J Geom Anal (2018) 28:2961–2983
https://doi.org/10.1007/s12220-017-9943-8

Intrinsic Structures of Certain Musielak–Orlicz Hardy Spaces

Jun Cao[1] · Liguang Liu[2] · Dachun Yang[3] · Wen Yuan[3]

Received: 18 July 2017 / Published online: 12 October 2017

Abstract For any $p \in (0, 1]$, let $H^{\Phi_p}(\mathbb{R}^n)$ be the Musielak–Orlicz Hardy space associated with the Musielak–Orlicz growth function Φ_p, defined by setting, for any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$,

$$\Phi_p(x, t) := \begin{cases} \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}} & \text{when } n(1/p-1) \notin \mathbb{N} \cup \{0\}, \\ \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N} \cup \{0\}, \end{cases}$$

which is the sharp target space of the bilinear decomposition of the product of the Hardy space $H^p(\mathbb{R}^n)$ and its dual. Moreover, $H^{\Phi_1}(\mathbb{R}^n)$ is the prototype appearing in the real-variable theory of general Musielak–Orlicz Hardy spaces. In this article,

✉ Liguang Liu
liuliguang@ruc.edu.cn

Jun Cao
caojun1860@zjut.edu.cn

Dachun Yang
dcyang@bnu.edu.cn

Wen Yuan
wenyuan@bnu.edu.cn

1 Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, People's Republic of China

2 Department of Mathematics, School of Information, Renmin University of China, Beijing 100872, People's Republic of China

3 Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China

the authors find a new structure of the space $H^{\Phi_p}(\mathbb{R}^n)$ by showing that, for any $p \in (0, 1]$, $H^{\Phi_p}(\mathbb{R}^n) = H^{\phi_0}(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ and, for any $p \in (0, 1)$, $H^{\Phi_p}(\mathbb{R}^n) = H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$, where $H^1(\mathbb{R}^n)$ denotes the classical real Hardy space, $H^{\phi_0}(\mathbb{R}^n)$ the Orlicz–Hardy space associated with the Orlicz function $\phi_0(t) := t/\log(e+t)$ for any $t \in [0, \infty)$, and $H^p_{W_p}(\mathbb{R}^n)$ the weighted Hardy space associated with certain weight function $W_p(x)$ that is comparable to $\Phi_p(x, 1)$ for any $x \in \mathbb{R}^n$. As an application, the authors further establish an interpolation theorem of quasilinear operators based on this new structure.

Keywords Hardy space · Musielak–Orlicz function · Muckenhoupt weight · Interpolation · Atom · Calderón–Zygmund decomposition

Mathematics Subject Classification Primary 42B30; Secondary 42B35 · 46E30 · 46B70

1 Introduction

The real-variable theory of the classical real Hardy spaces on the Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss [18] and later by Fefferman and Stein [7]. For any $p \in (0, \infty)$, the classical real Hardy space $H^p(\mathbb{R}^n)$ consists of all Schwartz distributions f such that

$$f^+ := \sup_{t\in(0,\infty)} |\phi_t * f| \in L^p(\mathbb{R}^n),$$

where ϕ is a function in the Schwartz class, $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$, and $\phi_t(\cdot) := t^{-n}\phi(t^{-1}\cdot)$. As one of the most important function spaces in harmonic analysis, $H^p(\mathbb{R}^n)$ has many applications in various fields of mathematics (see, for example, [6,7,16,17] and references therein). Later, the theory of $H^p(\mathbb{R}^n)$ was extended to the setting of the weighted Hardy spaces by García-Cuerva [8] and Strömberg, Torchinsky [20], and also to the setting of the Orlicz–Hardy space by Strömberg [19] and Janson [10]. Both of the latter two spaces can be viewed as special cases of more general Musielak–Orlicz Hardy spaces which were first introduced by Ky [13] (see also [23] for a complete survey of the real-variable theory of Musielak–Orlicz Hardy spaces).

The main aim of this article is to try to understand some intrinsic structure of the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ associated with the Musielak–Orlicz growth function

$$\Phi_p(x, t) := \begin{cases} \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}} & \text{when } n(1/p-1) \notin \mathbb{N}\cup\{0\}, \\ \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N}\cup\{0\}, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $t \in [0, \infty)$ (see [2,4,13]). The precise definition of $H^{\Phi_p}(\mathbb{R}^n)$ is as follows. In what follows, we use $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ to denote, respectively, the space of all Schwartz functions, equipped with the classical well-known topology, and its dual space, equipped with the weak-$*$ topology.

Definition 1.1 Let $p \in (0, 1]$ and Φ_p be as in (1.1).

(i) For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $m \in \mathbb{N}$, the *non-tangential grand maximal function* f_m^* of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\varphi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t,\, t\in(0,\infty)} |f * \varphi_t(y)|, \tag{1.2}$$

where $\varphi_t(\cdot) := t^{-n}\varphi(t^{-1}\cdot)$ for any $t \in (0, \infty)$, and

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \le m+1} \sup_{x\in\mathbb{R}^n} (1 + |x|)^{(m+2)(n+1)} |D^\alpha \varphi(x)| \le 1\right\}.$$

(ii) The *Musielak–Orlicz Lebesgue space* $L^{\Phi_p}(\mathbb{R}^n)$ is defined to be the space of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\Phi_p}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi_p(x, |f(x)|/\lambda)\, dx \le 1\right\} < \infty.$$

With m being the largest integer not greater than $n(1/p-1)$, the *Musielak–Orlicz Hardy space* $H^{\Phi_p}(\mathbb{R}^n)$ is defined to be the space of all Schwartz distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^{\Phi_p}(\mathbb{R}^n)} := \|f_m^*\|_{L^{\Phi_p}(\mathbb{R}^n)} < \infty,$$

where f_m^* is as in (1.2). In what follows, we simply write f_m^* as f^*.

Notice that the non-tangential grand maximal function f^* in Definition 1.1 can also be used to characterize the Hardy space $H^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Indeed, one has $\|f\|_{H^p(\mathbb{R}^n)} \sim \|f^*\|_{L^p(\mathbb{R}^n)}$ for any $p \in (0, 1]$ and any $f \in \mathcal{S}'(\mathbb{R}^n)$ with the equivalent positive constants independent of f.

One of the main motivations for us to study the aforementioned Musielak–Orlicz Hardy spaces $H^{\Phi_p}(\mathbb{R}^n)$ for any $p \in (0, 1]$ comes from the bilinear decomposition of the product of functions in Hardy space $H^p(\mathbb{R}^n)$ and its dual space. When $p = 1$, Bonami et al. [4] established the following sharp bilinear decomposition of the product of the Hardy space $H^1(\mathbb{R}^n)$ and its dual space $\mathrm{BMO}(\mathbb{R}^n)$

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n), \tag{1.3}$$

where $H^{\log}(\mathbb{R}^n)$ is just the Musielak–Orlicz Hardy space $H^{\Phi_1}(\mathbb{R}^n)$. The precise meaning of (1.3) is that there exist two bounded bilinear operators $S : H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T : H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \to H^{\log}(\mathbb{R}^n)$ such that the product, defined in the sense of $\mathcal{S}'(\mathbb{R}^n)$, of any $f \in H^1(\mathbb{R}^n)$ and $g \in \mathrm{BMO}(\mathbb{R}^n)$, denoted by $f \times g$, can be written as

$$f \times g = S(f,\ g) + T(f,\ g).$$

Furthermore, there exists a positive constant C such that, for any $(f, g) \in H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$,

$$\|S(f, g)\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{H^1(\mathbb{R}^n)}\|g\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

and

$$\|T(f, g)\|_{H^{\log}(\mathbb{R}^n)} \le C\|f\|_{H^1(\mathbb{R}^n)}\left[\|g\|_{\mathrm{BMO}(\mathbb{R}^n)} + \left|\fint_{B(\vec{0}_n,1)} g(x)\,dx\right|\right];$$

here and hereafter, $\vec{0}_n$ denotes the *origin* of $\mathbb{R}^n$ and $B(\vec{0}_n, 1)$ the *open unit ball* of $\mathbb{R}^n$ at $\vec{0}_n$. Moreover, it was proved in [4] that the space $H^{\log}(\mathbb{R}^n)$ is optimal in the sense that any vector space $\mathcal{Y} \subset H^{\log}(\mathbb{R}^n)$ adapted to the bilinear decomposition (1.3) satisfies $\mathcal{Y}^* = (H^{\log}(\mathbb{R}^n))^*$. The above result of [4] also answers a conjecture raised in [5]. Based on this result, Ky [13] further developed a general real-variable theory of Musielak–Orlicz Hardy spaces (see also [23] for a complete survey). Thus, the space $H^{\log}(\mathbb{R}^n)$ plays a role as a prototype in the study of the real-variable theory of general Musielak–Orlicz Hardy spaces.

Recently, the result of [4] was extended to the case $p \in (0, 1)$ in [2]. Indeed, when $p \in (0, 1)$, it was proved in [2] that $H^{\Phi_p}(\mathbb{R}^n)$ is the optimal function space that is adapted to the bilinear decomposition of the product of elements from the Hardy space $H^p(\mathbb{R}^n)$ and its dual space $\mathfrak{C}_{1/p-1}(\mathbb{R}^n)$

$$H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n). \tag{1.4}$$

The precise meaning of (1.4) is as follows: there exist two bounded bilinear operators

$$S:\ H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \quad \text{and}$$
$$T:\ H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n) \to H^{\Phi_p}(\mathbb{R}^n)$$

such that the product, defined in the sense of $\mathcal{S}'(\mathbb{R}^n)$, of any $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_{1/p-1}(\mathbb{R}^n)$, denoted by $f \times g$, can be written as

$$f \times g = S(f,\ g) + T(f,\ g)$$

and, furthermore, there exists a positive constant C such that, for any $(f, g) \in H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n)$,

$$\|S(f, g)\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{H^p(\mathbb{R}^n)}\|g\|_{\mathfrak{C}_{1/p-1}(\mathbb{R}^n)}$$

and

$$\|T(f,g)\|_{H^{\Phi_p}(\mathbb{R}^n)} \le C\|f\|_{H^p(\mathbb{R}^n)}\left[\|g\|_{\mathfrak{C}_{1/p-1}(\mathbb{R}^n)} + \fint_{B(\vec{0}_n,1)} |g(x)|\,dx\right].$$

Also, the target space $H^{\Phi_p}(\mathbb{R}^n)$ is sharp in the sense that any vector space $\mathcal{Y} \subset H^{\Phi_p}(\mathbb{R}^n)$ adapted to the bilinear decomposition (1.4) satisfies $\mathcal{Y}^* = (H^{\Phi_p}(\mathbb{R}^n))^*$ (see [2]).

It should be mentioned that the study of the bilinear decomposition of the product of elements from the Hardy space $H^p(\mathbb{R}^n)$ and its dual space can help us improve the boundedness of many nonlinear qualities such as the div-curl product and the weak Jacobian (see [3,4,6]) as well as the endpoint boundedness of commutators (see [12,15]).

Motivated by the aforementioned results of [2,4,13], it is interesting to provide a better understanding of the structure of the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$. It is easy to observe that, for any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, one has

$$\Phi_p(x,\ t) \sim \begin{cases} \dfrac{t}{1+[t(1+|x|)^n]^{1-p}} & \text{when } n(1/p-1) \notin \mathbb{N}\cup\{0\}, \\ \dfrac{t}{1+[t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N}, \\ \dfrac{t}{\log(e+t)+\log(e+|x|)} & \text{when } p=1 \end{cases} \tag{1.5}$$

with the equivalent positive constants independent of x and t. Based on (1.5), for any $x \in \mathbb{R}^n$ and $t \in [0,\ \infty)$, we consider the Orlicz function

$$\phi_0(t) := \frac{t}{\log(e+t)} \tag{1.6}$$

and the weight function

$$W_p(x) := \begin{cases} \dfrac{1}{(1+|x|)^{n(1-p)}} & \text{when } n(1/p-1) \notin \mathbb{N}\cup\{0\}, \\ \dfrac{1}{(1+|x|)^{n(1-p)}\left[\log(e+|x|)\right]^p} & \text{when } n(1/p-1) \in \mathbb{N}, \\ \dfrac{1}{\log(e+|x|)} & \text{when } p=1. \end{cases} \tag{1.7}$$

Let $H^{\phi_0}(\mathbb{R}^n)$ and $H^p_{W_p}(\mathbb{R}^n)$ be, respectively, the *Orlicz–Hardy space* associated with ϕ_0 and the *weighted Hardy space* associated with W_p, which are defined in the same way as Definition 1.1(ii), but with $\|f\|_{L^{\Phi_p}(\mathbb{R}^n)}$ therein replaced, respectively, by

$$\|f\|_{L^{\phi_0}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \phi_0(|f(x)|/\lambda)\,dx \le 1\right\}$$

and

$$\|f\|_{L^p_{W_p}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p\, W_p(x)\, dx\right\}^{1/p}.$$

We refer the reader to [8,10,19,20] for more properties on general Orlicz–Hardy spaces and weighted Hardy spaces.

Recall that, in [1], for any two quasi-Banach spaces A_0 and A_1, the pair (A_0, A_1) is said to be *compatible* if there exists a Hausdorff space $\mathbb{X}$ such that $A_0 \subset \mathbb{X}$ and $A_1 \subset \mathbb{X}$. For any compatible pair (A_0, A_1) of quasi-Banach spaces, the *sum space* $A_0 + A_1$ is defined by setting

$$A_0 + A_1 := \{a \in \mathbb{X} : \ \exists\, a_0 \in A_0 \text{ and } a_1 \in A_1 \text{ such that } \ a = a_0 + a_1\} \tag{1.8}$$

equipped with the *quasi-norm*

$$\|a\|_{A_0+A_1} := \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : \ a = a_0 + a_1,\ a_0 \in A_0 \text{ and } a_1 \in A_1\right\}.$$

In what follows, we use $H^{\phi_0}(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ (resp., $H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$) to denote the sum space, defined as in (1.8), with $\mathbb{X} := \mathcal{S}'(\mathbb{R}^n)$, $A_0 := H^{\phi_0}(\mathbb{R}^n)$ (resp., $A_0 := H^1(\mathbb{R}^n)$) and $A_1 := H^p_{W_p}(\mathbb{R}^n)$.

The main result of this article is the following representation of $H^{\Phi_p}(\mathbb{R}^n)$ for any $p \in (0, 1]$ as the sum of a (Orlicz–)Hardy space and a weighted Hardy space.

Theorem 1.2 *Let $p \in (0, 1]$. Define Φ_p, ϕ_0, and W_p as in* (1.1)*,* (1.6)*, and* (1.7)*, respectively. Then*

(i) the space $H^{\Phi_p}(\mathbb{R}^n)$ and $H^{\phi_0}(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ coincide with equivalent quasi-norms;
(ii) for any $p \in (0, 1)$, the space $H^{\Phi_p}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

The new structure of the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ established in Theorem 1.2 enables us to reduce the study of many properties of $H^{\Phi_p}(\mathbb{R}^n)$ to the corresponding ones of the Orlicz–Hardy space $H^{\phi_0}(\mathbb{R}^n)$ when $p \in (0, 1]$ (or the Hardy space $H^1(\mathbb{R}^n)$ when $p \in (0, 1)$) and the weighted Hardy space $H^p_{W_p}(\mathbb{R}^n)$, where the latter three kinds of Hardy-type spaces are well studied in various literatures; see, for example, [7,8,10,16,19,20] and references therein. A major task in the proof of Theorem 1.2 is decomposing every $f \in H^{\Phi_p}(\mathbb{R}^n)$ into the sum of two parts, which belongs to the desired sum space. We obtain this decomposition using the atomic characterization of $H^{\Phi_1}(\mathbb{R}^n)$ when $p = 1$ and the Calderón–Zygmund decomposition of $H^{\Phi_p}(\mathbb{R}^n)$ when $p \in (0, 1)$. The main trick is that we use different selection principles in different decompositions and these selection principles are based on the norm estimates for the characteristic functions of the balls, which are established in Sect. 2.

As an application of Theorem 1.2, we consider a concrete problem of the interpolation of quasilinear operators. Recall that the following definition of quasilinear operators is from [9]. Let T be an operator defined on some quasi-Banach space A and taking values in the set of all complex-valued finite almost everywhere measurable functions on $\mathbb{R}^n$. Such an operator T is said to be *quasilinear* if there exists a positive constant C such that, for any $f, g \in A$ and $\lambda \in \mathbb{C}$,

$$|T(f)| = |\lambda|\,|f| \quad \text{and} \quad |T(f+g)| \le C\,(|f| + |g|)\,.$$

Theorem 1.3 *Let $p \in (0, 1]$. Let Φ_p, ϕ_0, and W_p be as in (1.1), (1.6), and (1.7), respectively. Assume that T is a quasilinear operator bounded on $H^p_{W_p}(\mathbb{R}^n)$. Then*

(i) if T is bounded on $H^{\phi_0}(\mathbb{R}^n)$, then T is bounded on $H^{\Phi_p}(\mathbb{R}^n)$;
(ii) if $p \in (0, 1)$ and T is bounded on $H^1(\mathbb{R}^n)$, then T is bounded on $H^{\Phi_p}(\mathbb{R}^n)$.

This article is organized as follows. In Sect. 2, we establish several technical lemmas which are needed in the proof of Theorems 1.2 and 1.3. Section 3 is devoted to the proof of Theorem 1.2. Finally, using Theorem 1.2, we prove Theorem 1.3 in Sect. 4.

At the end of this section, we make some conventions on the notation. Throughout this article, let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, denote by $B(x, r)$ the ball with center x and radius r, that is $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. For any ball $B \subset \mathbb{R}^n$, we always denote by c_B its center and by r_B its radius and, for any $\lambda \in (0, \infty)$, by λB the ball with center c_B and radius λr_B. For any set $E \subset \mathbb{R}^n$, χ_E denotes its *characteristic function*. We use C to denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $s \in \mathbb{R}$, let $\lfloor s \rfloor$ be the largest integer not greater than s. We always use α to denote a multi-index $(\alpha_1, \ldots, \alpha_n)$ with every α_i being a non-negative integer.

2 Several Technical Lemmas

In this section, we present several technical lemmas which serve as preparations to prove Theorems 1.2 and 1.3. To this end, we begin with recalling some notions used in [13].

Definition 2.1 For any $p \in (0, \infty)$, an *Orlicz function* ϕ (which means that ϕ is nondecreasing and satisfies $\phi(0) = 0$, $\phi(t) > 0$ for $t \in (0, \infty)$ and $\lim_{t\to\infty} \phi(t) = \infty$) is said to be of *positive lower* (resp., *upper*) *type* p if there exists a positive constant C such that, for any $t \in [0, \infty)$ and $s \in (0, 1]$ (resp., $s \in [1, \infty)$),

$$\phi(st) \le Cs^p \phi(t).$$

Definition 2.2 A function $\phi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a *Musielak–Orlicz function* if the function $\phi(x, \cdot) : [0, \infty) \to [0, \infty)$ is an Orlicz function for any $x \in \mathbb{R}^n$, and the function $\phi(\cdot, t)$ is a measurable function for any $t \in [0, \infty)$.

Definition 2.3 Let ϕ be a Musielak–Orlicz function. For any given $p \in (0, \infty)$, the function ϕ is said to be of *positive uniformly lower* (resp., *upper*) *type* p if there exists a positive constant C such that, for any $x \in \mathbb{R}^n$, $t \in [0, \infty)$, and $s \in (0, 1]$ (resp., $s \in [1, \infty)$),

$$\phi(x, st) \leq Cs^p\phi(x, t).$$

Definition 2.4 Let ϕ be a Musielak–Orlicz function and $q \in [1, \infty)$. The function ϕ is said to satisfy the *uniformly Muckenhoupt* $\mathbb{A}_q(\mathbb{R}^n)$ *condition*, namely $\phi \in \mathbb{A}_q(\mathbb{R}^n)$, if

$$[\phi]_{\mathbb{A}_q(\mathbb{R}^n)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \sup\limits_{B\subset\mathbb{R}^n} \left[\frac{1}{|B|}\int_B \phi(x,t)\,dx\right]\left[\frac{1}{|B|}\int_B \{\phi(x,t)\}^{\frac{-1}{q-1}}\,dx\right]^{q-1} & \text{when } q \in (1,\infty), \\ \sup\limits_{t\in(0,\infty)} \sup\limits_{B\subset\mathbb{R}^n} \left[\frac{1}{|B|}\int_B \phi(x,t)\,dx\right]\left[\sup\limits_{x\in B}\{\phi(x,t)\}^{-1}\right] & \text{when } q = 1 \end{cases}$$

is finite, where the second suprema are taken over all balls B of $\mathbb{R}^n$. Let

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

Remark 2.5 Let $p \in (0, 1]$, Φ_p be as in (1.1), ϕ_0 as in (1.6), and W_p as in (1.7).

(i) We know (see [13] for the case $p = 1$ and [2] for the case $p \in (0, 1)$) that Φ_p is a Musielak–Orlicz function of uniformly upper 1 and of uniformly lower type p, and belongs to the uniformly Muckenhoupt weight class $\mathbb{A}_1(\mathbb{R}^n)$.

(ii) Notice that $\phi_0(t) \sim \Phi_1(0, t)$ and $W_p(x) \sim \Phi_p(x, 1)$, where the equivalent positive constants are independent of x and t. From these and (i) of this remark, it follows immediately that ϕ_0 is of upper and lower types 1, and W_p belongs to the usual Muckenhoupt weight class $A_1(\mathbb{R}^n)$. In particular, there exists a positive constant C such that, for any ball B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B W_p(x)\,dx \leq C \operatorname*{essinf}_{y\in B} W_p(y). \tag{2.1}$$

For any $p \in (0, 1]$, the next lemma provides an $L^{\Phi_p}(\mathbb{R}^n)$-norm estimate for χ_B, which was proved in [13] when $p = 1$ and [2] when $p \in (0, 1)$.

Lemma 2.6 *Let $p \in (0, 1]$, $\alpha = 1/p - 1$, and $B = B(c_B, r_B)$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$.*

(i) If $p = 1$, then

$$\|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)} \sim \frac{|B|}{\log(e + 1/|B|) + \sup_{x\in B}\left[\log(e + |x|)\right]} \sim \frac{|B|}{|\log r_B| + \log(e + |c_B|)},$$

where the equivalent constants are positive and independent of B.

(ii) If $p \in (0,\ 1)$*, then*

$$\|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)} \sim \Psi_\alpha(B)|B|,$$

where

$$\Psi_\alpha(B) := \begin{cases} \min\left\{1, \left(\dfrac{r_B}{1+|c_B|}\right)^{n\alpha}\right\} & \text{when } n\alpha \notin \mathbb{N}, \\ \min\left\{1, \left(\dfrac{r_B}{1+|c_B|}\right)^{n\alpha}\right\} \dfrac{1}{\log(1+|c_B|+r_B)} & \text{when } n\alpha \in \mathbb{N}, \end{cases}$$

and the equivalent constants are positive and independent of B.

For any $p \in (0,\ 1]$ and any ball $B \subset \mathbb{R}^n$, we still need to consider the Orlicz norm $\|\cdot\|_{L^{\phi_0}(\mathbb{R}^n)}$ and the weighted Lebesgue norm $\|\cdot\|_{L^p_{W_p}(\mathbb{R}^n)}$ estimates of the characteristic function χ_B.

Lemma 2.7 *Let* ϕ_1 *be as in* (1.1)*,* W_1 *as in* (1.7)*, and* ϕ_0 *as in* (1.6)*. Define* ${\phi_0}^{-1}$ *to be the inverse function of* ϕ_0*. For any* $t \in (0,\ \infty)$*, let*

$$\rho(t) := \frac{t^{-1}}{{\phi_0}^{-1}(t^{-1})}. \tag{2.2}$$

Then, for any ball $B \subset \mathbb{R}^n$*,*

(i) $\|\chi_B\|_{L^{\phi_0}(\mathbb{R}^n)} = |B|\rho(|B|) \sim \frac{|B|}{\log(e+1/|B|)}$;

(ii) $\|\chi_B\|_{L^1_{W_1}(\mathbb{R}^n)} \sim \frac{|B|}{\sup_{x\in B}[\log(e+|x|)]}$;

(iii) $\|\chi_B\|^{-1}_{L^{\Phi_1}(\mathbb{R}^n)} \sim \|\chi_B\|^{-1}_{L^{\phi_0}(\mathbb{R}^n)} + \|\chi_B\|^{-1}_{L^1_{W_1}(\mathbb{R}^n)}$,

where the equivalent constants in (i), (ii), and (iii) are positive and independent of B.

Proof We first prove (i). Recall that the equivalence $\|\chi_B\|_{L^{\phi_0}(\mathbb{R}^n)} \sim \frac{|B|}{\log(e+1/|B|)}$ was established in [22, Lemma 7.13]. Thus, to finish the proof of (i), it remains to establish the first equality of (i). Indeed, from the definition of ρ, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_0\left(\frac{\chi_B(x)}{|B|\rho(|B|)}\right) dx &= \int_{\mathbb{R}^n} \phi_0\left({\phi_0}^{-1}\left(|B|^{-1}\right)\chi_B(x)\right) dx \\ &= \int_B \phi_0\left({\phi_0}^{-1}\left(|B|^{-1}\right)\right) dx = 1,\end{aligned}$$

which immediately implies the first equality of (i) and hence (i) holds true.

We now prove (ii). By the fact that W_1 belongs to the Muckenhoupt weight class $A_1(\mathbb{R}^n)$ (see (2.1)), we know that

$$\|\chi_B\|_{L^1_{W_1}(\mathbb{R}^n)} = |B|\left[\frac{1}{|B|}\int_B W_1(x)\,dx\right] \sim |B| \inf_{x\in B} W_1(x)$$

$$\sim |B| \inf_{x\in B}\left[\frac{1}{\log(e+|x|)}\right] \sim \frac{|B|}{\sup_{x\in B}\log(e+|x|)},$$

which implies that (ii) holds true.

Finally, (iii) follows directly from (i) and (ii) of this lemma and Lemma 2.6(i). This finishes the proof of Lemma 2.7. □

Lemma 2.8 *Let $p \in (0,\ 1)$, Φ_p and W_p be, respectively, as in (1.1) and (1.7). Then, for any ball $B \subset \mathbb{R}^n$,*

$$\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)},$$

where the equivalent constants are positive and independent of B.

Proof Denote by c_B the center of B and by r_B its radius. By Lemma 2.6(ii), we have

$$\|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)} \sim \begin{cases} |B|\min\left\{1,\ \left(\dfrac{r_B}{1+|c_B|}\right)^{n(1/p-1)}\right\} & \text{when } n(1/p-1)\notin\mathbb{N}, \\ |B|\min\left\{1,\ \left(\dfrac{r_B}{1+|c_B|}\right)^{n(1/p-1)}\right\}\dfrac{1}{\log(e+r_B+|c_B|)} & \text{when } n(1/p-1)\in\mathbb{N}. \end{cases} \tag{2.3}$$

To estimate $\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)}$, we consider the following three cases.

Case (i) $|c_B| \geq 2r_B$. In this case, for any $x \in B$, it is easy to see that $|x| \sim |c_B|$. By this and (2.3), we conclude that, when $n(1/p-1)\notin\mathbb{N}$,

$$\begin{aligned}\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} &= \left\{\int_B \frac{1}{(1+|x|)^{n(1-p)}}\,dx\right\}^{1/p} \\ &\sim \left\{\int_B \frac{1}{(1+|c_B|)^{n(1-p)}}\,dx\right\}^{1/p} \\ &\sim \frac{|B|^{1/p}}{(1+|c_B|)^{n(1/p-1)}} \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)},\end{aligned}$$

as desired. Similarly, when $n(1/p-1)\in\mathbb{N}$, we have

$$\begin{aligned}\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} &= \left\{\int_B \frac{1}{(1+|x|)^{n(1-p)}\left[\log(e+|x|)\right]^p}\,dx\right\}^{1/p} \\ &\sim \frac{|B|^{1/p}}{(1+|c_B|)^{n(1/p-1)}\log(e+|c_B|)} \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)}.\end{aligned}$$

Case (ii) $|c_B| < 2r_B < 1$. In this case, for any $x \in B$, we have $|x| \leq |x-c_B|+|c_B| < r_B+|c_B| < 2$. Thus, whenever $n(1/p-1)$ is an integer or not, we always have

$$\inf_{x\in B} W_p(x) \sim 1.$$

From this and the fact that $W_p \in A_1(\mathbb{R}^n)$ (see (2.1)), it follows that

$$\begin{aligned}\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} &= |B|^{1/p}\left[\frac{1}{|B|}\int_B W_p(x)\,dx\right]^{1/p}\\ &\sim |B|^{1/p}\left[\inf_{x\in B} W_p(x)\right]^{1/p} \sim |B|^{1/p} \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)},\end{aligned}$$

as desired.

Case (iii) $|c_B| < 2r_B$ and $2r_B \ge 1$. In this case, for any $x \in B$, we have $|x| \le |x - c_B| + |c_B| < r_B + |c_B| < 3r_B$, so that $1 + |x| \lesssim r_B$ and hence

$$\begin{aligned}\inf_{x\in B} W_p(x) &= \begin{cases} \inf\limits_{x\in B} \dfrac{1}{(1+|x|)^{n(1-p)}} & \text{when } n(1/p-1)\notin\mathbb{N},\\ \inf\limits_{x\in B} \dfrac{1}{(1+|x|)^{n(1-p)}[\log(e+|x|)]^p} & \text{when } n(1/p-1)\in\mathbb{N}\end{cases}\\ &\sim \begin{cases} \dfrac{1}{|B|^{1-p}} & \text{when } n(1/p-1)\notin\mathbb{N},\\ \dfrac{1}{|B|^{1-p}[\log(e+r_B)]^p} & \text{when } n(1/p-1)\in\mathbb{N}.\end{cases}\end{aligned}$$

Consequently,

$$\begin{aligned}\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} &= |B|^{1/p}\left[\frac{1}{|B|}\int_B W_p(x)\,dx\right]^{1/p} \sim |B|^{1/p}\left[\inf_{x\in B} W_p(x)\right]^{1/p}\\ &\gtrsim \begin{cases} |B| & \text{when } n(1/p-1)\notin\mathbb{N},\\ \dfrac{|B|}{\log(e+r_B)} & \text{when } n(1/p-1)\in\mathbb{N}\end{cases}\\ &\sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)}.\end{aligned}$$

Also, when $n(1/p-1)\notin\mathbb{N}$, we have

$$\begin{aligned}\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} &\le \left[\int_{|x|<3r_B} \frac{1}{(1+|x|)^{n(1-p)}}\,dx\right]^{1/p}\\ &\sim \frac{|B|^{1/p}}{(1+3r_B)^{n(1/p-1)}} \sim |B| \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)}.\end{aligned}$$

Meanwhile, when $n(1/p-1)\in\mathbb{N}$, we obtain

$$\begin{aligned}\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} &\le \left[\int_{|x|<3r_B} \frac{1}{(1+|x|)^{n(1-p)}[\log(e+|x|)]^p}\,dx\right]^{1/p}\\ &= \left[\sum_{j=1}^{\infty}\int_{2^{-j}3r_B\le|x|<2^{-j+1}3r_B} \frac{1}{(1+|x|)^{n(1-p)}[\log(e+|x|)]^p}\,dx\right]^{1/p}\end{aligned}$$

$$\lesssim \left[\sum_{j=1}^{\infty} \frac{(2^{-j}r_B)^n}{(1+2^{-j}r_B)^{n(1-p)}[\log(e+2^{-j}r_B)]^p}\right]^{1/p}$$

$$\lesssim \left[\sum_{j=1}^{\infty} \frac{j^p(2^{-j}r_B)^{np}}{[\log(e+r_B)]^p}\right]^{1/p} \lesssim \frac{|B|}{\log(e+r_B)} \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)},$$

where we used the following estimate:

$$\frac{\log(e+r_B)}{\log(e+2^{-j}r_B)} \le \log(e+2^j) \lesssim j.$$

Altogether, we find that $\|\chi_B\|_{L^p_{W_p}(\mathbb{R}^n)} \sim \|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)}$ in the case $|c_B| < 2r_B$ and $2r_B \ge 1$.

Summarizing the above three cases, we conclude that (ii) holds true. This finishes the proof of Lemma 2.8. □

From Lemmas 2.6, 2.7, and 2.8, we deduce some interesting properties on the Musielak–Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ for any $p \in (0, 1]$. To be precise, we first recall the following definition of H^{Φ_p}-atoms from [13,14].

Definition 2.9 Let $p \in (0,\ 1]$, Φ_p be as in (1.1), $q \in (1, \infty]$, and $s \in \mathbb{Z}_+ \cap [\lfloor n(1/p - 1)\rfloor,\ \infty)$.

(I) For any $r \in [1, \infty]$ and any set $E \subset \mathbb{R}^n$, the *space* $L^r_{\Phi_p}(E)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that $\operatorname{supp} f \subset E$ and

$$\|f\|_{L^r_{\Phi_p}(E)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \left[\dfrac{1}{\Phi_p(E,t)} \displaystyle\int_E |f(x)|^r \Phi_p(x,t)\,dx\right]^{1/r}, & r \in [1,\infty); \\ \|f\|_{L^\infty(E)}, & r = \infty \end{cases}$$

is finite, where $\Phi_p(E,t) := \int_E \Phi_p(x,\ t)\,dx$ for any $t \in [0,\ \infty)$.

(II) A function a is called a $(\Phi_p,\ q,\ s)$-*atom* if there exists a ball $B \subset \mathbb{R}^n$ such that
  (i) $\operatorname{supp} a \subset B$;
  (ii) $\|a\|_{L^q_{\Phi_p}(B)} \le \|\chi_B\|^{-1}_{L^{\Phi_p}(\mathbb{R}^n)}$;
  (iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le s$.

(III) The *atomic Musielak–Orlicz Hardy space* $H^{\Phi_p,q,s}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_j \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$ and $\{a_j\}_j$ is a sequence of $(\Phi_p,\ q,\ s)$-atoms, respectively, associated with balls $\{B_j\}_j$, satisfying

$$\sum_j \Phi_p\left(B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{\Phi_p}(\mathbb{R}^n)}}\right) < \infty.$$

Moreover, let

$$\Lambda_{\Phi_p}(\{\lambda_j a_j\}_j) := \inf\left\{\lambda \in (0, \infty) : \sum_j \Phi_p\left(B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^{\Phi_p}(\mathbb{R}^n)}}\right) \le 1\right\}.$$

Then the *quasi-norm* of $f \in H^{\Phi_p, q, s}(\mathbb{R}^n)$ is defined by setting

$$\|f\|_{H^{\Phi_p, q, s}(\mathbb{R}^n)} := \inf\left\{\Lambda_{\Phi_p}(\{\lambda_j a_j\}_j)\right\}, \tag{2.4}$$

where the infimum is taken over all the decompositions of f as above.

The following atomic characterization of $H^{\Phi_p}(\mathbb{R}^n)$ follows from a general theory of the atomic characterization of Musielak–Orlicz Hardy spaces established in [13, Theorem 3.1].

Lemma 2.10 *Let $p \in (0, 1]$, Φ_p be as in (1.1), $q \in (1, \infty]$, and $s \in \mathbb{Z}_+ \cap [\lfloor n(1/p - 1)\rfloor, \infty)$. Then the spaces $H^{\Phi_p}(\mathbb{R}^n)$ and $H^{\Phi_p, q, s}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Remark 2.11 Let $p \in (0, 1]$, $q \in (1, \infty)$, and $s \in \mathbb{Z}_+ \cap [\lfloor n(1/p - 1)\rfloor, \infty)$. Assume that ϕ_0 and W_p are as in (1.6) and (1.7), respectively. Following Definition 2.9(II), if we replace $\Phi_p(x, t)$ therein by t^p, $t^p W_p(x)$, and $\phi_0(t)$, then we obtain the definitions of (p, q, s)*-atoms*, $(p, q, s)_{W_p}$*-atoms*, and (ϕ_0, q, s)*-atoms*, respectively. Correspondingly, we follow Definition 2.9(III) to introduce the atomic Hardy spaces $H^{p, q, s}(\mathbb{R}^n)$, $H^{p, q, s}_{W_p}(\mathbb{R}^n)$, and $H^{\phi_0, q, s}(\mathbb{R}^n)$ by replacing the quasi-norm in (2.4), respectively, by

$$\|f\|_{H^{p, q, s}(\mathbb{R}^n)} := \inf\left\{\left[\sum_{j\in\mathbb{N}} |\lambda_j|^p\right]^{1/p}\right\},$$

$$\|f\|_{H^{p, q, s}_{W_p}(\mathbb{R}^n)} := \inf\left\{\left[\sum_{j\in\mathbb{N}} |\lambda_j|^p\right]^{1/p}\right\},$$

and

$$\|f\|_{H^{\phi_0, q, s}(\mathbb{R}^n)} := \inf\left\{\Lambda_{\phi_0}(\{\lambda_j a_j\}_j)\right\},$$

where

$$\Lambda_{\phi_0}(\{\lambda_j a_j\}_j) := \inf\left\{\lambda \in (0, \infty) : \sum_j |B_j| \phi_0\left(\frac{|\lambda_j|}{\lambda |B_j| \rho(B_j))}\right) \le 1\right\}$$

with ρ as in (2.2). Then, from [13, Theorem 3.1], it also follows that

$$\begin{cases} H^p(\mathbb{R}^n) = H^{p,q,s}(\mathbb{R}^n) \\ H^p_{W_p}(\mathbb{R}^n) = H^{p,q,s}_{W_p}(\mathbb{R}^n) \\ H^{\phi_0}(\mathbb{R}^n) = H^{\phi_0,q,s}(\mathbb{R}^n) \end{cases}$$

with equivalent quasi-norms. See also [7,8,11,16,17,20,21] and references therein for more discussions on these three kinds of Hardy-type spaces.

From these and Lemmas 2.6, 2.7, and 2.8, we deduce the following proposition, which provides the basis to prove Theorem 1.2.

Proposition 2.12 *Let $p \in (0, 1]$, $q \in (1, \infty)$, and $s \in \mathbb{Z}_+ \cap [\lfloor n(1/p - 1)\rfloor, \infty)$. Let Φ_p, ϕ_0, and W_p be as in (1.1), (1.6), and (1.7), respectively. Then, for any ball $B \subset \mathbb{R}^n$ with center $c_B \in \mathbb{R}^n$ and radius $r_B \in (0, \infty)$, the following assertions are true:*

- *(i) any (ϕ_0, ∞, s)-atom or $(1, \infty, s)_{W_1}$-atom associated with the ball B is also a (Φ_1, ∞, s)-atom associated with the same ball B;*
- *(ii) if $r_B < 1$ and $|c_B| < 1/r_B$, then any (Φ_1, ∞, s)-atom associated with the ball B is also a (ϕ_0, ∞, s)-atom associated with the same ball B;*
- *(ii) if $r_B < 1$ and $|c_B| \geq 1/r_B$, or $r_B \geq 1$, then any (Φ_1, ∞, s)-atom associated with the ball B is also a $(1, \infty, s)_{W_1}$-atom associated with the same ball B;*
- *(iv) when $p \in (0, 1)$, any (Φ_p, ∞, s)-atom associated with the ball B is also a $(p, \infty, s)_{W_p}$-atom associated with the same ball B, and vise versa.*

Proof Let χ_B be the characteristic function of the ball B. By Lemma 2.7(iii), we know that

$$\|\chi_B\|^{-1}_{L^{\phi_0}(\mathbb{R}^n)} \lesssim \|\chi_B\|^{-1}_{L^{\Phi_1}(\mathbb{R}^n)}$$

and

$$\|\chi_B\|^{-1}_{L^1_{W_1}(\mathbb{R}^n)} \lesssim \|\chi_B\|^{-1}_{L^{\Phi_1}(\mathbb{R}^n)}.$$

By these and the definitions of (Φ_1, ∞, s)-atom, (ϕ_0, ∞, s)-atom, and $(1, \infty, s)_{W_1}$-atom, we know that any (ϕ_0, ∞, s)-atom or $(1, \infty, s)_{W_1}$-atom associated with the ball B is also a (Φ_1, ∞, s)-atom associated with B. Hence, (i) holds true.

Now we show (ii). If $r_B < 1$ and $|c_B| < 1/r_B$, then, for any $x \in B$, it holds true that $|x| \leq |x - c_B| + |c_B| < 1 + \frac{1}{r_B}$, which implies that

$$\sup_{x\in B} \log(e + |x|) \leq \log(e + 1 + 1/r_B) \sim \log(e + 1/|B|).$$

This, together with Lemmas 2.6(i) and 2.7(i), shows that

$$\|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)} \sim \frac{|B|}{\log(e + 1/|B|) + \sup_{x\in B} \log(e + |x|)}$$

$$\sim \frac{|B|}{\log(e+1/|B|)} \sim \|\chi_B\|_{L^{\phi_0}(\mathbb{R}^n)}.$$

Then, by the definitions of (Φ_1, ∞, s)-atom and (ϕ_0, ∞, s)-atom, we know that any (Φ_1, ∞, s)-atom associated with the ball B is also a (ϕ_0, ∞, s)-atom associated with B. This finishes the proof of (ii).

To prove (iii), we claim that, if $1/|c_B| \le r_B < 1$ or $r_B \ge 1$, then

$$\log(e+1/|B|) + \sup_{x\in B} \log(e+|x|) \sim \sup_{x\in B} \log(e+|x|). \tag{2.5}$$

Indeed, if $r_B \ge 1$, then (2.5) holds true immediately. If $1/|c_B| \le r_B < 1$, then

$$\log(e+1/|B|) \lesssim \log(e+|c_B|) \lesssim \sup_{x\in B} \log(e+|x|),$$

whence leading to (2.5). Thus, we conclude that

$$\begin{aligned}\|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)} &\sim \frac{|B|}{\log(e+1/|B|) + \sup_{x\in B}\log(e+|x|)}\\ &\sim \frac{|B|}{\sup_{x\in B}\left[\log(e+|x|)\right]} \sim \|\chi_B\|_{L^1_{W_1}(\mathbb{R}^n)}.\end{aligned}$$

Then, applying the definitions of (Φ_1, ∞, s)-atom and $(1, \infty, s)_{W_1}$-atom, we see that any (Φ_1, ∞, s)-atom associated with the ball B is also a $(1, \infty, s)_{W_1}$-atom associated with B. This finishes the proof of (iii).

To show (iv), for any $p \in (0, 1)$, by the definitions of (Φ_p, ∞, s)-atom and $(p, \infty, s)_{W_p}$-atom as well as Lemma 2.8, we immediately conclude that a function a on $\mathbb{R}^n$ is a (Φ_p, ∞, s)-atom associated with B if and only if a is a $(p, \infty, s)_{W_p}$-atom associated with the same ball B. Thus, (iv) holds true, which completes the proof of Proposition 2.12. $\square$

We end this section by recalling the following two lemmas, established in [13], on the Calderón–Zygmund decomposition of the elements of Musielak–Orlicz Hardy spaces.

Lemma 2.13 *Let $p \in (0, 1]$, $q \in (1, \infty)$, and Φ_p be as in (1.1). Then $L^q_{\Phi_p(\cdot,1)}(\mathbb{R}^n) \cap H^{\Phi_p}(\mathbb{R}^n)$ is dense in $H^{\Phi_p}(\mathbb{R}^n)$.*

Lemma 2.14 *Let $p \in (0, 1]$, $q \in (1, \infty)$, $s \in \mathbb{Z}_+ \cap [\lfloor n(1/p-1)\rfloor, \infty)$, and Φ_p be as in (1.1). For any $f \in L^q_{\Phi_p(\cdot,1)}(\mathbb{R}^n) \cap H^{\Phi_p}(\mathbb{R}^n)$, there exist family $\{\Lambda_k\}_{k\in\mathbb{Z}}$ of index set with elements of countable numbers, $\{g^k\}_{k\in\mathbb{Z}}$, and $\{b^k_i\}_{k\in\mathbb{Z}, i\in\Lambda_k} \subset \mathcal{S}'(\mathbb{R}^n)$ such that*

(i) for any $k \in \mathbb{Z}$, $f = g^k + \sum_{i\in\Lambda_k} b^k_i$ in $\mathcal{S}'(\mathbb{R}^n)$;

(ii) $f = \sum_{k\in\mathbb{Z}} (g^{k+1} - g^k)$ in $\mathcal{S}'(\mathbb{R}^n)$;

(iii) *for any* $k \in \mathbb{Z}$, *there exists a family* $\{h_i^k\}_{i\in\Lambda_k} \subset L^\infty(\mathbb{R}^n)$ *such that* $g^{k+1} - g^k = \sum_{i\in\Lambda_k} h_i^k$ *in* $\mathcal{S}'(\mathbb{R}^n)$*;*

(iv) *for any* $k \in \mathbb{Z}$ *and* $i \in \Lambda_k$, h_i^k *satisfies*

(a) $\operatorname{supp} h_i^k \subset B_i^k$, *where* $B_i^k := 18\widetilde{B}_i^k$ *and* $\{\widetilde{B}_i^k\}_{i\in\Lambda_k}$ *is a Whitney covering of* Ω_k *with*

$$\Omega_k := \left\{x \in \mathbb{R}^n : \ f^*(x) > 2^k\right\},$$

where f^* *denotes the non-tangential maximal function of* f *as in* (1.2) *with* m *therein equaling* $\lfloor n(1/p-1)\rfloor$*;*

(b) $\|h_i^k\|_{L^\infty(\mathbb{R}^n)} \le c2^k$, *where* c *is a positive constant independent of* k, i, *and* f*;*

(c) *for any multi-index* α *satisfying* $|\alpha| \le s$, *it holds true that* $\int_{\mathbb{R}^n} x^\alpha h_i^k(x)\,dx = 0$.

3 Proof of Theorem 1.2

Based on the technical lemmas established in Sect. 2, we now prove Theorem 1.2 by considering two cases: $p = 1$ and $p \in (0, 1)$. We point out that these two cases are based on different selection principles to obtain the desired sum space.

Proof of Theorem 1.2 *in the case* $p = 1$. We first establish the inclusion $H^{\phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n) \subset H^{\Phi_1}(\mathbb{R}^n)$. For any $f \in H^{\phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n)$, let $f_0 \in H^{\phi_0}(\mathbb{R}^n)$ and $f_1 \in H^1_{W_1}(\mathbb{R}^n)$ satisfy $f = f_0 + f_1$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{H^{\phi_0}(\mathbb{R}^n)+H^1_{W_1}(\mathbb{R}^n)} \sim \|f_0\|_{H^{\phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^1_{W_1}(\mathbb{R}^n)}.$$

Using (1.1), we know that, for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$,

$$\Phi_1(x,\ t) \lesssim \min\left\{\frac{t}{\log(e+t)},\ \frac{t}{\log(e+|x|)}\right\},$$

which, combined with the grand maximal function characterizations of these Hardy-type spaces, shows that

$$\begin{aligned}\|f\|_{H^{\Phi_1}(\mathbb{R}^n)} &\lesssim \|f_0\|_{H^{\Phi_1}(\mathbb{R}^n)} + \|f_1\|_{H^{\Phi_1}(\mathbb{R}^n)} \lesssim \|f_0\|_{H^{\phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^1_{W_1}(\mathbb{R}^n)}\\ &\sim \|f\|_{H^{\phi_0}(\mathbb{R}^n)+H^1_{W_1}(\mathbb{R}^n)}.\end{aligned}$$

This immediately implies the inclusion $H^{\phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n) \subset H^{\Phi_1}(\mathbb{R}^n)$.

We now prove the converse inclusion $H^{\Phi_1}(\mathbb{R}^n) \subset H^{\phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n)$. Without loss of generality, we may assume that $f \in H^{\Phi_1}(\mathbb{R}^n)$ and $\|f\|_{H^{\Phi_1}(\mathbb{R}^n)} = 1$; otherwise, we use $\widetilde{f} := f/\|f\|_{H^{\Phi_1}(\mathbb{R}^n)}$ to replace f in the same argument as below.

Let $s \in \mathbb{Z}_+$ and $s \geq n(1/p-1)$. By Lemma 2.10, we know that there exist $\{a_j\}_{j\in\mathbb{N}}$ of $(\Phi_1,\ \infty,\ s)$-atoms and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \tag{3.1}$$

in $H^{\Phi_1}(\mathbb{R}^n)$ and hence in $\mathcal{S}'(\mathbb{R}^n)$, and $\Lambda_{\Phi_1}(\{\lambda_j a_j\}_j) \leq 2$. Since Φ_1 is of uniformly lower type 1 and of uniformly upper type 1, it follows easily that, for any $x \in \mathbb{R}^n$ and $s, t \in (0, \infty)$,

$$\Phi_1(x, st) \sim s\Phi_1(x, t).$$

By this, the fact $\Lambda_{\Phi_1}(\{\lambda_j a_j\}_j) \leq 2$, and [13, Lemma 4.2(i)], we conclude that

$$\begin{aligned} 1 \geq \sum_j \Phi_1\left(B_j, \frac{|\lambda_j|}{2\|\chi_{B_j}\|_{L^{\Phi_1}(\mathbb{R}^n)}}\right) &= \sum_j \int_{B_j} \Phi_1\left(x,\ \frac{|\lambda_j|}{2\|\chi_{B_j}\|_{L^{\Phi_1}(\mathbb{R}^n)}}\right) dx \\ &\sim \sum_j |\lambda_j| \int_{B_j} \Phi_1\left(x,\ \frac{1}{\|\chi_{B_j}\|_{L^{\Phi_1}(\mathbb{R}^n)}}\right) dx \\ &\sim \sum_j |\lambda_j|. \end{aligned} \tag{3.2}$$

For any $j \in \mathbb{N}$, assume that a_j is supported on a ball $B_j := B(c_j,\ r_j)$, where $c_j \in \mathbb{R}^n$ and $r_j \in (0, \infty)$. Define

$$\mathrm{I}_0 := \left\{j\ :\ r_j < 1 \text{ and } |c_j| < \frac{1}{r_j}\right\} \quad \text{and} \quad \mathrm{I}_1 := \left\{j\ :\ r_j < 1 \text{ and } |c_j| \geq \frac{1}{r_j}, \text{ or } r_j \geq 1\right\}.$$

It is easy to see that $\mathrm{I}_0 \cap \mathrm{I}_1 = \emptyset$. We now write the decomposition in (3.1) as

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j = \sum_{j\in\mathrm{I}_0} \lambda_j a_j + \sum_{j\in\mathrm{I}_1} \lambda_j a_j =: f_0 + f_1. \tag{3.3}$$

Thus, by Proposition 2.12, we know that, for any $j \in \mathrm{I}_0$, a_j is a $(\phi_0,\ \infty,\ s)$-atom associated with the ball B_j and, for any $j \in \mathrm{I}_1$, a_j is a $(1,\ \infty,\ s)_{W_1}$-atom associated with B_j.

We now show $f_0 \in H^{\phi_0}(\mathbb{R}^n)$. For any $j \in I_0$, by $r_j < 1$ and Lemma 2.7(i), we know that

$$|B_j|\phi_0\left(\frac{1}{|B_j|\rho(|B_j|)}\right) \sim |B_j|\phi_0\left(\frac{\log(e+1/|B_j|)}{|B_j|}\right) \sim \frac{\log(e+1/|B_j|)}{\log\left(e+\frac{\log(e+1/|B_j|)}{|B_j|}\right)} \lesssim 1,$$

which, together with the fact that ϕ_0 is of lower type 1, further implies that

$$\sum_{j\in \mathrm{I}_0}|B_j|\phi_0\left(\frac{|\lambda_j|}{\sum_{j\in \mathrm{I}_0}|\lambda_j||B_j|\rho(|B_j|)}\right)\lesssim \sum_{j\in \mathrm{I}_0}\frac{|\lambda_j|}{\sum_{j\in \mathrm{I}_0}|\lambda_j|}|B_j|\phi_0\left(\frac{1}{|B_j|\rho(|B_j|)}\right)\lesssim 1.$$

From this and (3.2), it follows that

$$\Lambda_{\phi_0}\left(\{\lambda_j a_j\}_{j\in \mathrm{I}_0}\right)=\inf\left\{\lambda\in(0,\ \infty):\ \sum_{j\in \mathrm{I}_0}|B_j|\phi_0\left(\frac{|\lambda_j|}{\lambda|B_j|\rho(|B_j|)}\right)\le 1\right\}$$
$$\lesssim \sum_{j\in \mathrm{I}_0}\left|\lambda_j\right|\lesssim 1.$$

Thus, $f_0\in H^{\phi_0}(\mathbb{R}^n)$ and $\|f_0\|_{H^{\phi_0}(\mathbb{R}^n)}\lesssim \Lambda_{\phi_0}\left(\{\lambda_j a_j\}_{j\in \mathrm{I}_0}\right)\lesssim 1\sim\|f\|_{H^{\Phi_1}(\mathbb{R}^n)}$.

For f_1, using the atomic characterization of $H^1_{W_1}(\mathbb{R}^n)$ stated in Remark 2.11, (3.2), and (3.3), we find that $f_1\in H^1_{W_1}(\mathbb{R}^n)$ and

$$\|f_1\|_{H^1_{W_1}(\mathbb{R}^n)}\lesssim \sum_{j\in \mathrm{I}_1}\left|\lambda_j\right|\lesssim 1\sim\|f\|_{H^{\Phi_1}(\mathbb{R}^n)}.$$

Thus, we conclude that for any $f\in H^{\Phi_1}(\mathbb{R}^n)$ there exist $f_0\in H^{\phi_0}(\mathbb{R}^n)$ and $f_1\in H^1_{W_1}(\mathbb{R}^n)$ such that $f=f_0+f_1$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|f_0\|_{H^{\phi_0}(\mathbb{R}^n)}+\|f_1\|_{H^1_{W_1}(\mathbb{R}^n)}\lesssim \|f\|_{H^{\Phi_1}(\mathbb{R}^n)}$. This finishes the proof of the converse inclusion $H^{\Phi_1}(\mathbb{R}^n)\subset H^{\phi_0}(\mathbb{R}^n)+H^1_{W_1}(\mathbb{R}^n)$ and hence of Theorem 1.2 in the case $p=1$. □

We now turn to the proof of Theorem 1.2 in the case $p\in(0,\ 1)$.

Proof of Theorem 1.2 *in the case* $p\in(0,\ 1)$. Let $p\in(0,\ 1)$. For any $x\in\mathbb{R}^n$ and $t\in(0,\ \infty)$, using (1.1) and (1.5), we observe that

$$\Phi_p(x,\ t)\lesssim\min\left\{\phi_0(t),\ t^pW_p(x)\right\}\qquad\text{and}\qquad\phi_0(t)\le t.$$

From these observations, we argue as in the case $p=1$ and can obtain the inclusions

$$H^1(\mathbb{R}^n)+H^p_{W_p}(\mathbb{R}^n)\subset H^{\phi_0}(\mathbb{R}^n)+H^p_{W_p}(\mathbb{R}^n)\subset H^{\Phi_p}(\mathbb{R}^n)$$

with desired norm estimates.

It remains to prove $H^{\Phi_p}(\mathbb{R}^n)\subset H^1(\mathbb{R}^n)+H^p_{W_p}(\mathbb{R}^n)$. Due to similarity, we only consider the case $n(1/p-1)\in\mathbb{N}$. Consider first the case $f\in L^q_{\Phi_p(\cdot,1)}(\mathbb{R}^n)\cap H^{\Phi_p}(\mathbb{R}^n)$. Without loss of generality, we may also assume that $\|f\|_{H^{\Phi_p}(\mathbb{R}^n)}=1$.

Let $q\in(1,\ \infty)$ and $s\in\mathbb{Z}_+\cap[\lfloor n(1/p-1)\rfloor,\ \infty)$. Applying Lemma 2.14, there exist families $\{\Lambda_k\}_{k\in\mathbb{Z}}$ of index sets, $\{h_i^k\}_{k\in\mathbb{Z},i\in\Lambda_k}$ of functions in $L^\infty(\mathbb{R}^n)$, and $\{B_i^k\}_{k\in\mathbb{Z},i\in\Lambda_k}$ of balls such that

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\Lambda_k}h_i^k\qquad\text{in }\ \mathcal{S}'(\mathbb{R}^n).\tag{3.4}$$

Define

$$\mathrm{E} := \left\{x \in \mathbb{R}^n : \ f^*(x) < (1+|x|)^{-n}[\log(e+|x|)]^{-p/(1-p)}\right\}, \tag{3.5}$$

where f^* denotes the non-tangential maximal function as in (1.2) with $m := \lfloor n(1/p-1)\rfloor$. For any $k \in \mathbb{Z}$ and $i \in \Lambda_k$, define

$$B^k_{i,\mathrm{E}} := B^k_i \cap \mathrm{E} \quad \text{ and } \quad B^k_{i,\mathrm{E}^\complement} := B^k_i \cap \mathrm{E}^\complement.$$

Let

$$\mathrm{I}_0 := \left\{(k,i) : \left|B^k_{i,\mathrm{E}}\right| \geq \frac{1}{2}\left|B^k_i\right|\right\} \quad \text{ and } \quad \mathrm{I}_1 := \left\{(k,i) : \left|B^k_{i,\mathrm{E}^\complement}\right| \geq \frac{1}{2}\left|B^k_i\right|\right\}.$$

It is easy to see that $\mathrm{I}_0 \cap \mathrm{I}_1 = \emptyset$ and

$$\sum_{k\in\mathbb{Z}}\sum_{i\in\Lambda_k} = \sum_{(k,i)\in\mathrm{I}_0} + \sum_{(k,i)\in\mathrm{I}_1}.$$

Fix $k_0 \in \mathbb{Z}$. By Lemma 2.14, it holds true that

$$\sum_{(k_0,i)\in\mathrm{I}_0} |B^{k_0}_i| \leq 2\sum_{(k_0,i)\in\mathrm{I}_0} \left|B^{k_0}_{i,\mathrm{E}}\right| \lesssim \left|\left\{x \in \mathrm{E} : \ f^*(x) > 2^{k_0}\right\}\right|. \tag{3.6}$$

Similarly, for any $(k_0,i) \in \mathrm{I}_1$, using $W_p \in A_1(\mathbb{R}^n)$ (see (2.1)) and $|B^{k_0}_{i,\mathrm{E}^\complement}| \geq \frac{1}{2}|B^{k_0}_i|$, we obtain

$$\frac{W_p\left(B^{k_0}_i\right)}{W_p\left(B^{k_0}_{i,\mathrm{E}^\complement}\right)} \lesssim \frac{\left|B^{k_0}_i\right|}{\left|B^{k_0}_{i,\mathrm{E}^\complement}\right|} \lesssim 1.$$

This, combined with the same argument as in (3.6), implies that

$$\sum_{(k_0,i)\in\mathrm{I}_1} W_p\left(B^{k_0}_i\right) \lesssim \sum_{(k_0,i)\in\mathrm{I}_1} W_p\left(B^{k_0}_{i,\mathrm{E}^\complement}\right) \lesssim W_p\left(\left\{x \in \mathrm{E}^\complement : \ f^*(x) > 2^{k_0}\right\}\right). \tag{3.7}$$

We split the decomposition in (3.4) into

$$f = \sum_{(k,i)\in\mathrm{I}_0} h^k_i + \sum_{(k,i)\in\mathrm{I}_1} h^k_i =: \sum_{(k,i)\in\mathrm{I}_0} \lambda^{(0)}_{k,i} a^{(0)}_{k,i} + \sum_{(k,i)\in\mathrm{I}_1} \lambda^{(1)}_{k,i} a^{(1)}_{k,i} =: f_0 + f_1,$$

where

$$\begin{cases} \lambda_{k,i}^{(0)} := c2^k |B_i^k|; \\ \lambda_{k,i}^{(1)} := c2^k [W_p(B_i^k)]^{1/p}; \\ a_{k,i}^{(0)} := h_i^k / \lambda_{k,i}^{(0)}; \\ a_{k,i}^{(1)} := h_i^k / \lambda_{k,i}^{(1)} \end{cases}$$

and c is the same as in (b) of Lemma 2.14(iv) and is independent of $k,\ i$, and f. By Remark 2.11 and Lemma 2.14, it is easy to see that $a_{k,i}^{(0)}$ is a $(1,\ \infty,\ s)$-atom associated with the ball B_i^k and $a_{k,i}^{(1)}$ is a $(p,\ \infty,\ s)_{W_p}$-atom associated with B_i^k. From (3.5) and (3.6), it follows that

$$\begin{aligned} \sum_{(k,i)\in \mathrm{I}_0} \left|\lambda_{k,i}^{(0)}\right| &\lesssim \sum_{k\in\mathbb{Z}} 2^k \left|\left\{x \in \mathrm{E} :\ f^*(x) > 2^k\right\}\right| \\ &\lesssim \int_{\mathrm{E}} f^*(x)\,dx \\ &\sim \int_{\mathrm{E}} \frac{f^*(x)}{1 + [f^*(x)(1+|x|)^n]^{1-p}[\log(e+|x|)]^p}\,dx \\ &\lesssim \int_{\mathbb{R}^n} \Phi_p\left(x,\ f^*(x)\right)\,dx \lesssim 1, \end{aligned}$$

where the last inequality follows from the assumption $\|f\|_{H^{\Phi_p}(\mathbb{R}^n)} = 1$. Further, using the atomic characterization of $H^1(\mathbb{R}^n)$, we know that $f_0 \in H^1(\mathbb{R}^n)$ and $\|f_0\|_{H^1(\mathbb{R}^n)} \lesssim 1 \sim \|f\|_{H^{\Phi_p}(\mathbb{R}^n)}$.

For f_1, using (3.7), we find that

$$\begin{aligned} \sum_{(k,i)\in \mathrm{I}_1} \left|\lambda_{k,i}^{(1)}\right|^p &\lesssim \sum_{k\in\mathbb{Z}} 2^{kp} W_p\left(\left\{x \in \mathrm{E}^{\complement} :\ f^*(x) > 2^k\right\}\right) \\ &\lesssim \int_{\mathrm{E}^{\complement}} \left[f^*(x)\right]^p W_p(x)\,dx \\ &\sim \int_{\mathrm{E}^{\complement}} \frac{f^*(x)}{1 + [f^*(x)(1+|x|)^n]^{1-p}[\log(e+|x|)]^p}\,dx \\ &\lesssim \int_{\mathbb{R}^n} \Phi_p\left(x,\ f^*(x)\right)\,dx \lesssim 1. \end{aligned}$$

Then, using the atomic characterization of $H^p_{W_p}(\mathbb{R}^n)$ in Remark 2.11, we know that $f_1 \in H^p_{W_p}(\mathbb{R}^n)$ and $\|f_1\|_{H^p_{W_p}(\mathbb{R}^n)} \lesssim 1 \sim \|f\|_{H^{\Phi_p}(\mathbb{R}^n)}$.

Summarizing the above estimates gives us that $L^q_{\Phi_p(\cdot,1)}(\mathbb{R}^n) \cap H^{\Phi_p}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ and, for any $f \in L^q_{\Phi_p(\cdot,1)}(\mathbb{R}^n) \cap H^{\Phi_p}(\mathbb{R}^n)$, there exist $f_0 \in$

$H^1(\mathbb{R}^n)$ and $f_1 \in H^p_{W_p}(\mathbb{R}^n)$ such that $f = f_0 + f_1$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f_0\|_{H^1(\mathbb{R}^n)} + \|f_1\|_{H^p_{W_p}(\mathbb{R}^n)} \lesssim \|f\|_{H^{\Phi_p}(\mathbb{R}^n)}. \tag{3.8}$$

For a general $f \in H^{\Phi_p}(\mathbb{R}^n)$, by Lemma 2.13, there exist $\{f_l\}_{l\in\mathbb{N}} \subset L^q_{\Phi_p(\cdot,1)}(\mathbb{R}^n) \cap H^{\Phi_p}(\mathbb{R}^n)$ such that $f = \sum_{l\in\mathbb{N}} f_l$ in $H^{\Phi_p}(\mathbb{R}^n)$ and

$$\|f_l\|_{H^{\Phi_p}(\mathbb{R}^n)} \le 2^{2-l}\|f\|_{H^{\Phi_p}(\mathbb{R}^n)}$$

(see also [13, p. 138] for this fact). Applying the previous argument to each f_l with $l \in \mathbb{N}$, we find $f_{l,0} \in H^1(\mathbb{R}^n)$ and $f_{l,1} \in H^p_{W_p}(\mathbb{R}^n)$ such that $f_l = f_{l,0} + f_{l,1}$ in $\mathcal{S}'(\mathbb{R}^n)$, and

$$\left\|f_{l,0}\right\|_{H^1(\mathbb{R}^n)} + \left\|f_{l,1}\right\|_{H^p_{W_p}(\mathbb{R}^n)} \lesssim \|f_l\|_{H^{\Phi_p}(\mathbb{R}^n)} \lesssim 2^{-l}\|f\|_{H^{\Phi_p}(\mathbb{R}^n)}.$$

Define $f_0 := \sum_{l\in\mathbb{N}} f_{l,0}$ and $f_1 := \sum_{l\in\mathbb{N}} f_{l,1}$. It follows that $f_0 \in H^1(\mathbb{R}^n)$ and $f_1 \in H^p_{W_p}(\mathbb{R}^n)$ with

$$\|f_0\|_{H^1(\mathbb{R}^n)} \le \sum_{l\in\mathbb{N}} \left\|f_{l,0}\right\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^{\Phi_p}(\mathbb{R}^n)}$$

and

$$\|f_1\|^p_{H^p_{W_p}(\mathbb{R}^n)} \le \sum_{l\in\mathbb{N}} \left\|f_{l,1}\right\|^p_{H^p_{W_p}(\mathbb{R}^n)} \lesssim \sum_{l\in\mathbb{N}} 2^{-lp}\|f\|^p_{H^{\Phi_p}(\mathbb{R}^n)} \lesssim \|f\|^p_{H^{\Phi_p}(\mathbb{R}^n)}.$$

Altogether, we obtain $f = f_0 + f_1$ in $\mathcal{S}'(\mathbb{R}^n)$ and (3.8). This proves the inclusion $H^{\Phi_p}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ in the case $n(1/p-1) \in \mathbb{N}$.

The proof of $H^{\Phi_p}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$ in the case $n(1/p-1) \notin \mathbb{N}$ is similar to the previous proof for the case $n(1/p-1) \in \mathbb{N}$, but now instead of (3.5) we define the set E as follows:

$$\mathrm{E} := \left\{x \in \mathbb{R}^n :\ f^*(x) < (1+|x|)^{-n}\right\}.$$

The details are omitted. This finishes the proof of Theorem 1.2 when $p \in (0, 1)$. $\square$

Remark 3.1 Let $p \in (0, 1)$ and B be a ball in $\mathbb{R}^n$. For any $s \in \mathbb{Z}_+$ and $s \ge \lfloor n(1/p - 1)\rfloor$, we know from Proposition 2.12(iv) that a function a on $\mathbb{R}^n$ is a (Φ_p, ∞, s)-atom associated with B if and only if a is a $(p, \infty, s)_{W_p}$-atom associated with the same ball B. On the other hand, Theorem 1.2 implies that $H^p_{W_p} \subsetneqq H^{\Phi_p}(\mathbb{R}^n)$. However, for any $p \in (0, 1)$, from Lemma 2.8 and the definitions of the dual spaces of $H^{\Phi_p}(\mathbb{R}^n)$ and

$H^p_{W_p}(\mathbb{R}^n)$ (see [13, Theorem 3.2] for $p \in (n/(n+1), 1)$ and [14, Theorem 3.5] for general $p \in (0, 1)$), we deduce that

$$\left[H^{\Phi_p}(\mathbb{R}^n)\right]^* = \left[H^p_{W_p}(\mathbb{R}^n)\right]^*.$$

This shows that the difference between $H^{\Phi_p}(\mathbb{R}^n)$ and $H^p_{W_p}(\mathbb{R}^n)$ is very small.

4 Proof of Theorem 1.3

Applying Theorem 1.2, we prove Theorem 1.3 in this section.

Proof of Theorem 1.3 We first prove (i). Let $p \in (0, 1]$ and $f \in H^{\Phi_p}(\mathbb{R}^n)$. By Theorem 1.2, we know that there exist $f_0 \in H^{\phi_0}(\mathbb{R}^n)$ and $f_1 \in H^p_{W_p}(\mathbb{R}^n)$ such that $f = f_0 + f_1$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{H^{\Phi_p}(\mathbb{R}^n)} \sim \|f_0\|_{H^{\phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^p_{W_p}(\mathbb{R}^n)}.$$

Since T is quasilinear and bounded on $H^{\phi_0}(\mathbb{R}^n)$ and $H^p_{W_p}(\mathbb{R}^n)$, it follows, from (1.1), that

$$\begin{aligned}\|T(f)\|_{H^{\Phi_p}(\mathbb{R}^n)} &\lesssim \|T(f_0)\|_{H^{\Phi_p}(\mathbb{R}^n)} + \|T(f_1)\|_{H^{\Phi_p}(\mathbb{R}^n)}\\ &\lesssim \|T(f_0)\|_{H^{\phi_0}(\mathbb{R}^n)} + \|T(f_1)\|_{H^p_{W_p}(\mathbb{R}^n)}\\ &\lesssim \|f_0\|_{H^{\phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^p_{W_p}(\mathbb{R}^n)} \sim \|f\|_{H^{\Phi_p}(\mathbb{R}^n)},\end{aligned}$$

which implies that T is bounded on $H^{\Phi_p}(\mathbb{R}^n)$. This shows (i).

The proof of (ii) is similar to that of (i) by applying the equivalence established in Theorem 1.2(ii)

$$H^{\Phi_p}(\mathbb{R}^n) = H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n),$$

the details being omitted. This finishes the proof of Theorem 1.3. □

Acknowledgements This project was supported by the National Natural Science Foundation of China (Grant Nos. 11501506, 11771446, 11471042, 11571039, and 11671185). Jun Cao was partially supported by the Natural Science Foundation of Zhejiang University of Technology (Grant No. 2014XZ011).

References

1. Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften, vol. 223. Springer, Berlin (1976)
2. Bonami, A., Cao, J., Ky, L.D., Liu, L., Yang, D., Yuan, W.: A complete solution to bilinear decompositions of products of Hardy and Campanato spaces. Preprint
3. Bonami, A., Feuto, J., Grellier, S.: Endpoint for the DIV-CURL lemma in Hardy spaces. Publ. Mat. **54**, 341–358 (2010)

4. Bonami, A., Grellier, S., Ky, L.D.: Paraproducts and products of functions in BMO($\mathbb{R}^n$) and $H^1(\mathbb{R}^n)$ through wavelets. J. Math. Pures Appl. **97**(9), 230–241 (2012)
5. Bonami, A., Iwaniec, T., Jones, P., Zinsmeister, M.: On the product of functions in BMO and H^1. Ann. Inst. Fourier (Grenoble) **57**, 1405–1439 (2007)
6. Coifman, R.R., Lions, P.-L., Meyer, Y., Semmes, S.: Compensated compactness and Hardy spaces. J. Math. Pures Appl. **72**(9), 247–286 (1993)
7. Fefferman, C., Stein, E.M.: H^p spaces of several variables. Acta Math. **129**, 137–195 (1972)
8. García-Cuerva, J.: Weighted H^p spaces. Dissertationes Math. (Rozprawy Mat.) **162**, 1–63 (1979)
9. Grafakos, L.: Classical Fourier Analysis, Second edition, Graduate Texts in Mathematics, vol. 249. Springer, New York (2008)
10. Janson, S.: Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation. Duke Math. J. **47**, 959–982 (1980)
11. Jiang, R., Yang, D.: New Orlicz-Hardy spaces associated with divergence form elliptic operators. J. Funct. Anal. **258**, 1167–1224 (2010)
12. Ky, L.D.: Bilinear decompositions and commutators of singular integral operators. Trans. Am. Math. Soc. **365**, 2931–2958 (2013)
13. Ky, L.D.: New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integr. Equ. Operator Theor. **78**, 115–150 (2014)
14. Liang, Y., Yang, D.: Musielak-Orlicz Campanato spaces and applications. J. Math. Anal. Appl. **406**, 307–322 (2013)
15. Liu, L., Chang, D.-C., Fu, X., Yang, D.: Endpoint boundedness of commutators on spaces of homogeneous type. Appl. Anal. **96**, 2408–2433 (2017)
16. Lu, S.: Four Lectures on Real H^p Spaces. World Scientific Publishing Co. Inc, River Edge, NJ (1995)
17. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series, vol. 43, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ (1993)
18. Stein, E.M., Weiss, G.: On the theory of harmonic functions of several variables. I. The theory of H^p-spaces. Acta Math. **103**, 25–62 (1960)
19. Strömberg, J.-O.: Bounded mean oscillation with Orlicz norms and duality of Hardy spaces. Indiana Univ. Math. J. **28**, 511–544 (1979)
20. Strömberg, J.-O., Torchinsky, A.: Weighted Hardy Spaces. Lecture Notes in Mathematics 1381. Springer, Berlin (1989)
21. Viviani, B.: An atomic decomposition of the predual of $BMO(\rho)$. Rev. Mat. Iberoamericana **3**, 401–425 (1987)
22. Yang, D., Yang, S.: Local Hardy spaces of Musielak-Orlicz type and their applications. Sci. China Math. **55**, 1677–1720 (2012)
23. Yang, D., Liang, Y., Ky, L.D.: Real-Variable Theory of Musielak-Orlicz Hardy Spaces. Lecture Notes in Mathematics 2182. Springer, Cham (2017)