4CSLL5
Parameter Estimation (Supervised and Unsupervised)

Martin Emms

September 21, 2018

Parameter Estimation

Common-sense and relative frequency

Suppose a 2-sided 'coin' Z, one side labelled 'a', other side labelled 'b'

$P(Z = a)$: probability of giving 'a' when tossed – currently not known

$P(Z = b)$: probability of giving 'b' when tossed – currently not known

Suppose you have data **d** recording 100 tosses of Z

if there were (50 a, 50 b) in **d**, 'common-sense' says $P(Z = a) = 50/100$

if there were (30 a, 70 b) in **d**, 'common-sense' says $P(Z = a) = 30/100$

ie. you 'define' or 'estimate' the probability by the *relative frequency*

Data likelihood

assuming the tosses of Z are all independent, can work out the probability of the observed data **d** *if* Z's probabilities had particular values.

let θ_a and θ_b stand for $P(Z = a)$ and $P(Z = b)$

let $\#(a)$ be the number of 'a' outcomes in the sequence **d**

let $\#(b)$ be the number of 'b' outcomes in the sequence **d**

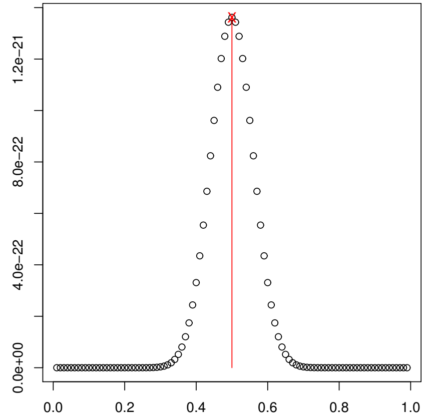
the probability of **d**, assuming the probability settings θ_a and θ_b is

$$p(\mathbf{d}) = \theta_a^{\#(a)} \times \theta_b^{\#(b)} \tag{1}$$

different settings of θ_a and θ_b will give different values for $p(\mathbf{d})$

following slides investigate this empirically

$p(\mathbf{d})$ for 50 a, 50 b

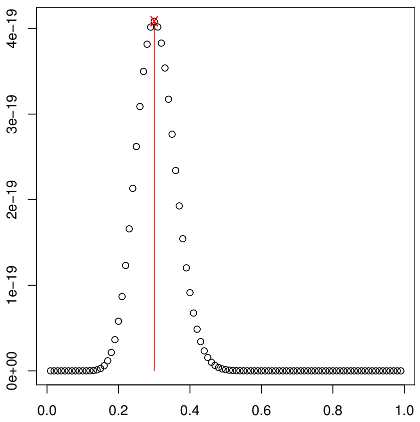


as θ_a is varied, data prob $p(\mathbf{d})$ varies

max occurs at $\theta_a = 0.5$

which is $\frac{50}{50+50}$

$p(\mathbf{d})$ for 30 a, 70 b

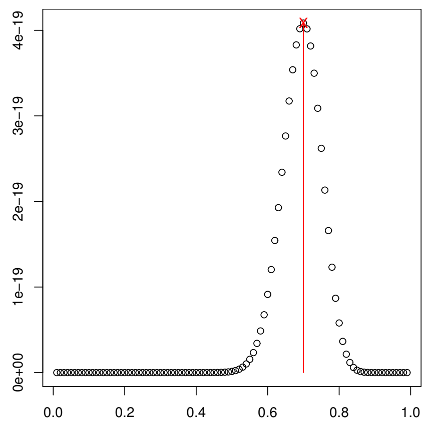


as θ_a is varied, data prob $p(\mathbf{d}; \theta_a, \theta_b)$ varies

max occurs at $\theta_a = 0.3$

which is $\frac{30}{30+70}$

$p(\mathbf{d})$ for 70 a, 30 b



as θ_a is varied, data prob $p(\mathbf{d}; \theta_a, \theta_b)$ varies

max occurs at $\theta_a = 0.7$

which is $\frac{70}{70+30}$

- in each case, it looks like the max of the data probability occured at the value given by the relative frequency
- this suggests that in these cases,

Max. Likelihood Estimator

if you wanted to find θ_a (and θ_b) that maximise the data probability, that is you want

$$\arg\max_{\theta_a,\theta_b} p(\mathbf{d};\theta_a,\theta_b)$$

then the relative frequencies would give the answer, that is

$$\theta_a = \frac{\#(a)}{\#(a)+\#(b)} \quad \theta_b = \frac{\#(b)}{\#(a)+\#(b)}$$

- technically expressed as: the relative frequency is a *maximum likelihood estimator* of the parameters

on reflection, if you have to set parameters given data, it makes a lot of sense to set the parameters to whatever values make the data as likely as possible

formula for $p(\mathbf{d};\theta_a,\theta_b)$ is (1), repeated below

$$p(\mathbf{d};\theta_a,\theta_b) = \theta_a^{\#(a)} \times \theta_b^{\#(b)}$$

and because $\theta_b = 1-\theta_a$ can really write this in terms of just parameter θ_a

$$p(\mathbf{d};\theta_a) = \theta_a^{\#(a)} \times (1-\theta_a)^{\#(b)}$$

Looking at some pics suggested a formula for the value of θ_a that maximises this. Can we actually *derive* this formula?

Yes $\Rightarrow$ take the log of this – the **log-likelihood** and use calculus to maximize *that* w.r.t. θ_a – this turns out to be (relatively) easy

Define $L(\theta_a)$ as $log(P(\mathbf{d};\theta_a))$. Then you get

$$L(\theta_a) = \#(a)\log\theta_a + \#(b)\log(1-\theta_a)$$

need to take derivative wrt to θ_a and set to 0, which is

$$\frac{dL(\theta_a)}{d\theta_a} = \frac{\#(a)}{\theta_a} - \frac{\#(b)}{1-\theta_a} = 0 \quad\Longrightarrow\quad \theta_a = \frac{\#(a)}{\#(a)+\#(b)} = \frac{\#(a)}{100}$$

so in this scenario of 100 tosses of Z, we have proven that **the relative frequency is always going to the maximum likelihood estimator**

now want to consider slightly more complex scenario

a more complex scenario

suppose D repetitions of
toss disc Z, to choose *one* of two coins A or B
then toss chosen coin 10 times

Suppose 9 repetitions gave

d	Z				**X**: tosses of chosen coin							H counts
1	A	H	H	H	H	H	H	H	H	T	T	(8H)
2	B	T	T	H	T	T	T	H	T	T	T	(2H)
3	A	H	T	H	H	T	H	H	H	H	T	(7H)
4	A	H	T	H	H	H	T	H	H	H	H	(8H)
5	B	T	T	T	T	T	T	H	T	T	T	(1H)
6	A	H	H	T	H	H	H	H	H	H	H	(9H)
7	A	T	H	H	T	H	H	H	H	H	T	(7H)
8	A	H	H	H	H	H	H	T	H	H	H	(9H)
9	B	H	H	T	T	T	T	T	H	T	T	(3H)

Let θ_a be Z's probability of giving A
Let $\theta_{h|a}$ be A's probability of giving H
Let $\theta_{h|b}$ be B's probability of giving H

'common sense' calculation of θ_a, $\theta_{h|a}$ and $\theta_{h|b}$

for θ_a, need (*count of Z = A cases*)/(*count of all Z cases*), ie.

$$est(\theta_a) = \frac{\sum_{d:Z=A} 1}{D} = \frac{6}{9} = 0.66 \tag{2}$$

for $\theta_{h|a}$, need
(*count of H when A chosen*)/(*count of all tosses when A chosen*), ie.

$$est(\theta_{h|a}) = \frac{\sum_{d:Z=A} \#(d,h)}{\sum_{d:Z=A} 10} = \frac{48}{60} = \frac{4}{5} = 0.8 \tag{3}$$

for $\theta_{h|b}$, need
(*count of H when B chosen*)/(*count of all tosses when B chosen*), ie.

$$est(\theta_{h|b}) = \frac{\sum_{d:Z=B} \#(d,h)}{\sum_{d:Z=B} 10} = \frac{6}{30} = \frac{1}{5} = 0.2 \tag{4}$$

to make the comparision with the hidden variable version which will come up later, its worth noting that we can formulate all the restricted sums $\sum_{d:Z=A}(\Phi(d))$ with *unrestricted sums* if we put a so-called Kronecker-delta indicator function inside the sum $\sum_d(\delta(d,A)\Phi(d))$ where $\delta(d,A)=1$ if datum d had $Z=A$, and is 0 otherwise.

$$est(\theta_a) = \frac{\sum_d \delta(d,A)}{D} \tag{5}$$

$$est(\theta_{h|a}) = \frac{\sum_d \delta(d,A)\#(d,h)}{\sum_d \delta(d,A)10} \tag{6}$$

$$est(\theta_{h|b}) = \frac{\sum_d \delta(d,B)\#(d,h)}{\sum_d \delta(d,B)10} \tag{7}$$

it turns out that in this scenario also, the 'common-sense', relative-frequency answers are also *maximum likelihood estimators* ie. values which maximise the probability of the data, and again it is (relatively) easy to show this by taking logs and using calculus.

the formula for $p(\mathbf{d};\theta_a,\theta_b,\theta_{h|a},\theta_{t|a},\theta_{h|b},\theta_{t|b})$

$$p(\mathbf{d}) = \prod_{d:Z=a}[\theta_a\theta_{h|a}^{\#(d,h)}\theta_{t|a}^{\#(d,t)}]\prod_{d:Z=b}[\theta_b\theta_{h|b}^{\#(d,h)}\theta_{t|b}^{\#(d,t)}]$$

and its log comes out as

$$\sum_{d:Z=a}[log\theta_a + \#(d,h)log\theta_{h|a} + \#(d,t)log\theta_{t|a}] + \sum_{d:Z=b}[log\theta_b + \#(d,h)log\theta_{h|b} + \#(d,t)log\theta_{t|b}]$$

call this $L(\theta_a,\theta_{h|a},\theta_{h|b})$

$$\sum_{d:Z=a}[log\theta_a + \#(d,h)log\theta_{h|a} + \#(d,t)log\theta_{t|a}] + \sum_{d:Z=b}[log\theta_b + \#(d,h)log\theta_{h|b} + \#(d,t)log\theta_{t|b}]$$

$L(\theta_a,\theta_{h|a},\theta_{h|b})$ – repeated above – can be split into 3 separate terms, $L(\theta_a) + L(\theta_{h|a}) + L(\theta_{h|b})$ concerning Z, A and B

$$L(\theta_a) = [\sum_{d:Z=a} 1]log\theta_a + [\sum_{d:Z=b} 1]log(1-\theta_a) \tag{8}$$

$$L(\theta_{h|a}) = [\sum_{d:Z=a} \#(d,h)]log\theta_{h|a} + [\sum_{d:Z=a} \#(d,t)]log(1-\theta_{h|a}) \tag{9}$$

$$L(\theta_{h|b}) = [\sum_{d:Z=b} \#(d,h)]log\theta_{h|b} + [\sum_{d:Z=b} \#(d,t)]log(1-\theta_{h|b}) \tag{10}$$

and this means that when you take the derivatives of $L(\theta_a,\theta_{h|a},\theta_{h|b})$ wrt. θ_a, $\theta_{h|a}$ and $\theta_{h|b}$ in each case you can just look at one of the above terms. They are all really of the same form being $N(log(p)) + M(log(1-p))$, the same form as seen in the first simple scenario, and it has maximum value at $p = \frac{N}{N+M}$

hence

$$\frac{\partial L(\theta_a)}{\partial \theta_a} = 0 \implies \theta_a = \frac{\sum_{d:Z=a} 1}{\sum_{d:Z=a} 1 + \sum_{d:Z=b} 1}$$

$$\frac{\partial L(\theta_{h|a})}{\partial \theta_{h|a}} = 0 \implies \theta_{h|a} = \frac{\sum_{d:Z=a} \#(d,h)}{\sum_{d:Z=a} \#(d,h) + \sum_{d:Z=a} \#(d,t)}$$

$$\frac{\partial L(\theta_{h|b})}{\partial \theta_{h|b}} = 0 \implies \theta_{h|b} = \frac{\sum_{d:Z=b} \#(d,h)}{\sum_{d:Z=b} \#(d,h) + \sum_{d:Z=b} \#(d,t)}$$

finally the denominators of these turn into D, $\sum_{d:Z=a} 10$ and $\sum_{d:Z=b} 10$ respectively and so are exactly the 'common sense' formulae we started with in (2), (3), (4)