FUZZY CONTROL:

An Introduction

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FUZZY CONTROL

Control Theory?

- The term *control* is generally defined as a mechanism used to guide or regulate the operation of a machine, apparatus or constellations of machines and apparatus.

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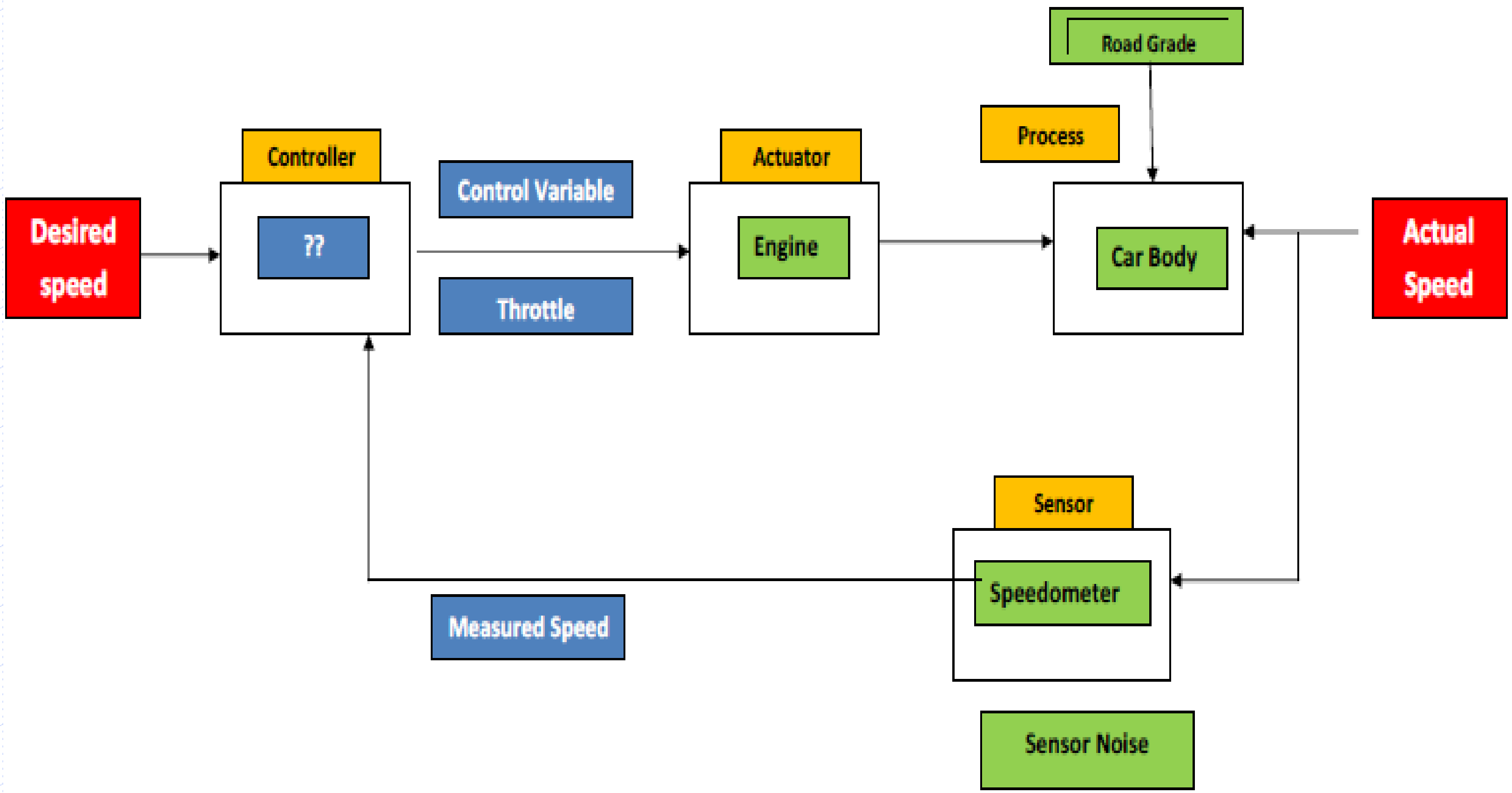
Control Theory?



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Control Theory?

- Often the notion of control is inextricably linked with *feedback*: a process of returning to the input of a device a fraction of the output signal. Feedback can be *negative*, whereby feedback opposes and therefore reduces the input, and feedback can be *positive* whereby feedback reinforces the input signal.

FUZZY CONTROL

CONTROL THEORY?

- 'Feedback control' is thus a mechanism for guiding or regulating the operation of a system or subsystems by returning to the input of the (sub)system a fraction of the output.

FUZZY CONTROL

CONTROL THEORY?

- The machinery or apparatus etc., to be guided or regulated is denoted by *S*, the input by *W* and the output by *y*, and the feedback controller by *C*. The input to the controller is the so-called *error* signal *e* and the purpose of the controller is to guarantee a desired response of the output *y*.

FUZZY CONTROL

DEFINITIONS

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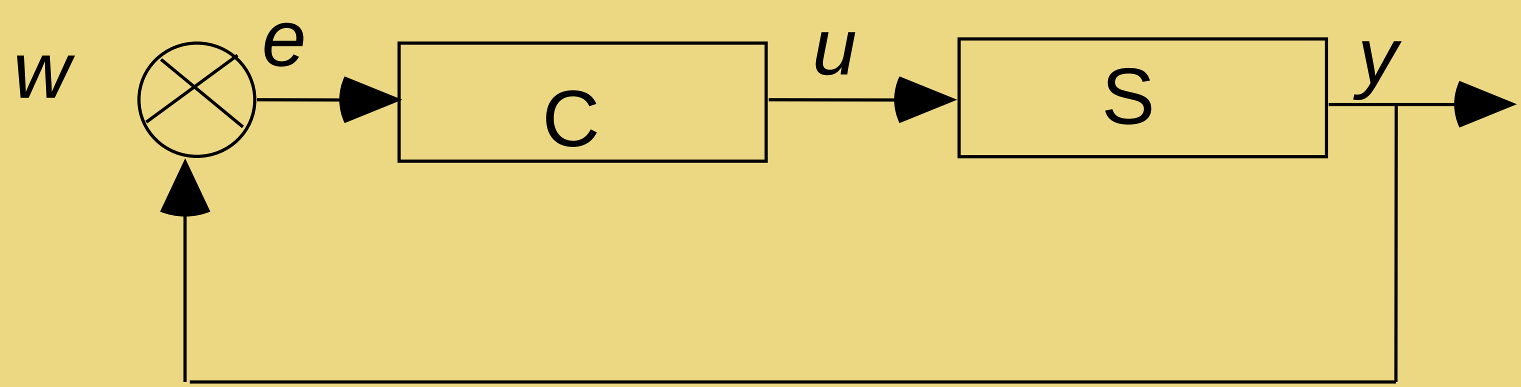

The machinery or apparatus etc., to be guided or regulated is denoted by *S*, the input by *W* and the output by *y*, and the feedback controller by *C*. The input to the controller is the so-called *error* signal *e* and the purpose of the controller is to guarantee a desired response of the output *y*.

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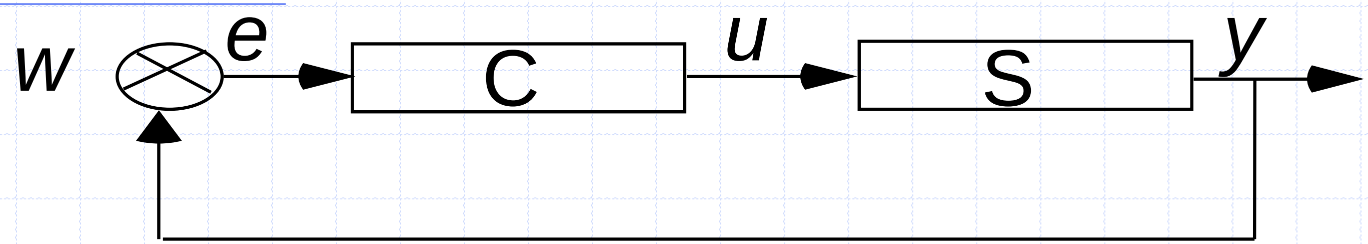
DEFINITIONS



One can intuitively argue that the control signal, *u*, in part, is

(a) Proportional to the error;

(b) Proportional to both the magnitude of the error and the duration of the error

(c) Proportional to the relative changes in the error values over time

FUZZY CONTROL

DEFINITIONS: Conventional Control and Fuzzy Control

- **In the case of classical operations of process control one has to solve the non-linear function u. Furthermore, it is very important that one also finds the proportionality constants K_I, K_D, and K_P**
- **In the case of fuzzy controller, the non-linear function is represented by a fuzzy mapping, typically acquired from human beings**

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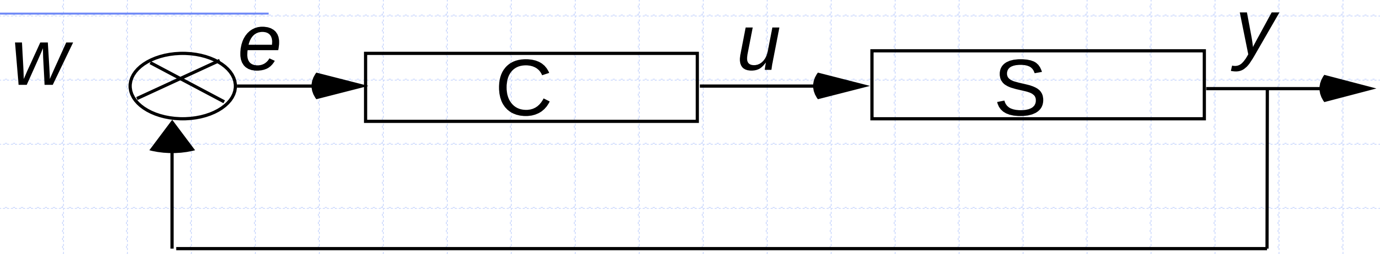
DEFINITIONS



One can intuitively argue that the control signal, *u*, in part,

(a) Proportional to the error;

(b) Proportional to the both the magnitude of the error and the duration of the error

(c) Proportional to the relative changes in the error values over time

The above intuition can be expressed more formally as an algebraic equation involving three proportionality constants –K_P, K_I and K_D

$$u(t) = K_P\, e(t) + K_I \int_0^t e(\tau)\,d\tau + K_D \frac{de(t)}{dt}$$

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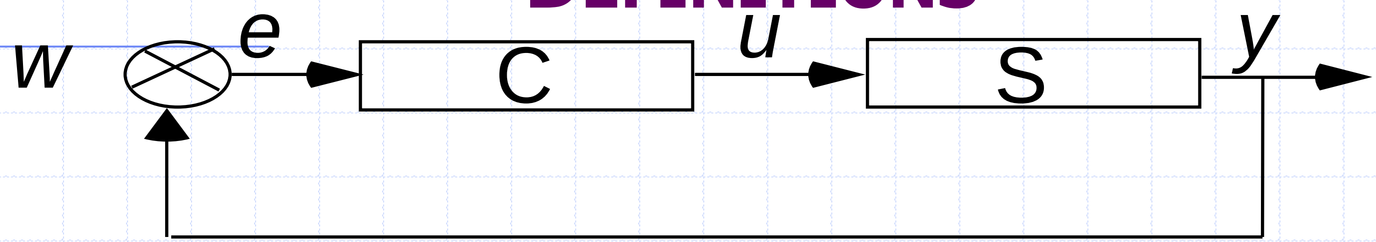
DEFINITIONS

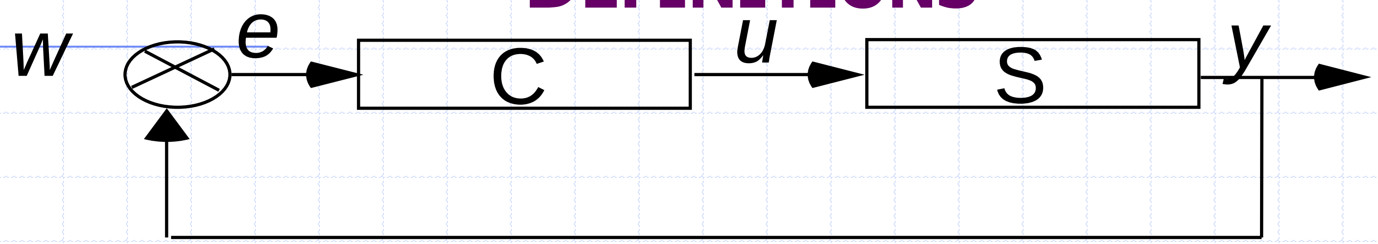


$$u(t) = K_P\, e(t) + K_I \int_0^t e(\tau)\,d\tau + K_D \frac{de(t)}{dt}$$

Value	determines reaction to the
Proportional (K_p)	current error
Integral (K_I)	sum of recent errors
Derivative (K_D)	rate at which the error has been changing

Fuzzy Control

Definitions



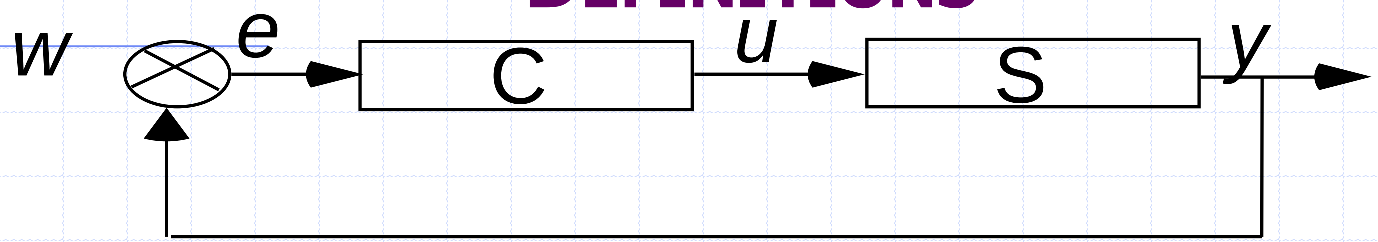
$$u(t) = K_P\, e(t) + K_I \int_0^t e(\tau)\,d\tau + K_D \frac{de(t)}{dt}$$

Value	determines reaction to the
Proportional	current error
Integral	sum of recent errors
Derivative	rate at which the error has been changing

The weighted sum of these three actions is used to adjust the process via a control element such as the position of a control valve or the power supply of a heating element.

Fuzzy Control

Definitions



$$u(t) = K_P\, e(t) + K_I \int_0^t e(\tau)\,d\tau + K_D \frac{de(t)}{dt}$$

$$u(t) = \sum_{i=P,D,I} K_i x_i$$

where

$$x_p = e(t);\; x_I = \int_0^t e(\tau)\,d\tau;\; x_D = \frac{de(t)}{dt}$$

FUZZY CONTROL

FUZZY CONTROLLERS

•A knowledge-based system for closed-loop control is a control system which enhances the performance, reliability, and robustness of control by incorporating knowledge which cannot be accommodated in the analytic model upon which the design of a control algorithm is based, and that is usually taken care of manual modes of operation, or by other safety and ancillary logic mechanisms.

Driankov,D., Hellendoorn, H., & Reinfrank, M. (1996). *An Introduction to Fuzzy Control. (2nd Edition).* Heidelberg: Springer-Verlag

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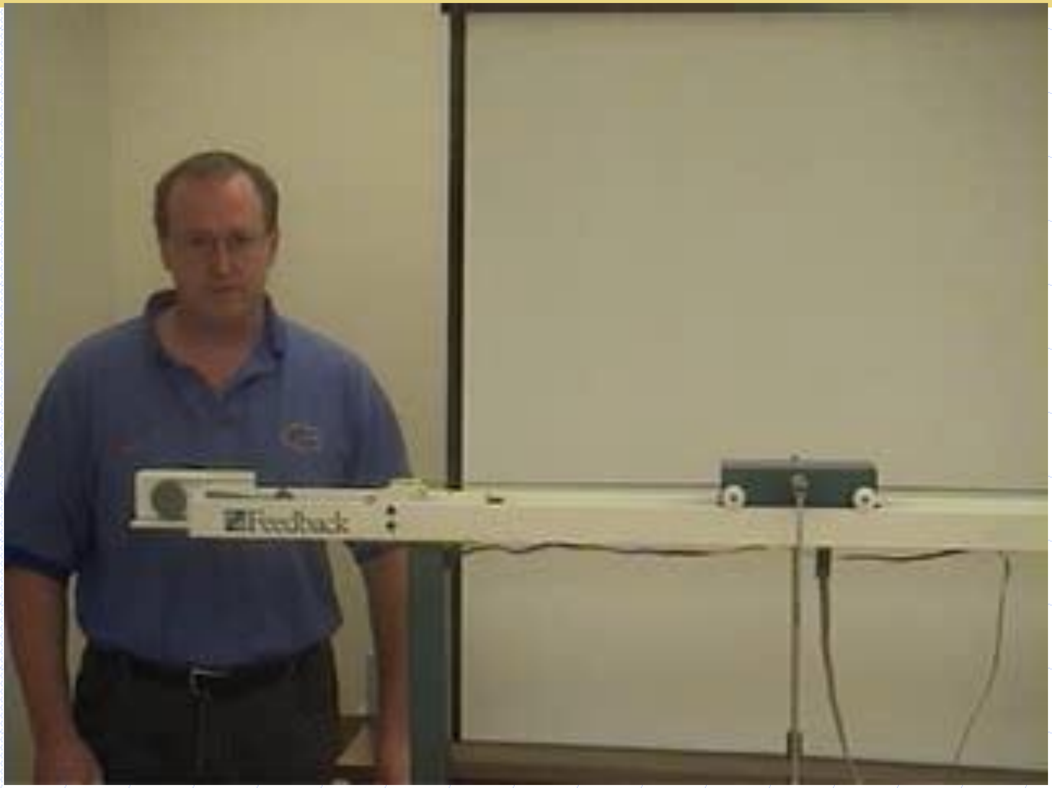
•A pendulum in control – linear and rotatory motion control: inverted pendulum uprighting and balancing with linear cart motion



Please have a look at quadrocopter pole balancing demos

http://www.video_demos.colostate.edu/controls/CSU_Controls_Lab/cart_inverted_pendulum.wmv

FUZZY CONTROL

DEFINITIONS: Conventional Control and Fuzzy Control

- In the case of classical operations of process control one has to solve the non-linear function *u*. Furthermore, it is very important that one also finds the proportionality constants K_I, K_D, and K_P
- In the case of fuzzy controller, the non-linear function is represented by a fuzzy mapping, typically acquired from human beings

FUZZY CONTROL

DEFINITIONS: Conventional Control and Fuzzy Control

'Conventional control theory uses a mathematical model of a process to be controlled and specifications of the desired closed-loop behavior to design a controller. This approach may fall short if the model of the process:

(a) is difficult to obtain, or

(b) is (partly) unknown, or

(c) is highly nonlinear.

(Babuska & Mamdani, accessed 16th Nov. 2007*)

*http://www.scholarpedia.org/article/Fuzzy_Control

FUZZY CONTROL

DEFINITIONS: Conventional Control and Fuzzy Control

'Conventional control theory uses a mathematical model of a process to be controlled and specifications of the desired closed-loop behavior to design a controller. This approach may fall short if the model of the process is difficult to obtain, (partly) unknown, or highly nonlinear. The design of controllers for seemingly easy everyday tasks such as driving a car or grasping a fragile object continues to be a challenge for robotics, while these tasks are easily performed by human beings. Yet, humans do not use mathematical models nor exact trajectories for controlling such processes.' (Babuska & Mamdani, accessed 16th Nov. 2007*)

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FUZZY CONTROL

FUZZY CONTROLLERS

- Here are some *heuristics* for making decisions in a feedback control loop:

IF the error is positive (negative)
& the change in error is approximately zero
THEN the change in control is positive (negative);

IF the error is approximately zero
& the change in error is positive (negative)
THEN the change in control is positive (negative);

IF the error and change in error are approximately zero
THEN the change in control is approximately zero.

FUZZY CONTROL

DEFINITIONS: Conventional Control and Fuzzy Control

- Logical rules with vague predicates can be used to derive inference from vague formulated data.
- The idea of linguistic control algorithms was a generalisation of the human experience to use linguistic rules with vague predicates in order to formulate control actions.

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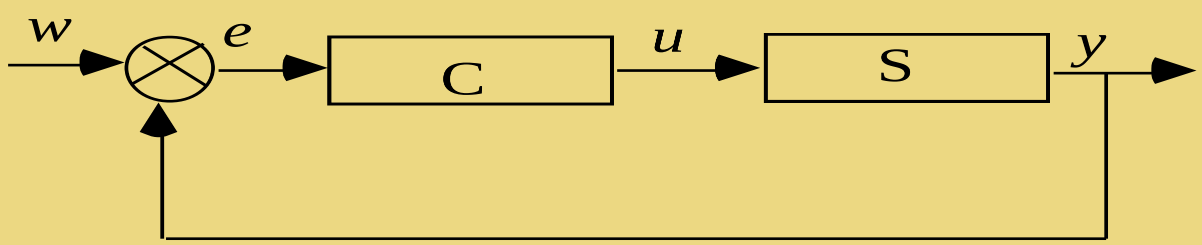
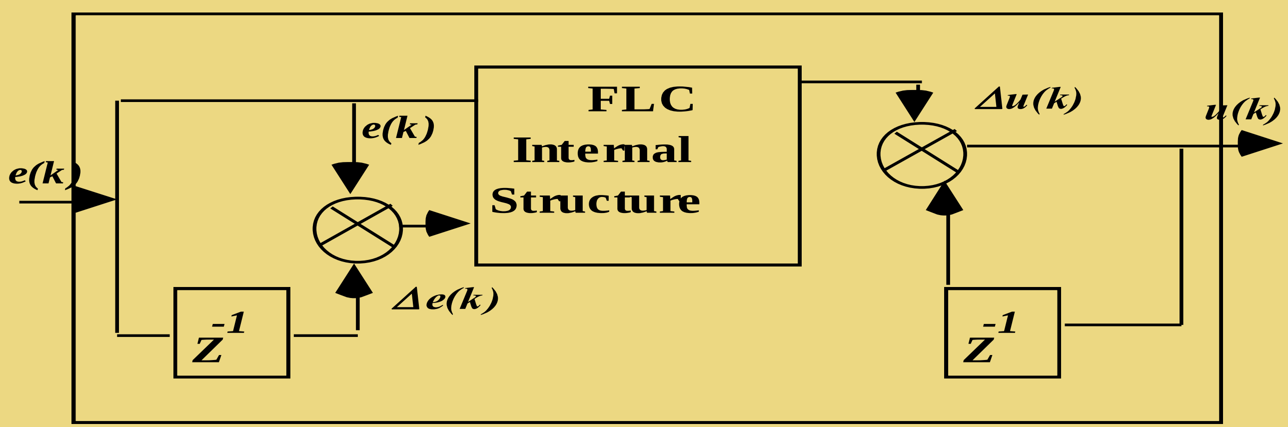
DEFINITIONS: Conventional Control and Fuzzy Control



Conventional Control System



Fuzzy-logic based Control System

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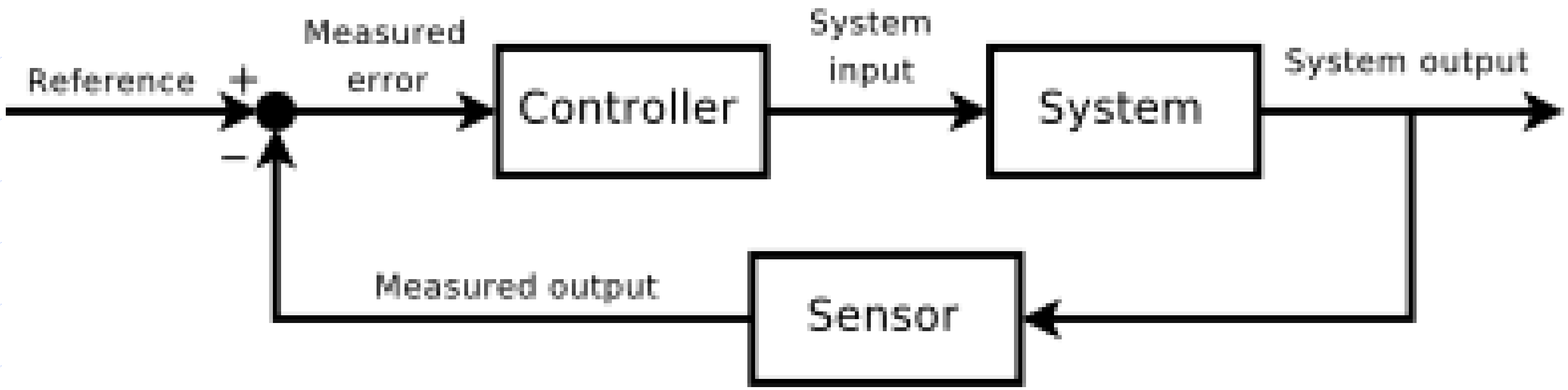
DEFINITIONS: Conventional Control and Fuzzy Control



http://en.wikipedia.org/wiki/Control_theory

FUZZY CONTROL

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- A fuzzy controller is a device that is intended to modelise some vaguely known or vaguely described *process*.

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- A knowledge-based system for closed-loop control is a control system which enhances the performance, reliability, and robustness of control by incorporating knowledge which cannot be accommodated in the analytic model upon which the design of a control algorithm is based, and that is usually taken care of manual modes of operation, or by other safety and ancillary logic mechanisms.

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FUZZY CONTROL

DEFINITIONS: Fuzzy Control

There are two types of fuzzy controllers:

Controller-type	Typical Operation
Mamdani (linguistic) controller with either fuzzy or singleton consequents.	*Direct* closed-loop controller
Takagi-Sugeno (TS) or Takagi-Sugeno-Kang controller	*Supervisory* controller – as a self tuning device

*http://www.scholarpedia.org/article/Fuzzy_Control

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- **The controller can be used with the process in two modes: *Feedback* mode when the fuzzy controller will act as a control device; and *feedforward* mode where the controller can be used as a prediction device.**

- All inputs to, and outputs from, the controller are in the form of linguistic variables. In many ways, a fuzzy controller maps the input variables into a set of output linguistic variables.

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- **Usually, a plant, process, vehicle, or any other <u>object</u> to be controlled is called a system (S).**

- The feedback controller is expected to 'guarantee a desired response', or output *y*.

Yager, R R & Filev, D P. (1994) *Essentials of Fuzzy Modeling and Control.* Chichester: John Wiley & Sons Ltd

FUZZY CONTROL

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- **Regulation** is a process described in the control theory literature as a process for 'keeping the output y close to the setpoint (reference input) w, despite the presence of disturbances, fluctuations of the system parameters, and noise measurements'. (Error $e=w-y$)

- **A controller is implemented using the *control algorithm*.**

Fuzzy Control

Fuzzy Controllers

A controller is implemented using the *control algorithm.*

Vehicle dynamics: Vehicle moving with velocity v(t) and control u(t):

$$\tau \frac{dv}{dt} + v(t) = Ku(t)$$

The solution of the above equation for K=2km/hour and τ=15 seconds:

$$v(t) = 0.936\, v(t-1) + 0.128\, u(t-1).$$

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The principal message in the fuzzy control literature is that "the control algorithm is a knowledge-based algorithm, described by the methods of fuzzy logic' (Yager and Filev, 1994:111)

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FUZZY CONTROLLERS

•A typical fuzzy logic controller is described by the relationship between change of control ($u(k)$) on the one hand and the error ($e(k)$) and change in the error on the other hand

$$\Delta e(k) = e(k) - e(k-1).$$

Such a control law is formalised as:

$$\Delta u(k) = F(e(k), \Delta e(k)).$$

FUZZY CONTROL
FUZZY CONTROLLERS

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FUZZY CONTROL

FUZZY CONTROLLERS

- Here are some *heuristics* for making decisions in a feedback control loop:

System Responsiveness

IF the error is positive (negative)
& the change in error is approximately zero
THEN the change in control is positive (negative);

Reduction in overshooting

IF the error is approximately zero
& the change in error is positive (negative)
THEN the change in control is positive (negative);

Steady State Control

IF the error and change in error are approximately zero
THEN the change in control is approximately zero.

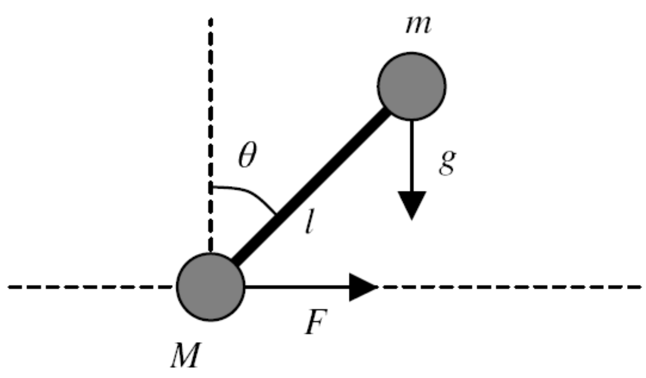
FUZZY CONTROL

Balancing the Cartpole

The Cartpole Problem is often used to illustrate the use of fuzzy logic.

Basically, we have a pole of length ***l***, with a mass ***m*** at its head and mass ***M*** at its base, has to be kept upright. The application of a force ***F*** is required to control the pole. These two masses are connected by a weightless shaft. The base can be moved on a horizontal axis.

The angle of the pole in relation to the vertical axis (θ), and the angular velocity ($d\theta/dt$) are two OUTPUT variables



Kruse, R., Gebhardt, J., & Klawonn (1994). *Foundations of fuzzy systems*. Chichester: John Wiley & Sons Ltd

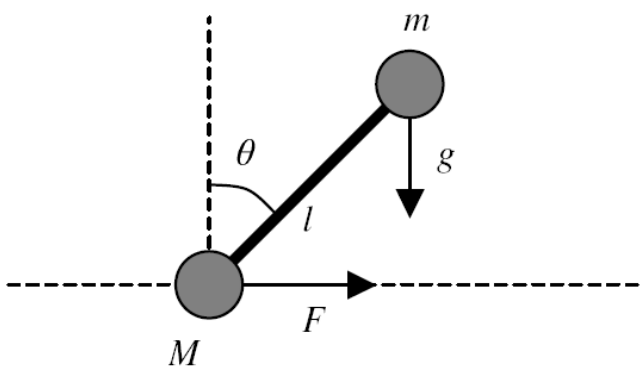
FUZZY CONTROL

Balancing the Cartpole

The Cartpole Problem is often used to illustrate the use of fuzzy logic.

The task is to determine the FORCE (F) necessary to balance the pole.

The key variables are the angle of the pole in relation to the vertical axis (θ), and the angular velocity ($d\theta/dt$)



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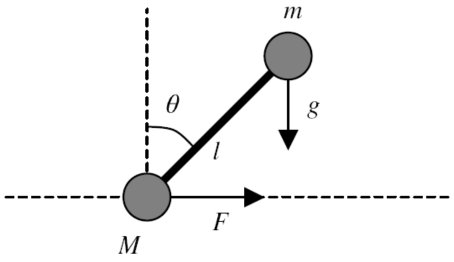
FUZZY CONTROL

Balancing the Cart pole

The control engineering solution for computing *F* is a second-order differential equation:

$$l*(M+m)g*\sin^2\theta*\frac{d^2\theta}{dt^2}$$
$$+m*l*\sin\theta*\cos\theta*\left(\frac{d\theta}{dt}\right)^2$$
$$-g*(M+m)*\sin\theta$$
$$=-F*\cos\theta$$



Fuzzy Control

Balancing the Cart pole

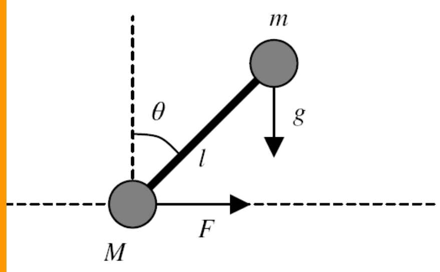
The control engineering solution for computing *F* is the computation of the angle (or rather, a periodic function of θ), the angular velocity, and the angular accelaration.

$$\Psi_1 * \sin^2\theta * angular\ accelaration$$

$$\Psi_2 * \frac{1}{2}\sin(2\theta) * (angular\ velocity)^2$$

$$\Psi_3 * \sin(\theta)$$

$$= -F * \cos(\theta)$$



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Balancing the Cart pole

It is possible to approach the control of the cart pole without the use of the differential equation by rules like:

IF

θ is approximately_zero (a_z) &

$d\theta/dt$ is approximately_zero (a_z)

THEN

F is approximately zero (a_z)

<u>A linguistic rule!</u>

FUZZY CONTROL

Balancing the Cart pole

It is possible to approach the control of the cart pole, without the use of the differential equation, by rules that can be expressed by a decision matrix that relates the angle at which the cartpole was tilting and change in the angle over time (angular velocity) to the force to be applied.

FUZZY CONTROL

Balancing the Cart pole

Consider the so-called *coarse* fuzzy partition of the linguistic variable *angle* (θ) expressed through the linguistic terms *negative* (n_θ), *approximately zero* (az_θ) and *positive* (p_θ).

The same can be said about a fuzzy partition of the angular velocity (Ǿ) and for the applied force (F): negative ($n_{Ǿ}$), approximately zero ($az_{Ǿ}$) and positive ($p_{Ǿ}$); negative velocity (n_F), approximately zero (az_F) and positive (p_F);

FUZZY CONTROL

Balancing the Cart pole

Note that three linguistic variables have their own term sets – a positive value of angle θ may not be the same in magnitude as the angular velocity $\dot{\theta}$ or that of force F.

FUZZY CONTROL

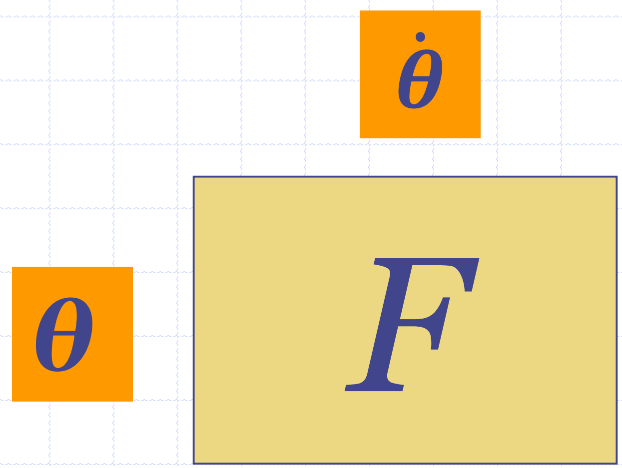
Balancing the Cart pole

There are nine possible rules in the coarsely partitioned fuzzy sets of angle, angular velocity and force. These are shown in the 4x4 matrix on the left hand side; the right hand side diagram shows the symbolic relationship between the three variables.

Angular velocity

Angle

	$n_{\acute{\o}}$	$az_{\dot{\theta}}$	$p_{\dot{\theta}}$
n_{θ}	n_F	n_F	az_F
az_{θ}	n_F	az_F	p_F
p_{θ}	az_F	p_F	p_F



FUZZY CONTROL
Balancing the Cartpole

For many control problems, like our cartpole balancing systems, given a crisp input value, a fully automatic control system requires the calculation of an appropriate crisp value.

The linguistic rules, comprising expressions *approximately zero, positive yet small* (*P_s*) have to be rendered computable.

FUZZY CONTROL

Balancing the Cartpole

The coarse rule base shown above has to be refined further to fine tune the control of the cartpole.

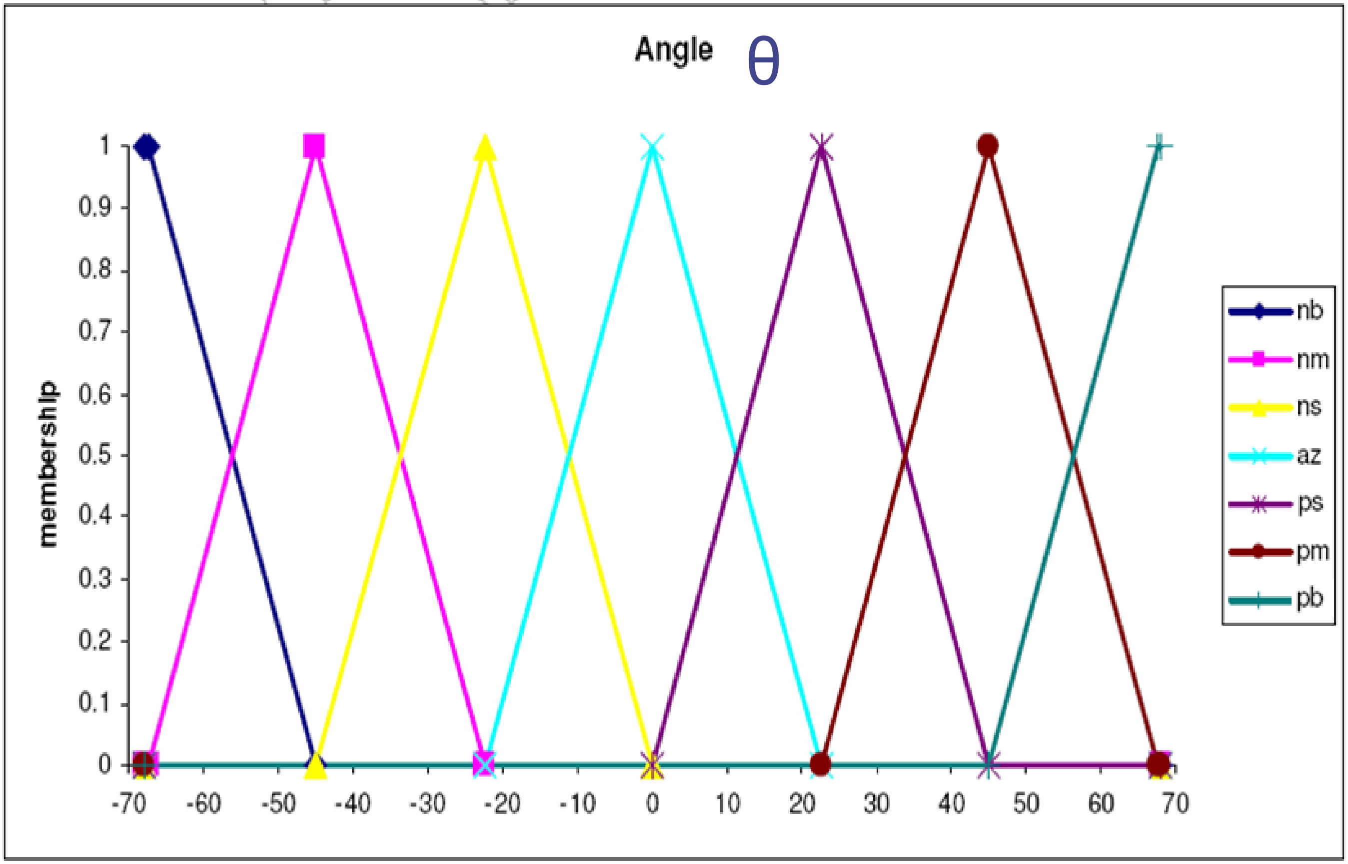
More terms are added to the existing term sets, comprising *negative*, *positive* and *approximately zero*, with terms like *negative big*, *negative medium* and *negative small*, *positive big*, *positive medium* and *positive small*.

Nine terms in all for each of the three variables.

FUZZY CONTROL

Balancing the Cartpole

A finer fuzzy partition for the linguistic variable θ



FUZZY CONTROL

Balancing the Cartpole

Triangular Membership

$$f(x:\alpha,\beta,\delta)=\begin{cases}0, & x\le\alpha\\ \dfrac{x-\alpha}{\beta-\alpha}, & \alpha\le x\le\beta\\ \dfrac{\delta-x}{\delta-\beta}, & \beta\le x\le\delta\\ 0 & \delta\le x\end{cases}$$

Fuzzy Control

Balancing the Cartpole

Theta =36, Theta_dot=-2.25

Fuzzification:
Theta is either positive_small or positive_medium;
Theta_dot is either approximately_zero or is negative+small

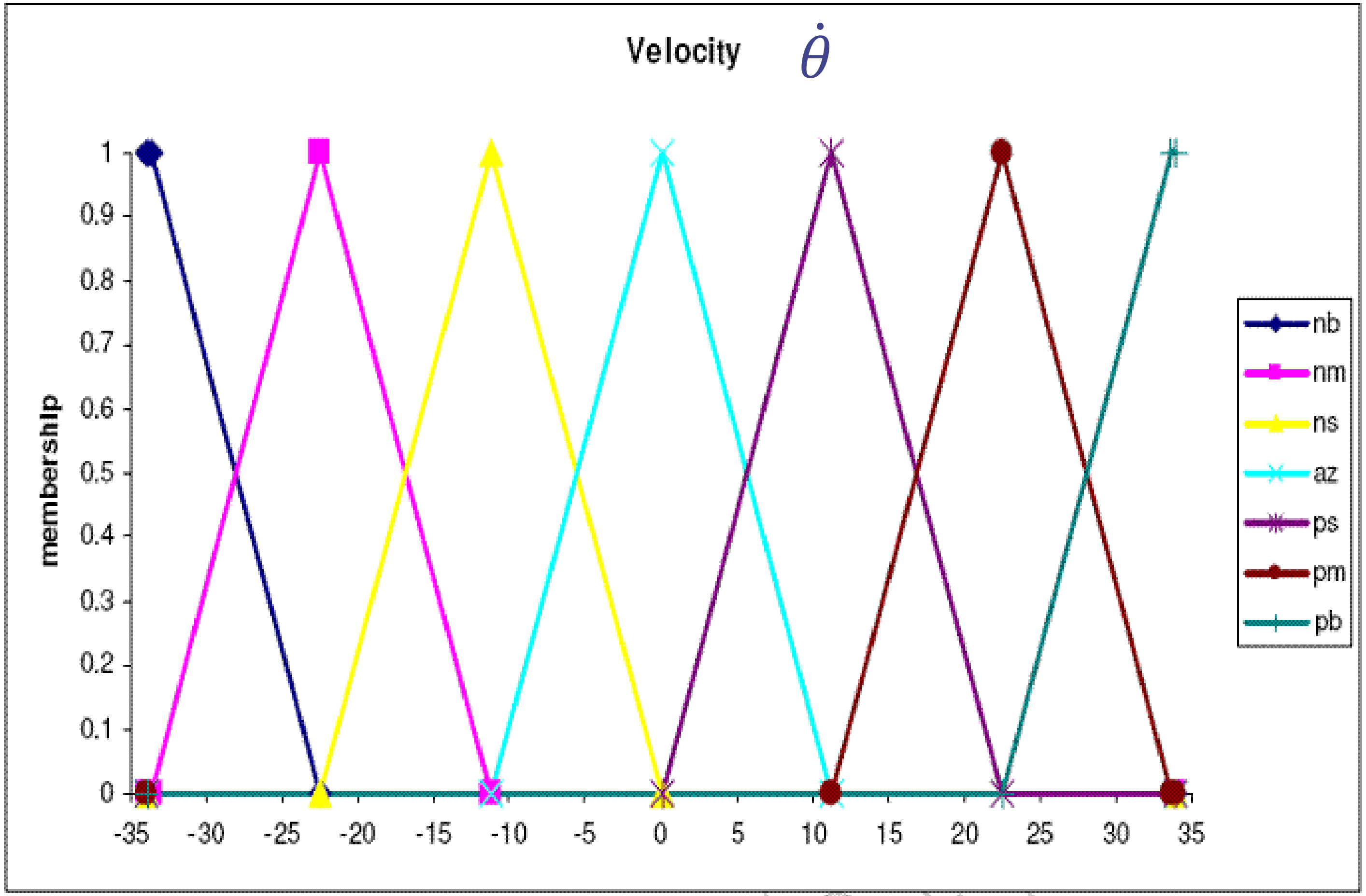
Theta=36 is neither negative or positive big nor is it negative or positive small

θ

FUZZY CONTROL

Balancing the Cartpole

A finer fuzzy partition for the linguistic variable $\dot{\theta}$



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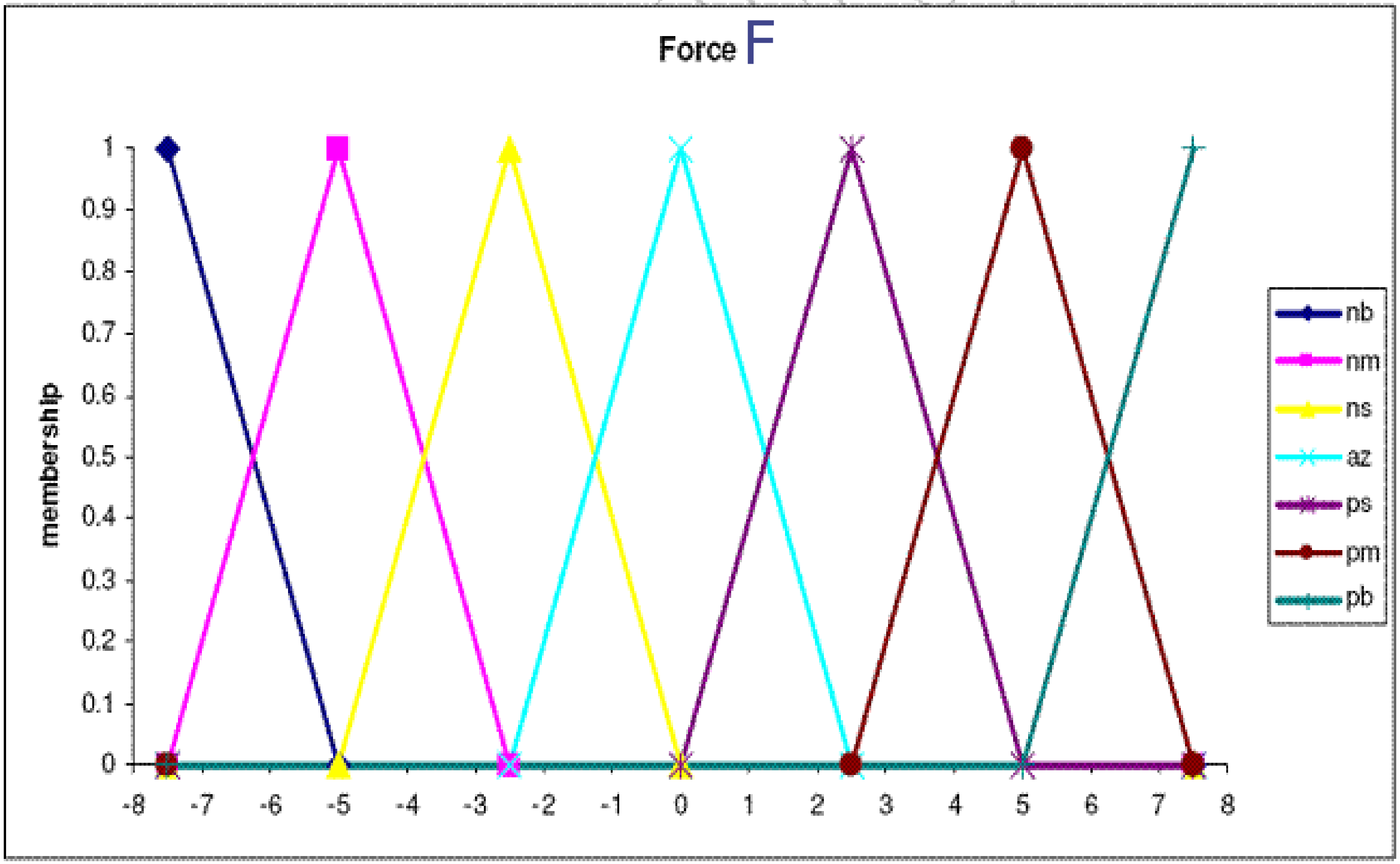
Balancing the Cartpole

A finer fuzzy partition for the linguistic variable F



FUZZY CONTROL

Balancing the Cartpole: The control matrix

A finer fuzzy partition of the linguistic variables leads to a 19 rule knowledge base

	n_b	n_m	n_s	a_z	p_s	p_m	p_b
n_b			p_s	p_b			
n_m				p_m			
n_s	n_m		n_s	p_s			
a_z	n_b	n_m	n_s	a_z	p_s	p_m	p_b
p_s				n_s	p_s		p_m
p_m				n_m			
p_b				n_b	n_s		

FUZZY CONTROL

Balancing the Cartpole: The control matrix

θ

$\dot{\theta}$

F

The change in angle, and the rate at which it changes, has to be accompanied by a change in the force that balances the cartpole.

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Balancing the Cartpole: The rule base

Rule			
#1	IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is a_z	THEN	$\underline{\underline{F}}$ is a_z
#2	IF $\underline{\underline{\theta}}$ is P_s and $\underline{\underline{\dot{\theta}}}$ is P_s	THEN	$\underline{\underline{F}}$ is P_s
#3	IF $\underline{\underline{\theta}}$ is P_s and $\underline{\underline{\dot{\theta}}}$ is a_z	THEN	$\underline{\underline{F}}$ is P_s
#4	IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is n_s	THEN	$\underline{\underline{F}}$ is P_s
#5	IF $\underline{\underline{\theta}}$ is n_s and $\underline{\underline{\dot{\theta}}}$ is n_b	THEN	$\underline{\underline{F}}$ is P_s

FUZZY CONTROL

Balancing the Cartpole: The rule base

Rule		
#6	IF $\underline{\underline{\theta}}$ is P_b and $\underline{\underline{\dot{\theta}}}$ is P_s	THEN $\underline{\underline{F}}$ is P_m
#7	IF $\underline{\underline{\theta}}$ is P_m and $\underline{\underline{\dot{\theta}}}$ is a_z	THEN $\underline{\underline{F}}$ is P_m
#8	IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is n_m	THEN $\underline{\underline{F}}$ is P_m
#9	IF $\underline{\underline{\theta}}$ is P_b and $\underline{\underline{\dot{\theta}}}$ is a_z	THEN $\underline{\underline{F}}$ is P_b
#10	IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is n_b	THEN $\underline{\underline{F}}$ is P_b

FUZZY CONTROL

Balancing the Cartpole: The rule base

#11	IF $\underline{\underline{\theta}}$ is P_s and $\underline{\underline{\dot{\theta}}}$ is P_b	THEN	$\underline{\underline{F}}$ is n_s
#12	IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is P_s	THEN	$\underline{\underline{F}}$ is n_s
#13	IF $\underline{\underline{\theta}}$ is n_s and $\underline{\underline{\dot{\theta}}}$ is n_s	THEN	$\underline{\underline{F}}$ is n_s
#14	IF $\underline{\underline{\theta}}$ is n_s and $\underline{\underline{\dot{\theta}}}$ is a_z	THEN	$\underline{\underline{F}}$ is n_s

FUZZY CONTROL

Balancing the Cartpole: The rule base

#15 IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is P_m THEN $\underline{\underline{F}}$ is n_m

#16 IF $\underline{\underline{\theta}}$ is n_m and $\underline{\underline{\dot{\theta}}}$ is a_z THEN $\underline{\underline{F}}$ is n_m

#17 IF $\underline{\underline{\theta}}$ is n_b and $\underline{\underline{\dot{\theta}}}$ is n_s THEN $\underline{\underline{F}}$ is n_m

#18 IF $\underline{\underline{\theta}}$ is a_z and $\underline{\underline{\dot{\theta}}}$ is P_b THEN $\underline{\underline{F}}$ is n_b

#19 IF $\underline{\underline{\theta}}$ is n_b and $\underline{\underline{\dot{\theta}}}$ is a_z THEN $\underline{\underline{F}}$ is n_b

FUZZY CONTROL

Balancing the Cartpole: The membership functions

Let us look at a simple controller. The intervals of the fuzzy sets related to the control variables, both input and output, are spread over equal intervals.

The universe of discourse for the angle of the pole is defined over the interval [-90°,+90°] and the fuzzy membership function for the angle θ, μ_θ is defined over this universe of discourse. The angular velocity $\dot{\theta}$ is defined over the interval [-45°,+45°] and the force F is defined over the interval [-10,+10]. A typical simple controller is controlled by triangular functions. For the angular membership function the width is 45°, the angular velocity the width is 22.5° and the force has a width of 5.

FUZZY CONTROL

Balancing the Cartpole: The membership functions

	n_b	n_m	n_s	a_z	p_s	p_m	p_b
θ_{max}	-67.5	-45.0	-22.5	0	22.5	45.0	67.5
$\theta^{(1)}_{min}$	-45.0	-22.5	0	-22.5	0	22.5	45.0
$\theta^{(2)}_{min}$	-90.0	-67.5	-45.0	22.5	45.0	67.5	90.0
$\dot{\theta}_{max}$	-33.75	-22.5	-11.25	0	11.25	22.5	33.75
$\dot{\theta}^{(1)}_{min}$	-22.5	-11.25	0	-11.25	0	11.25	22.5
$\dot{\theta}^{(2)}_{min}$	-45	-33.75	-22.5	11.25	22.5	33.75	45
F_{max}	-7.5	-5.0	-2.5	0	2.5	5.0	7.5
$F^{(1)}_{min}$	-5.0	-2.5	0	-2.5	0	2.5	5.0
$F^{(2)}_{min}$	-10.0	-7.5	-5.0	2.5	5.0	7.5	10.0

FUZZY CONTROL

Balancing the Cartpole: An example

Consider the case when the input variables are:

$$\theta = \quad 36^o$$

$$\dot{\theta} = \quad -2.25^o / \sec.$$

When we look at the rule base, only rules 3 & 7 will fire:

#3 IF $\underline{\underline{\theta}}$ is P_s and $\underline{\underline{\dot{\theta}}}$ is a_z THEN $\underline{\underline{F}}$ is P_s

#7 IF $\underline{\underline{\theta}}$ is P_m and $\underline{\underline{\dot{\theta}}}$ is a_z THEN $\underline{\underline{F}}$ is P_m

FUZZY CONTROL

Balancing the Cartpole: Fuzzification

Consider the case when the input variables are:

$$\theta = 36^o$$
$$\dot{\theta} = -2.25^o / \sec.$$

For θ the relevant linguistic terms are *positive small (Rule#3)* & *positive medium (Rule#7)* and the corresponding membership functions are:

$$\mu_{ps}(\theta) = \begin{cases} 1/22.5\theta & if \quad 0 \le \theta \le 22.5 \\ 1 & if \quad \theta = 22.5 \\ -1/22.5\theta + 2 & if \quad 22.5 \le \theta \le 45 \end{cases}$$

$$\mu_{pm}(\theta) = \begin{cases} 1/22.5\theta - 1 & if \quad 22.5 \le \theta \le 45 \\ 1 & if \quad \theta = 45 \\ -1/22.5\theta + 3 & if \quad 45 \le \theta \le 67.5 \end{cases}$$

FUZZY CONTROL

Balancing the Cartpole: Inference

For the angular velocity $\dot{\theta}$ the relevant linguistic term is *approximately zero* (for Rules#3&7) and the corresponding membership functions are:

$$\mu_{az}(\dot{\theta}) = \begin{cases} 1/11.25\,\dot{\theta}+1 & if \quad -11.25 \le \dot{\theta} \le 0 \\ 1 & if \quad \dot{\theta} = 0 \\ -1/11.25\,\dot{\theta}+1 & if \quad 0 \le \dot{\theta} \le 11.25 \end{cases}$$

FUZZY CONTROL

Balancing the Cartpole: Inference

From the control matrix (given above) we infer that the relevant linguistic terms for *F* are *positive small* (Rule#3) and *positive medium* (Rule#7):

$$\mu_{ps}(F) = \begin{cases} 2/5F & if \quad 0 \le F \le 2.5 \\ 1 & if \quad F = 2.5 \\ -2/5F + 2 & if \quad 2.5 \le F \le 5 \end{cases}$$

$$-----------------------------$$

$$\mu_{pm}(F) = \begin{cases} 2/5F - 1 & if \quad 2.5 \le F \le 5 \\ 1 & if \quad F = 5 \\ -2/5F + 3 & if \quad 5 \le F \le 7.5 \end{cases}$$

FUZZY CONTROL

Balancing the Cartpole: Inference

The premise of Rule#3 has been fulfilled to the following extent:

$$\mu_{ps}(\theta) = -1/22.5\theta + 2 \quad 22.5 \le \theta \le 45,$$

$$\therefore \quad \mu_{ps}(36) = -36/22.5 + 2 = 0.4$$

$$\mu_{az}(\dot{\theta}) = 1/11.25\dot{\theta} + 1 \quad if \quad -11.25 \le \dot{\theta} \le 0$$

$$\therefore \mu_{az}(\dot{\theta}) = -2.25/11.25 + 1 = 0.8$$

From the angular (θ) premise we have 0.4 and from the angular velocity (Ǿ) premise we have 0.8. As the two premises are conjunctive then we have to take the minimum of the two ➔min{0.4,0.8}=0.4

FUZZY CONTROL

Balancing the Cartpole: Inference

The output fuzzy set of Rule#3 is determined by applying an α-level cut to fuzzy set associated with the output – the force *F* fuzzy set in our case. The cut is to be applied at 0.4:

$$\mu_{\theta = 36^\circ \,\&\, \dot{\theta} = -2.25^\circ/\text{second}}^{output(R_3)}(F) = \begin{cases} 2/5F & if & 0 \le F \le 1 \\ 0.4 & if & 1 \le F \le 4 \\ -2/5F + 2 & if & 4 \le F \le 5 \\ 0 & otherwise & \end{cases}$$

FUZZY CONTROL

Balancing the Cartpole: Inference

The premise of Rule#7 has been fulfilled to the following extent:

$$\mu_{pm}(\theta) = 1/22.5\theta - 1 \quad 22.5 \le \theta \le 45,$$

$$\therefore \quad \mu_{pm}(36) = 36/22.5 - 1 = 0.6$$

$$\mu_{az}(\dot{\theta}) = 1/11.25\dot{\theta} + 1 \quad if \quad -11.25 \le \dot{\theta} \le 0$$

$$\therefore \mu_{az}(\dot{\theta}) = -2.25/11.25 + 1 = 0.8$$

From the angular premise we have 0.6 and from the angular velocity premise we have 0.8. As the two premises are conjunctive then we have to take the minimum of the two ➔min{0.6,0.8}=0.6

FUZZY CONTROL

Balancing the Cartpole: Inference

The output fuzzy set of Rule#7 is determined by applying an α-level cut to fuzzy set associated with the output – the force F fuzzy set in our case. The cut is to be applied at 0.6:

$$\mu^{output(R_7)}_{\theta=36^\circ \,\&\, \dot{\theta}=-2.25^\circ/\sec}(F)=\begin{cases} 2/5F-1 & \text{if} & 2.5\le F\le 4 \\ 0.6 & \text{if} & 4\le F\le 6 \\ -2/5F+3 & \text{if} & 6\le F\le 7.5 \\ 0 & \text{otherwise} & \end{cases}$$

FUZZY CONTROL

Balancing the Cartpole: Composition

The task now is to combine all the fuzzy sets obtained from all the rules into ONE fuzzy set by determining the MAXIMUM – the union of all the obtained fuzzy sets. In the case of the example of the cartpole only two fuzzy sets, associated with rules 3 and 7, have a non-zero output and all others provide the null fuzzy set as their output:

$$\mu_{\substack{output\left(\bigcup_{i=1}^{19} R_i\right) \\ \theta = 36^\circ \,\&\, \dot{\theta} = -2.25^\circ/\sec}}(F) = \begin{cases} 2/5F & if \quad 0 \le F \le 1 \\ 0.4 & if \quad 1 \le F \le 3.5 \\ 2/5F - 1 & if \quad 3.5 \le F \le 4 \\ 0.6 & if \quad 4 \le F \le 6 \\ -2/5F + 3 & if \quad 6 \le F \le 7.5 \end{cases}$$

FUZZY CONTROL

Balancing the Cartpole: Defuzzification

In order to obtain a single crisp value (η)from the output fuzzy set of all the rules we can use any of the three methods:

(1)The Maximum Criterion Model: Choose any value where the output fuzzy set (of all the rules) achieves a maximum. This means that any value of the force *F* between 4 and 6 will do to steady the cartpole:

$$\mu_{output\left(\bigcup_{i=1}^{19} R_i\right)}(F) = 0.6 \quad if \quad 4 \le F \le 6$$

$$\theta = 36^\circ \;\&\; \dot{\theta} = -2.25^\circ/\sec$$

$$=========== OUTPUT ======================$$

$$\eta = [4,6]$$

FUZZY CONTROL

Balancing the Cartpole: Defuzzification

(2)The Mean of Maximum Method: Here we have to obtain the mean of the the output fuzzy set over the values where it achieves the maximum. This restricts the mean to the output values of 0.6 that is between the values of the force *F* between 4 and 6.

$$\mu^{output}_{\theta=36^\circ \,\&\, \dot{\theta}=-2.25^\circ/\sec}(F)=0.6 \quad if \quad 4\leq F\leq 6$$

$$==============\ OUTPUT\ ==================$$

$$\eta=\frac{1}{\sum_{F\ \varepsilon\ Max(\mu^{output}_{\theta=36^\circ \& \dot{\theta}=-2.25^\circ/\text{second}})}\mu(F)}\sum_{F\ \varepsilon\ Max(\mu^{output}_{\theta=36^\circ \& \dot{\theta}=-2.25^\circ/\text{second}})}F.\mu(F)$$

$$\eta\approx\frac{4*0.6+4.5*0.6+5*0.6+5.5*0.6+6*0.6}{(0.6+0.6+0.6+0.6+0.6)}=\frac{15}{3}=5$$

FUZZY CONTROL

Balancing the Cartpole: Defuzzification

(3)The Centre of Gravity (COG) or Centre of Area (COA) Method: Here we have to obtain the mean of the the output fuzzy set over the entire range of the output fuzzy set. The COA is to be computed over the entire range of values of the output fuzzy set

$$\eta = \frac{1}{\int_{F \varepsilon [0,7.5]} \mu^{output}_{\theta = 36^\circ \,\&\, \dot{\theta} = -2.25^\circ/\sec}(F)\,dF}$$

$$* \int_{F \varepsilon [0,7.5]} F * \mu^{output}_{\theta = 36^\circ \,\&\, \dot{\theta} = -2.25^\circ/\sec}(F)\,dF$$

FUZZY CONTROL

Balancing the Cartpole: Defuzzification

(3)The Centre of Gravity (COG) or Centre of Area (COA) Method: The computation of the output fuzzy set with different values of *F* are as follows (Note that μ(F)=0 if F<=0 or F>=7.5:

F	μ(F)	F* μ(F)=
0.5	0.2	0.1
1	0.4	0.4
1.5	0.4	0.6
2	0.4	0.8
3	0.4	1.2
3.5	0.4	1.4
4	0.6	2.4
4.5	0.6	2.7
5	0.6	3
5.5	0.6	3.3
6.5	0.6	3.9
7	0.6	4.2
SUM	6.2	24.5

FUZZY CONTROL

Balancing the Cartpole: Defuzzification

(3)The Centre of Gravity (COG) or Centre of Area (COA) Method: The computation of the output fuzzy set with different values of *F* are as follows (Note that μ(F)=0 if F<=0 or F>=7.5:

$$\eta \approx \frac{1}{(0+0.2+0.4+0.4+0.4+0.4+0.4+0.4+0.6+0.6+0.6+0.6+0.6+0.6+0.6+0)}$$
$$*(0*0+0.5*0.2+1*0.4+1.5*0.4+2*0.4+2.5*0.4+3*0.4+3.5*0.4+$$
$$4*0.6+4.5*0.6+5*0.6+5.5*0.6+6*0.6+6.5*0.6+7*0.6+7.5*0)$$
$$\eta \approx 24.5/6.2 = 3.95$$

FUZZY CONTROL

The applications of cartpole balancing

The training of a good goalkeeper!

https://www.youtube.com/watch?v=CIF2SBVY-J0