CS4004/CS4504: FORMAL VERIFICATION

Lecture 2: Propositional Logic

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LOGICAL STATEMENTS

Propositions are **declarative statements**.

- → "Alice is an engineer"
- → "The sum of 3 and 5 is 8"
- → "The train is late"

Declarative statements can be *declared* to be either **true** or **false** (but not both). These are the truth-values of propositions.

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Not all statements are declarative. The following cannot be declared true/false.

- “Let’s go to the cinema” (proposal)
- “Where is Soli?” (question)
- “Fantastic!” (exclamation)
- “It will probably rain tomorrow” (likelihood)

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Propositional logic involves **only** declarative statements.

Complex propositions can be constructed by simple ones using operators.

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We can examine whether such propositions are true or false when we know the values of the basic propositions.

→ The train is not late ("the train is late" is **false**),

→ Bob is not late to work ("Bob late to work" is **false**),

→ Is p true? (without knowing whether there were any taxis in the station)

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We can examine the **necessary** values of the basic propositions that would make complex compositions be true/false.

- → *p* is true,
- → Bob is not late to work,
- → The train is late
- → Are there any taxis in the station?

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English (or any human language) is imprecise and subtle (verb tenses, etc.) and error prone.

A more mathematical language for logic would make the above arguments clear ("calculemus!" – stay tuned).

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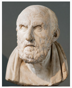
  - **conjunction**, **disjunction**, **negation**, different forms of implication



Chrysippus of Soli – 3rd c. BC

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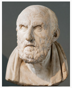

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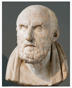

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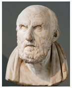
    - Believed all human ideas are made of small number of basic ideas (alphabet of human thought); complex ideas derived from basic ones with combinations similar to arithmetic multiplication.
    - Arguments would be solved by calculating "Calculemus!".
    - Symbolic rules for **conjunction**, **disjunction**, **negation**, ...



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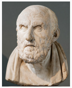

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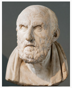
- **Boole** and **De Morgan** redescovered and improved Leibniz calculus of logic (1847) – modern form of Propositional Logic.
- Developed more in the 20th c. (e.g., **Gerhard Gentzen**) influenced by other logicians/philosophers (e.g., Russell, Wittgenstein...)



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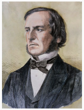

George Boole, Augustus De Morgan – 19th c.

Images from wikipedia

- Declarative statements have **no intrinsic truth-value**.
- They are simply a string of symbols, representing the declarative statements.
- A priori, all declarative statements (**propositions**) could be either `True` or `False` (but not both).
  - "Logic is interesting"
  - "This program terminates"
  - "Bob is male"
  - "Bob is female"
  - "Gates graduated Harvard"

SYMBOLIC PROPOSITIONAL LOGIC

- We will consider some declarative statements (**propositions**) to be atomic or **indecomposable**.
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$$p_1, p_2, \ldots, q_1, q_2, \ldots, r_1, r_2, \ldots$$

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- We form complex propositions using the following operators
  - **Negation**: symbols: $\neg$ $\sim$ examples: $\neg p$ $\sim q$
  - **disjunction**: symbols: $\vee$ `or` examples: $p \vee q$ p' `or` q'
  - **conjunction**: symbols: $\wedge$ `and` examples: $p \wedge q$ p' `and` q'
  - **implication** or **conditional**: symbols: $\rightarrow$ `implies` examples: $p \rightarrow q$ p' `implies` q'
  - **parantheses**: symbols: ([ ])
- We will call atomic and complex propositions formulas

EXAMPLES OF SYMBOLIC PROPOSITIONS

p: "The train is late"
q: "There are taxis in the station"
r: "Bob is late to work"

$$(p \wedge \neg q) \rightarrow r$$

p': "It is warm"
q': "It is humid"
r': "It is raining"

$$q' \rightarrow p'$$

$$(p' \wedge q') \rightarrow (r' \vee p')$$

$$(p' \texttt{ and } q') \texttt{ implies } (r' \texttt{ or } p')$$

To not use so many parentheses we use **binding priorities**:

$\neg$ binds more tightly than

$\wedge$ and $\vee$ which bind more tightly than

$\rightarrow$ which is right-associative

eg:

$$p_1 \wedge \neg q_1 \rightarrow r_1 \vee (p_2 \wedge \neg q_2) \rightarrow r_2$$

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Remember: everything so far is just syntax and syntactical conventions

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MORE PRECISE SYNTAX

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Formulas are strings from the alphabet
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We have to be more rigorous:

We use a **formal grammar** that unambiguously accepts exactly the well-formed terms of our logic

To do this we need to define grammar (or meta-) variables that stand for **any term derivable from the grammar**. Variables:

$$A, C, B, \ldots$$

(The book uses the greek letters ϕ, ψ, χ)

Definition

The **logical formulas** of Propositional Logic are exactly those accepted by the following grammar in Backus Naur Form (BNF):

$$A, C, B ::= p \quad | \quad (\neg A) \quad | \quad (A \wedge A) \quad | \quad (A \vee A) \quad | \quad (A \rightarrow A)$$

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It is useful to think of Propositional Formulas as syntax trees where

- → all the **leafs** are atomic propositions
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We also bring back **binding priorities**:

$\neg$ binds more tightly than
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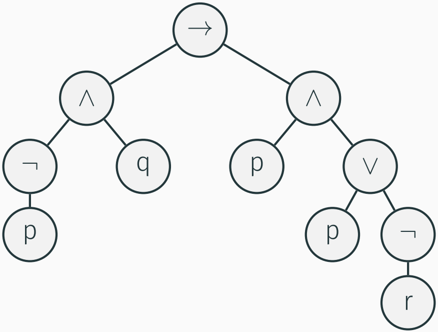
EXAMPLE

What is the parse tree of the formula:

$$(((\neg p) \land q) \to (p \land (q \lor (\neg r))))$$

What is the parse tree of the formula:

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$



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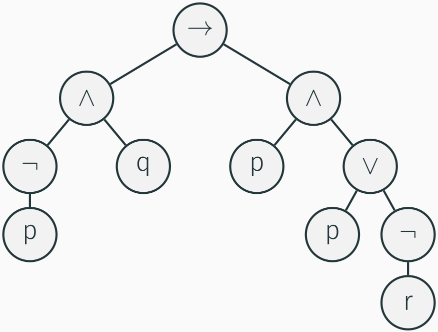

With implicit binding priorities:

$$\neg p \land q \to (p \land (q \lor \neg r))$$

Write in Symbolic Propositional Logic

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2. “If the sun shines today then it won’t shine tomorrow”

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Can you parse the following formulas assuming the implicit binding?

3 $p \vee q \rightarrow \neg p \wedge q$

3 $p \wedge q \wedge q$

Semantics: the meaning of formulas

Now we are going to look at the truth values of propositional logic formulas A. We will write this as $\mathrm{sem}(A)$

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- The meaning of each logical operator is a **predefined function** which maps the truth values of its parameters to the truth value of the formula obtained by applying the operator its parameters.

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- → The meaning of each logical operator is a **predefined function** which maps the truth values of its parameters to the truth value of the formula obtained by applying the operator its parameters.
- → The valuation or model of a fomula A is an **assignment** of each propositional atom in A to a truth value.

The **meaning** of p: “It is warm” can be any of the following:

- $\text{sem}(p) = \mathsf{T}$
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SEMANTICS OF ATOMIC FORMULAS

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A **model** of the formula $A = p$ can be any of the following

- $p \mapsto \mathsf{T}$
- $p \mapsto \mathsf{F}$

The meaning of operator $\neg$ is a function $f_\sim$ which takes one argument. This function is uniquely identified by the truth table:

A	$\neg A$
T	F
F	T

Here the A colum lists all possible truth values of A. The $\neg A$ column the result of applying the $f_\sim$ to these values; these are the truth values of the formula $\neg A$.

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- $p \mapsto$ T
- $p \mapsto$ F

The meaning of operators $\wedge$ and $\vee$ are two functions with two argumens:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
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A	B	$A \vee B$
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SEMANTICS OF COMPLEX FORMULAS

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A **model** of the formula $A = \neg p \wedge (q \vee p)$ can be any of the following

- $p \mapsto$ T, $q \mapsto$ T
- $p \mapsto$ T, $q \mapsto$ F
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The meaning of operator $\rightarrow$ is a function with two arguments:

A	B	$A \rightarrow B$
T	T	T
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F	T	T
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We can write truth tables of composed operators using multiple columns

e.g.: $\neg A \vee \neg B \to B$

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg A \vee \neg B \to B$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	T
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Write the truth table of $\neg A \vee B \rightarrow B$

EXERCISE

Write the truth table of $\neg A \vee B \rightarrow B$

A	B	$\neg A$	$\neg A \vee B$	$\neg A \vee B \rightarrow B$
T	T			
T	F			
F	T			
F	F			

Complete the following truth table

A	B	$\neg A$	$\neg A \vee B$	$\neg A \vee B \to A$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	F
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Create the truth table of the formula:

$$(p \to \neg q) \to (q \vee \neg p)$$

Create the truth table of the formula:

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Give a model that makes the formula true.

Create the truth table of the formula:

$$(p \rightarrow \neg q) \rightarrow (q \vee \neg p)$$

Give a model that makes the formula true.

Give a model that makes the formula false.

Definition

A is satisfiable when it has **a model** which makes it true.

A is falsifiable when it has **a model** which makes it false.

A is valid or a tautology when it has **no model** which makes it false.

A is invalid or a contradiction when it has **no model** which makes it true.

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Q: show that if $A_1 \wedge (A_2 \wedge A_3)$ is satisfiable then $(A_1 \wedge A_2) \wedge A_3$ is satisfiable.

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Q: show that if $A_1 \wedge (A_2 \wedge A_3)$ is satisfiable then $(A_1 \wedge A_2) \wedge A_3$ is satisfiable.

Q: Let $(A_1 \wedge A_2) \to A_3$ be valid. Is it necessary that A_3 is satisfiable?