta_0, q_0} = A_1$ and $(A_0, A_3)_{\theta_1, q_1} = A_2$. Then

$$(A_1, A_2)_{\theta, q} = (A_0, A_3)_{\lambda, q}.$$

3. Proofs of Theorems 1.6 and 1.7

In this section, we give the proofs of Theorems 1.6 and 1.7. To prove Theorem 1.6, we need the following proposition.

Proposition 3.1: Let $w \in A_\infty(\mathbb{R}^n)$, $p_0 \in (0, 1]$ and $p_1 \in (1, \infty)$ such that $w \in A_{p_1}(\mathbb{R}^n)$. Then, for all $\theta \in (0, 1)$, $q \in (0, \infty]$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$\left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q} = T_w^{p,q}(\mathbb{R}_+^{n+1}).$$

Proof: We first show the inclusion that

$$\left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q} \subset T_w^{p,q}(\mathbb{R}_+^{n+1}). \tag{3.1}$$

Indeed, let $F \in \left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q}$. By the fact that

$$\left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q} \subset T_w^{p_0}(\mathbb{R}_+^{n+1}) + T_w^{p_1}(\mathbb{R}_+^{n+1}),$$

we know that there exist $F_0 \in T_w^{p_0}(\mathbb{R}_+^{n+1})$ and $F_1 \in T_w^{p_0}(\mathbb{R}_+^{n+1})$ such that

$$F = F_0 + F_1. \tag{3.2}$$

This implies that

$$\mathcal{A}(F) \le \mathcal{A}(F_0) + \mathcal{A}(F_1) \in L_w^{p_0}(\mathbb{R}^n) + L_w^{p_1}(\mathbb{R}^n),$$

which, together with the definition of Peetre's K-functional, shows that for all $t \in (0, \infty)$,

$$\begin{aligned}
&K(t, \mathcal{A}(F_0) + \mathcal{A}(F_1);\, L_w^{p_0}(\mathbb{R}^n),\, L_w^{p_1}(\mathbb{R}^n)) \\
&\quad \le \|\mathcal{A}(F_0)\|_{L_w^{p_0}(\mathbb{R}^n)} + t\|\mathcal{A}(F_0)\|_{L_w^{p_0}(\mathbb{R}^n)} = \|F_0\|_{T_w^{p_0}(\mathbb{R}_+^{n+1})} + t\|F_1\|_{T_w^{p_1}(\mathbb{R}_+^{n+1})}.
\end{aligned}$$

Thus, by the arbitrariness of the decomposition in (3.2), we see that

$$K(t, \mathcal{A}(F_0) + \mathcal{A}(F_1);\, L_w^{p_0}(\mathbb{R}^n),\, L_w^{p_1}(\mathbb{R}^n)) \le K(t, F;\, T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})). \tag{3.3}$$

Now, we first assume that $q \in (0, \infty)$. In this case, by (3.3), the fact that

$$(L_w^{p_0}(\mathbb{R}^n),\, L_w^{p_1}(\mathbb{R}^n))_{\theta,q} = L_w^{p,q}(\mathbb{R}^n)$$

(see, e.g. [42, Theorem 16]) and Definition 1.3, we obtain

$$\begin{aligned}
\|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})} &= \|\mathcal{A}(F)\|_{L_w^{p,q}(\mathbb{R}^n)} \le \|\mathcal{A}(F_0) + \mathcal{A}(F_1)\|_{L_w^{p,q}(\mathbb{R}^n)} \\
&\sim \|\mathcal{A}(F_0) + \mathcal{A}(F_1)\|_{(L_w^{p_0}(\mathbb{R}^n), L_w^{p_1}(\mathbb{R}^n))_{\theta,q}} \\
&\sim \left\{\int_0^\infty \left[t^{-\theta} K(t, \mathcal{A}(F_0) + \mathcal{A}(F_1);\, L_w^{p_0}(\mathbb{R}^n),\, L_w^{p_1}(\mathbb{R}^n))\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\lesssim \left\{\int_0^\infty \left[t^{-\theta} K\left(t, F;\, T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}}
\end{aligned}$$

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$$\sim \|F\|_{(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1}))_{\theta,q}} < \infty,$$

which immediately implies that $F \in T_w^{p,q}(\mathbb{R}_+^{n+1})$. This shows that (3.1) holds true for all $q \in (0, \infty)$.

The case $q = \infty$ is similar to the case when $q \in (0, \infty)$, we only need to modify the corresponding representation of the quasi-norm for $L_w^{p,\infty}(\mathbb{R}^n)$, the details being omitted. Thus, (3.1) holds true.

We now prove the inclusion

$$T_w^{p,q}(\mathbb{R}_+^{n+1}) \subset \left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q}.$$

Since $p_0 \in (0, 1]$, $p_1 \in (1, \infty)$ and $w \in A_{p_1}(\mathbb{R}^n)$, by Lemma 2.6, we only need to show

$$T_w^{p,q}(\mathbb{R}_+^{n+1}) \subset \left(T_{w,\frac{1}{3}}^{p_0}(\mathbb{R}_+^{n+1}),\, T_{w,\frac{1}{3}}^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q}. \tag{3.4}$$

To this end, we consider two cases on the size of q.

Case (i) $q \in (0, \infty)$. In this case, for all $F \in T_w^{p,q}(\mathbb{R}_+^{n+1})$, let f, f_w^*, F_0 and F_1 be as in Lemma 2.11 with $\beta = \frac{1}{3}$. By Lemma 2.11, we know that

$$F = F_0 + F_1 \in T_{w,\frac{1}{3}}^{p_0}(\mathbb{R}_+^{n+1}) + T_{w,\frac{1}{3}}^{p_1}(\mathbb{R}_+^{n+1}).$$

Using the definition of Peetre's K-functional and Minkowski's inequality, we see that

$$\begin{aligned}
&\|F\|_{(T_{w,\frac{1}{3}}^{p_0}(\mathbb{R}_+^{n+1}),\, T_{w,\frac{1}{3}}^{p_1}(\mathbb{R}_+^{n+1}))_{\theta,q}} \\
&= \left\{\int_0^\infty \left[t^{-\theta} K\left(t, F;\, T_{w,\frac{1}{3}}^{p_0}(\mathbb{R}_+^{n+1}),\, T_{w,\frac{1}{3}}^{p_1}(\mathbb{R}_+^{n+1})\right)\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\lesssim \left\{\int_0^\infty \left[t^{-\theta}\|F_0\|_{T_{w,\frac{1}{3}}^{p_0}(\mathbb{R}_+^{n+1})}\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}} + \left\{\int_0^\infty \left[t^{1-\theta}\|F_1\|_{T_{w,\frac{1}{3}}^{p_1}(\mathbb{R}_+^{n+1})}\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&=: \mathrm{I} + \mathrm{J}.
\end{aligned} \tag{3.5}$$

For I, by Lemmas 2.10(i) and 2.9(i) and the change of variable (let $u := t^\alpha$), we know that

$$\begin{aligned}
\mathrm{I} &\sim \left\{\int_0^\infty \left(t^{-\theta}\left\{\int_0^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_0)\right)_w^*(s)\right]^{p_0} ds\right\}^{\frac{1}{p_0}}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\sim \left\{\int_0^\infty \left(t^{-\theta}\left\{\int_0^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_0)\,\chi_O\right)_w^*(s)\right]^{p_0} ds\right\}^{\frac{1}{p_0}}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\lesssim \left\{\int_0^\infty t^{-\theta q}\left\{\int_0^{t^\alpha} [(\mathcal{A}(F))_w^*(s)]^{p_0}\, ds\right\}^{\frac{q}{p_0}} \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\sim \alpha^{-\frac{1}{q}}\left\{\int_0^\infty u^{-\frac{\theta q}{\alpha}}\left\{\int_0^u [(\mathcal{A}(F))_w^*(s)]^{p_0}\, ds\right\}^{\frac{q}{p_0}} \frac{du}{u}\right\}^{\frac{p_0}{q}\frac{1}{p_0}}.
\end{aligned}$$

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To continue the estimates of I, we further consider two cases. If $q \geq p_0$, then, by Lemma 2.12(i), we know that

$$\mathrm{I} \lesssim \alpha^{-\frac{1}{q}} \left(\frac{\frac{q}{p_0}}{\frac{\theta q}{\alpha}}\right)^{\frac{1}{p_0}} \left\{\int_0^\infty \left\{u\left[\left(\mathcal{A}(F)\right)_w^*(u)\right]^{p_0}\right\}^{\frac{q}{p_0}} u^{-\frac{\theta q}{\alpha}-1}\,\mathrm{d}u\right\}^{\frac{1}{q}}$$
$$\sim \alpha^{-\frac{1}{q}} \left(\frac{\alpha}{p_0\theta}\right)^{\frac{1}{p_0}} \left\{\int_0^\infty \left[u^{\frac{1}{p_0}-\frac{\theta}{\alpha}} \left(\mathcal{A}(F)\right)_w^*(u)\right]^q \frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}}. \tag{3.6}$$

Observe that $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p}$. By an elementary calculation, we find that $\theta = \frac{\frac{1}{p}-\frac{1}{p_0}}{\frac{1}{p_1}-\frac{1}{p_0}}$. Thus, we have

$$\frac{\theta}{\alpha} = \frac{\frac{1}{p}-\frac{1}{p_0}}{\frac{1}{p_1}-\frac{1}{p_0}}\left(\frac{1}{p_0}-\frac{1}{p_1}\right) = \frac{1}{p_0} - \frac{1}{p},$$

which, combined with (3.6), shows that

$$\mathrm{I} \lesssim \alpha^{-\frac{1}{q}} \left(\frac{\alpha}{p_0\theta}\right)^{\frac{1}{p_0}} \left\{\int_0^\infty \left[u^{\frac{1}{p}} \left(\mathcal{A}(F)\right)_w^*(u)\right]^q \frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}} \sim \alpha^{-\frac{1}{q}} \left(\frac{\alpha}{p_0\theta}\right)^{\frac{1}{p_0}} \|F\|_{T_w^{p,q}(\mathbb{R}^{n+1}_+)}. \tag{3.7}$$

This is a desired estimate.

If $q < p_0$, take $\widetilde{p} \in (0, q)$. By Lemma 2.3(ii), we know that $L_w^{p_0,\widetilde{p}}(\mathbb{R}^n) \subset L_w^{p_0}(\mathbb{R}^n)$ and

$$\|f\|_{L_w^{p_0}(\mathbb{R}^n)} \leq \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \|f\|_{L_w^{p_0,\widetilde{p}}(\mathbb{R}^n)},$$

which, together with Definition 1.3, Lemmas 2.10(i), 2.9(i) and 2.12(i), and the change of variables (let $u := t^\alpha$), we find that

$$\mathrm{I} \sim \left\{\int_0^\infty \left[t^{-\theta}\|\mathcal{A}^{(\frac{1}{3})}(F_0)\|_{L_w^{p_0}(\mathbb{R}^n)}\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}$$
$$\lesssim \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left\{\int_0^\infty \left[t^{-\theta}\|\mathcal{A}^{(\frac{1}{3})}(F_0)\|_{L_w^{p_0,\widetilde{p}}(\mathbb{R}^n)}\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}$$
$$\sim \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left[\int_0^\infty t^{-\theta q}\left\{\int_0^\infty \left[s^{\frac{1}{p_0}}\left(\mathcal{A}^{(\frac{1}{3})}(F_0)\right)_w^*(s)\right]^{\widetilde{p}} \frac{\mathrm{d}s}{s}\right\}^{\frac{q}{\widetilde{p}}} \frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$\sim \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left[\int_0^\infty t^{-\theta q}\left\{\int_0^\infty \left[s^{\frac{1}{p_0}}\left(\mathcal{A}^{(\frac{1}{3})}(F_0)\chi_O\right)_w^*(s)\right]^{\widetilde{p}} \frac{\mathrm{d}s}{s}\right\}^{\frac{q}{\widetilde{p}}} \frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$\lesssim \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left[\int_0^\infty t^{-\theta q}\left\{\int_0^{t^\alpha} \left[\left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(s)\right]^{\widetilde{p}} s^{\frac{\widetilde{p}}{p_0}-1}\,\mathrm{d}s\right\}^{\frac{q}{\widetilde{p}}} \frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$\lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left[\int_0^\infty \left\{\int_0^u \left[\left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(s)\right]^{\widetilde{p}} s^{\frac{\widetilde{p}}{p_0}-1}\,\mathrm{d}s\right\}^{\frac{q}{\widetilde{p}}} u^{-\frac{\theta q}{\alpha}-1}\,\mathrm{d}u\right]^{\frac{1}{q}}$$

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$$\lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left(\frac{\frac{q}{\widetilde{p}}}{\frac{\theta q}{\alpha}}\right)^{\frac{1}{\widetilde{p}}} \left\{\int_0^\infty \left(u\left[\left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(u)\right]^{\widetilde{p}} u^{-\frac{\theta q}{\alpha}-1}\right)^{\frac{q}{\widetilde{p}}} u^{-\frac{\theta q}{\alpha}-1}\,\mathrm{d}u\right\}^{\frac{1}{q}}$$

$$\sim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left(\frac{q\alpha}{\widetilde{p}\theta}\right)^{\frac{1}{\widetilde{p}}} \left\{\int_0^\infty \left[u^{\frac{1}{p_0}-\frac{\theta}{\alpha}} \left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(u)\right]^q \frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}},$$

which, combined with the definitions of α and θ, further implies that

$$\begin{aligned} \mathrm{I} &\lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left(\frac{q\alpha}{\widetilde{p}\theta}\right)^{\frac{1}{\widetilde{p}}} \left\{\int_0^\infty \left[u^{\frac{1}{p}} \left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(u)\right]^q \frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}} \\ &\sim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{p}}{p_0}\right)^{\frac{p_0-\widetilde{p}}{p_0\widetilde{p}}} \left(\frac{q\alpha}{\widetilde{p}\theta}\right)^{\frac{1}{\widetilde{p}}} \|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}. \end{aligned} \tag{3.8}$$

This is a desired estimate.

We now turn to the estimates of J. By Minkowski's inequality, we first write that

$$\begin{aligned} \mathrm{J} &\sim \left\{\int_0^\infty \left[t^{1-\theta} \left\|\mathcal{A}^{(\frac{1}{3})}(F_1)\right\|_{L_w^{p_1}(\mathbb{R}^n)}\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\int_0^\infty \left[t^{1-\theta} \left\|\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_O\right\|_{L_w^{p_1}(\mathbb{R}^n)}\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\ &\quad + \left\{\int_0^\infty \left[t^{1-\theta} \left\|\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\right\|_{L_w^{p_1}(\mathbb{R}^n)}\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} =: \mathrm{J}_1 + \mathrm{J}_2. \end{aligned} \tag{3.9}$$

For J_1, by Lemmas 2.3(i) and 2.10(ii), [37, Proposition 1.4.5(4)], and the change of variable (let $u := t^\alpha$), we know that

$$\begin{aligned} \mathrm{J}_1 &\sim \left[\int_0^\infty t^{q(1-\theta)} \left\{\int_0^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_O\right)_w^*(s)\right]^{p_1} \mathrm{d}s\right\}^{\frac{q}{p_1}} \frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}} \\ &\lesssim \left\{\int_0^\infty t^{q(1-\theta)} \left[\int_0^{t^\alpha} \mathrm{d}s\right]^{\frac{q}{p_1}} \left[(\mathcal{A}(F))_w^*(t^\alpha)\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\ &\sim \left\{\int_0^\infty t^{q(1-\theta)} t^{\frac{\alpha q}{p_1}} \left[(\mathcal{A}(F))_w^*(t^\alpha)\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\ &\sim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left\{\int_0^\infty u^{\frac{q}{\alpha}(1-\theta)+\frac{q}{p_1}} \left[(\mathcal{A}(F))_w^*(u)\right]^q \frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}}. \end{aligned}$$

Moreover, since $\theta = \frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}$ and $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$, it follows that $\frac{1-\theta}{\alpha} = \frac{1}{p} - \frac{1}{p_1}$, which further implies that

$$\mathrm{J}_1 \lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left\{\int_0^\infty \left[u^{\frac{1}{p}} (\mathcal{A}(F))_w^*(u)\right]^q \frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}} \sim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}. \tag{3.10}$$

To estimate J_2, we consider two cases on q and p_1.

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If $q \geq p_1$, then, by Lemmas 2.3(i) and 2.9(ii), and [37, Proposition 1.4.5(4)], we first write

$$\begin{aligned}
\mathrm{J}_2 &\sim \left\{\int_0^\infty t^{(1-\theta)q}\left\{\int_0^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\right)_w^*(s)\right]^{p_1}\,\mathrm{d}s\right\}^{\frac{q}{p_1}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\int_0^\infty t^{(1-\theta)q}\left\{\int_0^{t^\alpha} \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\right)_w^*(t^\alpha)\right]^{p_1}\,\mathrm{d}s\right\}^{\frac{q}{p_1}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}\\
&\quad+\left\{\int_0^\infty t^{(1-\theta)q}\left\{\int_{t^\alpha}^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\right)_w^*(s)\right]^{p_1}\,\mathrm{d}s\right\}^{\frac{q}{p_1}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} =: \mathrm{K}_1+\mathrm{K}_2. \qquad (3.11)
\end{aligned}$$

Similar to the corresponding estimates of J_1 in (3.10), we conclude that

$$\mathrm{K}_1 \lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}. \qquad (3.12)$$

For K_2, by the change of variable, [37, Proposition 1.4.5(4)], Lemmas 2.10(ii) and 2.12(ii), and the fact that $\frac{1-\theta}{\alpha}=\frac{1}{p}-\frac{1}{p_1}$, we see that

$$\begin{aligned}
\mathrm{K}_2 &\lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left\{\int_0^\infty u^{\frac{(1-\theta)q}{\alpha}}\left\{\int_u^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(s)\right]^{p_1}\,\mathrm{d}s\right\}^{\frac{q}{p_1}}\frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}}\\
&\lesssim \left(\frac{\frac{q}{p_1}}{\frac{(1-\theta)q}{\alpha}}\right)^{\frac{1}{p_1}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left\{\int_0^\infty \left[u^{[\frac{1-\theta}{\alpha}-\frac{1}{p_1}]}\left(\mathcal{A}^{(\frac{1}{3})}(F)\right)_w^*(u)\right]^q\frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}}\\
&\lesssim \left[\frac{\alpha}{p_1(1-\theta)}\right]^{\frac{1}{p_1}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}, \qquad (3.13)
\end{aligned}$$

which, together with (3.11) through (3.13), implies that

$$\mathrm{J}_2 \lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\max\left\{1,\left[\frac{\alpha}{p_1(1-\theta)}\right]^{\frac{1}{p_1}}\right\}\|F\|_{T_w^{p,q}(\mathbb{R}_+^{n+1})}. \qquad (3.14)$$

This is a desired estimate in the case when $q \geq p_1$.

We now consider the case when $q < p_1$. In this case, take $\widetilde{\widetilde{p}} \in (0, q)$. By Lemma 2.3(ii), we know that $L_w^{p_1,\widetilde{\widetilde{p}}}(\mathbb{R}^n) \subset L^{p_1}(\mathbb{R}^n)$ and

$$\|f\|_{L_w^{p_1}(\mathbb{R}^n)} \leq \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}}\|f\|_{L_w^{p,\widetilde{\widetilde{p}}}(\mathbb{R}^n)}.$$

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By this, combined with Definition 1.3, we know that

$$\begin{aligned}
\mathrm{J}_2 &\lesssim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty \left[t^{1-\theta}\|\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\|_{L^{p_1,\widetilde{\widetilde{p}}}(\mathbb{R}^n)}\right]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\
&\sim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty t^{(1-\theta)q}\left(\int_0^\infty s^{\frac{\widetilde{\widetilde{p}}}{p_1}}\left[\left(\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\right)_w^*(s)\right]^{\widetilde{\widetilde{p}}}\frac{\mathrm{d}s}{s}\right)^{\frac{q}{p_1}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\
&\lesssim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty t^{(1-\theta)q}\left(\int_0^{t^\alpha} s^{\frac{\widetilde{\widetilde{p}}}{p_1}}\left[\left(\mathcal{A}^{(\frac{1}{3})}(F_1)\,\chi_{O^{\complement}}\right)_w^*(s)\right]^{\widetilde{\widetilde{p}}}\frac{\mathrm{d}s}{s}\right)^{\frac{q}{p_1}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\
&\quad + \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty t^{(1-\theta)q}\left(\int_{t^\alpha}^\infty \cdots \frac{\mathrm{d}s}{s}\right)^{\frac{q}{p_1}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} =: \widetilde{\mathrm{K}}_1 + \widetilde{\mathrm{K}}_2.
\end{aligned}$$

To estimate $\widetilde{\mathrm{K}}_1$, by [37, Proposition 1.4.5(4)], Lemma 2.9(ii), the change of variable and the fact $\frac{1-\theta}{\alpha} = \frac{1}{p} - \frac{1}{p_1}$, we know that

$$\begin{aligned}
\widetilde{\mathrm{K}}_1 &\lesssim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty t^{(1-\theta)q}\left(\int_0^{t^\alpha} s^{\frac{\widetilde{\widetilde{p}}}{p_1}}\frac{\mathrm{d}s}{s}\right)^{\frac{q}{p_1}}\left[\left(\mathcal{A}(F)\,\chi_{O^{\complement}}\right)_w^*(t^\alpha)\right]^q\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\
&\sim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left\{\int_0^\infty \left[u^{\frac{1-\theta}{\alpha}+\frac{1}{p_1}}\left(\mathcal{A}(F)\right)_w^*(u)\right]^q\frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}} \\
&\sim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \|F\|_{T_w^{p,q}(\mathbb{R}^{n+1}_+)}.
\end{aligned}$$

For $\widetilde{\mathrm{K}}_2$, using [37, Proposition 1.4.5(4)], Lemmas 2.9(ii) and 2.12(ii), and the change of variable, we find that

$$\begin{aligned}
\widetilde{\mathrm{K}}_2 &\lesssim \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty t^{(1-\theta)q}\left(\int_{t^\alpha}^\infty s^{\frac{\widetilde{\widetilde{p}}}{p_1}}\left[\left(\mathcal{A}(F)\right)_w^*(s)\right]^{\widetilde{\widetilde{p}}}\frac{\mathrm{d}s}{s}\right)^{\frac{q}{\widetilde{\widetilde{p}}}}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \\
&\sim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left\{\int_0^\infty \left[\int_u^\infty s^{\frac{\widetilde{\widetilde{p}}}{p_1}-1}\left[\left(\mathcal{A}(F)\right)_w^*(s)\right]^{\widetilde{\widetilde{p}}}\,\mathrm{d}s\right]^{\frac{q}{\widetilde{\widetilde{p}}}} u^{\frac{(1-\theta)q}{\alpha}}\frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}} \\
&\lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left[\frac{\alpha}{\widetilde{\widetilde{p}}(1-\theta)}\right]^{\frac{1}{\widetilde{\widetilde{p}}}} \left\{\int_0^\infty \left[u^{\frac{1}{p_1}-\frac{1-\theta}{\alpha}}\left(\mathcal{A}(F)\right)_w^*(u)\right]^q\frac{\mathrm{d}u}{u}\right\}^{\frac{1}{q}} \\
&\sim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}} \left[\frac{\alpha}{\widetilde{\widetilde{p}}(1-\theta)}\right]^{\frac{1}{\widetilde{\widetilde{p}}}} \|F\|_{T_w^{p,q}(\mathbb{R}^{n+1}_+)},
\end{aligned}$$

which, combined with (3.15) and (3.15), shows that

$$\mathrm{J}_2 \lesssim \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \max\left\{\left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}},\ \left(\frac{\widetilde{\widetilde{p}}}{p_1}\right)^{\frac{p_1-\widetilde{\widetilde{p}}}{p_1\widetilde{\widetilde{p}}}}\left[\frac{\alpha}{\widetilde{\widetilde{p}}(1-\theta)}\right]^{\frac{1}{\widetilde{\widetilde{p}}}}\right\} \|F\|_{T_w^{p,q}(\mathbb{R}^{n+1}_+)}. \tag{3.15}$$

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Thus, combining (3.5), (3.7), (3.8), (3.9), (3.10), (3.14) and (3.15), we conclude that

$$\|F\|_{(T^{p_0}(\mathbb{R}^{n+1}_+,), T^{p_1}(\mathbb{R}^{n+1}_+,))_{\theta,q}} \lesssim \|F\|_{T^{p,q}(\mathbb{R}^{n+1}_+)},$$

which shows that (3.4) holds true in Case (i).

Case (ii) $q = \infty$. In this case, let $F_0 := F\chi_{\widehat{O}_{\frac{1}{3}}}$ and $F_1 := F\chi_{\mathbb{R}^{n+1}_+ \setminus \widehat{O}_{\frac{1}{3}}}$. By an argument similar to that used in the proof of Lemma 2.11, we see that $F_0 \in T^{p_0}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+)$ and $F_1 \in T^{p_1}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+)$. Thus, $F = F_0 + F_1 \in T^{p_0}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+) + T^{p_1}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+)$. Moreover, by the definition of K-functionals, we know that

$$\begin{aligned}
&\|F\|_{(T^{p_0}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+), T^{p_1}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+))_{\theta,\infty}} \\
&\quad = \sup_{t\in(0,\infty)} t^{-\theta} K\left(t, F;\ T^{p_0}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+),\ T^{p_1}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+)\right) \\
&\quad \le \sup_{t\in(0,\infty)} t^{-\theta} \|F_0\|_{T^{p_0}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+)} + \sup_{t\in(0,\infty)} t^{1-\theta} \|F_1\|_{T^{p_1}_{w,\frac{1}{3}}(\mathbb{R}^{n+1}_+)} =: A + B.
\end{aligned} \tag{3.16}$$

For A, by Lemma 2.3(i), [37, Proposition 1.4.5(4)], Lemmas 2.9(i) and 2.10(i), and the fact that $\frac{\theta}{\alpha} = \frac{1}{p_0} - \frac{1}{p}$, we find that

$$\begin{aligned}
A &= \sup_{t\in(0,\infty)} t^{-\theta} \left\{ \int_0^\infty \left[\left(\mathcal{A}^{(\frac{1}{3})}(F_0) \right)^*_w (s) \right]^{p_0} ds \right\}^{\frac{1}{p_0}} \\
&\le \sup_{t\in(0,\infty)} t^{-\theta} \left\{ \int_0^{t^\alpha} \left[\left(\mathcal{A}(F) \right)^*_w (s)\, s^{\frac{1}{p}} \right]^{p_0} s^{-\frac{p_0}{p}}\, ds \right\}^{\frac{1}{p_0}} \\
&= \|F\|_{T^{p,\infty}_w(\mathbb{R}^{n+1}_+)} \sup_{t\in(0,\infty)} t^{\alpha(\frac{1}{p_0} - \frac{1}{p}) - \theta} \le \|F\|_{T^{p,\infty}_w(\mathbb{R}^{n+1}_+)}.
\end{aligned}$$

The estimates for B are similar to those for J in (3.9) with minor modifications, the details being omitted. This, combined with (3.16) and (3.17), shows that

$$\|F\|_{(T^{p_0}_w(\mathbb{R}^{n+1}_+), T^{p_1}_w(\mathbb{R}^{n+1}_+))_{\theta,\infty}} \lesssim \|F\|_{T^{p,\infty}_w(\mathbb{R}^{n+1}_+)},$$

which immediately implies that (3.4) holds true in Case (ii) and hence completes the proof of Proposition 3.1. □

We now turn to the proof of Theorem 1.6.

Proof of Theorem 1.6: Let $\widetilde{p}_0 \in (0, \min\{1, p_0\})$ and $\widetilde{p}_1 \in (\max\{p_1, 1\}, \infty)$ such that $w \in A_{\widetilde{p}_1}(\mathbb{R}^n)$. By Proposition 3.1, we know that there exist $\theta_0, \theta_1 \in (0, 1)$, respectively, satisfying $\frac{1}{p_0} = \frac{1-\theta_0}{\widetilde{p}_0} + \frac{\theta_0}{\widetilde{p}_1}$ and $\frac{1}{p_1} = \frac{1-\theta_1}{\widetilde{p}_0} + \frac{\theta_1}{\widetilde{p}_1}$ such that

$$\left(T^{\widetilde{p}_0}_w(\mathbb{R}^{n+1}_+),\ T^{\widetilde{p}_1}_w(\mathbb{R}^{n+1}_+) \right)_{\theta_0, p_0} = T^{p_0}_w(\mathbb{R}^{n+1}_+)$$

and

$$\left(T^{\widetilde{p}_0}_w(\mathbb{R}^{n+1}_+),\ T^{\widetilde{p}_1}_w(\mathbb{R}^{n+1}_+) \right)_{\theta_0, p_1} = T^{p_1}_w(\mathbb{R}^{n+1}_+).$$

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Moreover, by the reiteration theorem (see Lemma 2.13), we know that

$$\left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta,q} = \left(T_w^{\widetilde{p}_0}(\mathbb{R}_+^{n+1}),\, T_w^{\widetilde{p}_1}(\mathbb{R}_+^{n+1})\right)_{\lambda,q}, \tag{3.17}$$

where $\lambda :=(1-\theta)\theta_0 + \theta\theta_1$. Using an elementary calculation, we see that

$$\frac{1-\lambda}{\widetilde{p}_0} + \frac{\lambda}{\widetilde{p}_1} = \frac{1-(1-\theta)\theta_0 - \theta\theta_1}{\widetilde{p}_0} + \frac{(1-\theta)\theta_0 + \theta\theta_1}{\widetilde{p}_1} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p},$$

which, combined with (3.17), implies that

$$\left(T_w^{p_0}(\mathbb{R}_+^{n+1}),\, T_w^{p_1}(\mathbb{R}_+^{n+1})\right)_{\theta_0,p_1} = T_w^{p}(\mathbb{R}_+^{n+1}).$$

This finishes the proof of Theorem 1.6. $\square$

We now turn to the proof of Theorem 1.7. To this end, we first recall some facts from [34]. Let $(\mathbb{X}_0, \mathbb{X}_1)$ and $(\mathbb{Y}_0, \mathbb{Y}_1)$ be two compatible pairs of quasi-Banach spaces. The pair $(\mathbb{Y}_0, \mathbb{Y}_1)$ is called a *retract* of $(\mathbb{X}_0, \mathbb{X}_1)$ if, for $i \in \{0, 1\}$, there exist two bounded linear operators E, bounded from $\mathbb{Y}_i$ to $\mathbb{X}_i$, and R, bounded from $\mathbb{X}_i$ to $\mathbb{Y}_i$, such that $R\circ E = I$ on $\mathbb{Y}_i$.

Kalton et al. [34, Lemma 7.11] proved the following theorem on the real interpolation of the quasi-Banach pair.

Theorem 3.2: [34] Let $(\mathbb{X}_0, \mathbb{X}_1)$ and $(\mathbb{Y}_0, \mathbb{Y}_1)$ be two compatible pairs of quasi-Banach spaces. Assume that $(\mathbb{Y}_0, \mathbb{Y}_1)$ is a retract of $(\mathbb{X}_0, \mathbb{X}_1)$. Then, for all $\theta \in (0, 1)$ and $q \in (0, \infty]$, $(\mathbb{Y}_0, \mathbb{Y}_1)_{\theta,q} = R\left((\mathbb{X}_0, \mathbb{X}_1)_{\theta,q}\right)$.

With the help of Theorems 1.6 and 3.2, we now prove Theorem 1.7.

Proof of Theorem 1.7: To prove this theorem, we first show that $(H_w^{p_0}(\mathbb{R}^n), H_w^{p_1}(\mathbb{R}^n))$ is a retract of $(T_w^{p_0}(\mathbb{R}_+^{n+1}), T_w^{p_1}(\mathbb{R}_+^{n+1}))$. Indeed, let ϕ be as in (1.10). For all $f \in L^2(\mathbb{R}^n)$, $F \in T_w^2(\mathbb{R}_+^{n+1})$ and $(x, t) \in \mathbb{R}_+^{n+1}$, let

$$Q_\phi(f)(x,t) := (\phi_t * f)(x) \tag{3.18}$$

and

$$\Pi_\phi(F)(x) := \int_0^\infty \phi_t * F(x,t)\,\frac{dt}{t}. \tag{3.19}$$

From [28, Proposition 5.2], it follows that, for all $p \in (0, 1]$ and $g \in H_w^{p,\infty}(\mathbb{R}^n)$, g is a distribution vanishes weakly at infinity. This, together with [21, p.50, Theorem 1.64], implies that the following Calderón reproducing formula holds true, namely,

$$g = \int_0^\infty \phi_t * \phi_t * g\,\frac{dt}{t} \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{3.20}$$

which, combined with (3.18) and (3.19), immediately shows that, for all $0 < p_0 < p_1 < \infty$ and $i \in \{0, 1\}$,

$$\Pi_\phi \circ Q_\phi = I \ \text{ on } H_w^{p_i}(\mathbb{R}^n).$$

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Thus, $(H_w^{p_0}(\mathbb{R}^n), H_w^{p_1}(\mathbb{R}^n))$ is a retract of $(T_w^{p_0}(\mathbb{R}_+^{n+1}), T_w^{p_1}(\mathbb{R}_+^{n+1}))$. By Theorems 1.6 and 3.2, we know that

$$(H_w^{p_0}(\mathbb{R}^n), H_w^{p_1}(\mathbb{R}^n))_{\theta,\infty} = \Pi_\phi\left((T_w^{p_0}(\mathbb{R}_+^{n+1}), T_w^{p_1}(\mathbb{R}_+^{n+1}))_{\theta,\infty}\right) = \Pi_\phi\left(T_w^{p,\infty}(\mathbb{R}_+^{n+1})\right).$$

To finish the proof of Theorem 1.7, we only need to show that

$$\Pi_\phi\left(T_w^{p,\infty}(\mathbb{R}_+^{n+1})\right) = H_w^{p,\infty}(\mathbb{R}^n).$$

Indeed, for any $f \in H_w^{p,\infty}(\mathbb{R}^n)$, by the square function characterization of the weighted weak Hardy space (see, e.g. [28]), we see that $Q_\phi(f) \in T_w^{p,\infty}(\mathbb{R}_+^{n+1})$, which, together with (3.20), implies that $f = \Pi_\phi \circ Q_\phi(f) \in \Pi_\phi(T_w^{p,\infty}(\mathbb{R}_+^{n+1}))$ and hence

$$H_w^{p,\infty}(\mathbb{R}^n) \subset \Pi_\phi(T_w^{p,\infty}(\mathbb{R}_+^{n+1})).$$

For the converse inclusion, let $F \in T_w^{p,\infty}(\mathbb{R}_+^{n+1})$. Since Π_ϕ is bounded from $T_w^{p_i}(\mathbb{R}_+^{n+1})$ to $H_w^{p_i}(\mathbb{R}^n)$ for $i \in \{0, 1\}$, we know that Π_ϕ is bounded from $T_w^{p,\infty}(\mathbb{R}_+^{n+1})$ to

$$(H_w^{p_0}, H_w^{p_1})_{\theta,\infty} \subset H_w^{p_0}(\mathbb{R}^n) + H_w^{p_1}(\mathbb{R}^n).$$

This shows that $\Pi_\phi(F) \in \mathcal{S}'(\mathbb{R}^n)$. Moreover, by using the weak atomic decomposition of the weak tent space $T_w^{p,\infty}(\mathbb{R}_+^{n+1})$ (see Lemma 2.4), we know that $F = \sum_{i\in\mathbb{Z}, j\in\mathbb{N}} \lambda_{i,j} A_{i,j}$ almost everywhere in $\mathbb{R}_+^{n+1}$, where $\{\lambda_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}} \subset \mathbb{C}$ satisfies

$$\sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{N}} |\lambda_{i,j}|^p\right\}^{\frac{1}{p}} \lesssim \|F\|_{T_w^{p,\infty}(\mathbb{R}_+^{n+1})}$$

and $\{A_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$ are the $T_w^p(\mathbb{R}_+^{n+1})$ tent atoms as in [38]. Moreover, from the proof of [38, Lemma 3.14], we deduce that for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\Pi_\phi(A_{i,j})$ is an $H_w^p(\mathbb{R}^n)$ atom up to a harmless constant multiple. Thus, by the weak atomic decomposition of the weighted weak Hardy space $H_w^{p,\infty}(\mathbb{R}^n)$ (see, e.g. [28, Theorem 2.15]), we conclude that $\Pi_\phi(F) \in H_w^{p,\infty}(\mathbb{R}^n)$, which implies that

$$\Pi_\phi(T_w^{p,\infty}(\mathbb{R}_+^{n+1})) \subset H_w^{p,\infty}(\mathbb{R}^n)$$

and hence completes the proof of Theorem 1.7. □

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