Model Predictive Control using MATLAB
6: Nonlinear MPC (NMPC)

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Overview

1. NMPC: Introduction
2. NMPC: Algorithm

NMPC: Introduction

System model

- Discrete-time nonlinear system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \tag{1}$$

- $k \in \mathbb{T} = \{0, 1, ..., N_T - 1\}$: discrete time instant.
- $\mathbf{x}_k \in \mathbb{X} \subseteq \mathbb{R}^n$: state vector.
- $\mathbf{u}_k \in \mathbb{U} \subseteq \mathbb{R}^m$: control input vector.
- $\mathbf{f} : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$: nonlinear mapping.
- $\mathbb{X}, \mathbb{U}$: constraint sets for the state and input vectors defined by

$$\begin{aligned} \mathbb{X} &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{F_x}\mathbf{x} \leq \mathbf{g_x}\} \\ \mathbb{U} &= \{\mathbf{u} \in \mathbb{R}^m : \mathbf{F_u}\mathbf{u} \leq \mathbf{g_u}\}. \end{aligned} \tag{2}$$

NMPC Problem

- Cost function

$$J_k = \mathbf{x}_{k+N|k}^T \mathbf{Q}_N \mathbf{x}_{k+N|k} + \sum_{i=k}^{k+N-1} \mathbf{x}_{i|k}^T \mathbf{Q} \mathbf{x}_{i|k} + \mathbf{u}_{i|k}^T \mathbf{R} \mathbf{u}_{i|k} \tag{3}$$

- Predicted state and control sequence

$$\begin{aligned} \mathbf{X}_k &= \left(\mathbf{x}_{k|k}, \mathbf{x}_{k+1|k}, ..., \mathbf{u}_{k+N|k}\right) \\ \mathbf{U}_k &= \left(\mathbf{u}_{k|k}, \mathbf{u}_{k+1|k}, ..., \mathbf{u}_{k+N-1|k}\right) \end{aligned} \tag{4}$$

Problem 1 (NMPC)

For the nonlinear system with the current state $\boldsymbol{x}_{k|k} = \boldsymbol{x}_k$, compute the control sequence $\boldsymbol{U}_k$ by solving the optimization problem

$$\begin{aligned} &\underset{\boldsymbol{U}_k}{inf}\ J_k \quad \textit{subject to} \\ &\quad \boldsymbol{U}_k \in \mathbb{U}^N, \quad \boldsymbol{X}_k \in \mathbb{X}^{N+1}, \quad k \in \mathbb{T} \\ &\quad \boldsymbol{x}_{i+1|k} = \boldsymbol{f}(\boldsymbol{x}_{i|k}, \boldsymbol{u}_{i|k}), \quad k \in \mathbb{T}, i = k, ..., k+N-1. \end{aligned} \tag{5}$$

NMPC: Algorithm

NMPC: Algorithm

- Using $\mathbf{X}_k$ and $\mathbf{U}_k$ the cost function and constraints for the NMPC can be rewritten as

$$J_k = \mathbf{X}_k^T \mathbf{Q}_{\mathbf{X}} \mathbf{X}_k + \mathbf{U}_k^T \mathbf{R}_{\mathbf{U}} \mathbf{U}_k \tag{6}$$

and

$$\begin{aligned} \mathbf{F}_{\mathbf{X}} \mathbf{X}_k &\leq \mathbf{g}_{\mathbf{X}} \\ \mathbf{F}_{\mathbf{U}} \mathbf{U}_k &\leq \mathbf{g}_{\mathbf{U}} \\ \mathbf{f}_{eq}(\mathbf{X}_k, \mathbf{U}_k) &= 0 \end{aligned} \tag{7}$$

where

$$\mathbf{f}_{eq}(\mathbf{X}_k, \mathbf{U}_k) = \begin{bmatrix} \mathbf{x}_{k|k} - \mathbf{x}_k \\ \mathbf{x}_{k+1|k} - \mathbf{f}(\mathbf{x}_{k|k}, \mathbf{u}_{k|k}) \\ \vdots \\ \mathbf{x}_{k+N|k} - \mathbf{f}(\mathbf{x}_{k+N-1|k}, \mathbf{u}_{k+N-1|k}) \end{bmatrix} \tag{8}$$

NMPC: Algorithm

- Define $\mathbf{z} = \begin{bmatrix} \mathbf{X_k} \\ \mathbf{U_k} \end{bmatrix}$ and $\mathbf{H}, \mathbf{F}, \mathbf{g}$ as in LMPC, the optimization problem is obtained as

$$\begin{aligned} \underset{\mathbf{z}}{inf} \quad & \mathbf{z}^T \mathbf{H} \mathbf{z} \quad subject\ to \\ & \mathbf{F}\mathbf{z} \leq \mathbf{g} \\ & \mathbf{f}_{eq}(\mathbf{z}) = 0 \end{aligned} \tag{9}$$

- Here the equality constraint is nonlinear which makes the optimization problem a nonlinear programming problem.
- The control input with MPC is

$$\mathbf{u}_k = [\mathbf{U}_k^*]_1 = \mathbf{u}_{k|k}^*. \tag{10}$$

NMPC: Algorithm

Algorithm 1 : NMPC

Require $\mathbf{f}, N_T, N, n, m, \mathbf{Q}, \mathbf{R}, \mathbf{Q}_{N_T}, \mathbf{F}_{\mathbf{x}}, \mathbf{g}_{\mathbf{x}}, \mathbf{F}_{\mathbf{u}}, \mathbf{g}_{\mathbf{u}}$

Initialize $\mathbf{x}_0, \mathbf{z}_0$

Construct $\mathbf{Q}_X, \mathbf{R}_{\mathbf{U}}, \mathbf{H}, \mathbf{F}, \mathbf{g}$

for $k = 0$ *to* $N_T - 1$ **do**

    $\mathbf{x}_k = [\mathbf{X}]_{k+1}$ (obtain $\mathbf{x}_k$ from measurement/estimation)

    Compute $\mathbf{z}^* = \begin{bmatrix} \mathbf{X}_k^* \\ \mathbf{U}_k^* \end{bmatrix}$ by solving the optimization problem

    Apply $\mathbf{u}_k = [\mathbf{U}_k^*]_1$ to the system

    Update $\mathbf{z}_0 = \mathbf{z}^*$

end for

Thank you