

CS4004/CS4504: FORMAL VERIFICATION

Lecture 5: Propositional Logic

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Last lecture:

→ **Syntactic entailment**: $A_1 \dots A_n \vdash B$

Meaning: from $A_1 \dots A_n$ we can derive B using the inference rules of propositional logic

Different than semantic entailment $A_1 \dots A_n \models B$ and syntactic entailment $A \rightarrow B$. (there is a connection between these entailments)

→ Inference rules are defined using the system of **natural deduction**

Conjunction introduction and elimination rules:

$$\frac{A_1 \quad \dots \quad A_n}{B} \text{ RULE NAME}$$

These are **simple natural deduction rules**

Simple inference rules: “given formulas derive a formula,” e.g.:

$$\frac{A_1 \quad A_2}{A_1 \wedge A_2} \wedge i \qquad \frac{A_1 \wedge A_2}{A_1} \wedge e_1 \qquad \frac{A_1 \wedge A_2}{A_2} \wedge e_2$$

Proof of $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$.

1	$(p \wedge q) \wedge r$	premise
2	$s \wedge t$	premise
3	$p \wedge q$	$\wedge e_1$ 1
4	q	$\wedge e_2$ 3
5	s	$\wedge e_1$ 2
6	$q \wedge s$	$\wedge i$ 4, 5



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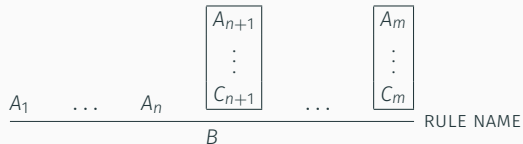
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Proof of $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$.

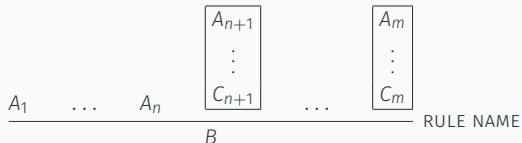
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Complex inference rules: “given **proofs and formulas** derive a formula”



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Meaning of $\boxed{\begin{array}{c} A_i \\ \vdots \\ C_i \end{array}}$: “assume A_i has been established in your proof and prove C_i .”

Example: disjunction rules.

DISJUNCTION

The introduction rules of \vee are simple rules.

- given a number of formulas as premises...(?)
- produce a formula $A_1 \vee A_2$.

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- and given some proofs... (?)
- produce a formula B

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$$\frac{A_1}{A_1 \vee A_2} \vee i_1 \qquad \frac{A_2}{A_1 \vee A_2} \vee i_2 \qquad \frac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \vee e$$

PROOF WITH DISJUNCTION

Prove $p \vee q \vdash q \vee p$

Proof.



$$\frac{A_1}{A_1 \vee A_2} \vee i_1$$

$$\frac{A_2}{A_1 \vee A_2} \vee i_2$$

$$\frac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \wedge e$$

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Prove $p \vee q \vdash q \vee p$

Proof.

1 $p \vee q$ premise

□

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Prove $p \vee q \vdash q \vee p$

Proof.

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2

□

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4	q	assumption
5	$q \vee p$	$\vee i_1$ 4

□

$$\begin{array}{c}
 \frac{A_1}{A_1 \vee A_2} \vee i_1 \quad \frac{A_2}{A_1 \vee A_2} \vee i_2 \quad \frac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \wedge e
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6	$q \vee p$	$\vee e$ 1, 2-3, 4-5

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$$\frac{A_1}{A_1 \vee A_2} \vee i_1$$

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$$\frac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \wedge e$$

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Prove $p \vee q \vdash q \vee p$

Proof.

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Boxes are subproofs, from (A) given assumptions and (B) the formulas we know before the subproof we derive an intermediate conclusion. (Here we did not have to use (B))

$$\begin{array}{c}
 \frac{A_1}{A_1 \vee A_2} \vee i_1 \quad \frac{A_2}{A_1 \vee A_2} \vee i_2 \quad \frac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \wedge e
 \end{array}$$

Prove $(p \vee q) \vee r \vdash p \vee (q \vee r)$. (Hint: nested boxes)

$$\begin{array}{c}
 \frac{A_1}{A_1 \vee A_2} \vee i_1 \qquad \frac{A_2}{A_1 \vee A_2} \vee i_2 \qquad \frac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \vee e
 \end{array}$$

Prove $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$ and $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$.

$$\begin{array}{c}
 \frac{A_1 \quad A_2}{A_1 \wedge A_2} \wedge i \qquad \frac{A_1 \wedge A_2}{A_1} \wedge e_1 \qquad \frac{A_1 \wedge A_2}{A_2} \wedge e_2 \\
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 \end{array}$$

IMPLICATION

First: elimination of \rightarrow (aka **modus ponens**)

$$\frac{A \quad A \rightarrow B}{B} \rightarrow e$$

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From:

p : it rained

$p \rightarrow q$: if it rained then the street is wet

we derive:

q : the street is wet

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we derive:

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From:

p : the program input is an array

$p \rightarrow q$: if the program input is an array then the program output is a sorted array

we derive:

q : the program output is a sorted array

Show $p \rightarrow q \rightarrow r, p, q \vdash r$

Show $p \rightarrow q \rightarrow r, p, p \rightarrow q \vdash r$

$$\frac{A \quad A \rightarrow B}{B} \rightarrow e$$

If we prove $A_1, \dots, A_n \vdash B$ then we can use in our proofs a **derivable rule** (aka a theorem)

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The following is a derivable rule:

$$\frac{A_1 \rightarrow A_2 \quad \neg A_2}{\neg A_1} \text{ MT}$$

this is called **modus tollens**.

Introduction of \rightarrow :

$$\frac{\begin{array}{|c|} A \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow i$$

“If we can prove B by assuming A , then A implies B .”

Prove: $p \rightarrow q \vdash \neg q \rightarrow \neg p$

Example statement: If it rained then the road is wet. Therefore, if the road is not wet then it did not rain.

$$\begin{array}{c}
 \frac{A \quad A \rightarrow B}{B} \rightarrow e \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B} \rightarrow i \qquad \frac{A_1 \rightarrow A_2 \quad \neg A_2}{\neg A_1} \text{ MT}
 \end{array}$$