On Stabilization of Continuous-time and Discrete-time Symmetric Bilinear Systems by Constant Controls*

Lin Tie
School of Automation Science and Electrical Engineering
Beihang University (Beijing University of Aeronautics and Astronautics), 100191, Beijing, P. R. China
Email: tielinllc@gmail.com

***Abstract*—In this paper, the stabilization problems of two-dimensional symmetric bilinear systems by constant controls are considered. Both the continuous-time case and the discrete-time case are studied and necessary and sufficient conditions for stabilization of the systems are presented. It is shown that if the control is multiple then the continuous-time system is stabilizable if and only if its discrete-time counterpart is stabilizable.**

***Index Terms*—Bilinear systems, Stabilization, Constant controls.**

I. Introduction

Given the capability of modeling a large number of processes in the real world, bilinear systems have received considerable attention [1-3]. From the theoretical point of view, such systems own a simple nonlinear structure and hence are thought of to be better understood than most other nonlinear systems. Furthermore, bilinear systems have particular advantages in control, optimization, identification, etc. It is reasonable to say that bilinear systems comprise an important class as well as a simple class of nonlinear systems.

Many contributions are available on the stabilization problems of bilinear systems, both in the continuous-time case [4-23] and the discrete-time case [24-26]. Most of the contributions are for the bilinear systems with drift (Ax) [6,8-10,13,16-26] or with linear control part (bu) [4,6,8,10,13,16,23-25] and use linear or nonlinear state feedback control, while the problem of constructing constant stabilizing controls is less considered. Although the topic of designing stabilizing controllers for bilinear systems have been extensively investigated in the literature, it remains important and challenging as most existing results are limited to special subclasses of bilinear systems. For the bilinear systems with neither drift nor linear control part, we have

$$\begin{aligned}\dot{x}(t) &= (u_1(t)B_1 + \cdots + u_m(t)B_m)x(t) \\ &= \sum_{i=1}^{m} u_i(t)B_i x(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\begin{bmatrix} u_1(t) & \cdots & u_m(t) \end{bmatrix}^T \in \Omega \subset \mathbb{R}^m$, and $B_1, \ldots, B_m \in \mathbb{R}^{n\times n}$ are always assumed to be linearly independent. System (1) can be derived from autonomous linear systems if considering the parameters as control inputs. System (1) can also be seen from switched linear systems if $u_1(t), \ldots, u_m(t)$ satisfy a switching law. In fact, bilinear systems belong to nonlinear systems but are the most close to linear systems among nonlinear systems. They form a transitional class between the linear and the general nonlinear systems. If $u_1(t), \ldots, u_m(t)$ are symmetric (namely $u_i(t) = \sigma$ and $u_i(t) = -\sigma$ can both be obtained from the control set Ω) and piecewise constant, then system (1) is called *symmetric bilinear systems*. Among bilinear systems, unconstrained symmetric bilinear systems ($\Omega = \mathbb{R}^m$) have the most complete theory. In this paper we let $\Omega = \mathbb{R}^m$. Using Euler discretization to (1) yields system (1)'s discrete-time counterpart [3,27,28]

$$\begin{aligned}x(k+1) &= (I + u_1(k)B_1 + \cdots + u_m(k)B_m)x(k) \\ &= \left(I + \sum_{i=1}^{m} u_i(k)B_i\right)x(k)\end{aligned} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ and $u_1(k), \ldots, u_m(k) \in \mathbb{R}$.

The stabilization problems of systems (1) and (2) are not well studied[1]. Available results are quite few. In this paper, we consider the problem of stabilizing systems (1) and (2) by constant controls. In particular, we focus on the systems in dimension two and give necessary and sufficient conditions. It is shown that if $m > 1$ then system (1) is stabilizable if and only if system (2) is stabilizable. The study on the continuous-time case is based on the controllability and that on the discrete-time case is based on the single-input system. Thus the paper is organized as follows. Section II deals with the single-input cases. Section III investigates the multi-input cases and presents necessary and sufficient conditions for stabilization. Section IV gives the conclusion remarks of the paper.

*This work was supported by the China Postdoctoral Science Foundation funded project under Grants 2011M500216, 2012T50035 and the National Nature Science Foundation of China under Grants 61203231, 61203022.

[1]For controllability of systems (1) and (2) one can go to [28].

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II. Single-input Cases

The stabilization problems of two-dimensional systems (1) and (2) with single input are easy to solve by using the linear systems theory.

Theorem 1. The bilinear system

$$\dot{x}(t) = u(t) Bx(t) \tag{3}$$

is stabilizable by constant controls if and only if

$$\det B > 0 \text{ and } \mathrm{tr}(B) \neq 0,$$

where $x(t) = [x_1(t)\ x_2(t)]^T \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, and $B \in \mathbb{R}^{2\times 2}$.

Proof. Necessity:

Note that a nonsingular transformation to system (3) does not change its stabilization. We only need to consider B in canonical form. If

$$\det B = 0,$$

write

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It can be seen that $\dot{x}_2(t) \equiv 0$ for both of the above two cases and thus the system is not stabilizable.

If

$$\det B < 0,$$

write

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$$

where $\lambda_1, \lambda_2 > 0$. In this case, solving the system (3) yields

$$\begin{aligned} x(t) &= e^{\int_0^t u(\tau)d\tau B} x(0) \\ &= \begin{bmatrix} e^{\lambda_1 \int_0^t u(\tau)d\tau} & 0 \\ 0 & e^{-\lambda_2 \int_0^t u(\tau)d\tau} \end{bmatrix} x(0) \\ &= \begin{bmatrix} e^{\lambda_1 \int_0^t u(\tau)d\tau} x_1(0) \\ e^{-\lambda_2 \int_0^t u(\tau)d\tau} x_2(0) \end{bmatrix}. \end{aligned}$$

Then it follows

$$\begin{aligned} & x_1^{\lambda_2}(t)\, x_2^{\lambda_1}(t) \\ = \; & e^{\lambda_1\lambda_2 \int_0^t u(\tau)d\tau} x_1^{\lambda_2}(0)\, e^{-\lambda_1\lambda_2 \int_0^t u(\tau)d\tau} x_2^{\lambda_1}(0) \\ = \; & x_1^{\lambda_2}(0)\, x_2^{\lambda_1}(0), \end{aligned}$$

which implies that the system cannot be stabilized if $x_1(0), x_2(0) \neq 0$ as $x_1^{\lambda_2}(t)\, x_2^{\lambda_1}(t)$ will be a constant for any $t > 0$.

If

$$\det B > 0 \text{ and } \mathrm{tr}(B) = 0,$$

then B has a pair of pure image eigenvalues and we can write

$$B = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$$

where $\lambda \neq 0$. Therefore,

$$e^{\int_0^t u(\tau)d\tau B} = \begin{bmatrix} \cos\lambda \int_0^t u(\tau)\,d\tau & \sin\lambda \int_0^t u(\tau)\,d\tau \\ -\sin\lambda \int_0^t u(\tau)\,d\tau & \cos\lambda \int_0^t u(\tau)\,d\tau \end{bmatrix}$$

and

$$\|x(t)\|_2 = \left\| e^{\int_0^t u(\tau)d\tau B} x(0) \right\|_2 = \|x(0)\|_2,$$

which implies that the system cannot be stabilized.

Sufficiency:

If

$$\det B > 0 \text{ and } \mathrm{tr}(B) \neq 0,$$

then B has either a pair of complex conjugate eigenvalues with the real part nonzero or two real eigenvalues with the same sign. Therefore, we can always choose a constant v such that vB's eigenvalues are in the open left half-plane. By using $u(t) = v$ the system is exponentially stabilized. □

Theorem 2. The bilinear system

$$x(k+1) = (I + u(k) B)\, x(k) \tag{4}$$

is stabilizable by constant controls if and only if

$$\det B < 0 \text{ or } \det B > 0 \text{ and } \mathrm{tr}(B) \neq 0, \tag{5}$$

where $x(k) \in \mathbb{R}^2$, $u(k) \in \mathbb{R}$, and $B \in \mathbb{R}^{2\times 2}$.

Proof. Necessity:

If $\det B = 0$, we can make a similar analysis to the system (4) as shown for system (3) in this case and prove that it is not stabilizable.

If

$$\det B > 0 \text{ and } \mathrm{tr}(B) = 0,$$

then B has a pair of pure image eigenvalues and we can write

$$B = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$$

where $\lambda \neq 0$. Since

$$\|(I + u(k) B)\|_2 = \sqrt{1 + (u(k)\lambda)^2} \geq 1,$$

any initial state cannot be transferred to a state with its Euclidean norm less than that of the initial state. Thus the system cannot be stabilized.

Sufficiency:

If

$$\det B < 0 \text{ or } \det B > 0 \text{ and } \mathrm{tr}(B) \neq 0,$$

we can write

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \sigma & \mu \\ -\mu & \sigma \end{bmatrix} \tag{6}$$

where $\lambda_1, \lambda_2, \lambda, \sigma$ are nonzero. For the first case in (6), let

$$v_1 = \frac{-1+\varepsilon}{\lambda_1},\ v_2 = \frac{-1+\varepsilon}{\lambda_2}$$

where $\varepsilon > 0$ is small enough, then

$$\left(I + v_2 B\right)\left(I + v_1 B\right) = \varepsilon \left[\begin{array}{cc} 1 - \frac{\lambda_1}{\lambda_2} + \varepsilon\frac{\lambda_1}{\lambda_2} & 0 \\ 0 & 1 - \frac{\lambda_2}{\lambda_1} + \varepsilon\frac{\lambda_2}{\lambda_1} \end{array}\right]$$

has all the eigenvalues inside the unit circle and the system can be stabilized by

$$u\left(k\right) = \left\{\begin{array}{l} v_1 \text{ for } k = 0, 2, \ldots, 2m, \ldots \\ v_2 \text{ for } k = 1, 3, \ldots, 2m+1, \ldots \end{array}\right. .$$

For the second case in (6), by

$$u\left(k\right) = \frac{-1+\varepsilon}{\lambda}$$

the system can be stabilized as

$$I + u\left(k\right)B = \left[\begin{array}{cc} \varepsilon & \frac{-1+\varepsilon}{\lambda} \\ 0 & \varepsilon \end{array}\right]$$

has all the eigenvalues inside the unit circle, where $0 < |\varepsilon| < 1$. Finally, for the last case in (6), by $u\left(k\right) = v$ where

$$v \in \left\{\begin{array}{ll} \left(0, \frac{2\sigma}{\sigma^2+\lambda^2}\right) & \text{if } \sigma > 0 \\ \left(\frac{2\sigma}{\sigma^2+\lambda^2}, 0\right) & \text{if } \sigma < 0 \end{array}\right. ,$$

the system can be stabilized for the same reason. □

The stabilization problems of single-input systems (3) and (4) have been solved. In next section we consider the multi-input systems. In particular, the continuous-time case will be studied by the controllability and the discrete-time case will be investigated based on the single-input system.

III. Multi-input Cases

A. Continuous-time Systems

*Lemma 1***.** For any $a_1, a_2 \in \mathbb{R}$,

$$\det\left(a_1 B_1 + a_2 B_2\right) \equiv 0 \tag{7}$$

if and only if B_1, B_2 are singular, have a common real eigenvector, and can be simultaneously transformed into

$$PB_1P^{-1} = \left[\begin{array}{cc} \lambda_1 & b_1 \\ 0 & 0 \end{array}\right], \; PB_2P^{-1} = \left[\begin{array}{cc} \lambda_2 & b_2 \\ 0 & 0 \end{array}\right], \tag{8}$$

where $B_1, B_2 \in \mathbb{R}^{2\times 2}$ and $P \in \mathbb{R}^{2\times 2}$ is nonsingular.

*Proof***.** For sufficiency, if there exists a nonsingular matrix P such that (8) holds, then it is easy to see that B_1, B_2 are singular and $P^{-1}\left[1\ 0\right]^T$ is a common real eigenvector of them. Furthermore, we have

$$\det\left(a_1 B_1 + a_2 B_2\right) = \det\left(a_1 PB_1P^{-1} + a_2 PB_2P^{-1}\right) \equiv 0.$$

For necessity, let $a_1 = 0, a_2 \neq 0$ and $a_1 \neq 0, a_2 = 0$ respectively, then from (7) we know that B_1, B_2 are both singular. Without loss of generality, let

$$B_1 = \left[\begin{array}{cc} \lambda_1 & b_1 \\ 0 & 0 \end{array}\right].$$

Furthermore, denote B_2 by

$$\left[\begin{array}{cc} \lambda_2 & b_2 \\ b_3 & b_4 \end{array}\right]$$

where $b_4\lambda_2 - b_3 b_2 = 0$ as B_2 is singular. Let $a_1 = 1, a_2 = 1$ in (7). We can deduce

$$b_4\lambda_1 - b_3 b_1 = 0.$$

If b_3, b_4 are zero, then we are through. Otherwise, $\left[b_4\ \ -b_3\right]^T$ is a common real eigenvector of B_1, B_2. So far we have proven that B_1, B_2 are singular and have a common real eigenvector. Thus there exists a nonsingular matrix P such that

$$PB_1P^{-1} = \left[\begin{array}{cc} \lambda_1 & b_1 \\ 0 & 0 \end{array}\right]$$

and $P^{-1}\left[1\ 0\right]^T$ is the common real eigenvector of B_1, B_2. Then

$$B_2P^{-1}\left[1\ 0\right]^T = P^{-1}\left[1\ 0\right]^T \Rightarrow PB_2P^{-1}\left[1\ 0\right]^T = \left[1\ 0\right]^T$$

and PB_2P^{-1} must have the following form

$$\left[\begin{array}{cc} \lambda_2 & b_2 \\ 0 & 0 \end{array}\right]$$

since B_2 is singular. □

We need the following conclusion on controllability.

Proposition 1 ***[28].*** The bilinear system

$$\dot{x}\left(t\right) = \left(u_1\left(t\right)B_1 + u_2\left(t\right)B_2\right)x\left(t\right) \tag{9}$$

is controllable if and only if B_1, B_2 have no common real eigenvector, where $x(t) = \left[x_1(t)\ x_2(t)\right]^T \in \mathbb{R}^2$, $u_1\left(t\right), u_2\left(t\right) \in \mathbb{R}$, and $B_1, B_2 \in \mathbb{R}^{2\times 2}$ are linearly independent.

*Theorem 3***.** System (9) is stabilizable by constant controls if and only if

$$\det\left(a_1 B_1 + a_2 B_2\right) \neq 0 \text{ for some } a_1, a_2 \in \mathbb{R}. \tag{10}$$

*Proof***.** *Necessity:*
If for any $a_1, a_2 \in \mathbb{R}$

$$\det\left(a_1 B_1 + a_2 B_2\right) \equiv 0,$$

from Lemma 1 we can write

$$B_1 = \left[\begin{array}{cc} \lambda_1 & b_1 \\ 0 & 0 \end{array}\right], \; B_2 = \left[\begin{array}{cc} \lambda_2 & b_2 \\ 0 & 0 \end{array}\right]$$

without loss of generality. Then $\dot{x}_2(t) \equiv 0$ for $t > 0$ and the system is not stabilizable.
Sufficiency:
If B_1, B_2 do not have a common real eigenvector (condition (10) thus holds), then system (9) is controllable. Thus for any given initial state ξ, we can have $v_1\left(t\right), v_2\left(t\right)$ and $T > 0$ such that

$$x\left(T\right) = e^{\int_0^T\left(v_1(\tau)B_1 + v_2(\tau)B_2\right)d\tau}\xi = e^{-a}\xi$$

where $v_1(t), v_2(t)$ are piecewise constant functions on $[0, T)$ and a is a positive number. Now let $u_1(t) = v_1(t - kT), u_2(t) = v_2(t - kT)$ on $[kT, (k+1)T)$ for $k = 0, 1, \ldots$. Then it follows

$$x((k+1)T) = e^{-a} x(kT) = e^{-(k+1)a} \xi,$$

which implies that the system (9) is stabilized.

If B_1, B_2 have a common real eigenvector, named ζ, using $P = \begin{bmatrix} \zeta & \eta \end{bmatrix}$ we can make $P^{-1}B_1P$, $P^{-1}B_2P$ upper triangular matrices, where η is any vector linearly independent with ζ. Thus B_1, B_2 are both assumed to be upper triangular, i.e.

$$B_1 = \begin{bmatrix} \lambda_1 & b_1 \\ 0 & \lambda_2 \end{bmatrix}, \ B_2 = \begin{bmatrix} \mu_1 & b_2 \\ 0 & \mu_2 \end{bmatrix}.$$

From condition (10) we have one of λ_2, μ_2 nonzero. Assume λ_2 is nonzero and we can get

$$B_2 - \frac{\mu_2}{\lambda_2} B_1 = \begin{bmatrix} \mu_1 - \frac{\mu_2}{\lambda_2}\lambda_1 & b_2 - \frac{\mu_2}{\lambda_2} b_1 \\ 0 & 0 \end{bmatrix}.$$

If $\mu_1 - \frac{\mu_2}{\lambda_2}\lambda_1 \neq 0$, we can certainly choose a nonzero number c such that

$$\left(\lambda_1 + c\left(\mu_1 - \frac{\mu_2}{\lambda_2}\lambda_1\right)\right)\lambda_2 > 0.$$

Let $u_1(t) = -\text{sign}(\lambda_2)\left(1 - \frac{\mu_2}{\lambda_2}c\right), u_2(t) = -\text{sign}(\lambda_2)\, c$. We have

$$\dot{x}(t) = (u_1(t) B_1 + u_2(t) B_2)\, x(t) =$$

$$\begin{bmatrix} -\left|\lambda_1 + c\left(\mu_1 - \frac{\mu_2}{\lambda_2}\lambda_1\right)\right| & b_1 + c\left(b_2 - \frac{\mu_2}{\lambda_2} b_1\right) \\ 0 & -|\lambda_2| \end{bmatrix} x(t)$$

which is a stable linear system. If $\mu_1 - \frac{\mu_2}{\lambda_2}\lambda_1 = 0$, then $b_2 - \frac{\mu_2}{\lambda_2} b_1 \neq 0$ (as B_1, B_2 are linearly independent) and $\lambda_1 \neq 0$ (otherwise it contradicts condition (10)). For any given state $\xi = [\xi_1\ \xi_2]^T$, if $\xi_2 = 0$, let $u_1(t) = -\text{sign}(\lambda_1)\left(1 - \frac{\mu_2}{\lambda_2}\right), u_2(t) = -\text{sign}(\lambda_1)$. We have

$$\begin{aligned} \dot{x}_1(t) &= -|\lambda_1|\, x_1(t), \\ \dot{x}_2(t) &= 0 \end{aligned}$$

as $x_2(0) = \xi_2 = 0$, which implies that the system (9) is stabilized. If $\xi_2 \neq 0$ and if $\xi_1 \neq 0$, let

$$\begin{aligned} u_1(t) &= \text{sign}\left(\frac{\xi_1}{\left(b_2 - \frac{\mu_2}{\lambda_2} b_1\right)\xi_2}\right)\frac{\mu_2}{\lambda_2}, \\ u_2(t) &= -\text{sign}\left(\frac{\xi_1}{\left(b_2 - \frac{\mu_2}{\lambda_2} b_1\right)\xi_2}\right) \end{aligned}$$

on $[0, T_1)$ where

$$T_1 = \left|\frac{\xi_1}{\left(b_2 - \frac{\mu_2}{\lambda_2} b_1\right)\xi_2}\right|.$$

We have

$$x(T_1) = e^{\int_0^{T_1}(u_1(\tau)B_1 + u_2(\tau)B_2)d\tau}\xi = \begin{bmatrix} 0 \\ \xi_2 \end{bmatrix}.$$

Then let

$$\begin{aligned} u_1(t) &= -\text{sign}(\lambda_2 b_2 - \mu_2 b_1)\, b_2, \\ u_2(t) &= \text{sign}(\lambda_2 b_2 - \mu_2 b_1)\, b_1 \end{aligned}$$

on $[T_1, +\infty)$. The system (9) can be written as

$$\begin{aligned} \dot{x}_1(t) &= 0, \\ \dot{x}_2(t) &= -|\lambda_2 b_2 - \mu_2 b_1|\, x_2(t) \end{aligned}$$

where $t \in [T_1, +\infty)$. This implies that the system (9) is stabilized; finally, if $\xi_2 \neq 0$ and $\xi_1 = 0$, by letting

$$\begin{aligned} u_1(t) &= -\text{sign}(\lambda_2 b_2 - \mu_2 b_1)\, b_2, \\ u_2(t) &= \text{sign}(\lambda_2 b_2 - \mu_2 b_1)\, b_1 \end{aligned}$$

we can finish the proof. □

Theorem 4. Consider system (1) with $n = 2, m \geq 2$. Then the system is stabilizable by constant controls if and only if there exist $a_1, \ldots, a_m \in \mathbb{R}$ such that

$$\det(a_1 B_1 + \cdots + a_m B_m) \neq 0. \tag{11}$$

Proof. For necessity, if for any $a_1, \ldots, a_m \in \mathbb{R}$

$$\det(a_1 B_1 + \cdots + a_m B_m) \equiv 0,$$

we can show that there exist a nonsingular matrix P such that

$$P B_i P^{-1} = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \tag{12}$$

for $i = 1, \ldots, m$ as we prove Lemma 1. Then the system can be seen to be unstabilizable.

For sufficiency, if there exist $a_1, \ldots, a_m$ such that (11) holds, at least one of $a_1, \ldots, a_m$ is nonzero. Assume $a_1 \neq 0$. If $a_2 B_2 + \cdots + a_m B_m \neq 0$, then let $u_2(t) = a_2 \tilde{u}_2(t), \ldots, u_m(t) = a_m \tilde{u}_2(t)$ where $\tilde{u}_2(t) \in \mathbb{R}$. The system (1) is rewritten as

$$\dot{x}(t) = (u_1(t) B_1 + \tilde{u}_2(t)(a_2 B_2 + \cdots + a_m B_m))\, x(t). \tag{13}$$

By Theorem 3 system (13) is stabilizable as

$$\det(a_1 B_1 + 1 \times (a_2 B_2 + \cdots + a_m B_m)) \neq 0.$$

If $a_2 B_2 + \cdots + a_m B_m = 0$, then B_1 is nonsingular. Let $u_3(t) = 0, \ldots, u_m(t) = 0$. Then system (1) is stabilizable due to Theorem 3 as $\det(a_1 B_1 + a_2 B_2) \neq 0$ for $a_1 \neq 0, a_2 = 0$. □

Remark 1. To justify whether there exist $a_1, \ldots, a_m$ such that (11) holds is quite easy. If one of $B_1, \ldots, B_m$ is nonsingular, we are done. Otherwise, find some B_i with $\text{tr}(B_i) \neq 0$; transform it into canonical form and make the same transformation to the others. The rest is to check whether the transformed matrices are all of the form as given in (12).

B. Discrete-time Systems

Theorem 5. The bilinear system

$$x(k) = (I + u_1(k) B_1 + u_2(k) B_2) x(k) \tag{14}$$

is stabilizable by constant controls if and only if condition (10) is satisfied, where $x(k) \in \mathbb{R}^2$, $u_1(k), u_2(k) \in \mathbb{R}$, and $B_1, B_2 \in \mathbb{R}^{2\times 2}$ are linearly independent.

Proof. Necessity:
If for any $a_1, a_2 \in \mathbb{R}$

$$\det(a_1 B_1 + a_2 B_2) \equiv 0,$$

then from Lemma 1 we can write

$$B_1 = \begin{bmatrix} \lambda_1 & b_1 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} \lambda_2 & b_2 \\ 0 & 0 \end{bmatrix}$$

without loss of generality. As $x_2(k+1) - x_2(k) = 0$ for $k = 0, 1, \ldots$, the system is not stabilizable.
Sufficiency:
We study the following two cases:

1. at least one of B_1, B_2 is nonsingular;

2. both of B_1, B_2 are singular.

For **Case 1**, condition (10) is satisfied. Assume B_1 is nonsingular. If B_1 satisfies condition (5), then by letting $u_2(k) = 0$ the system (14) is stabilizable in view of Theorem 2. Otherwise, if $\det B_1 > 0$ and $\mathrm{tr}(B_1) = 0$, B_1 has a pair of pure imaginary eigenvalues and can be written as

$$\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$$

where $\lambda \neq 0$. If $\mathrm{tr}(B_2) \neq 0$, by letting $u_2(k) = \varepsilon u_1(k)$ where ε is a sufficiently small number the system (14) is stabilizable as $\det(B_1 + \varepsilon B_2)$ can be nonzero and $\mathrm{tr}(B_1 + \varepsilon B_2) = \varepsilon \mathrm{tr}(B_2) \neq 0$; if $\mathrm{tr}(B_2) = 0$, let

$$B_2 = \begin{bmatrix} \mu & b_1 \\ b_2 & -\mu \end{bmatrix}.$$

Consider the following inequality

$$s^2 + (b_1 - b_2) s - b_1 b_2 - \mu^2 < 0. \tag{15}$$

Since

$$(b_1 - b_2)^2 - 4\left(-b_1 b_2 - \mu^2\right) = (b_1 + b_2)^2 + 4\mu^2 > 0,$$

(15) has solutions (if $(b_1 + b_2)^2 + 4\mu^2 = 0$, then $\mu = 0, b_2 = -b_1$ which implies that B_1, B_2 are linearly independent). Choose c such that $s = c$ is a solution to (15). We have

$$\det\left(\frac{c}{\lambda} B_1 + B_2\right) = c^2 + (b_1 - b_2) c - b_1 b_2 - \mu^2 < 0.$$

Thus by letting $u_1(k) = \frac{c}{\lambda} u_2(k)$ the system (14) can be shown to be stabilizable.
For **Case 2**, without loss of generality, write

$$B_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \tag{16}$$

where $\lambda \neq 0$ and $b_1 b_4 - b_2 b_3 = 0$. For B_1 being the former in (16), we have

$$\det(cB_1 + B_2) = c b_4 \lambda.$$

If $b_4 = 0$, then one of b_2, b_3 equals zero which contradicts condition (10). Hence $b_4 \neq 0$. Let $u_1(k) = -\mathrm{sign}(b_4\lambda)\, u_2(k)$. Then

$$\begin{aligned} x(k) &= (I + u_1(k) B_1 + u_2(k) B_2) x(k) \\ &= (I + u_2(k)(-\mathrm{sign}(b_4\lambda) B_1 + B_2)) x(k) \end{aligned}$$

where

$$\det(-\mathrm{sign}(b_4\lambda) B_1 + B_2) = -|b_4\lambda| < 0.$$

By Theorem 2 the system (14) is stabilizable. For B_1 being the latter in (16), we can make a similar analysis. $\square$

Theorem 6. Consider system (2) with $n = 2, m \geq 2$. Then the system is stabilizable by constant controls if and only if condition (11) holds.

The proof of Theorem 6 is omitted since it is similar to that of Theorem 4.

Remark 2. From Theorems 5 and 6, the two-dimensional system (1) is stabilizable if and only if its discrete-time counterpart is stabilizable when the control is multiple. More interestingly, in [28] it has been shown that the two-dimensional system (1) is controllable if and only if its discrete-time counterpart is controllable. As a result, the continuous-time systems and the discrete-time systems has the same controllability and stabilizability, which may be useful in practice.

Finally, we give some sufficient conditions for stabilization of systems (1) and (2), which are easy to verify.

Corollary 1. Consider system (1) (system (2)) with $n = 2, m \geq 2$. Then the system is stabilizable by constant controls if one of $B_1, \ldots, B_m$ is nonsingular.

Corollary 2. Consider system (1) (system (2)) with $n = 2, m \geq 2$. Then the system is stabilizable by constant controls if $B_1, \ldots, B_m$ do not have a common eigenvector.

Either one of $B_1, \ldots, B_m$ is nonsingular or $B_1, \ldots, B_m$ do not have a common eigenvector makes condition (11) hold.

IV. CONCLUSIONS

In this paper, the stabilization problems of two-dimensional continuous-time and discrete-time symmetric bilinear systems by constant controls are studied. Necessary and sufficient conditions are derived. It is shown that if the control is multiple then the continuous-time system is stabilizable if and only if its discrete-time counterpart is stabilizable. Future work should consider the stabilization problems of symmetric bilinear systems with high dimensions.

References

[1] C. Bruni, G. D. Pillo, and G. Koch, "Bilinear systems: an appealing class of 'nearly linear' systems in theory and applications," *IEEE Trans. Automat. Contr.*, vol. 19, 1974, pp. 334-348.

[2] R. R. Mohler and W. J. Kolodziej, "An overview of bilinear system theory and applications," *IEEE Trans. Systems, Man, and Cybernetics*, vol. 10, 1980, pp. 683-688.

[3] D. L. Elliott, *Bilinear Control Systems: Matrices in Action*. Dordrecht: Springer, 2009.

[4] T. Ionescu and R. V. Monopoli, "On the stabilization of bilinear systems via hyperstability," *IEEE Trans. Automat. Contr.*, vol. 2, 1975, pp. 280-284.

[5] M. Slemrod, "Stabilization of bilinear control systems with applications to nonconservative problems in elasticity," *SIAM J. Contr. Optimiz.*, vol. 16, 1978, pp. 131-141.

[6] R. Longchamp, "Controller design for bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 25, 1980, pp. 547–548.

[7] I. Derese and E. Noldus, "Design of linear feedback laws for bilinear systems," *Int. J. Contr.*, vol. 31, 1980, pp. 219-237.

[8] P. -O. Gutman, "Stabilizing controllers for bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 26, 1981, pp. 917-922.

[9] E. P. Ryan and N. J. Buckingham, "On asymptotically stabilizing feedback control of bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 28, 1983, pp. 863-864.

[10] D. E. Kodishek and K. S. Narendra, "Stabilizability of second-order bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 28, 1983, pp. 987-989.

[11] O. M. Grasselli and S. Longhi, "On the stabilization of a class of bilinear systems," *Int. J. Contr.*, vol. 37, 1983, pp. 413-420.

[12] S. G. Tzafestas and K. E. Anagnostou, "Stabilization of singularly perturbed strictly bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 29, 1984, pp. 943-946.

[13] R. Genesio and A. Tesi, "The output stabilization of SISO bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 33, 1988, pp. 950-952.

[14] S. Čelikovský, "On the stabilization of the homogeneous bilinear systems," *Syst. Contr. Lett.*, vol. 21, 1993, pp. 503-510.

[15] R. Chabour, G. Sallet, and J. C. Vivalda, "Stabilization of nonlinear two dimensional system: a bilinear approach," *Math. Control Signals Systems*, vol. 6, 1996, pp. 224–246.

[16] A. Y. Khapalov and R. R. Mohler, "Asymptotic stabilization of the bilinear time-invariant system via piecewise-constant feedback," *Syst. Contr. Lett.*, vol. 33, 1998, pp. 47–54.

[17] M. -S. Chen and S. -T. Tsao, "Exponential stabilization of a class of unstable bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 45, 2000, pp. 989-992.

[18] O. Chabour and J. C. Vivalda, "Remark on local and global stabilization of homogeneous bilinear systems," *Syst. Contr. Lett.*, vol. 41, 2000, pp. 141-143.

[19] H. Jerbi, "Global feedback stabilization of new class of bilinear systems," *Syst. Contr. Lett.*, vol. 42, 2001, pp. 313-320.

[20] S. Hanba and Y. Miyasato, "Output feedback stabilization of bilinear systems using dead-beat observers," *Automatica*, vol. 37, 2001, pp. 915-920.

[21] Z. D. Wang, H. Qiao, and K. J. Burnham, "On stabilization of bilinear uncertain time-delay stochastic systems with Markovian jumping parameters," *IEEE Trans. Automat. Contr.*, vol. 47, 2002, pp. 640-646.

[22] S. Hanba, "Stabilization of continuous-time homogeneous bilinear systems by constant inputs through nonlinear minimization with applications to the static output feedback stabilization problem," *Int. J. Robust Nonlinear Control*, vol. 18, 2008, pp. 705–713.

[23] F. Amato, C. Cosentino, A. S. Fiorillo, and A. Merola, "Stabilization of bilinear systems via linear state-feedback control," *IEEE Trans. on Circuits and Systems II*, vol. 56, 2009, pp. 76-80.

[24] J. -S. Chiou, F. -C. Kung, and T. -H. S. Li, "Robust stabilization of a class of singularly perturbed discrete bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 45, 2000, pp. 1187-1191.

[25] M. Bacic, M. Cannon, and B. Kouvaritakis, "Constrained control of SISO bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 48, 2003, pp. 1443-1447.

[26] G. P. Lu, G. Feng, and Z. -P. Jiang, "Saturated feedback stabilization of discrete-time descriptor bilinear systems," *IEEE Trans. Automat. Contr.*, vol. 52, 2007, pp. 1700-1704.

[27] L. Tie, K. -Y. Cai, and Y. Lin, "On uncontrollable discrete-time bilinear systems which are 'nearly' controllable," *IEEE Trans. Automat. Contr.*, vol. 55, 2010, pp. 2853-2858.

[28] L. Tie, K. -Y. Cai, and Y. Lin, "A linear algebraic criterion for controllability of both continuous-time and discrete-time symmetric bilinear systems," *J. the Franklin Institute*, vol. 350, 2013, pp. 898-910.