Calculation of the least $\mathcal{L}_1$ measure for switched linear systems via similarity transformation

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***Abstract*—A strategy of similarity transformation is proposed to calculate the least $\mathcal{L}_1$ measure for continuous-time switched linear systems. The similarity transformation we performed on matrix A is PAP^{-1}, where the similarity transformation matrix P is a row operator multiplying a row of matrix A by a nonzero constant. A sequence of minimum of matrix set measure corresponding to a series of transformations is obtained. Furthermore, the sequence is convergent to a constant which can be used to estimate the largest divergence rate. These transformations are easily applicable because of their simple forms. A numerical simulation shows the effectiveness of the proposed method.**

I. INTRODUCTION

A switched system is a dynamical system which is composed of finite subsystems described by differential or difference equations and a switching rule that coordinates the switchings among the subsystems. The stability issue on switched system has attracted increasing attention, see, e.g., the papers [1], [2], [3], the recent book [4], and the references therein.

For switched dynamical systems, the switching may be unknown or induced by uncertainties, such as external disturbances. In these cases, in order to keep the system working, the system is expected to be stable under arbitrary switching. Here we define guaranteed stability as the stability of switched systems under arbitrary switching [4]. It is a well-established fact that guaranteed stability is equivalent to the existence of a common Lyapunov function [5]. However, to our knowledge, there is generally no systematic way or computational algorithm to find a common Lyapunov function.

For continuous-time switched linear systems, matrix measure has been proven to be an efficient tool on the study of stability analysis. It was proved that the largest possible divergence rate captures the worst-case convergence rate of state trajectories [6]. Furthermore, the largest divergence rate is exactly the least possible common matrix set measure of the subsystems [7]. Therefore, some recent research efforts focused on using matrix measure for the stability analysis. In [8], it was derived that, for a matrix, the extreme measure is exactly equal to the largest real part of the eigenvalues, which is independent of matrix norm. Therefore, a matrix is stable if and only if there exists a negative matrix measure. In [9], it was proved that switched time-variant linear system is guaranteed stable if all the matrix measures of subsystem are negative. In [10], an algorithm based on the sum-of-squares (SOS) technique was introduced to approximate the switched system's extreme measure which characterizes the largest divergence rate. In [11], it was shown that a necessary and sufficient condition for asymptotic stability of system $\dot{x} = Ax$ is that there exists a full column rank matrix $\varphi \in \mathcal{R}^{m\times n}, m \geq n$ and an $m \times m$-matrix Γ with strictly negative row dominating diagonal, such that $A^T\varphi = \varphi\Gamma$. For the system $\dot{x} = \sum_{i=1}^{m} \omega_i A_i x$, where $\sum_{i=1}^{m} \omega_i = 1, \omega_i \geq 0$, Blanchini [12] derived that the asymptotic stability of the system is equivalent to the existence of the matrix $X_{n\times r}$, $(r \geq n)$, and $H^{(k)}$ with $||H^{(k)}||_1 < 0$ such that, $A_k X = XH^{(k)}$, $k = 1, 2, \cdots, m$. The above result was restated as that, for the switched linear system, it is asymptotically stable if and only if the subsystem matrices are simultaneously generalized similar to a matrix with a negative $\mathcal{L}_1$ measure induced by the $\mathcal{L}_1$ norm [7]. This transformation has been proven to be a rigorous and efficient method for studying the guaranteed stability of switched linear systems [4]. Unfortunately, applying the above result is still difficult because, in general, the numerical search for the matrix φ is not tractable [2].

Motivated by the above-mentioned works, in this paper, we introduce a similarity transformation method to study the guaranteed stability of a switched linear system. The similarity transformation matrices considered in this work are all elementary matrices obtained by multiplying a row of identity matrix by a constant. Under a series of transformations, a sequence of minimum of matrix set measure is obtained. Furthermore, it can be established that the sequence is decreasing and has a lower bound. Therefore the sequence is convergent to a constant which can be served as an estimation of the largest divergence rate. These transformations are easily applicable because of their simple forms. A numerical example is used to demonstrate the effectiveness of the proposed method.

Notation: For a matrix $A = [a_{ij}]_{n\times n}$, we define a column

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sum of the form $\sum_{i=1}^{n} \alpha_{ij}, j = 1, 2, \cdots, n$, where

$$\alpha_{ij} = \begin{cases} a_{ij}, & \text{if } i = j, \\ |a_{ij}|, & \text{otherwise.} \end{cases}$$

In this work, the similarity transformation is successively carried out row by row. For a matrix $A = [a_{ij}]_{n\times n}$, the similarity transformation on the i-th row is called the i-th similarity transformation. The similarity transformations from the first row to the n-th row are called a group of transformation. $X_i^l = \mathrm{diag}(1, \cdots, k_i^l, \cdots, 1), l = 1, 2, \cdots, i = 1, 2, \cdots, n$ is referred to an elementary matrix. Superscript l tells the l-th group of similarity transformation and subscript i indicates the i-th similarity transformation in the l-th group. $X^l A (X^l)^{-1}$ represents a group of similarity transformations, where $X^l = \prod_{i=1}^{n} X_i^l$. New matrices obtained by a series of similarity transformations are written as, $(A_1)_i^l = X_i^l (A_1)_{(i-1)}^l (X_i^l)^{-1}, (A_2)_i^l = X_i^l (A_2)_{(i-1)}^l (X_i^l)^{-1}, i = 1, 2, \cdots, n$. f_{ij}^l denotes the j-th column sum of $A_i^l, j = 1, \cdots, n$. Let's take $A = [a_{ij}]_{3\times 3}$ as an example

$$\begin{aligned} f_{11}^l &= a_{11} + |\frac{k_1^l k_1^{l-1} \ldots k_1^1}{k_2^{l-1} \cdots k_2^1} a_{21}| + |\frac{k_1^l k_1^{l-1} \ldots k_1^1}{k_3^{l-1} \cdots k_3^1} a_{31}|, \\ f_{12}^l &= |\frac{k_2^{l-1} \ldots k_2^1}{k_1^l k_1^{l-1} \cdots k_1^1} a_{12}| + a_{22} + |\frac{k_2^{l-1} \ldots k_2^1}{k_3^{l-1} \cdots k_3^1} a_{32}|, \\ f_{13}^l &= |\frac{k_3^{l-1} \ldots k_3^1}{k_1^l k_1^{l-1} \cdots k_1^1} a_{13}| + |\frac{k_3^{l-1} \ldots k_3^1}{k_2^{l-1} \cdots k_2^1} a_{23}| + a_{33}. \end{aligned}$$

Obviously, each column sum f_{ij}^l is a function of the corresponding independent variable k_i^l. The point at which two column sums intersect is called intersection point. Certainly this point has (x, y) coordinates which satisfies both column sum functions. The longitudinal coordinate of the intersection represents the same value of these two column sums. In this paper, we concentrate on those intersection points of straight lines $f_{ij}^l, j = 1, 2, \cdots, n, j \neq i$, with the curve f_{ii}^l. The first intersection point is then referred to that point whose abscissa is the minimum abscissa of all of these intersection points. For example, the intersection points of the straight lines f_{12}^1 and f_{13}^1 with the curve f_{11}^1 in the right half-plane are shown in Fig. 1.

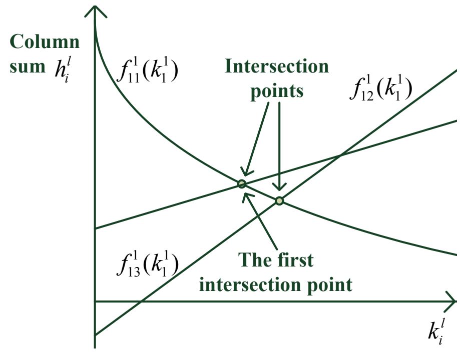

Fig. 1. The intersection points of the straight lines f_{12}^1 and f_{13}^1 with the curve f_{11}^1

II. Preliminaries

Consider a class of switched linear systems in the form of

$$\dot{x}(t) = A_{\sigma(t)} x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state vector, $\sigma(t)$ is a switching signal taking values from the index set $M = \{1, 2\}, A_i \in \mathbb{R}^{n\times n}$, $i \in M$ are real constant matrices. For brevity, we term the switched linear system as system $\mathbf{A} = \{A_1, A_2\}$.

The object of this paper is to find the least $\mathcal{L}_1$ measure of a switched linear system. In order to do so, some basic definitions are list as follows.

***Definition* 1:** For system $\mathbf{A}$, the largest divergence rate, also known as the largest Lyapunov exponent, is defined as

$$\varrho(\mathbf{A}) = \limsup_{t\to\infty,\, \sigma\in\Xi, |x|=1} \frac{\ln|\phi(t; 0, x, \sigma)|}{t}, \tag{2}$$

where $\phi(t; 0, x, \sigma)$ is the state evolution at time t with $x(0) = x$ under switching signal σ for continuous-time switched linear systems, and Ξ is the set of switching signals.

***Definition* 2:** (Matrix Measure) For any vector norm $|\cdot|$ in $\mathbb{R}^n$ and the corresponding induced norm $\|\cdot\|$ in $\mathbb{R}^{n\times n}$, the matrix measure is defined as

$$\mu_{|\cdot|}(\mathbf{A}) = \lim_{\tau\to 0^+} \frac{\|I + \tau\mathbf{A}\| - 1}{\tau}. \tag{3}$$

For briefness, the subscript $|\cdot|$ will be dropped.

For any vector norm $|\cdot|$ in $\mathbb{R}^n$, the definition of induced matrix set measure is given as follows.

***Definition* 3:** (Matrix Set Measure) Fix a vector norm $|\cdot|$ in $\mathbb{R}^n$ and the corresponding induced norm $\|\cdot\|$ in $\mathbb{R}^{n\times n}$. For a given set of matrices $\mathbf{A} = \{A_1, A_2\}$, the induced matrix set measure is

$$\mu(\mathbf{A}) = \max\{\mu(A_1), \mu(A_2)\}.$$

In this paper, the vector norm is corresponding to the $\mathcal{L}_1$ norm. Therefore the matrix measure is referred to the $\mathcal{L}_1$ measure induced by $\mathcal{L}_1$ norm unless otherwise stated.

III. Main results

We aim to get the least $\mathcal{L}_1$ measure via similarity transformation PAP^{-1}, where similarity transformation matrix P is an elementary matrix. It is well known that, to perform an elementary row or column operation on A is equal to multiplying a corresponding elementary matrix on the left or right of A. The elementary matrices in this paper are all obtained by multiplying the i-th row of identity matrix by a constant.

***Proposition* 1:** The minimum of matrix set measure after each transformation has a common lower bound.

Proof: Suppose $A_1 = [a_{ij}]_{n\times n}$ and $A_2 = [b_{ij}]_{n\times n}$ are $n \times n$ matrices. Perform an elementary row and column operation simultaneously on matrices A_1 and A_2, namely as, $X_1^1 A_1 (X_1^1)^{-1}, X_1^1 A_2 (X_1^1)^{-1}$. More explicitly, using components, the above transformation takes the form,

$$X_1^1 A_1 (X_1^1)^{-1} = \begin{bmatrix} a_{11} & k_1^1 a_{12} & \cdots & k_1^1 a_{1n} \\ \frac{1}{k_1^1} a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k_1^1} a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$X_1^1 A_2 (X_1^1)^{-1} = \begin{bmatrix} b_{11} & k_1^1 b_{12} & \cdots & k_1^1 b_{1n} \\ \frac{1}{k_1^1} b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k_1^1} b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix},$$

where k_1^1 is a nonzero constant to be determined such that the matrix set measure of $(A_1)_1^1$ and $(A_2)_1^1$ reaches its minimum.

Consequently, the corresponding column sums are as follow,

$$\begin{aligned} f_{11}^1 &= a_{11} + |\frac{1}{k_1^1} a_{21}| + \cdots + |\frac{1}{k_1^1} a_{n1}|, \\ f_{12}^1 &= |k_1^1 a_{12}| + a_{22} + \cdots + |a_{n2}|, \\ &\vdots \\ f_{1n}^1 &= |k_1^1 a_{1n}| + |a_{2n}| + \cdots + a_{nn}, \\ g_{11}^1 &= b_{11} + |\frac{1}{k_1^1} b_{21}| + \cdots + |\frac{1}{k_1^1} b_{n1}|, \\ g_{12}^1 &= |k_1^1 b_{12}| + b_{22} + \cdots + |b_{n2}|, \\ &\vdots \\ g_{1n}^1 &= |k_1^1 b_{1n}| + |b_{2n}| + \cdots + b_{nn}. \end{aligned} \tag{4}$$

We seek to minimize $\max\left\{f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1\right\}$ on $R\backslash\{0\}$. From the definition of matrix set measure, we obtain,

$$\mu(A_1, A_2) = \max\Big\{\mu(A_1), \mu(A_2)\Big\}.$$

Therefore,

$$\begin{aligned} &\mu((A_1)_1^1, (A_2)_1^1) \\ =& \max\Big\{\max\left\{f_{11}^1, \cdots, f_{1n}^1\right\}, \max\left\{g_{11}^1, \cdots, g_{1n}^1\right\}\Big\} \\ =& \max\left\{f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1\right\}. \end{aligned}$$

The expressions of column sums show that,

$$f_{1j}^1 \geq a_{jj}, \quad g_{1j}^1 \geq b_{jj}, \quad j = 1, \cdots, n.$$

Hence the following inequality holds true,

$$\begin{aligned} \beta_1^1 &\triangleq \max\left\{f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1\right\} \\ &\geq \max\left\{a_{11}, \cdots, a_{nn}, b_{11}, \cdots, b_{nn}\right\}. \end{aligned}$$

We know that the similarity transformations don't change the diagonal elements of matrices $(A_1)_i^l$ and $(A_2)_i^l$. Using a similar idea as in the former part of the proof, we could reach the conclusion that performing an arbitrary similarity transformation, with similarity transformation matrix being elementary matrix, simultaneously on the matrices $(A_1)_i^l$ and $(A_2)_i^l, l = 1, 2, \cdots, i = 1, 2, \cdots, n$ has the same result which can be expressed as

$$\begin{aligned} \beta_i^l &\triangleq \max\left\{f_{i1}^l, \cdots, f_{in}^l, g_{i1}^l, \cdots, g_{in}^l\right\} \\ &\geq \max\left\{a_{11}, \cdots, a_{nn}, b_{11}, \cdots, b_{nn}\right\}. \end{aligned}$$

Therefore

$$\begin{aligned} h_i^l &\triangleq \min\left\{\beta_i^l\right\} \geq \min\max\left\{a_{11}, \cdots, a_{nn}, b_{11}, \cdots, b_{nn}\right\} \\ &= \max\left\{a_{11}, \cdots, a_{nn}, b_{11}, \cdots, b_{nn}\right\}. \end{aligned}$$

That is, the minimum of matrix set measure after each transformation has a common lower bound. ■

***Proposition* 2:** $f(x) = |\frac{a_1}{x}| + b_1$ and $g(x) = |\frac{a_2}{x}| + b_2$ have at most one intersection point when $x > 0$.

Proof: Obviously, simple calculation can lead to the result. It is worth noting that, $f(x)$ and $g(x)$ may have no intersection point or $f(x)$ and $g(x)$ coincide with each other. For consistency, we define the abscissa of the intersection point equals to $+\infty$ accordingly. ■

Proposition 1 provides the boundness of the matrix set measure which is very important in the development of the stability analysis. The task in this paper is to find elementary transformation matrices X_i^l, such that the sequence of matrix set measure is decreasing and finally reaches the minimum. We do this by an iterative procedure. With the simple form of X_i^l and $(X_i^l)^{-1}$, applying once similarity transformation affects only one corresponding row and column. We first discuss the general case that each column and row of A_1 or A_2 have at least one nonzero off-diagonal elementary. We investigate the minimum of matrix set measure and obtain the following results.

***Theorem* 1:** For the i-th transformation in the l-th group

$$\begin{bmatrix} X_i^l & \\ & X_i^l \end{bmatrix} \begin{bmatrix} (A_1)_{i-1}^l & \\ & (A_2)_{i-1}^l \end{bmatrix} \begin{bmatrix} (X_i^l)^{-1} & \\ & (X_i^l)^{-1} \end{bmatrix},$$

$i = 1, 2, \cdots, n, l = 1, 2, \cdots$, the minimum of matrix set measure

$$h_i^l = \min\max\Big\{f_{i1}^l, \cdots, f_{in}^l, g_{i1}^l, \cdots, g_{in}^l\Big\}$$

is attained at the first intersection of the maximum curve $\max\left\{f_{ii}^l, g_{ii}^l\right\}$ with straight lines $f_{ij}^l, g_{ij}^l, j = 1, 2, \cdots, n$, $j \neq i$.

Proof: Without loss of generality, let's discuss the first transformation,

$$\begin{bmatrix} X_1^1 & \\ & X_1^1 \end{bmatrix} \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} (X_1^1)^{-1} & \\ & (X_1^1)^{-1} \end{bmatrix}.$$

Observe that $f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1$ are all y-axis symmetric. Using the symmetry, the proof will be shown for $x > 0$. The proof for $x < 0$ is essentially the same. Therefore, the interval we discussed here is $(0, +\infty)$, unless otherwise stated.

Notice that f_{11}^1, g_{11}^1 are both decreasing, while $f_{12}^1, \cdots, f_{1n}^1$, $g_{12}^1, \cdots, g_{1n}^1$ are all increasing. From Proposition 2, we know that f_{11}^1 and g_{11}^1 have at most one intersection. Denote the abscissa of this intersection as k. Then we will investigate the maximum of f_{11}^1 and g_{11}^1 in two cases.

Case 1. $f_{11}^1 < g_{11}^1$ when $x < k$; $f_{11}^1 = g_{11}^1$ when $x = k$; $f_{11}^1 > g_{11}^1$ when $x > k$.

Case 2. $f_{11}^1 > g_{11}^1$ when $x < k$; $f_{11}^1 = g_{11}^1$ when $x = k$; $f_{11}^1 < g_{11}^1$ when $x > k$.

Without loss of generality, we just discuss the former case. The maximum of f_{11}^1 and g_{11}^1 is

$$\overline{f_{11}^1} = \max\left\{f_{11}^1, g_{11}^1\right\} \quad = \quad \begin{cases} g_{11}^1, & \text{if } x \le k, \\ f_{11}^1, & \text{otherwise.} \end{cases}$$

Assume that abscissas of intersection points of straight lines $f_{12}^1, \cdots, f_{1n}^1, g_{12}^1, \cdots, g_{1n}^1$ with a curve $\overline{f_{11}^1}$ are $x_1, x_2, \cdots, x_{2n-2}$ respectively and $x_1 \le x_2 \le \cdots \le x_{2n-2}$. The value of the minimum of matrix set measure is discussed with three situations.

Case 1. Abscissas of intersection points are all less than the constant k. Notice that $\overline{f_{11}^1} = g_{11}^1$ is a decreasing function. Therefore, $g_{11}^1(x_1) \ge g_{11}^1(x_2) \ge \cdots \ge g_{11}^1(x_{2n-2})$.

1) Since $\overline{f_{11}^1} = g_{11}^1$ is decreasing, x_1 is the minimum point in the interval $(0, x_1]$. While f_{12}^1 is an increasing function, x_1 is the maximum point in the interval $(0, x_1]$. Together with $f_{12}^1(x_1) = g_{11}^1(x_1)$, we have $\max\left\{f_{12}^1, g_{11}^1\right\} = g_{11}^1$, when $x \le x_1$.

2) Similarly, we have $g_{11}^1(x_1) \ge g_{11}^1(x_j) = f_{1,j+1}^1(x_j) \ge f_{1,j+1}^1(x_1), j = 1, 2, \cdots, n-1$. Utilizing the monotonicity of functions g_{11}^1 and $f_{1,j+1}^1, j = 1, 2, \cdots, n-1$, one can obtain that $\max\left\{f_{1,j+1}^1, g_{11}^1\right\} = g_{11}^1$, when $x \le x_1$.

3) Similarly, $g_{11}^1(x_1) \ge g_{11}^1(x_{j+n-1}) = g_{1,j+1}^1(x_{j+n-1}) \ge g_{1,j+1}^1(x_1), j = 1, 2, \cdots, n-1$. Utilizing the monotonicity of functions g_{11}^1 and $g_{1,j+1}^1, j = 1, 2, \cdots, n-1$, it is not difficult to get that $\max\left\{g_{1,j+1}^1, g_{11}^1\right\} = g_{11}^1$, when $x \le x_1$.

To sum up, $\beta_1^1 = g_{11}^1$ is a decreasing function and x_1 is the minimum point when $x \le x_1$. While $\beta_1^1 = \max\left\{f_{12}^1, \cdots, f_{1n}^1, g_{12}^1, \cdots, g_{1n}^1\right\}$ is an increasing function and x_1 is the minimum point when $x \ge x_1$. Hence, x_1 is the minimum point of β_1^1 in the interval $(0, +\infty)$.

Case 2. The proof is essentially similar as the Case 1 if abscissas of intersection points are all greater than the constant k. One can obtain that x_1 is the minimum point of β_1^1 in the interval $(0, +\infty)$.

Case 3. The constant k lies among the abscissas of intersection points. Suppose that $x_1 \le x_2 \cdots \le x_m \le k \le \cdots \le x_{2n-2}$. Since $\overline{f_{11}^1} = \max\left\{f_{11}^1, g_{11}^1\right\}$, we have, $g_{11}^1(x_1) \ge \cdots \ge g_{11}^1(x_m) \ge g_{11}^1(k) = f_{11}^1(k) \ge f_{11}^1(x_{m+1}) \ge \cdots \ge f_{11}^1(x_{2n-2})$.

1) Since $\overline{f_{11}^1} = g_{11}^1$ when $x < k$ and g_{11}^1 is decreasing, x_1 is the minimum point in the interval $(0, x_1]$. While f_{12}^1 is an increasing function, x_1 is the maximum point in the interval $(0, x_1]$. Together with $f_{12}^1(x_1) = g_{11}^1(x_1)$, we have $\max\left\{f_{12}^1, g_{11}^1\right\} = g_{11}^1$, when $x \le x_1$.

2) Similarly, we derive $g_{11}^1(x_1) \ge g_{11}^1(x_j) = f_{1,j+1}^1(x_j) \ge f_{1,j+1}^1(x_1), j = 1, 2, \cdots, n-1$. Utilizing the monotonicity of functions g_{11}^1 and $f_{1,j+1}^1, j = 1, 2, \cdots, n-1$, one can establish that $\max\left\{f_{1,j+1}^1, g_{11}^1\right\} = g_{11}^1$, when $x \le x_1$.

3) Similarly, $g_{11}^1(x_1) \ge g_{11}^1(k) = f_{11}^1(k) \ge f_{11}^1(x_{j+n-1}) = g_{1,j+1}^1(x_{j+n-1}) \ge g_{1,j+1}^1(x_1), j = 1, 2, \cdots, n-1$. Utilizing the monotonicity of functions g_{11}^1 and $g_{1,j+1}^1$, we can establish that $\max\left\{g_{1,j+1}^1, g_{11}^1\right\} = g_{11}^1$, when $x \le x_1$.

To summarize, $\beta_1^1 = g_{11}^1$ is a decreasing function and x_1 is the minimum point when $x \le x_1$. While $\beta_1^1 = \max\left\{f_{12}^1, \cdots, f_{1n}^1, g_{12}^1, \cdots, g_{1n}^1\right\}$ is an increasing function and x_1 is the minimum point when $x \ge x_1$. Hence, x_1 is the minimum point of β_1^1 in the interval $(0, +\infty)$.

Base on the discussion above, the minimum of the matrix set measure $\min\max\left\{f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1\right\}$ is equal to the value of function β_1^1 at $x = x_1$, i.e., $\beta_1^1(x_1) = \min\max\left\{f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1\right\}$. Thus, the minimum of the matrix set measure of $(A_1)_1^1$ and $(A_2)_1^1$ is at the first intersection of straight lines $f_{12}^1, \cdots, f_{1n}^1, g_{12}^1, \cdots, g_{1n}^1$ with the maximum curve $\overline{f_{11}^1}$. In other words, there exists a constant $k_1^1 = x_1$ such that $h_1^1 = \min\beta_1^1(x) = \beta_1^1(k_1^1)$.

Using a similar idea as in the former part of the proof, we could reach the conclusion that the minimum of matrix set measure of $(A_1)_i^l$ and $(A_2)_i^l$ obtained by performing similarity transformation simultaneously on the matrices $(A_1)_{i-1}^l$ and $(A_2)_{i-1}^l$ should be at the first intersection point of the maximum curve $\max\left\{f_{ii}^l, g_{ii}^l\right\}$ with straight lines $f_{ij}^l, g_{ij}^l, j = 1, 2, \cdots, n, j \ne i$. ■

***Remark* 1:** The abscissa of the first intersection represents the nonzero constant k_i^l corresponding to the elementary matrix X_i^l, the longitudinal coordinate of the first intersection represents the minimum of the matrix set measure. Fig. 2 shows a graphical explanation of the proof.

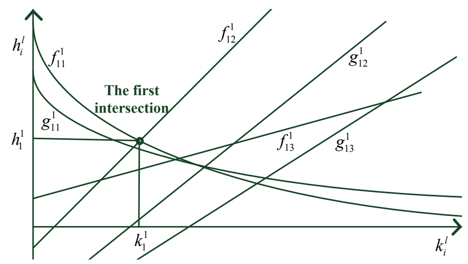

Fig. 2. A graphical explanation of the proof of Theorem 1

The theorem provides an approach for calculating the minimum of matrix set measure after each transformation. To establish the property of the sequence of the minimum of matrix set measure, we obtain the following conclusion.

***Theorem* 2:** Each similarity transformation reduces the minimum of matrix set measure of $(A_1)_i^l$ and $(A_2)_i^l, i = 1, 2, \cdots, n, l = 1, 2, \cdots$. That is, the sequence of the minimum of matrix set measure $\{h_i^l\}, i = 1, 2, \cdots, n, l = 1, 2, \cdots$ is decreasing.

Proof: Taking $k_1^1 = 1$ in the equations (4), we derive

$$\min\max\left\{f_{11}^1(1), \cdots, f_{1n}^1(1), g_{11}^1(1), \cdots, g_{1n}^1(1)\right\} = h_0,$$

which is the initial matrix set measure of the given matrices A_1 and A_2. From Theorem 1, the minimum of matrix set measure h_1^1 is the first intersection corresponding to k_1^1 which is the minimum abscissa of the intersection points of straight lines $f_{12}^1,\cdots,f_{1n}^1,g_{12}^1,\cdots,g_{1n}^1$ with a curve $\max\left\{f_{11}^1,g_{11}^1\right\}$. Let x_1 denote the abscissa of the first intersection point. Furthermore, β_1^1 is decreasing in the interval $(0,x_1]$ and increasing in $[x_1,+\infty)$. There are two cases occurring in the following discussion if $x_1\neq 1$.

Case 1. $x_1<1$

We know that $\beta_1^1(x)$ is increasing in the interval $[x_1,+\infty)$, therefore $h_1^1=\beta_1^1(x_1)\leq\beta_1^1(1)=h_0$.

Case 2. $x_1>1$

Notice that $\beta_1^1(x)$ is decreasing in the interval $(0,x_1]$. Therefore $h_1^1=\beta_1^1(x_1)\leq\beta_1^1(1)=h_0$.

Obviously, if $x_1=1$, we get $h_1^1=\beta_1^1(x_1)=\beta_1^1(1)=h_0$. Therefore, $h_1^1\leq h_0$.

Using a similar idea as in the former part of the proof, $h_i^l, l=1,2,\cdots,i=1,2,\cdots,n$ can be found by performing corresponding similarity transformations. The above results generalize a sequence of minimum of matrix set measure $\left\{h_i^l\right\}, l=1,2,\cdots,i=1,2,\cdots,n$. Moreover, $h_n^l\leq\cdots\leq h_n^1\leq\cdots\leq h_1^1\leq h_0$. That is, the sequence of the minimum of matrix set measure $\left\{h_i^l\right\}, l=1,2,\cdots,i=1,2,\cdots,n$ is decreasing. ■

One of the problems arising in the study now is to determine under what condition can get or approximate the least $\mathcal{L}_1$ measure, which is addressed in the following result.

***Theorem* 3:** The sequence of minimum of matrix set measure $\left\{h_i^l\right\}, l=1,2,\cdots,i=1,2,\cdots,n$ is convergent. Moreover, if n consecutive constants $k_i^l=1$, the sequence is no longer strictly decreasing. That is, if n consecutive matrices $X_i^l=I$, where l may tend to $+\infty$, the least $\mathcal{L}_1$ measure is reached.

Proof: From Proposition 1 and Theorem 2, we can directly get that $\{h_i^l\}, l=1,2,\cdots,i=1,2,\cdots,n$ is decreasing and has a lower bound. Then, the first statement of the theorem follows.

If n consecutive matrices $X_i^l=I$, without loss of generality, suppose

$$X^l=X_n^l\cdots X_1^l=I.$$

Denote

$$\bar{X}^l=\begin{bmatrix} X^l & \\ & X^l\end{bmatrix}.$$

Therefore

$$\begin{bmatrix} (A_1)_n^l & \\ & (A_2)_n^l\end{bmatrix}=\bar{X}^l\begin{bmatrix} (A_1)_n^{l-1} & \\ & (A_2)_n^{l-1}\end{bmatrix}(\bar{X}^l)^{-1}.$$

Hence the minimum of matrix set measure of $(A_1)_n^l$ and $(A_2)_n^l$, denoted as h_n^l, is equal to h_n^{l-1}. ■

The above result shows that the minimum of matrix set measure is no longer strictly decreasing. That is, the least $\mathcal{L}_1$ measure is reached.

***Remark* 2:** It is worthwhile mentioning that Theorem 3 is just a sufficient condition. More precisely, after a series of transformations, some minimums of the matrix set measure may be attained at the first intersection of the maximum curve $\max\left\{f_{ii}^l,g_{ii}^l\right\}$ with a constant function. Hence there may exist the case that the sequence is not strictly decreasing while not all of the k_i^l are equal to 1. Let's take a 3rd-order system as an example,

$$A_1=\begin{bmatrix} -4 & 3 & 0\\ 0 & -5 & -1\\ 1 & 0 & -3\end{bmatrix}.$$

Proceeding as before, we have $k_1^1=k_2^1=k_3^1=0.5$, and the minimum of matrix measure is $h_1^1=h_2^1=h_3^1=-2$, as shown in Fig. 3. Moreover, $(A_1)_1^2=A_1$. That is, the matrix obtain by using a group of transformation is exactly equal to the given initial matrix, which can be treated as a circle. Therefore, the minimum of matrix measure is no longer strictly decreasing and the least $\mathcal{L}_1$ measure is equal to -2.

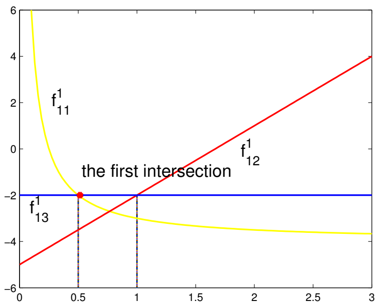

Fig. 3. The first intersection of the maximum curve with a constant function

***Remark* 3:** If all the elements of the given matrices A_1 and A_2 are nonzero, the column sums $f_{ij}^l,g_{ij}^l, l=1,2,\cdots,j=1,2,\cdots,n, j\neq i$ are all strictly increasing and f_{ii}^l and g_{ii}^l are both strictly decreasing. Hence the sequence of minimum of matrix set measure $\left\{h_i^l\right\}, l=1,2,\cdots,i=1,2,\cdots,n$ is strictly decreasing when $k_i^l\neq 1$. Therefore the above sufficient condition is also necessary.

Now we will discuss some trivial cases.

Case 1. Assume off-diagonal elements of a same column of A_1 and A_2 are all zero. Without loss of generality, let off-diagonal elements of the first column of A_1 and A_2 be all zero. Then $f_{11}^1=a_{11}, g_{11}^1=b_{11}$. There are two cases occurring in the following discussion.

Case 1.1. All the off-diagonal elements of the first row of $A_1=[a_{ij}]_{n\times n}$ and $A_2=[b_{ij}]_{n\times n}$ are zero. The expressions of the column sums (4) indicate that performing the first row and column of elementary operations on matrices A_1 and A_2 contributes nothing to the matrix set measure of A_1 and A_2. In this case, we reduce the $n\times n$ matrix to $(n-1)\times(n-1)$ one by removing the first column and row of A_1 and A_2 simultaneously, which is solved as discussed in Theorem 1.

Case 1.2. The off-diagonal elements of the first row of A_1 and A_2 have at least one nonzero element. Notice that $f_{12}^1,\cdots,f_{1n}^1,g_{12}^1,\cdots,g_{1n}^1$ are all increasing. If $\max\left\{f_{11}^1,g_{11}^1\right\}$ has intersections with $f_{12}^1,\cdots,f_{1n}^1,g_{12}^1,\cdots,g_{1n}^1$, the calculation of the minimum of the matrix set measure is exactly the same as the general case. Otherwise, the minimum of the matrix set measure is equal to the maximum of values of functions $f_{12}^1,\cdots,f_{1n}^1,g_{12}^1,\cdots,g_{1n}^1$ at $k_1^1=0$.

After the first transformation, the off-diagonal elements of the first column and row are all zero which is then solved as discussed in Case 1.1.

Case 2. Assume off-diagonal elements of a same row of A_1 and A_2 are all zero. Without loss of generality, let the off-diagonal elements of the first row of A_1 and A_2 be all zero. Therefore f_{1j}^1 and $g_{1j}^1, j = 2, \cdots, n$ are all constant functions. The off-diagonal elements of the first column of A_1 and A_2 have at least one nonzero element. If $\max\{f_{11}^1, g_{11}^1\}$ has intersections with $f_{12}^1, \cdots, f_{1n}^1, g_{12}^1, \cdots, g_{1n}^1$, the minimum of the matrix set measure is exactly the same as the general case. Otherwise the minimum of the matrices measure $\min\max\{f_{11}^1, \cdots, f_{1n}^1, g_{11}^1, \cdots, g_{1n}^1\}$ tends to $\max\{a_{11}, b_{11}\}$. That is, the minimum abscissa tends to $+\infty$. After the transformation, the off-diagonal elements of the first column and row are all zero which is then solved as discussed in Case 1.1.

***Remark* 4:** All the conclusions obtained above can be extended to a set of matrices $\mathbf{A} = \{A_1, A_2, \cdots, A_m\}$. Note that the proof is exactly the same, and we will omit here to save space.

IV. Example

To verify the effectiveness of the proposed method, consider a switched linear system with two 4th-order subsystems given as follow,

$$\begin{aligned} A_1 &= \begin{bmatrix} -5 & 2 & 0.61 & 2 \\ 1.5 & -4.8 & 0.2 & 1.3 \\ 1 & 0.6 & -2.5 & 0.5 \\ 0.9 & 3 & 0.5 & -5 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -4 & 1 & 0.5 & -1.2 \\ -1 & -3.8 & 0.8 & 2.5 \\ 0.2 & 1 & -7 & 0.4 \\ 0.6 & -2.2 & 2.7 & -5.6 \end{bmatrix}. \end{aligned}$$

It's easy to verify that $\max_{1\le j\le 4}\{\operatorname{Re}\lambda_j(A_1), \operatorname{Re}\lambda_j(A_2)\} = -0.9106$. The matrix set measure of A_1 and A_2 induced by $\mathcal{L}_1$ norm is $\mu(A_1, A_2) = 0.8$. Therefore, the results in [8] and [9] cannot be used to test the guaranteed asymptotically stability. Using the method in [10], we obtain that the upper bound of the extreme measure is 1.5608, which fails the stability test. Our criterion is applied as follows. Proceeding as before, the result is list in Table I. After 8 groups of similarity transformations, the least $\mathcal{L}_1$ measure is reached since there are consecutive $k_2^8 = k_3^8 = k_4^8 = k_1^9 = 1$. Therefore, the least $\mathcal{L}_1$ measure is $\mu((A_1)_1^9, (A_2)_1^9) = -0.8598$. Thus the system is guaranteed asymptotically stable. The extreme measure lies in $[-0.9106, -0.8598]$ which can be served as an estimation of the largest divergence rate. The accuracy exceeds 94.42%.

TABLE I
MINIMUM OF MATRIX SET MEASURE h_i^l AND k_i^l

m	k_i^l	h_i^l	m	k_i^l	h_i^l
1	0.6632	0.1265	18	0.9985	-0.8580
2	1.0000	0.1265	19	1.0000	-0.8580
3	0.5584	-0.1385	20	0.9990	-0.8586
4	0.7280	-0.7656	21	0.9993	-0.8591
5	0.9711	-0.7865	22	0.9995	-0.8592
6	0.9526	-0.8035	23	1.0000	-0.8592
7	1.0000	-0.8035	24	0.9997	-0.8594
8	0.9712	-0.8223	25	0.9998	-0.8596
9	0.9786	-0.8374	26	0.9998	-0.8596
10	0.9858	-0.8422	27	1.0000	-0.8596
11	1.0000	-0.8422	28	0.9999	-0.8597
12	0.9905	-0.8482	29	0.9999	-0.8597
13	0.9936	-0.8527	30	1.0000	-0.8598
14	0.9954	-0.8542	31	1.0000	-0.8598
15	1.0000	-0.8542	32	1.0000	-0.8598
16	0.9969	-0.8561	33	1.0000	-0.8598
17	0.9979	-0.8575			

V. Conclusion

In this work, a similarity transformation method was presented to study the stability of continuous-time switched linear systems. Similarity transformation matrices are all elementary matrices obtained by multiplying the i-th row of identity matrix by a constant. The first intersection was defined to characterize the minimum of matrix set measure after each transformation. Under a series of transformations, we obtained a sequence of minimum of matrix set measure which is bounded and decreasing. Therefore the sequence was convergent to a constant which can be served as an estimation of the largest divergence rate. These transformations are easily applicable because of their simple forms. A numerical simulation was presented to show the effectiveness of the proposed method.

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