Model Predictive Control using MATLAB
2: Linear MPC (LMPC)

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Overview

LMPC: Introduction

System model

- Discrete-time linear time-invariant (LTI) system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{1}$$

- $k \in \mathbb{T} = \{0, 1, ..., N_T - 1\}$: discrete time instant.
- $\mathbf{x}_k \in \mathbb{X} \subseteq \mathbb{R}^n$: state vector.
- $\mathbf{u}_k \in \mathbb{U} \subseteq \mathbb{R}^m$: control input vector.
- $\mathbf{A} \in \mathbb{R}^{n \times n}$: system matrix.
- $\mathbf{B} \in \mathbb{R}^{n \times m}$: input matrix.
- $\mathbb{X}$, $\mathbb{U}$: constraint sets for the states and control inputs defined by

$$\begin{aligned} \mathbb{X} &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{F_x}\mathbf{x} \leq \mathbf{g_x}\} \\ \mathbb{U} &= \{\mathbf{u} \in \mathbb{R}^m : \mathbf{F_u}\mathbf{u} \leq \mathbf{g_u}\}. \end{aligned} \tag{2}$$

Constrained LQR (CLQR) Probelm

- Cost function

$$J = \mathbf{x}_{N_T}^T \mathbf{Q}_{N_T} \mathbf{x}_{N_T} + \sum_{k=0}^{N_T-1} \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k \tag{3}$$

  where $\mathbf{Q}_{N_T} \geq 0, \mathbf{Q} > 0$, $\mathbf{R} > 0$ are the weighting matrices.

- State and control sequence

$$\begin{aligned} \mathbf{X} &= \left(\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{N_T}\right) \\ \mathbf{U} &= \left(\mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_{N_T-1}\right) \end{aligned} \tag{4}$$

Problem 1 (CLQR)

For the LTI system with the initial state $\boldsymbol{x}_0$, compute the control sequence $\boldsymbol{U}$ by solving the optimization problem

$$\begin{aligned} &\underset{\boldsymbol{U}}{inf}\ J \quad \textit{subject to} \\ &\quad \boldsymbol{U} \in \mathbb{U}^{N_T}, \quad \boldsymbol{X} \in \mathbb{X}^{N_T+1} \\ &\quad \boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k, \quad k \in \mathbb{T}. \end{aligned} \tag{5}$$

LMPC Problem

- Cost function

$$J_k = \mathbf{x}_{k+N|k}^T \mathbf{Q}_N \mathbf{x}_{k+N|k} + \sum_{i=k}^{k+N-1} \mathbf{x}_{i|k}^T \mathbf{Q} \mathbf{x}_{i|k} + \mathbf{u}_{i|k}^T \mathbf{R} \mathbf{u}_{i|k} \tag{6}$$

- Predicted state and control sequence

$$\begin{aligned} \mathbf{X}_k &= \left(\mathbf{x}_{k|k}, \mathbf{x}_{k+1|k}, ..., \mathbf{u}_{k+N|k}\right) \\ \mathbf{U}_k &= \left(\mathbf{u}_{k|k}, \mathbf{u}_{k+1|k}, ..., \mathbf{u}_{k+N-1|k}\right) \end{aligned} \tag{7}$$

Problem 2 (LMPC)

For the LTI system with the current state $\boldsymbol{x}_{k|k} = \boldsymbol{x}_k$, compute the control sequence $\boldsymbol{U}_k$, by solving the optimization problem

$$\begin{aligned} &\underset{\boldsymbol{U}_k}{inf}\ J_k \quad \textit{subject to} \\ &\quad \boldsymbol{U}_k \in \mathbb{U}^N, \ \ \boldsymbol{X}_k \in \mathbb{X}^{N+1}, \quad k \in \mathbb{T} \\ &\quad \boldsymbol{x}_{i+1|k} = \boldsymbol{A}\boldsymbol{x}_{i|k} + \boldsymbol{B}\boldsymbol{u}_{i|k}, \quad k \in \mathbb{T}, i = k, ..., k+N-1. \end{aligned} \tag{8}$$

LMPC: Algorithm

LMPC: Algorithm

- The solution of the state equation gives

$$\begin{bmatrix} \mathbf{x}_{k|k} \\ \mathbf{x}_{k+1|k} \\ \vdots \\ \mathbf{x}_{k+N|k} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{A} \\ \vdots \\ \mathbf{A}^N \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}^{N-1}\mathbf{B} & \mathbf{A}^{N-2}\mathbf{B} & \dots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{k|k} \\ \mathbf{u}_{k+1|k} \\ \vdots \\ \mathbf{u}_{k+N-1|k} \end{bmatrix} \tag{9}$$

- By defining the following matrices

$$\mathbf{X}_k = \begin{bmatrix} \mathbf{x}_{k|k} \\ \mathbf{x}_{k+1|k} \\ \vdots \\ \mathbf{x}_{k+N|k} \end{bmatrix}, \mathbf{U}_k = \begin{bmatrix} \mathbf{u}_{k|k} \\ \mathbf{u}_{k+1|k} \\ \vdots \\ \mathbf{u}_{k+N-1|k} \end{bmatrix}, \ \mathbf{A}_{\mathbf{X}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{A} \\ \vdots \\ \mathbf{A}^N \end{bmatrix}, \mathbf{B}_{\mathbf{U}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}^{N-1}\mathbf{B} & \mathbf{A}^{N-2}\mathbf{B} & \dots & \mathbf{B} \end{bmatrix} \tag{10}$$

the equation (9) is rewritten as

$$\mathbf{X}_k = \mathbf{A}_{\mathbf{X}}\mathbf{x}_k + \mathbf{B}_{\mathbf{U}}\mathbf{U}_k \tag{11}$$

- Similarly, by defining

$$\mathbf{Q_X} = \begin{bmatrix} \mathbf{Q} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}_N \end{bmatrix}, \quad \mathbf{R_U} = \begin{bmatrix} \mathbf{R} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{R} \end{bmatrix} \tag{12}$$

the cost function can be represented in terms of $\mathbf{X}_k$ and $\mathbf{U}_k$ as

$$J_k = \mathbf{X}_k^T \mathbf{Q_X} \mathbf{X}_k + \mathbf{U}_k^T \mathbf{R_U} \mathbf{U}_k \tag{13}$$

- And by defining

$$\mathbf{F_X} = \begin{bmatrix} \mathbf{F_x} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{F_x} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{F_x} \end{bmatrix}, \quad \mathbf{g_X} = \begin{bmatrix} \mathbf{g_x} \\ \mathbf{g_x} \\ \vdots \\ \mathbf{g_x} \end{bmatrix}, \quad \mathbf{F_U} = \begin{bmatrix} \mathbf{F_u} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{F_u} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{F_u} \end{bmatrix}, \quad \mathbf{g_U} = \begin{bmatrix} \mathbf{g_u} \\ \mathbf{g_u} \\ \vdots \\ \mathbf{g_u} \end{bmatrix} \tag{14}$$

the constraints are represented in terms of $\mathbf{X}_k$ and $\mathbf{U}_k$ as

$$\begin{aligned} \mathbf{F_X}\mathbf{X}_k &\leq \mathbf{g_X} \\ \mathbf{F_U}\mathbf{U}_k &\leq \mathbf{g_U} \end{aligned} \tag{15}$$

LMPC: Algorithm

- Define

$$\mathbf{z} = \begin{bmatrix} \mathbf{X_k} \\ \mathbf{U_k} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{Q_X} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_U} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F_X} & \mathbf{0} \\ \mathbf{0} & \mathbf{F_U} \end{bmatrix}$$
$$\mathbf{g} = \begin{bmatrix} \mathbf{g_X} \\ \mathbf{g_U} \end{bmatrix}, \quad \mathbf{F}_{eq} = \begin{bmatrix} \mathbf{I} & -\mathbf{B_U} \end{bmatrix}, \quad \mathbf{g}_{eq} = \mathbf{A_X}\mathbf{x}_k \tag{16}$$

- Using this the MPC optimization problem is represented as

$$\begin{aligned} \inf_{\mathbf{z}} \quad & \mathbf{z}^T\mathbf{H}\mathbf{z} \quad \textit{subject to} \\ & \mathbf{F}\mathbf{z} \leq \mathbf{g} \\ & \mathbf{F}_{eq}\mathbf{z} = \mathbf{g}_{eq} \end{aligned} \tag{17}$$

  which is a quadratic programming problem.

- The control input with MPC is

$$\mathbf{u}_k = [\mathbf{U}_k^*]_1 = \mathbf{u}_{k|k}^*. \tag{18}$$

LMPC: Algorithm

Algorithm 1 : LMPC

Require $\mathbf{A}, \mathbf{B}, N_T, N, n, m, \mathbf{Q}, \mathbf{R}, \mathbf{Q}_{N_T}, \mathbf{F}_{\mathbf{x}}, \mathbf{g}_{\mathbf{x}}, \mathbf{F}_{\mathbf{u}}, \mathbf{g}_{\mathbf{u}}$
Initialize $\mathbf{x}_0, \mathbf{z}_0$
Construct $\mathbf{A}_{\mathbf{X}}, \mathbf{B}_{\mathbf{U}}, \mathbf{Q}_X, \mathbf{R}_{\mathbf{U}}, \mathbf{H}, \mathbf{F}, \mathbf{g}$
for $k = 0$ *to* $N_T - 1$ **do**
    $\mathbf{x}_k = [\mathbf{X}]_{k+1}$ (obtain $\mathbf{x}_k$ from measurement/estimation)
    Compute $\mathbf{F}_{eq}, \mathbf{g}_{eq}$
    Compute $\mathbf{z}^* = \begin{bmatrix} \mathbf{X}_k^* \\ \mathbf{U}_k^* \end{bmatrix}$ by solving the optimization problem
    Apply $\mathbf{u}_k = [\mathbf{U}_k^*]_1$ to the system
    Update $\mathbf{z}_0 = \mathbf{z}^*$
end for

Thank you