On the effect of adding frequency-response-constrained input channels on the achievable performance of discrete-time control systems

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***Abstract*—This paper deals with the effect on the control performance, of adding frequency-response-constrained control channels to a stable, discrete-time and linear time-invariant (LTI) MIMO plant. We focus on the control tracking problem with step references, in which a 2-norm based performance index is considered. In order to introduce the frequency-response limitations of the additional channels, we assume that the constraint is on the energy of the stationary deviations of the additional input signals filtered by a previously selected filter. The main result of this work is a closed form expression that quantifies the benefits on the control performance of square MIMO plants when adding input channels with those limitations.**

I. Introduction

Since Bode's seminal work [1], several studies on performance bounds computation have been reported in the literature, see e.g. [4], [6], [7], [18], [19], [21] . Those studies have mainly dealt with optimal unconstrained tracking and regulation problems in the feedback control of linear systems; they consider different set-ups. The results of those works are consistent with previous results about fundamental limitations (see, e.g., [8], [5], and the references therein), and they show that the closed loop performance is limited, in different ways, by the non-minimum phase (NMP) zeros, unstable poles and time delays.

In control engineering, adding control input channels to the plant is often used as a method to improve closed loop control performance [9], [13]–[15], [20]. A natural question that arises from the above strategy is how much the achievable closed loop performance is improved when the additional channels are non-ideal. Results related to the performance bound of stable, discrete-time (input) augmented plants can be found in [16]. That work presents a closed form expression that quantifies the benefits of adding *unconstrained* input channels on the best achievable performance, using as a performance index the 2-norm of the tracking error for step references. The result in [16] shows that the performance improvement depends explicitly on the NMP zeros of the original plant which are not zeros of the augmented one. Results for the *constrained* case can be found in [10], which studies, in a state variable framework, the effect of additional input channels on the regulation performance of continuous-time MIMO systems, using a quadratic performance index that penalizes the control energy on *all* channels. Based on a numerical approach, [10] aims to characterize the performance bounds in terms of linear matrix inequalities (LMI) and Riccati equations.

This paper studies the effect of adding frequency-response-constrained input channels on the achievable performance of stable discrete-time square MIMO systems. We focus on tracking problems for step references subject to constraints on the additional input channels, in which the constraint is such that allows to introduce the frequency-response limitation of the additional signals. The main result of this paper is a closed form expression that quantifies the performance improvement of adding control channels, which highlight how it is affected by the plant dynamical features and the additional channels limitations.

It should be noted that our results extend those of the unconstrained case [16]. Moreover, unlike [10], we here derive closed form results, under the assumption that only the additional channels have frequency-response limitations.

This work is organized as follows. Section II introduces preliminaries definitions and the notation used in this paper. In Section III we formulate the performance improvement problem when adding additional input channels with frequency-response limitations. In Section IV, we present preliminary results related to the zero structure of augmented plants. Section V presents this paper main results, characterizing the performance improvement in closed form. An example in Section VI illustrates those main results. Finally conclusions are drawn in Section VII.

II. Preliminaries

We will use boldface for matrices and vectors, and standard fonts for scalars. Denote by $\mathbb{R}$ and $\mathbb{C}$ the set of real numbers and complex numbers, respectively. Given $x \in \mathbb{C}$, $\bar{x}$ and $|x|$ denotes its conjugate and magnitude, respectively. We denote by $\mathbb{C}^{n\times m}$ ($\mathbb{R}^{n\times m}$) the set of complex (real) $n \times m$ matrices. Given $\mathbf{A} \in \mathbb{C}^{n\times m}$, then rank$\{\mathbf{A}\}$, $\mathbf{A}^T$, $\mathbf{A}^H$ and $\mathbf{A}^\dagger$ denotes its rank, transpose, hermitian (conjugate transpose) and

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pseudo-inverse, respectively. If $\mathbf{A}$ is square, $\det\{\mathbf{A}\}$ and $\mathbf{A}^{-1}$ denote its determinant and inverse. We use $\mathbf{0}_{n\times m}$ to refer to an $n\times m$ matrix full of zeros.

We denote by $\mathcal{R}$ the set of rational functions in the indeterminate z with coefficients in $\mathbb{R}$, and by $\mathcal{R}_p$ the subset of $\mathcal{R}$ containing all proper transfer functions. For a transfer matrix $\mathbf{M}[z]\in\mathcal{R}^{n\times m}$, we define $\mathbf{M}[z]^{\sim}\triangleq\mathbf{M}[\bar{z}^{-1}]^H$, which reduces to $\mathbf{M}[z]^{\sim}\triangleq\mathbf{M}[z^{-1}]^T$ in the real-rational case. We say that a transfer matrix $\mathbf{M}[z]\in\mathcal{R}^{n\times p}$ is unitary if and only if $\mathbf{M}[z]^{\sim}\mathbf{M}[z]=\mathbf{I}_p$, where $\mathbf{I}_p$ denotes the $p\times p$ identity matrix. The set of all $n\times m$ matrix functions measurable over the unit circle will be denoted by $\mathcal{L}_2^{n\times m}$. The norm induced by the usual inner product in $\mathcal{L}_2$ is called 2-norm and will be denoted by $\|\cdot\|_2$ [11], [22]. We denote by $\mathcal{RH}_2^{n\times m}$ the set of $n\times m$ transfer matrices which are real-rational, stable and strictly proper, and by $\mathcal{RH}_2^{\perp\, n\times m}$ the set of all $n\times m$ real-rational transfer matrices which are analytic in $|z|\leq 1$. Finally, we will denote by $\mathcal{RH}_\infty^{n\times m}$ the set of $n\times m$ transfer matrices which are real-rational, stable and proper.

It is said that a zero of a transfer matrix is non-minimum phase (NMP) if is located outside or on the unit circle (including zeros at infinity). A transfer function $\mathbf{P}[z]\in\mathcal{R}$, with no poles at $z=c$, has a zero at $z=c$ if $\operatorname{rank}\{\mathbf{P}[c]\}<$ normal rank$\{\mathbf{P}[z]\}$, where normal rank$\{\mathbf{P}[z]\}$ is the maximum allowable rank of $\mathbf{P}[z]$ for some $z\in\mathbb{C}$ [22]. If $\mathbf{P}[z]$ has a zero at $z=c$, then there exists a non-zero vector $\boldsymbol{\eta}_c$ such that $\boldsymbol{\eta}_c^H\mathbf{P}[c]=0$. If this vector is unitary, then $\boldsymbol{\eta}_c$ is called unitary left direction associated with the zero at $z=c$.

If $\mathbf{P}[z]\in\mathcal{RH}_\infty^{n\times m}$ is right-invertible, has d zeros at infinity, q finite NMP zeros strictly outside the unit circle denoted by $z_1,\ldots,z_q$, with $|z_i|\neq 1$ (and possibly additional zeros on the unit circle), then there exist a unitary matrix $\boldsymbol{\xi}_{\mathbf{P}}[z]^{-1}\in\mathcal{RH}_\infty^{n\times n}$ and a right-invertible matrix $\overline{\mathbf{P}}[z]\in\mathcal{RH}_\infty^{n\times m}$, with no zeros for $|z|>1$, such that $\mathbf{P}[z]\triangleq\boldsymbol{\xi}_{\mathbf{P}}[z]^{-1}\overline{\mathbf{P}}[z]$ [11], [22]. An explicit construction is given by $\boldsymbol{\xi}_{\mathbf{P}}[z]=\boldsymbol{\xi}_{\mathbf{P}}^{z}[z]\boldsymbol{\xi}_{\mathbf{P}}^{\infty}[z]$, where (see [19], [21])

$$\boldsymbol{\xi}_{\mathbf{P}}^{\infty}[z]\triangleq\prod_{i=1}^{d}\mathbf{L}_{\infty,d-i+1}[z]\,,\quad \boldsymbol{\xi}_{\mathbf{P}}^{z}[z]\triangleq\prod_{i=1}^{q}\mathbf{L}_{z,q-i+1}[z]\,,\quad (1)$$

and

$$\mathbf{L}_{\infty,i}[z]\triangleq z\boldsymbol{\eta}_{\infty,i}\boldsymbol{\eta}_{\infty,i}^H+\mathbf{U}_{\infty,i}\mathbf{U}_{\infty,i}^H\,,\quad (2)$$

$$\mathbf{L}_{z,i}[z]\triangleq\prod_{i=1}^{q}\frac{1-z_i}{1-\bar{z}_i}\frac{1-\bar{z}_i z}{z-z_i}\boldsymbol{\eta}_{z,i}\boldsymbol{\eta}_{z,i}^H+\mathbf{U}_{z,i}\mathbf{U}_{z,i}^H\,,\quad (3)$$

and where $\boldsymbol{\eta}_{\infty,i}$ denotes the unitary left direction associated to the first zero at infinity of $\mathbf{X}_i[z]$, with $\mathbf{X}_{i+1}[z]=\mathbf{L}_{\infty,i}[z]\mathbf{X}_i[z]$, $i=1,...,d$, $\mathbf{X}_1[z]=\mathbf{P}[z]$, and $\mathbf{U}_{\infty,i}$ is such that $[\,\boldsymbol{\eta}_{\infty,i}\quad\mathbf{U}_{\infty,i}\,]$ is unitary. Likewise, $\boldsymbol{\eta}_{z,i}$ denotes the unitary left direction associated to the zero at z_i of $\mathbf{Y}_i[z]$, with $\mathbf{Y}_{i+1}[z]=\mathbf{L}_{z,i}[z]\mathbf{Y}_i[z]$, $i=1,...,q$, $\mathbf{Y}_1[z]=\boldsymbol{\xi}_{\mathbf{P}}^{\infty}[z]\mathbf{P}[z]$ and $\mathbf{U}_{z,i}$ is such that $[\,\boldsymbol{\eta}_{z,i}\quad\mathbf{U}_{z,i}\,]$ is unitary. We note that $\boldsymbol{\xi}_{\mathbf{P}}[1]=\mathbf{I}_n$. We will refer to $\boldsymbol{\xi}_{\mathbf{P}}[z]$ constructed as above as a left interactor for $\mathbf{P}[z]$ [19].

III. Problem Formulation

Let us first consider the standard one degree-of-freedom feedback loop of Figure 1, where $\mathbf{G}[z]\in\mathcal{RH}_2^{n\times n}$ is a full

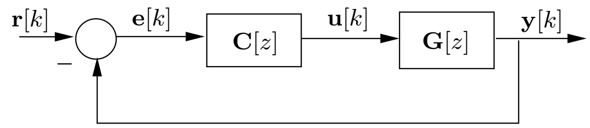

Fig. 1. Standard one degree-of-freedom control scheme for the square plant $\mathbf{G}[z]$.

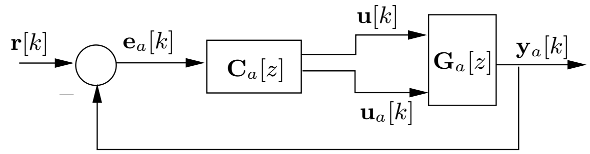

Fig. 2. Feedback control loop for the augmented plant $\mathbf{G}_a[z]$.

normal rank and stable square MIMO plant and $\mathbf{C}[z]\in\mathcal{R}_p^{n\times n}$ is a proper LTI controller. The signals $\mathbf{r}[k]\in\mathbb{R}^n$, $\mathbf{e}[k]\in\mathbb{R}^n$, $\mathbf{u}[k]\in\mathbb{R}^n$ and $\mathbf{y}[k]\in\mathbb{R}^n$ are, respectively, the reference, tracking error, control input and controlled output. We focus on tracking problems for step references and, therefore, we adopt a quadratic criterion to measure the tracking error. Thus, we consider as a performance index

$$J\triangleq\sum_{k=0}^{\infty}\mathbf{e}[k]^T\mathbf{e}[k]\,,\quad (4)$$

with $\mathbf{r}[k]\triangleq\boldsymbol{\nu}\mu[k]$, where $\boldsymbol{\nu}\in\mathbb{R}^n$ and $\mu[k]\in\mathbb{R}$ is a unit step function. Assume that, in order to improve the closed loop performance of $\mathbf{G}[z]$, m additional control channels are added to that plant. We denote the resulting augmented plant by $\mathbf{G}_a[z]$, i.e.,

$$\mathbf{G}_a[z]\triangleq[\mathbf{G}[z]\quad\mathbf{F}[z]]\in\mathcal{RH}_2^{n\times(n+m)}\,,\quad (5)$$

where $\mathbf{F}[z]\in\mathcal{RH}_2^{n\times m}$ models the dynamics between the new m control signals and the n plant outputs. Figure 2 presents the feedback control loop for the augmented plant, where $\mathbf{u}_a[k]\in\mathbb{R}^m$ corresponds to the additional control signals, $\mathbf{e}_a[k]\in\mathbb{R}^n$ is the tracking error for the augmented plant and $\mathbf{C}_a[z]\in\mathcal{R}_p^{(n+m)\times n}$ is a LTI controller.

We assume that the additional control channels have frecuency-response limitations. In order to introduce these limitations, we assume that the additional input signals are constrained to satisfy

$$R_a\triangleq\sum_{k=0}^{\infty}\{\mathbf{m}[k]*(\mathbf{u}_a[k]-\mathbf{u}_{a,\infty})\}^T \times\{\mathbf{m}[k]*(\mathbf{u}_a[k]-\mathbf{u}_{a,\infty})\}\leq\gamma,\quad (6)$$

for some $\gamma\geq 0$, where $*$ denotes convolution, $\mathbf{u}_{a,\infty}\in\mathbb{R}^m$ is an arbitrary vector which corresponds to the stationary value of $\mathbf{u}_a[k]$, and $\mathbf{m}[k]\in\mathbb{R}^{m\times m}$ is a diagonal matrix such that each element of $\mathbf{M}[e^{j\omega}]$, with $\mathbf{M}[z]=\mathcal{Z}\{\mathbf{m}[k]\}\in\mathcal{RH}_\infty^{m\times m}$, is a high-pass filter. The matrix function $\mathbf{M}[e^{j\omega}]$ filters the stationary deviations of the additional input signals and, therefore, it is a previously selected filter that allows to introduce the frequency-response limitations of the additional control channels. We will thus penalize all additional control components having energy in the frequency band which is beyond the channel bandwidth.

Our interest is to minimize the energy of the tracking error $\mathbf{e}_a[k]$ over all stabilizing controllers, while satisfying (6). That is, we aim at finding

$$J_{a,\gamma}^{\rm opt} = \inf_{\substack{\mathbf{C}_a[z]\in\mathcal{K}_a \\ R_a\leq\gamma}} J_a \,, \tag{7}$$

where J_a is defined as J in (4), with $\mathbf{e}[k] = \mathbf{e}_a[k]$, and $\mathcal{K}_a$ denotes the set of all proper LTI controllers $\mathbf{C}_a[z]$ that render the closed loop of Figure 2 internally stable and well-posed [22].

We aim at quantifying the impact of adding frequency-response-constrained input channels on the performance of a square MIMO system. Based on the above, we propose to characterize

$$\mathcal{D}_\gamma \triangleq J^{\rm opt} - J_{a,\gamma}^{\rm opt}, \quad J^{\rm opt} \triangleq \inf_{\mathbf{C}[z]\in\mathcal{K}} J \,, \tag{8}$$

where J and $J_{a,\gamma}^{\rm opt}$ are as in (4) and (7), and $\mathcal{K}$ is the set of all proper LTI controllers that render the closed loop of Figure 1 internally stable and well-posed. It should be noted that (8) can be solved by using standard numerical methods [3]. Nevertheless, such procedure has the disadvantage that it does not allow one to characterize explicitly the benefits of additional control channels.

IV. Zero structure of augmented systems

In order to present our main result we will begin by recalling results in [16], where the zero structure of augmented plants is explored.

Lemma 1: Consider the augmented plant $\mathbf{G}_a[z]$ defined in (5), where $\mathbf{G}[z] \in \mathcal{RH}_2^{n\times n}$ has full normal rank and no zeros on the unit circle, and $\mathbf{F}[z] \in \mathcal{RH}_2^{n\times m}$. Then, the multi-set of zeros of the augmented plant $\mathbf{G}_a[z]$ is a subset of the multi-set of zeros of the square plant $\mathbf{G}[z]$ (including zeros at infinity). Moreover, if $z = c$ is a (finite or infinite) zero of both, $\mathbf{G}[z]$ and $\mathbf{G}_a[z]$, then the left unitary directions associated to that zero can be chosen to be equal in both cases.

It follows from Lemma 1 that one cannot add MIMO zeros by adding input channels to a square MIMO system. On the contrary, the addition of input channels may lead to the elimination of certain plant zeros.

Lemma 2: Consider the setup and assumptions of Lemma 1. Assume that $\mathbf{G}[z]$ has ℓ (finite and infinite) NMP zeros, and that $\mathbf{G}_a[z]$ has ℓ_a (finite and infinite) NMP zeros. Then, it is always possible to write a left interactor $\boldsymbol{\xi}_{\mathbf{G}}[z]$ for $\mathbf{G}[z]$ as

$$\boldsymbol{\xi}_{\mathbf{G}}[z] = \hat{\boldsymbol{\xi}}[z]\boldsymbol{\xi}_{\mathbf{G}_a}[z] \,, \tag{9}$$

where $\boldsymbol{\xi}_{\mathbf{G}_a}[z]$ and $\hat{\boldsymbol{\xi}}[z]$ are, respectively, left interactors for $\mathbf{G}_a[z]$ and $\boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{G}[z]$ (i.e., $\hat{\boldsymbol{\xi}}[z]$ extracts, from $\boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{G}[z]$, the $\ell - \ell_a$ NMP zeros that are zeros of $\mathbf{G}[z]$ but not of $\mathbf{G}_a[z]$).

Lemma 2 states that a left interactor for a transfer matrix $\mathbf{G}[z] \in \mathcal{RH}_\infty^{n\times n}$ that is augmented by adding control channels, can be constructed in a way such that a left interactor for the augmented transfer matrix appears as a pre-multiplicative factor. The matrix factor $\hat{\boldsymbol{\xi}}[z]$ in (9) captures the differences between the structure of NMP zeros of the original square plant $\mathbf{G}[z]$, and that of the augmented one $\mathbf{G}_a[z]$.

V. Performance improvement of adding additional channels

A. Unconstrained channels

In this section we will begin by assuming that the additional control channels are not constrained, i.e., that $\gamma = \infty$ in (7). The following corollary characterize D_∞ (see (8)).

Corollary 1: Consider the setup and the assumptions of Lemma 1. Construct a left interactor $\boldsymbol{\xi}_{\mathbf{G}}[z]$ for $\mathbf{G}[z]$ in a way such that (9) holds. Assume that the transfer matrix $\boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{G}[z]$ has q_Δ finite NMP zeros denoted by $z_1, \ldots, z_{q_\Delta}$ and d_Δ zeros at infinity. Then,

$$\mathcal{D}_\infty = \boldsymbol{\nu}^H \left[\sum_{i=1}^{d_\Delta} \boldsymbol{\eta}_{\infty_i}\boldsymbol{\eta}_{\infty_i}^H + \sum_{i=1}^{q_\Delta} \frac{|z_i|^2 - 1}{|1 - z_i|^2}\boldsymbol{\eta}_{z_i}\boldsymbol{\eta}_{z_i}^H \right] \boldsymbol{\nu} \,, \tag{10}$$

where $\boldsymbol{\nu}$ is the step reference direction, $\boldsymbol{\eta}_{\infty,i}$ is the unitary left direction of the first zero at infinity of $\mathbf{X}_i[z]$, with $\mathbf{X}_{i+1}[z]$ defined as in Section II, with $\mathbf{X}_1[z] = \boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{G}[z]$, and $\boldsymbol{\eta}_{z,i}$ is the unitary left direction associated to the zero at $z = z_i$ of $\mathbf{Y}_i[z]$, with $\mathbf{Y}_{i+1}[z]$ defined as in Section II, with $\mathbf{Y}_1[z] = \mathbf{X}_{d_\Delta+1}[z]$.

Proof: Immediate from Lemma 2 and the proof of Theorem 3.1 in [21] (See [16] for details). ∎

Corollary 1 allows one to quantify the benefits arising from adding *unconstrained* control channels to a square plant. It is seen that the performance improvement depends exclusively on the (finite and infinite) NMP zeros which are eliminated by the addition of new channels.

B. Constrained channels

This section gives a characterization for $\mathcal{D}_\gamma$ in (8) for finite γ. The Lagrangian [2], [17] associated with the optimization problem (7) is given by

$$L_a(\lambda) \triangleq J_a + \lambda R_a \,, \tag{11}$$

where $\lambda \geq 0$ is the Lagrange multiplier. Define

$$L_a^{\rm opt}(\lambda) \triangleq \inf_{\mathbf{C}_a[z]\in\mathcal{K}_a} L_a(\lambda) \,, \tag{12}$$

where $\mathcal{K}_a$ is as in (7). Also define

$$\mathbf{W}_\lambda[z] \triangleq \begin{bmatrix} \boldsymbol{\xi}_{G_a}[z]\mathbf{G}[z] & \boldsymbol{\xi}_{G_a}[z]\mathbf{F}[z] \\ \mathbf{0}_{m\times n} & \sqrt{\lambda}\,\mathbf{M}[z] \end{bmatrix} \in \mathcal{RH}_\infty^{(n+m)\times(n+m)} \,, \tag{13}$$

where $\boldsymbol{\xi}_{G_a}[z]$ is a left interactor for $\mathbf{G}_a[z]$, and $\mathbf{M}[z]$ is defined as in (6). From the above we have that

$$\det\{\mathbf{W}_\lambda[z]\} = \sqrt{\lambda}^m \det\{\boldsymbol{\xi}_{G_a}[z]\mathbf{G}[z]\}\det\{\mathbf{M}[z]\} \,. \tag{14}$$

From (14) it follows that the multi-set of (finite and infinite) NMP zeros of $\mathbf{W}_\lambda[z]$ corresponds to the union of the multi-set of NMP zeros of $\mathbf{M}[z]$, and the multi-set of NMP zeros of the square plant $\mathbf{G}[z]$ that are eliminated by adding additional control channels.

The following lemma gives a closed form expression for the optimal cost $L_a^{\rm opt}(\lambda)$ in (12). Before presenting the corresponding results, we will introduce the following assumptions.

Assumption 1:

1) The transfer matrix $\mathbf{M}[z]$ is stable, biproper and minimum phase, i.e., $\mathbf{M}[z]$, $\mathbf{M}[z]^{-1} \in \mathcal{RH}_\infty^{m\times m}$.

2) The multi-set of NMP zeros of $\mathbf{G}[z]$ that are not zeros of $\mathbf{G}_a[z]$ contains non-repeated finite elements and multiple infinite elements. (i.e., given assumption (1), $\mathbf{W}_\lambda[z]$ has multiple zeros at infinity, and finite NMP zeros with, at most, (algebraic) multiplicity one).

3) The stationary value of the additional control signals is rewritten as $\mathbf{u}_{a,\infty} = \mathbf{P}\boldsymbol{\nu}$, where $\mathbf{P} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{\nu}$ corresponds to the reference direction. □□

Lemma 3: Consider the setup and assumptions of Lemma 1 and suppose, in addition, that Assumption 1 holds. Denote by $c_1, \dots, c_{q_a}$ the finite NMP zeros of $\mathbf{G}_a[z]$, and assume that $\mathbf{G}_a[z]$ has d_a zeros at infinity. Consider the transfer matrix $\mathbf{W}_\lambda[z]$ defined in (13). Denote the finite NMP zeros of $\mathbf{W}_\lambda[z]$ by $z_1, ..., z_{q_\Delta}$, and assume that $\mathbf{W}_\lambda[z]$ has d_Δ zeros at infinity[1]. Define

$$\mathbf{E}_{\lambda,i} \triangleq \prod_{k=1}^{q_\Delta - i} \mathbf{L}_{z,q_\Delta-k+1}[z_i]\hat{\boldsymbol{\eta}}_{z,i}\hat{\boldsymbol{\eta}}_{z,i}^H \times \prod_{k=q_\Delta-i+2}^{q_\Delta} \mathbf{L}_{z,q_\Delta-k+1}[z_i] \in \mathbb{C}^{(n+m)\times(n+m)}, \quad (15)$$

where $\hat{\boldsymbol{\eta}}_{z,i}$ denotes the left unitary direction of the zero at $z = z_i$ of $\mathbf{Y}_i[z]$, with $\mathbf{Y}_{i+1}[z]$, $\mathbf{L}_{z,i}[z]$ and $\mathbf{L}_{\infty,i}[z]$ defined as in Section II, with $\mathbf{Y}_1[z] = \prod_{i=1}^{d_\Delta} \mathbf{L}_{\infty,d_\Delta-k+1}[z]\mathbf{W}_\lambda[z]$. Moreover, consider the following power series expansion

$$\frac{1}{z^{d_\Delta}}\prod_{i=1}^{d_\Delta}\mathbf{L}_{\infty,d_\Delta-i+1}[z]\begin{bmatrix}\mathbf{I}_n\\ \sqrt{\lambda}\,\mathbf{M}[z]\mathbf{P}\end{bmatrix} \triangleq \sum_{k=0}^{\infty}\mathbf{R}_{\lambda,k}z^{-k}, \quad (16)$$

with $\mathbf{R}_{\lambda,k} \in \mathbb{R}^{(n+m)\times n}$, $\forall k$. Then, the optimal value $L_a^{\rm opt}(\lambda)$ in (12) is given by

$$\begin{aligned} L_a^{\rm opt}(\lambda) &= J_{a,\infty}^{\rm opt} + \boldsymbol{\nu}^H \sum_{a=0}^{d_\Delta-2}\sum_{k=a+1}^{d_\Delta-1}\mathbf{R}_{\lambda,a}^T\mathbf{R}_{\lambda,k}(a-k)\boldsymbol{\nu} \\ &+\boldsymbol{\nu}^H\sum_{a,k=0}^{d_\Delta-1}\mathbf{R}_{\lambda,a}^T\left[\sum_{i=1}^{q_\Delta}\frac{|z_i|^2-1}{|1-z_i|^2}\hat{\boldsymbol{\eta}}_{z,i}\hat{\boldsymbol{\eta}}_{z,i}^H + (d_\Delta - a)\mathbf{I}_{n+m}\right]\mathbf{R}_{\lambda,k}\boldsymbol{\nu} \\ &+ 2\boldsymbol{\nu}^H\sum_{a=0}^{d_\Delta-1}\sum_{i=1}^{q_\Delta}\frac{|z_i|^2-1}{|1-z_i|^2}\mathbf{R}_{\lambda,a}^T\mathbf{E}_{\lambda,i}\sum_{k=d_\Delta}^{\infty}\mathbf{R}_{\lambda,k}z_i^{d_\Delta-k}\boldsymbol{\nu} \\ &+\boldsymbol{\nu}^H\sum_{i,k=1}^{q_\Delta}\frac{(|z_i|^2-1)(|z_k|^2-1)}{(z_i-1)(z_k-1)(z_iz_k-1)}\sum_{a=d_\Delta}^{\infty}\mathbf{R}_{\lambda,a}^T\mathbf{E}_{\lambda,i}^T\mathbf{E}_{\lambda,k}z_i^{d_\Delta-a} \\ &\times\sum_{b=d_\Delta}^{\infty}\mathbf{R}_{\lambda,b}z_k^{d_\Delta-b}\boldsymbol{\nu}, \end{aligned} \quad (17)$$

where $J_{a,\infty}^{\rm opt}$ is given by

$$J_{a,\infty}^{\rm opt} = \boldsymbol{\nu}^H\left[\sum_{z=1}^{d_a}\hat{\boldsymbol{\beta}}_{\infty,i}\hat{\boldsymbol{\beta}}_{\infty,i}^H + \sum_{i=1}^{q_a}\frac{|c_i|^2-1}{|1-c_i|^2}\hat{\boldsymbol{\beta}}_{c,i}\hat{\boldsymbol{\beta}}_{c,i}^H\right]\boldsymbol{\nu}, \quad (18)$$

where $\hat{\boldsymbol{\beta}}_{\infty,i}$ denotes the unitary left direction of the first zero at infinity of $\mathbf{X}_i[z]$, with $\mathbf{X}_{i+1}[z]$ defined as in Section II, with $\mathbf{X}_1[z] = \mathbf{G}_a[z]$, and $\hat{\boldsymbol{\beta}}_{c,i}$ denotes the unitary left direction of the zero at $z = c_i$ of $\mathbf{Y}_i[z]$, with $\mathbf{Y}_{i+1}[z]$ defined as in Section II, with $\mathbf{Y}_1[z] = \mathbf{X}_{d_a+1}[z]$.

Proof: From Parseval's relation and the Youla parameterization of stabilizing controllers [11], [22] it follows that

$$J_a = \left\|(\mathbf{I}_n - \mathbf{G}_a[z]\mathbf{Q}[z])\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2 \quad (19)$$

$$R_a = \left\|\mathbf{M}[z](\mathbf{K}\mathbf{Q}[z]-\mathbf{P})\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2, \quad (20)$$

where $\mathbf{K} \triangleq [\,\mathbf{0}_{m\times n}\ \ \mathbf{I}_m\,] \in \mathbb{R}^{m\times(n+m)}$, $\mathbf{P} \in \mathbb{R}^{m\times n}$ is defined as in Assumption 1 and $\boldsymbol{\nu} \in \mathbb{R}^n$ is the reference direction. Thus, (12) can be written as

$$L_a^{\rm opt}(\lambda) = \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{(n+m)\times n}}\left\{\left\|(\mathbf{I}_n - \mathbf{G}_a[z]\mathbf{Q}[z])\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2 + \lambda\left\|\mathbf{M}[z](\mathbf{K}\mathbf{Q}[z]-\mathbf{P})\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2\right\}. \quad (21)$$

Consider the transfer matrix $\mathbf{W}_\lambda[z]$ in (13) and define $\overline{\mathbf{W}}_\lambda[z] = \boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]\mathbf{W}_\lambda[z]$, where $\boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]$ is a left interactor for $\mathbf{W}_\lambda[z]$ and $\overline{\mathbf{W}}_\lambda[z] \in \mathcal{RH}_\infty^{(n+m)\times(n+m)}$ is right-invertible in $\mathcal{RH}_\infty$. Since $\boldsymbol{\xi}_{\mathbf{G}_a}[z]$ and $\boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]$ are unitary, it follows by proceeding as in the proof of Theorem 3.1 in [21] that (21) reduces to

$$L_a^{\rm opt}(\lambda) = \left\|(\boldsymbol{\xi}_{\mathbf{G}_a}[z]-\mathbf{I}_n)\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2 + \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{(n+m)\times n}} J_1, \quad (22)$$

with

$$J_1 \triangleq \left\|\left(\boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]\begin{bmatrix}\mathbf{I}_n\\ \sqrt{\lambda}\mathbf{M}[z]\mathbf{P}\end{bmatrix} - \overline{\mathbf{W}}_\lambda[z]\mathbf{Q}[z]\right)\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2. \quad (23)$$

Proceeding as in the proof of Theorem 3 in [19], if follows that the first term on the right hand side of (22) reduces to $J_{a,\infty}^{\rm opt}$ given by (18). On the other hand, using (16) it follows that J_1 can be written as

$$J_1 = \left\|\left(\mathbf{A}_1[z] + \mathbf{A}_2[z] - \overline{\mathbf{W}}_\lambda[z]\mathbf{Q}[z]\right)\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2, \quad (24)$$

where

$$\mathbf{A}_1[z] \triangleq \boldsymbol{\xi}_{\mathbf{W}_\lambda}^z[z]\sum_{k=0}^{d_\Delta-1}\mathbf{R}_{\lambda,k}z^{d_\Delta-k} \in \mathcal{RH}_2^{\perp\,(n+m)\times n} \quad (25)$$

$$\mathbf{A}_2[z] \triangleq \boldsymbol{\xi}_{\mathbf{W}_\lambda}^z[z]\sum_{k=d_\Delta}^{\infty}\mathbf{R}_{\lambda,k}z^{d_\Delta-k} \in \mathcal{RL}_2^{(n+m)\times n}, \quad (26)$$

with $\boldsymbol{\xi}_{\mathbf{W}_\lambda}^z[z] = \prod_{i=1}^{q_\Delta}\mathbf{L}_{z,q_\Delta-i+1}[z]$. Since Assumption 1 holds, we have that $\mathbf{W}_\lambda[z]$ has non-repeated finite NMP zeros and, therefore, $\mathbf{A}_2[z]$ is proper and has non-repeated unstable poles. By using partial fraction expansion, $\mathbf{A}_2[z]$ can be written as

[1]Our discussion following (14) justifies the use of the same symbols introduced in Corollary 1.

$\mathbf{A}_2[z] = \mathbf{A}_{21}[z] + \mathbf{A}_{22}[z]$, where $\mathbf{A}_{22}[z] \in \mathcal{RH}_2^{\perp(n+m)\times n}$ is given by

$$\mathbf{A}_{22}[z] \triangleq \sum_{i=1}^{q_\Delta} \frac{(1-z_i)(1-|z_i|^2)}{1-\bar{z_i}} \mathbf{E}_{\lambda,i} \sum_{k=d_\Delta}^{\infty} \mathbf{R}_{\lambda,k} {z_i}^{d_\Delta - k} \frac{1}{z-z_i}\,, \tag{27}$$

with $\mathbf{E}_{\lambda,i}$ defined as in (15), and $\mathbf{A}_2[z] \in \mathcal{RH}_\infty^{(n+m)\times n}$. Using the above and applying the standard orthogonal decomposition in $\mathcal{L}_2$, it follows that J_1 can be written as

$$J_1 = \left\| \left(\mathbf{C}_1[z] + \mathbf{C}_2[z]\right) \frac{\boldsymbol{\nu}}{z-1} \right\|_2^2 + \left\| \left(\mathbf{A}_1[1] + \mathbf{A}_{21}[1] + \mathbf{A}_{22}[z] - \overline{\mathbf{W}}_\lambda \mathbf{Q}[z]\right) \frac{\boldsymbol{\nu}}{z-1} \right\|_2^2, \tag{28}$$

where $\mathbf{C}_1[z] \in \mathcal{RH}_2^{\perp(n+m)\times n}$ and $\mathbf{C}_2[z]\mathcal{RH}_2^{\perp(n+m)\times n}$ are given by

$$\begin{aligned} \mathbf{C}_1[z] &\triangleq \mathbf{A}_1[z] - \mathbf{A}_1[1] \\ &= \sum_{k=0}^{d_\Delta - 1} \left(\boldsymbol{\xi}^z_{\mathbf{W}_\lambda}[z] z^{d_\Delta - k} - \mathbf{I}_{n+m} \right) \mathbf{R}_{\lambda,k}\,, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathbf{C}_2[z] &\triangleq \mathbf{A}_{21}[z] - \mathbf{A}_{21}[1] \\ &= \sum_{i=1}^{q_\Delta} \frac{1-|z_i|^2}{1-\bar{z_i}} \mathbf{E}_{\lambda,i} \sum_{k=d_\Delta}^{\infty} \mathbf{R}_{\lambda,k} {z_i}^{d_\Delta - k} \frac{1-z}{z-z_i}\,. \end{aligned} \tag{30}$$

Since $\overline{\mathbf{W}}_\lambda[z]$ is right-invertible in $\mathcal{RH}_\infty$, it is possible to select $\mathbf{Q}[z] \in \mathcal{RH}_\infty$ such that the second term on the right hand side of (28) vanishes. Thus, we have that

$$J_1^{\rm opt} = \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{(n+m)\times n}} J_1 = \left\| \left(\mathbf{C}_1[z] + \mathbf{C}_2[z]\right) \frac{\boldsymbol{\nu}}{z-1} \right\|_2^2. \tag{31}$$

Applying the definition of 2-norm, from the above it follows that

$$J_1^{\rm opt} = \frac{-\boldsymbol{\nu}^H}{2\pi j} \oint \left(\mathbf{C}_1[z] + \mathbf{C}_2[z]\right)^\sim \left(\mathbf{C}_1[z] + \mathbf{C}_2[z]\right) \frac{dz}{(z-1)^2} \boldsymbol{\nu}\,, \tag{32}$$

where the integral is over the unit circle, travelled counterclock-wise. Using the Residue Theorem and Lemma 1 in [12], it is straightforward to show that

$$J_1^{\rm opt} = -\boldsymbol{\nu}^H \left(\alpha_1 - 2\alpha_2 + \alpha_3\right) \boldsymbol{\nu}\,, \tag{33}$$

where

$$\boldsymbol{\alpha}_1 \triangleq \frac{1}{2\pi j} \oint \frac{\mathbf{C}_1[z]^\sim \mathbf{C}_1[z]}{(z-1)^2} dz = \sum_{a=0}^{d_\Delta - 2} \sum_{k=a+1}^{d_\Delta - 1} \mathbf{R}^T_{\lambda,a} \mathbf{R}_{\lambda,k} (k-a) + \sum_{a,k=0}^{d_\Delta - 1} \mathbf{R}^T_{\lambda,a} \left[\sum_{i=1}^{q_\Delta} \frac{1-|z_i|^2}{|1-z_i|^2} \hat{\boldsymbol{\eta}}_{z,i} \hat{\boldsymbol{\eta}}^H_{z,i} + (a - d_\Delta) \mathbf{I}_{n+m} \right] \mathbf{R}_{\lambda,k}\,,$$

$$\begin{aligned} \boldsymbol{\alpha}_2 &\triangleq \frac{1}{2\pi j} \oint \frac{\mathbf{C}_1[z]^\sim \mathbf{C}_2[z]}{(z-1)^2} dz \\ &= \sum_{a=0}^{d_\Delta - 1} \sum_{i=1}^{q_\Delta} \frac{1-|z_i|^2}{|1-z_i|^2} \mathbf{R}^T_{\lambda,a} \mathbf{E}_{\lambda,i} \sum_{k=d_\Delta}^{\infty} \mathbf{R}_{\lambda,k} z_i^{d_\Delta - k}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} \boldsymbol{\alpha}_3 &\triangleq \frac{1}{2\pi j} \oint \frac{\mathbf{C}_2[z]^\sim \mathbf{C}_2[z]}{(z-1)^2} dz \\ &= \sum_{i,k=1}^{q_\Delta} \frac{(|z_i|^2 - 1)(|z_k|^2 - 1)}{(\bar{z_i} - 1)(\bar{z_k} - 1)(1 - z_i z_k)} \\ &\quad \times \sum_{a=d_\Delta}^{\infty} \mathbf{R}^T_{\lambda,a} \mathbf{E}^T_{\lambda,i} \mathbf{E}_{\lambda,k} z_i^{d_\Delta - a} \sum_{b=d_\Delta}^{\infty} \mathbf{R}_{\lambda,b} z_k^{d_\Delta - b}\,, \end{aligned} \tag{35}$$

which proves our results. This completes the proof. ■

From Lemma 3 we see that the minimal value of the Lagrangian $L_a(\lambda)$ depends explicitly on the (finite and infinite) NMP zeros of the augmented plant $\mathbf{G}_a[z]$, and also on the (finite and infinite) NMP zeros of the square plant $\mathbf{G}[z]$ that are eliminated when adding the additional control channels. Indeed, the presence the first ones is in the first term on the right hand side of (17). In turn, the remaining terms in the right hand side of (17) depend on the eliminated NMP zeros. Thus, one concludes that $L_a^{\rm opt}(\lambda)$ depends on all the (finite and infinite) NMP zeros of the (original) square plant. On the other hand, we have that the effect from the filter structure $\mathbf{M}[z]$, which captures the frequency-response limitations of the additional channels, is embedded in the matrices $\mathbf{R}_{\lambda,k}$ and $\mathbf{E}_{\lambda,i}$.

The constraint (6) do not only restricts the additional control signals, but also determines the stationary value of the original control inputs $\mathbf{u}[k]$. Therefore, one should carefully choose the stationary value of the additional signals. For that reason, in the sequel we consider the following additional assumption.

Assumption 2: The matrix $\mathbf{P} \in \mathbb{R}^{m\times n}$ in Assumption 1 satisfies

$$J^{\rm opt} = \inf_{\mathbf{Q} \in \mathcal{RH}_\infty^{n\times n}} \left\| \left(\mathbf{I}_n - \mathbf{G}[z]\mathbf{Q}[z] - \mathbf{F}[z]\mathbf{P}\right) \frac{\boldsymbol{\nu}}{z-1} \right\|_2^2, \tag{36}$$

where J^{opt} is defined as in (8). □□

The following lemma characterizes the limiting behavior of $L_a^{\rm opt}(\lambda)$ as a function of the multiplier λ.

Corollary 2: Consider the setup and assumptions of Lemma 1 and suppose, in addition, that Assumption 2 holds. Then, $L_a^{\rm opt}(\lambda)$ is an increasing function of $\lambda \geq 0$, $\lim_{\lambda\to 0} L_a^{\rm opt}(\lambda) = J_{a,\infty}^{\rm opt}$, and $\lim_{\lambda\to\infty} L_a^{\rm opt}(\lambda) = J^{\rm opt}$, where $J_{a,\infty}^{\rm opt}$ and $J^{\rm opt}$ are as in (7) and (8).

Proof: The first two statements are immediate. On the other hand, note that $L_a(\lambda)$ can be written as

$$L_a(\lambda) = \left\| \left(\mathbf{I}_n - \mathbf{G}[z]\mathbf{Q}_G[z] - \mathbf{F}[z]\mathbf{Q}_F[z]\right) \frac{\boldsymbol{\nu}}{z-1} \right\|_2^2 + \lambda \left\| \mathbf{M}[z] \left(\mathbf{Q}_F[z] - \mathbf{P}\right) \frac{\boldsymbol{\nu}}{z-1} \right\|_2^2, \tag{37}$$

where $\mathbf{Q}_G[z] \in \mathcal{RH}_\infty^{n\times n}$ and $\mathbf{Q}_F[z] \in \mathcal{RH}_\infty^{m\times n}$. Thus, together with the fact that Assumption 2 holds, the third statement follows. ■

We are now ready to present the main results of this paper. To that end, we need to introduce some additional assumptions.

Assumption 3:

1) In addition of part (1) in Assumption 1, the multi-set of NMP zeros of $\mathbf{G}[z]$ that are not zeros of $\mathbf{G}_a[z]$ have one infinite element (i.e., $\mathbf{W}_\lambda[z]$ has one zero at infinity).
2) The constraint in (6) is not redundant, i.e., $\gamma < \gamma_{\max}$, where $\gamma_{\max}$ is the infimum of (6) when the controller $\mathbf{C}_a[z]$ is constrained to belong to $\mathcal{K}_a$ and to achieve $J_a = J_{a,\infty}^{\text{opt}}$.[2] □□

Theorem 1: Consider the setup and assumptions of Lemma 1 and suppose, in addition, that Assumptions 2 and 3 holds. Define the following matrices in $\mathbb{C}^{(n+m)\times(n+m)}$:

$$\mathbf{A}_{\lambda,i} \triangleq \mathbf{L}_{\infty,1}\left[\bar{z}_i^{-1}\right]\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times m}\\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m}\end{bmatrix}\mathbf{L}_{\infty,1}[\bar{z}_i]\mathbf{E}_{\lambda,i}^H\,, \tag{38}$$

$$\mathbf{A}_{\lambda,\infty} \triangleq \boldsymbol{\xi}_{\mathbf{W}_\lambda}[0]\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times m}\\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m}\end{bmatrix}\hat{\boldsymbol{\eta}}_{\infty,1}\hat{\boldsymbol{\eta}}_{\infty,1}^H\prod_{i=1}^{q_\Delta}\mathbf{L}_{z,i}[0]^\sim\,, \tag{39}$$

where $\boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]$ is a left interactor for $\mathbf{W}_\lambda[z]$, $\hat{\boldsymbol{\eta}}_{\infty,1}$ denotes the unitary left direction of the zero at infinity of $\mathbf{W}_\lambda[z]$, and $\mathbf{L}_{z,i}[z]$, $\mathbf{L}_{\infty,1}[z]$, $\mathbf{E}_{\lambda,i}$ and $z_1,\ldots,z_{q_\Delta}$ are as in Lemma 1. Define, also, the following transfer functions in $\mathcal{H}_2^{\perp(n+m)\times(n+m)}$:

$$\mathbf{K}_{\lambda,i}[z] \triangleq \frac{\mathbf{E}_{\lambda,i}^H\prod_{i=1}^{q_\Delta}\mathbf{L}_{z,q_\Delta-i+1}[z]}{(1-\bar{z}_i z)}\,, \tag{40}$$

$$\boldsymbol{\Gamma}_\lambda[z] \triangleq \mathbf{R}_{\lambda,0} + \sum_{k=1}^{q_\Delta}\frac{1-|z_k|^2}{1-\bar{z}_k}\mathbf{E}_{\lambda,k}\sum_{a=1}^{\infty}\mathbf{R}_{\lambda,a}z_k^{1-a}\frac{z-1}{z-z_k}\,, \tag{41}$$

where $\mathbf{R}_{\lambda,i}$ are also as in Lemma 1. Then, the optimal choice for the Lagrange multiplier is $\lambda=\lambda_\gamma$, where $\lambda_\gamma>0$ satisfies

$$\gamma = \left\|\left\{\begin{bmatrix}\mathbf{0}_{m\times n} & \mathbf{I}_m\end{bmatrix}\mathbf{W}_{\lambda_\gamma}[z]^{-1}\left(\mathbf{L}_{\infty,1}[z]^{-1}\times\sum_{k=1}^{\infty}\mathbf{R}_{\lambda_\gamma,k}z^{1-k} + \boldsymbol{\xi}_{\mathbf{W}_{\lambda_\gamma}}[z]^{-1}\boldsymbol{\Gamma}_{\lambda_\gamma}[z]\right) - \mathbf{P}\right\}\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2, \tag{42}$$

and $\mathcal{D}_\gamma = \mathcal{D}_\infty - \Delta_{\lambda_\gamma}$, where $\mathcal{D}_\infty$ was calculated in Corollary 1 and

$$\begin{aligned}\Delta_{\lambda_\gamma} \triangleq{}& -\boldsymbol{\nu}^H\mathbf{R}_{\lambda_\gamma,0}^T\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times m}\\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m}\end{bmatrix}\left[\sum_{i=1}^{q_\Delta}\frac{1-|z_i|^2}{|1-z_i|^2}\right.\\ &\times\left.\left(\hat{\boldsymbol{\eta}}_{z,i}\hat{\boldsymbol{\eta}}_{z,i}^H\mathbf{R}_{\lambda_\gamma,0} + \mathbf{E}_{\lambda_\gamma,i}\sum_{a=1}^{\infty}\mathbf{R}_{\lambda_\gamma,a}z_i^{1-a}\right) - \mathbf{R}_{\lambda_\gamma,0}\right]\boldsymbol{\nu}\\ &-\boldsymbol{\nu}^H\sum_{i,k=1}^{q_\Delta}\left[\frac{|z_i|^2-1}{|1-z_i|^2}\mathbf{R}_{\lambda_\gamma,0}^T\prod_{i=1}^{q_\Delta}\mathbf{L}_{z,q_\Delta-i+1}\left[\bar{z}_i^{-1}\right]\right.\\ &\left.+\frac{|z_i|^2-1}{\bar{z}_i-|z_i|^2}\frac{|z_k|^2-1}{1-z_k}\sum_{a=1}^{\infty}\mathbf{R}_{\lambda_\gamma,a}^T\bar{z}_k^{1-a}\mathbf{K}_{\lambda_\gamma,k}\left[\bar{z}_i^{-1}\right]\right]\\ &\times\mathbf{A}_{\lambda_\gamma,i}\boldsymbol{\Gamma}_{\lambda_\gamma}\left[\bar{z}_i^{-1}\right]\boldsymbol{\nu} - \boldsymbol{\nu}^H\boldsymbol{\Gamma}_{\lambda_\gamma}[\infty]^T\mathbf{A}_{\lambda_\gamma,\infty}\boldsymbol{\Gamma}_{\lambda_\gamma}[0]\boldsymbol{\nu}\geq 0\,.\end{aligned} \tag{43}$$

[2]Given $\mathbf{G}[z]$ with no zeros on the unit circle, the problem of finding $\gamma_{\max}$ is always feasible.

Proof: Since $\mathbf{G}[z]$ has no zeros on the unit circle and, Assumptions 1 and 3 holds, it follows from the proof of Lemma 3 that the infimum (28) is achievable upon choosing

$$\mathbf{Q}[z] = \mathbf{Q}_\lambda^{\text{opt}}[z] = \mathbf{W}_\lambda[z]^{-1}\left(\mathbf{L}_{\infty,1}[z]^{-1}\sum_{k=1}^{\infty}\mathbf{R}_{\lambda,k}z^{1-k} + \boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]^{-1}\mathbf{R}_{\lambda,0} - \boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]^{-1}\mathbf{C}_\lambda[z]\right), \tag{44}$$

where $\mathbf{C}_\lambda[z]$ is given by (30), with $d_\Delta=1$. Given the above, standard convex optimization results [2], [17] show that, for any $\gamma\geq 0$ satisfying Assumption 3, $J_{a,\gamma}^{\text{opt}}$ in (7) satisfies

$$J_{a,\gamma}^{\text{opt}} = J_a\left(\mathbf{Q}_{\lambda_\gamma}^{\text{opt}}[z]\right)\,, \tag{45}$$

where we have used (19), and $\lambda_\gamma>0$ satisfies (again, we use (19))

$$\gamma = \left\|\mathbf{M}[z]\left(\begin{bmatrix}\mathbf{0}_{m\times n} & \mathbf{I}_n\end{bmatrix}\mathbf{Q}_{\lambda_\gamma}^{\text{opt}}[z] - \mathbf{P}\right)\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2. \tag{46}$$

Now, (42) follows from (46), (44) and upon noting that $\boldsymbol{\Gamma}_\lambda[z] = \mathbf{R}_{\lambda,0} - \mathbf{C}_\lambda[z]$. Thus, we have that

$$\mathcal{D}_\gamma = J^{\text{opt}} - J_a\left(\mathbf{Q}_{\lambda_\gamma}^{\text{opt}}[z]\right). \tag{47}$$

In what follows we characterize $J_a\left(\mathbf{Q}_{\lambda_\gamma}^{\text{opt}}[z]\right)$ in (45) in closed form. From (16), it is straightforward to prove that $\mathbf{Q}_\lambda^{\text{opt}}[z]$ in (44) can be written as

$$\mathbf{Q}_\lambda^{\text{opt}}[z] = \mathbf{W}_\lambda[z]^{-1}\left(\left\{\boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]^{-1} - \mathbf{I}_{n+m}\right\}\mathbf{R}_{\lambda,0} - \boldsymbol{\xi}_{\mathbf{W}_\lambda}[z]^{-1}\mathbf{C}_\lambda + \begin{bmatrix}\mathbf{I}_m\\ \sqrt{\lambda}\mathbf{M}[z]\mathbf{P}\end{bmatrix}\right). \tag{48}$$

Then, by using the standard orthogonal decomposition in $\mathcal{L}_2$, it can be shown from (13), (19) and (48) that

$$J_a\left(\mathbf{Q}_{\lambda_\gamma}^{\text{opt}}[z]\right) = J_{a,\infty}^{\text{opt}} + J_1,\quad J_1 = \left\|\left(\mathbf{A}[z]-\mathbf{B}[z]\right)\frac{\boldsymbol{\nu}}{z-1}\right\|_2^2, \tag{49}$$

where $J_{a,\infty}^{\text{opt}}$ is given as in (18), and

$$\mathbf{A}[z] \triangleq \begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times m}\end{bmatrix}\left(\boldsymbol{\xi}_{\mathbf{W}_{\lambda_\gamma}}[z]^{-1} - \mathbf{I}_{n+m}\right)\mathbf{R}_{\lambda_\gamma,0}\in\mathcal{RH}_\infty^{n\times n}\,, \tag{50}$$

$$\mathbf{B}[z] \triangleq \begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times m}\end{bmatrix}\boldsymbol{\xi}_{\mathbf{W}_{\lambda_\gamma}}[z]^{-1}\mathbf{C}_{\lambda_\gamma}[z]\in\mathcal{RH}_\infty^{n\times n}\,. \tag{51}$$

Applying the definition of 2-norm it follows from (49) that

$$J_1 = -\frac{\boldsymbol{\nu}^H}{2\pi j}\oint\left\{\mathbf{A}[z]-\mathbf{B}[z]\right\}^\sim\left\{\mathbf{A}[z]-\mathbf{B}[z]\right\}\frac{dz}{(z-1)^2}\boldsymbol{\nu}\,, \tag{52}$$

where the integral is over the unit circle, travelled counter-clockwise. Since Assumption 1 holds, by using the Residue Theorem, Lemma 1 in [12] and the fact that $\boldsymbol{\Gamma}_{\lambda_\gamma}[z] = \mathbf{R}_{\lambda_\gamma,0} - \mathbf{C}_{\lambda_\gamma}[z]$, it can be shown that $J_1=\Delta_{\lambda_\gamma}$, with Δ_{λ_γ} given as in (43). Furthermore, given Assumption 2 holds, it is straightforward to show that $\Delta_{\lambda_\gamma}\geq 0$. Finally,

$\mathcal{D}_\gamma = \mathcal{D}_\infty - \Delta_{\lambda_\gamma}$ is obtained by noting that, by definition, $\mathcal{D}_\infty = J^{\rm opt} - J^{\rm opt}_{a,\infty}$. ■

Remark 1: Suppose that no infinite zero of $\mathbf{G}[z]$ is eliminated when adding additional control channels, i.e., that $\mathbf{W}_\lambda[z]$ has no zeros at infinity. Then, it follows from the proof of Theorem 1 that

$$\Delta_{\lambda_\Delta} = -\boldsymbol{\nu}^H \sum_{i,k,a=1}^{q_\Delta} \frac{|z_i|^2-1}{\bar{z}_i - |z_i|^2}\frac{|z_k|^2-1}{1-z_k}\frac{|z_a|^2-1}{1-\bar{z}_a}\frac{1-\bar{z}_i}{z_a\bar{z}_i-1}$$
$$\times \begin{bmatrix} \mathbf{I}_n \\ \sqrt{\lambda}\,\mathbf{M}[z_k]\mathbf{P}\end{bmatrix}^H \mathbf{K}_{\lambda_\gamma,k}\left[\bar{z}_i^{-1}\right]\mathbf{A}_{\lambda_\gamma,i}\mathbf{E}_{\lambda_\gamma,a}\begin{bmatrix} \mathbf{I}_n \\ \sqrt{\lambda}\,\mathbf{M}[z_a]\mathbf{P}\end{bmatrix}\boldsymbol{\nu}, \tag{53}$$

where $\mathbf{E}_{\lambda,i}$, $\mathbf{A}_{\lambda,i}$ and $\mathbf{K}_{\lambda,i}$ are as in Theorem 1, but with $\mathbf{L}_{\infty,1}[z] = \mathbf{I}_{n+m}$. □

Theorem 1 shows that the achievable performance improvement of a square MIMO plant that is augmented by adding frequency-response-constrained input channels, depends explicitly on the (finite and infinite) NMP zeros that are eliminated as a consequence of adding input channels, on the stationary value of the additional signals, on the structure of the filter $\mathbf{M}[z]$ and on the restriction level γ. Furthermore, the results of Theorem 1 allow us to quantify explicitly the effect of the additional channels restrictions on the performance improvement.

The next corollary give us some elementary facts about the limiting behavior of $\mathcal{D}_\gamma$.

Corollary 3: Consider the setup, notation and assumptions of Theorem 1. $\mathcal{D}_\gamma$ is an increasing function of γ in the interval $[0,\ \gamma_{\max}]$, $\lim_{\gamma\to\gamma_{\max}}\mathcal{D}_\gamma = \mathcal{D}_\infty$, and $\lim_{\gamma\to 0}\mathcal{D}_\gamma = 0$.

Proof: Standard results on the geometry of convex constrained optimization problems [2], [17] allow one to conclude that $J_a\left(\mathbf{Q}^{\rm opt}_{\lambda_\gamma}[z]\right)$, with λ_γ being a function of γ through (42), is decreasing in γ. Thus, since $\mathcal{D}_\gamma = J^{\rm opt} - J_a\left(\mathbf{Q}^{\rm opt}_{\lambda_\gamma}[z]\right)$ and $J^{\rm opt}$ does not depend on γ, our first claim follows. Our remaining claims are immediate by definition. ■

C. Discussion

As the results of the above section shown, the eliminated finite NMP zeros play a key role on the performance improvement $\mathcal{D}_\gamma$. However, how these zeros affect it does not follow easily from Theorem 1. Therefore, to gain insight into the results in Theorem 1, we next consider the following particularly case. Suppose that the addition of input channels eliminates only one finite NMP zero at $z = z_1$ from the original square plant $\mathbf{G}[z]$. From $\mathbf{W}_{\lambda_\gamma}[z]$ in (13) it follows that $\hat{\boldsymbol{\eta}}_{z,1} = [\,\alpha\boldsymbol{\eta}^H_{z,1}\ \ \bar{\boldsymbol{\eta}}^H_1\,]^H$ for some $\bar{\boldsymbol{\eta}}_1 \in \mathbb{R}^m$ and $|\alpha| < 1$ (here, we use the notation in Corollary 1 and Lemma 3.). Then, the above allows one to deduce from (53) that[3]

$$\mathcal{D}_{\lambda_\gamma} = P[z_1]\boldsymbol{\nu}\Big\{\boldsymbol{\eta}_{z,1}\boldsymbol{\eta}^H_{z,1} - \alpha^4\left(\mathbf{I}_n - [\boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{F}[z]]_{z=z_1}\mathbf{P}\right)^H$$
$$\times\,\boldsymbol{\eta}_{z,1}\boldsymbol{\eta}^H_{z,1}\left(\mathbf{I}_n - [\boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{F}[z]]_{z=z_1}\mathbf{P}\right)\Big\}\boldsymbol{\nu} \tag{54}$$

[3] We use $[\mathbf{X}[z]]_{z=a}$ to refer to $\mathbf{X}[a]$.

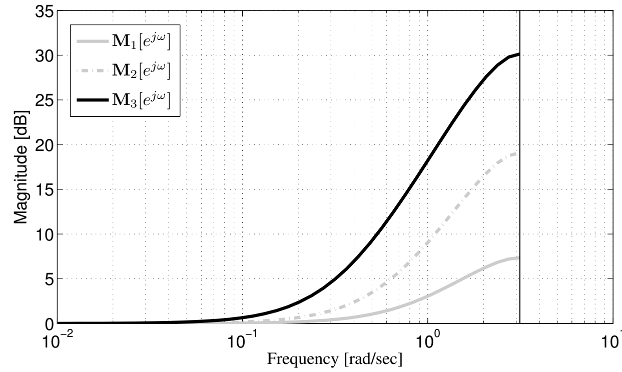

Fig. 3. Magnitude frequency response of the filters $\mathbf{M}_i[e^{j\omega}]$, with $i = 1, 2, 3$.

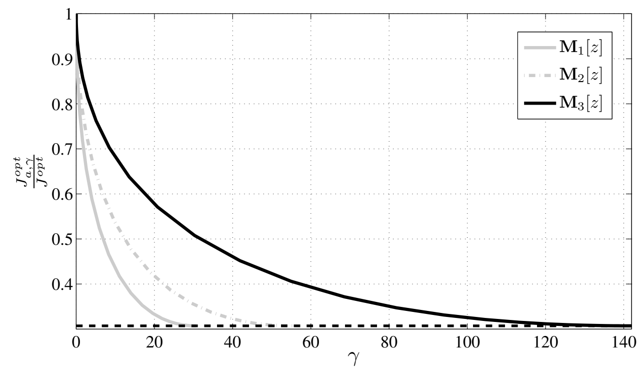

Fig. 4. Achievable performance ratio $\frac{J^{\rm opt}_{a,\gamma}}{J^{\rm opt}}$ as a function of $\gamma \in [0, \gamma_{\max}]$ and $\frac{J^{\rm opt}_{a,\infty}}{J^{opt}}$ (horizontal dotted line), for several filters $\mathbf{M}_i[z]$.

with $P[z] = \frac{|z|^2-1}{|1-z|^2}$. Moreover, it can be shown that

$$\boldsymbol{\eta}^H_{z,1}\left(\mathbf{I}_n - \mathbf{F}[1]\mathbf{P}\right)\boldsymbol{\nu} = \boldsymbol{\eta}^H_{z,1}\mathbf{G}[1]\mathbf{u}_{g,\infty}\,, \tag{55}$$

where $\mathbf{u}_{g,\infty} \in \mathbb{R}^n$ corresponds to the stationary value of the original input signals. It follows from (54), (55), and the facts that z_1 is a zero of $\boldsymbol{\xi}_{\mathbf{G}_a}[z]\mathbf{G}[z]$ and $\boldsymbol{\xi}_{\mathbf{G}_a}[1] = \mathbf{I}_n$, that when $z_1 \to 1$, $D_\gamma \to P[z_1]\,\boldsymbol{\nu}^H\boldsymbol{\eta}_{z,1}\boldsymbol{\eta}^H_{z,1}\boldsymbol{\nu}$. On the other hand, it is easy to see that $P[z] \to \infty$ when $z \to 1$ (resp., $P[z] \to 0$ when $z \to -1$). From the above it is immediate to see that, provided $\boldsymbol{\nu}$ is not orthogonal to $\boldsymbol{\eta}_{z,1}$, $\mathcal{D}_\gamma \to \infty$ when $z_1 \to 1$. Moreover, it is also true that $\mathcal{D}_\gamma \to 0$ when $z_1 \to -1$. The above result leads to conclude that it is highly beneficial to add input channels that eliminate NMP zeros of $\mathbf{G}[z]$ which are close to $z = 1$. On the contrary, the elimination of NMP zero close to $z = -1$ does not help to improve the achievable performance of the square plant.

VI. Numerical Example

Consider a square MIMO plant $\mathbf{G}[z]$ given by

$$\mathbf{G}[z] = \begin{bmatrix} \frac{0.1(z-2)}{z^3} & \frac{5}{(z-0.2)} \\ \frac{3(z-2)}{z^2} & \frac{3}{z^2}\end{bmatrix} \in \mathcal{RH}_2^{2\times 2}\,. \tag{56}$$

The plant $\mathbf{G}[z]$ has three NMP zeros: one finite zero located at $z = 2$, and two zeros at infinity. To improve the achievable performance when controlling $\mathbf{G}[z]$, it is proposed to add one input channel modelled by

$$\mathbf{F}[z] = \begin{bmatrix} \frac{2}{(z-0.2)z^3} \\ \frac{4}{z^4} \end{bmatrix} \in \mathcal{RH}_2^{2\times 1}\,. \tag{57}$$

The augmented plant $\mathbf{G}_a[z] = [\mathbf{G}[z] \quad \mathbf{F}[z]] \in \mathcal{RH}_2^{2\times 3}$ has two zeros at infinity. Thus, the addition of the control channel $\mathbf{F}[z]$ eliminates the finite NMP zeros of $\mathbf{G}[z]$. Moreover, we consider the following filters (see Figure 3)

$$\mathbf{M}_1[z] = \frac{5}{3}\frac{(z-0.4)}{z}\,, \tag{58}$$

$$\mathbf{M}_2[z] = \frac{4(z-0.5)^2)}{z^2}\,, \tag{59}$$

$$\mathbf{M}_3[z] = \frac{1}{0.09}\frac{(z-0.7)^2}{z^2}\,. \tag{60}$$

Figure 4 shows the ratio $\frac{J_{a,\gamma}^{\mathrm{opt}}}{J^{\mathrm{opt}}}$ as a function of $\gamma \in [0, \gamma_{\max}]$ for each filter $\mathbf{M}_i[z]$, $i = 1, 2, 3$, when the reference is given by $\mathbf{r}[k] = \boldsymbol{\nu}\mu[k]$, with $\boldsymbol{\nu} = [1 \quad -1]^T$, and the stationary value of the additional input signals is $\mathbf{u}_{a,\infty} = \mathbf{P}\boldsymbol{\nu}$, where $\mathbf{P} = [\,-2 \;\; 3.9759\,]$ satisfies Assumption 2.

Figure 4 shows, in agreement with Theorem 1, that in each case the achievable performance improvement is less significant when the constraint on the additional input channel is more stringent. Indeed, we see that $J_{a,\gamma}^{\mathrm{opt}}$ tends to J^{opt} when γ approaches 0. It can be also seen that $J_{a,\gamma}^{\mathrm{opt}}$ tends to $J_{a,\infty}^{\mathrm{opt}}$ when γ approaches $\gamma_{\max}$. On the other hand, it is seen clearly the effect of the high-pass filter $\mathbf{M}[e^{j\omega}]$ on the best achievable performance of the augmented plant.

VII. Conclusion

We have derived closed form expressions which quantify the performance improvement of square MIMO systems when frequency-response-constrained input channels are added to the plant. The results show that the performance benefit depends on the (finite and infinite) NMP zeros of the original plant being eliminated by adding input channels, the structure of the filter that introduces the additional channels limitations, and also depends on how restricted is the constraint on the additional input channels. It should be noted that our results can be easily extended to MIMO plants which are originally wide.

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