Dynamical Average Consensus in Networked Linear Multi-Agent Systems with Communication Delays

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***Abstract*—In this paper, dynamical average consensus problem is studied for the networked identical linear multi-agent systems. A double coupling dynamical consensus protocol is proposed which is composed of the feedback of the protocol variables and the state variables. The protocol updates its information according to its own protocol communication topology which may be different from the state communication topology of the agents, and both of these two communication topologies are assumed to be balanced digraphs. Firstly, double topology protocol is analyzed for the system without communication delays. Then the protocol is extended to the case with time-varying delays. It is proved that all the nodes in the network can achieve dynamical average consensus asymptotically for appropriate communication delays. Numerical examples are given in the end of the paper to demonstrate the effectiveness of the theoretical results.**

I. Introduction

Flicking through the journals of control in the past decades, it is not difficult to find that an abundance of literatures are concerning about the cooperation of multi-agent systems, that is, a group of auto agents are controlled to reach an agreement on each other. In practice, as a result of failures of communication links (due to limited or unreliable information exchanges) or newly appearing members of the network, the topology of the dynamical network might be changing. And the critical problem is that how to design a networked consensus protocol (especially the distributed consensus protocol) such that each member in the group can reach consensus based on the shared information. There are many outstanding works [1–6] which have been established by researchers from various disciplines in this filed.

Most of the aforementioned literatures are concerning about the consensus problem for multi-agents with single-integrator or double-integrators dynamics which are independent of the agents' dynamics. The final consensus states are promoted by the networked protocol, and this is a significant distinction from the synchronization of complex networks which mainly focus on the nodes' dynamic behaviors instead of the whole network's. Recently, lots of attentions have been put on the consensus problems of the multi-agent systems with rich nodes dynamics. As for the consensus problem of linear multi-agent systems, novel works can be found in [7–12] and the references cited therein. By constructing a dynamical output feedback coupling control strategy, Scardovi and Sepulchre [7] have investigated the synchronization of a network composed by identical linear state-space models with time-varying and directed interconnection structure. Based on the low gain method, the authors have studied the consensus of high-order linear systems in [8] by dynamic output protocol. Sufficient conditions have been given in [9] for the existence of a synchronizing feedback law for the coupled neutrally stable linear systems and coupled critically unstable linear systems. Based on the functional input-decoupled observers and a static feedback, the authors [10] have considered the consensus problem among identical linear systems. As for the consensus problem for the nonlinear multi-agent systems, we refer to [13, 14] where the authors have considered the second-order/the first-order consensus problems for multi-agent systems with nonlinear dynamics and directed topologies. With a virtual leader in the dynamic proximity network, in [15], the authors have investigated the second-order consensus of multiple nonlinear dynamical mobile agents.

By introducing a state protocol which updates its information according to the agents' state communication topology, the authors [12] have investigated the consensus problems for linear multi-agent systems based on a dynamical output protocol without delay. Under generalized interaction topologies, the coordination problem of double-integrator systems has been considered based on the Lyapunov methods [16]. Illuminated by the novel ideas from the above two papers, in this article, we intend to consider the average consensus problem for the linear multi-agent systems with an observed protocol composed by the feedback of the agents' state variables with state communication topology $\mathcal{G}_a$ and the protocol variables with the protocol communication topology $\mathcal{G}_b$. These two interaction topologies $\mathcal{G}_a$ and $\mathcal{G}_b$ might be different, and the protocol is independent of or dependent on the time-varying communication delays. Due to the existence of different topologies and the time-varying communication delays, it is hard to analyze the eigenvalues of the system transformation matrix by the pure algebraic method used before. Under such cases, we will employ the energy function method to analyze the convergence of the protocol.

This paper is organized as follows. In Section II, the system model is introduced and some necessary preliminaries are given which will be used throughout the paper. In Section III, the convergence is analyzed for the multi-agent systems with two different topology cases under protocol with/without time-varying delays. Illustrative examples are provided in Section IV to demonstrate the utility of the theoretical results. Finally, we conclude the paper in Section V.

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II. NOTATION AND PRELIMINARIES

Throughout this paper the following notations will be used. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all real $n \times m$ matrices, respectively. I_n denotes the $n \times n$ identity matrix and $\mathbf{1}_N \in \mathbb{R}^N$ is the column vector of all ones. The notation $P > Q$, where P and Q are symmetric matrices, means that the matrix $P - Q$ is positive definite. $\|\cdot\|$ denotes the Euclidean norm of a vector and the induced norm of a matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for the algebraic operations.

A directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to represent the communication topology in a networked multi-agent system, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the finite set of agents and $\mathcal{E}$ is the set of edges. The edge $e_{ij} = (i, j) \in \mathcal{E}$ indicates that the agent j can detect the information sent by the agent i. A directed path is a sequence of edges in the directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, and a graph with the property that $e_{ij} \in \mathcal{E}$ implies $e_{ji} \in \mathcal{E}$ is said to be undirected. Graph $\mathcal{G}$ is called strongly connected if between any pair of distinct nodes $i, j \in \mathcal{V}$, there is a directed path from node i to node j, and $\mathcal{G}$ is said to be weakly connected if by replacing all of its directed edges with undirected edges, it turns into a connected undirected graph. A directed tree is a digraph, where every node, except the root node, has exactly one parent node. A spanning tree of $\mathcal{G}$ is a directed tree whose node set is $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}$. For a digraph $\mathcal{G}$, the adjacency matrix $A \in \mathbb{R}^{N\times N}$ is defined as $a_{ij} \geq 0$, in which $a_{ij} = 1 \Leftrightarrow e_{ji} = (j, i) \in \mathcal{E}$ and $a_{ij} = 0$ if $e_{ji} \notin \mathcal{E}$. The Laplacian matrix of the digraph $\mathcal{G}$ is defined as $L = D - A$, where D is a diagonal matrix with $d_{ii} = \sum_{j\neq i} a_{ij}$.

Lemma 1: [18] All the eigenvalue of Laplacian matrix L have nonnegative real parts. 0 is an eigenvalues of L, with $\mathbf{1}_N$ as the corresponding right eigenvector. Furthermore, 0 is a simple eigenvalue of L if and only if the digraph $\mathcal{G}$ has a directed spanning tree.

Definition 1: [17] A square matrix $L \in \mathbb{R}^{N\times N}$ is said to be a balanced matrix if and only if $\mathbf{1}_{\mathrm{N}}^T L = 0$ and $L\mathbf{1}_{\mathrm{N}} = 0$.

It is known that each balanced matrix corresponds to a balanced digraph (every node of the directed graph $\mathcal{G}$ is balanced, i.e., its in-degree and out-degree are equal) [2].

Lemma 2: [17] Consider the matrix

$$L = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}_{N\times N}$$

The following statements hold:

(1) The eigenvalues of L are N with multiplicity $N-1$, and 0 with multiplicity 1. The vectors $\mathbf{1}_{\mathrm{N}}^T$ and $\mathbf{1}_{\mathrm{N}}$ are the left and the right eigenvectors of L associated with the zero eigenvalue, respectively.

(2) There exists an orthogonal matrix U such that

$$U^T L U = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & NI_{N-1} \end{bmatrix}$$

and U is the matrix of eigenvectors of L. For any balanced matrix $B \in \mathbb{R}^{N\times N}$,

$$U^T B U = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \Delta \end{bmatrix}$$

where Δ is an upper triangular Jordan block.

III. CONSENSUS PROTOCOLS

Consider a networked identical linear multi-agent system, which also can be viewed as the linearized model of some nonlinear system near the consensus manifold, the dynamics of the ith agent is described by

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i \\ y_i = Cx_i \end{cases} \tag{1}$$

for $i = 1, 2, \ldots, N$, where $x_i \in \mathbb{R}^n$ is the state vector and $u_i \in \mathbb{R}^p$ is the control input (or protocol), and $y_i \in \mathbb{R}^q$ is the measured output. In this paper, it is assumed that the pair (A, B, C) is controllable and detectable, which is a basic requirement for the linear control systems.

We say that the protocol u_i asymptotically solves the dynamical average consensus problem for the group of agents if for any initial states $x_i(0)$ $(i = 1, 2, \ldots, N)$,

$$\lim_{t\to\infty} \|x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(t)\| = 0 \quad i = 1, 2, \ldots, N.$$

In practice, usually each agent is equipped with a state perception sensor and a control input actuator. The information gathered by the state perception sensors is transformed to the measured outputs which will then be sent to the control input actuator. Based on the state measured outputs and the self-regulation values, the control input actuator generates the control input signals. The topologies formed by the state perception sensors and the control input actuators are named the state communication graph $\mathcal{G}_a$ and the protocol communication graph $\mathcal{G}_b$ respectively.

In this paper, the following dynamical observer-type consensus protocol based on the state communication topology $\mathcal{G}_a$ and protocol communication topology $\mathcal{G}_b$ is proposed:

$$\begin{cases} u_i = K\eta_i \\ \dot{\eta}_i = (A + BK)\eta_i - \gamma_1 F \sum_{j=1}^{N} a_{ij}(y_i - y_j) \\ \qquad + \gamma_2 F \sum_{j=1}^{N} b_{ij} C(\eta_i - \eta_j) \end{cases} \tag{2}$$

where $\eta_i \in \mathbb{R}^n$ is the protocol variable of the dynamic controller, and $K \in \mathbb{R}^{p\times n}, F \in \mathbb{R}^{n\times q}$ are the feedback gain matrices to be determined, $\gamma_1, \gamma_2 > 0$ are the coupling strengths of the state communication topology $\mathcal{G}_a$ and the protocol communication topology $\mathcal{G}_b$, respectively. a_{ij}, b_{ij} are the elements of adjacency matrices of the above two topologies.

Remark 1: In the above protocol, all agents exchange their state information and the protocol information via the state communication graph $\mathcal{G}_a$ and the protocol communication graph $\mathcal{G}_b$, respectively. Generally, the protocol communication graph is set to be complete in topology. However under the actual communication environment, $\mathcal{G}_b$ may not be complete due to the communication packet dropping or the false tripping caused by noises. Moreover, these two kinds of interaction topologies may be different, which means each agent can get its neighbors' state information by the communication $\mathcal{G}_a$ but it may not know what its neighbors will do to reach the common agreement.

Let $\xi_i = [x_i^T, \eta_i^T]^T$, and $\xi(t) = [\xi_1^T(t), \xi_2^T(t), \ldots, \xi_N^T(t)]^T$. Then under the protocol (2), system (1) can be written as the following augmented dynamical model:

$$\dot{\xi}(t) = (I_N \otimes \mathcal{A} - \gamma_1 L_a \otimes H_1 + \gamma_2 L_b \otimes H_2)\xi(t) \qquad (3)$$

where

$$\mathcal{A} = \left[\begin{smallmatrix} A & BK \\ 0 & A+BK \end{smallmatrix}\right], \; H_1 = \left[\begin{smallmatrix} 0 & 0 \\ FC & 0 \end{smallmatrix}\right], \; H_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & FC \end{smallmatrix}\right],$$

$L_a, L_b \in \mathbb{R}^{N\times N}$ are the Laplacian matrices of digraphs $\mathcal{G}_a$ and $\mathcal{G}_b$ respectively.

In the following, we only focus on the case that communication topologies are balanced digraphs. Under such case, let $e(t) = [(I_N - \frac{\mathbf{1}_\mathrm{N}\cdot\mathbf{1}_\mathrm{N}^T}{N}) \otimes I_{2n}]\xi(t) = \hat{M} \otimes I_{2n}\xi(t)$, where

$$\hat{M} = \begin{bmatrix} 1-\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ -\frac{1}{N} & 1-\frac{1}{N} & \cdots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N} & -\frac{1}{N} & \cdots & 1-\frac{1}{N} \end{bmatrix}$$

From Lemma 2, it is easy to see that 0 is a simple eigenvalue of $\hat{M}$ with $\mathbf{1}_\mathrm{N}$ as the corresponding right eigenvector and 1 is another eigenvalue with multiplicity $N-1$. Also, for any Balanced matrix L, we always have $\hat{M}L = L\hat{M}$. By the properties of kronecker product, it follows that:

$$\dot{e}(t) = [I_N \otimes \mathcal{A} - \gamma_1 L_a \otimes H_1 + \gamma_2 L_b \otimes H_2]e(t) \qquad (4)$$

It can be easily seen that system (1) reaches consensus if $\lim_{t\to\infty} e(t) = 0$.

A. Convergence Analysis for Protocol without Delays

The following presents sufficient conditions for the consensus problem of (1) under the dynamical protocol (2). The criteria reflect the relationship between the system matrices, the control gain matrices, the coupling strengths, and the communication topologies.

Theorem 1: For the system (1) under the protocol (2), suppose the state communication topology $\mathcal{G}_a$ and the protocol communication topology $\mathcal{G}_b$ are all balanced digraph with a directed spanning tree. For any given system structure (A, B, C), coupling strengths γ_1, γ_2, and suitable feedback gain matrices K, F, the consensus is asymptotically reached if there exists matrix $\mathcal{P} > 0$ such that:

$$\begin{aligned} \mathcal{Z} \doteq & I_N \otimes (\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A}) - \gamma_1(L_a^T \otimes (H_1^T\mathcal{P}) \\ & + L_a \otimes (\mathcal{P}H_1)) + \gamma_2(L_b^T \otimes (H_2^T\mathcal{P}) \\ & + L_b \otimes (\mathcal{P}H_2)) < 0. \end{aligned}$$

Proof: We just need to prove that the system (4) is asymptotically stable. Consider the Lyapunov function candidate $V(t) = e^T(t)[I_N \otimes \mathcal{P}]e(t)$, the derivative of $V(t)$ along the trajectories of (4) gives

$$\begin{aligned} \dot{V}(t) = & e^T(t)\left[I_N \otimes (\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A}) \right. \\ & - \gamma_1(L_a^T \otimes H_1^T\mathcal{P} + L_a \otimes \mathcal{P}H_1) \\ & \left. + \gamma_2(L_b^T \otimes H_2^T\mathcal{P} + L_b \otimes \mathcal{P}H_2)\right] e(t) \\ = & e^T(t)\mathcal{Z}e(t). \end{aligned} \qquad (5)$$

Under the condition $\mathcal{Z} < 0$, one can obtain the conclusion immediately. This completes the proof. ■

Theorem 2: For the system (1) under the protocol (2), suppose the state communication topology $\mathcal{G}_a$ is connected and undirected and the protocol communication topology $\mathcal{G}_b$ is complete. For any given system structure (A, B, C), coupling strengths γ_1, γ_2, and suitable feedback gains K, F, the protocol (2) solves the consensus problem if there exist $\tilde{\mathcal{P}} > 0$ such that:

$$\begin{aligned} \tilde{\mathcal{Z}}_i \doteq & \mathcal{A}^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}\mathcal{A} - \gamma_1\lambda_i(L_a)(H_1^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}H_1) \\ & + \gamma_2 N(H_2^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}H_2) < 0, \quad i = 2, 3 \ldots, N. \end{aligned}$$

Proof: Since the state communication topology $\mathcal{G}_a$ is undirected and the protocol communication topology $\mathcal{G}_b$ is complete, from the Lemma 2, there exist a orthogonal matrix $U = [\frac{\mathbf{1}_\mathrm{N}}{\sqrt{N}}, U_1] \in \mathbb{R}^{N\times N}$ formed by the eigenvectors of L_b such that $U^T L_b U = \mathrm{diag}\{0, \mathrm{N}I_{N-1}\}$ and $U^T L_a U = \mathrm{diag}\{0, \Delta\}$, where Δ is a upper triangular Jordan block. Let $\hat{\theta}(t) = \left(U^T \otimes I_{2n}\right) e(t) = [\hat{\theta}_1^T(t), \hat{\theta}_1^T(t), \ldots, \hat{\theta}_1^T(t)]^T$, it is easy to have $\hat{\theta}_1(t) = \left(\frac{\mathbf{1}_\mathrm{N}^T}{\sqrt{N}} \otimes I_{2n}\right) e(t) = \left(\frac{\mathbf{1}_\mathrm{N}^T}{\sqrt{N}} \otimes I_{2n}\right) \left[\left(I_N - \frac{\mathbf{1}_\mathrm{N}\cdot\mathbf{1}_\mathrm{N}^T}{N}\right) \otimes I_{2n}\right] \xi(t) = 0$. Define $\theta(t) = [\theta_1^T(t), \theta_2^T(t), \ldots, \theta_{N-1}^T(t)]^T = [\hat{\theta}_2^T(t), \hat{\theta}_3^T(t), \ldots, \hat{\theta}_N^T(t)]^T$, and the asymptotical stability of the system (4) is equivalent to the following system:

$$\dot{\theta}(t) = (I_{N-1} \otimes \mathcal{A} - \gamma_1 \Lambda \otimes H_1 + \gamma_2 \mathrm{N} I_{N-1} \otimes H_2)\theta(t) \qquad (6)$$

where $\Lambda = \mathrm{diag}\{\lambda_2(L_a), \lambda_3(L_a), \ldots, \lambda_N(L_a)\}$.

Consider the Lyapunov function candidate $V(t) = \theta^T(t)[I_{N-1} \otimes \tilde{\mathcal{P}}]\theta(t)$, the derivative of $V(t)$ along the trajectories of system (6) is

$$\begin{aligned} \dot{V}(t) = & \theta^T(t)\Big[I_{N-1} \otimes (\mathcal{A}^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}\mathcal{A}) - \gamma_1\Lambda \otimes (H_1^T\tilde{\mathcal{P}} \\ & + \tilde{\mathcal{P}}H_1) + \gamma_2 N I_{N-1} \otimes (H_2^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}H_2)\Big]\theta(t) \\ = & \sum_{i=1}^{N-1} \theta_i^T(t)\tilde{\mathcal{Z}}_i\theta_i(t) \end{aligned}$$

Under the conditions $\tilde{\mathcal{Z}}_i < 0, i = 2, \ldots, N$, it follows the conclusion holds. This completes the proof. ■

Remark 2: For the closed-loop system (4), due to the exist of the inconsistent double topologies $\mathcal{G}_a$ and $\mathcal{G}_a$, it is more difficult to analyze the stability of (4) by the eigenvalues analysis technique or the frequency-domain analysis method. One preferable choice is the Lyapunov function method utilized in this paper. Although the conditions derived are somewhat complex, with the help of the toolbox of Matlab they are highly effective.

B. Convergence Analysis for Protocol with Delays

In real communication networks, time delays are unavoidable due to the limited communication capacity of sensors or transmitting equipments. It is well known that time delays may result in oscillatory behaviors or network instability or be beneficial to reach agreement. In this subsection, the protocol with state and communication delays is considered:

$$\begin{cases} u_i = K\eta_i \\ \dot{\eta}_i = (A+BK)\eta_i(t) \\ \qquad - \gamma_1 F \sum_{j=1}^{N} a_{ij} C\left[x_i(t-\tau_1(t)) - x_j(t-\tau_1(t))\right] \\ \qquad + \gamma_2 F \sum_{j=1}^{N} b_{ij} C\left[\eta_i(t-\tau_2(t)) - \eta_j(t-\tau_2(t))\right] \end{cases} \tag{7}$$

where $\tau_1(t) \in [0, \tau_1]$ and $\tau_2(t) \in [0, \tau_2]$ are time-varying delays satisfying $\dot{\tau}_1(t) \le d_1$, $\dot{\tau}_2(t) \le d_2$ for all $t > 0$.

Under the protocol (7), system (1) can be augmented as:

$$\begin{aligned} \dot{\xi}(t) =& (I_N \otimes \mathcal{A})\xi(t) - \gamma_1 (L_a \otimes H_1)\xi(t-\tau_1(t)) \\ &+ \gamma_2 (L_b \otimes H_2)\xi(t-\tau_2(t)) \end{aligned} \tag{8}$$

Similarly, to investigate the consensus of system (8) under the protocol (7), we just need to consider the asymptotical stability of the following error system:

$$\begin{cases} \dot{e}(t) = (I_N \otimes \mathcal{A})e(t) - \gamma_1 (L_a \otimes H_1)e(t-\tau_1(t)) \\ \qquad + \gamma_2 (L_b \otimes H_2)e(t-\tau_2(t)) \\ e(t) = \phi(t) \in C([-\max\{\tau_1, \tau_2\}, 0], \mathbb{R}^{2nN}) \end{cases} \tag{9}$$

Theorem 3: For the system (1) under the protocol (7), suppose the state communication topology $\mathcal{G}_a$ and the protocol communication topology $\mathcal{G}_b$ are all balanced digraph with a directed spanning tree. For any given system structure (A, B, C), coupling strengths γ_1, γ_2, and suitable feedback gains K, F, the consensus is asymptotically reached if there exist $\mathcal{P} > 0, \mathcal{Q}_j > 0, \mathcal{R}_j > 0$ $(j = 1, 2)$ such that

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \mathcal{H}^T(I_N \otimes \mathcal{R}_1) & \mathcal{H}^T(I_N \otimes \mathcal{R}_2) \\ * & \Phi_{22} & 0 & \Phi_{24} & \Phi_{25} \\ * & * & \Phi_{33} & \Phi_{34} & \Phi_{35} \\ * & * & * & -\tau_1^{-1}(I_N \otimes \mathcal{R}_1) & 0 \\ * & * & * & * & -\tau_2^{-1}(I_N \otimes \mathcal{R}_2) \end{bmatrix} < 0$$

where

$$\begin{aligned} \Phi_{11} &= \mathcal{H}^T(I_N \otimes \mathcal{P}) + (I_N \otimes \mathcal{P})\mathcal{H} \\ &\quad + I_N \otimes (d_1 \mathcal{Q}_1 + d_2 \mathcal{Q}_2), \\ \Phi_{12} &= \gamma_1 L_a \otimes \mathcal{P}H_1 - (d_1 - 1)I_N \otimes \mathcal{Q}_1, \\ \Phi_{13} &= -\gamma_2 L_b \otimes \mathcal{P}H_2 - (d_2 - 1)I_N \otimes \mathcal{Q}_2, \\ \Phi_{22} &= (d_1 - 1)I_N \otimes \mathcal{Q}_1 - \tau_1^{-1} I_N \otimes \mathcal{R}_1, \\ \Phi_{24} &= \gamma_1 (L_a \otimes H_1)^T (I_N \otimes \mathcal{R}_1), \\ \Phi_{25} &= \gamma_1 (L_a \otimes H_1)^T (I_N \otimes \mathcal{R}_2), \\ \Phi_{33} &= (d_2 - 1)I_N \otimes \mathcal{Q}_2 - \tau_2^{-1} I_N \otimes \mathcal{R}_2, \\ \Phi_{34} &= -\gamma_2 (L_b \otimes H_2)^T (I_N \otimes \mathcal{R}_1), \\ \Phi_{35} &= -\gamma_2 (L_b \otimes H_2)^T (I_N \otimes \mathcal{R}_2), \\ \mathcal{H} &= I_N \otimes \mathcal{A} - \gamma_1 L_a \otimes H_1 + \gamma_2 L_b \otimes H_2. \end{aligned}$$

Proof: By denoting $\varphi(t) = e(t) - e(t - \tau_1(t)), \psi(t) = e(t) - e(t - \tau_2(t))$, the asymptotical stability of the system (9) is equivalent to the asymptotical stability of the following system:

$$\begin{aligned} \dot{e}(t) =& \left(I_N \otimes \mathcal{A} - \gamma_1 L_a \otimes H_1 + \gamma_2 L_b \otimes H_2\right) e(t) \\ &+ \gamma_1 (L_a \otimes H_1)\varphi(t) - \gamma_2 (L_b \otimes H_2)\psi(t). \end{aligned} \tag{10}$$

Considering the Lyapunov functional candidate $V(t) = V_1(t) + V_2(t) + V_3(t)$, where

$$\begin{aligned} V_1(t) &= e^T(t)(I_N \otimes \mathcal{P})e(t), \\ V_2(t) &= \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} e^T(s)(I_N \otimes \mathcal{Q}_j)e(s)\mathrm{d}s, \\ V_3(t) &= \sum_{j=1}^{2} \int_{t-\tau_j}^{t} (s - t + \tau_j)\dot{e}^T(s)(I_N \otimes \mathcal{R}_j)\dot{e}(s)\mathrm{d}s, \end{aligned}$$

the derivative of $V(t)$ along the trajectories of (10) gives

$$\begin{aligned} \dot{V}_1(t) =& e^T(t)(\mathcal{H}^T(I_N \otimes \mathcal{P}) + (I_N \otimes \mathcal{P})\mathcal{H})e(t) \\ &+ 2\gamma_1 e^T(t)[L_a \otimes (\mathcal{P}H_1)]\varphi(t) \\ &- 2\gamma_2 e^T(t)[L_b \otimes (\mathcal{P}H_2)]\psi(t), \\ \dot{V}_2(t) \le& e^T(t)(I_N \otimes (\mathcal{Q}_1 + \mathcal{Q}_2))e(t) \\ &+ (d_1 - 1)e^T(t - \tau_1(t))(I_N \otimes \mathcal{Q}_1)e(t - \tau_1(t)) \\ &+ (d_2 - 1)e^T(t - \tau_2(t))(I_N \otimes \mathcal{Q}_2)e(t - \tau_2(t)), \\ \dot{V}_3(t) =& \dot{e}^T(t)(I_N \otimes (\tau_1 \mathcal{R}_1 + \tau_2 \mathcal{R}_2))\dot{e}(t) \\ &- \int_{t-\tau_1}^{t} \dot{e}^T(s)(I_N \otimes \mathcal{R}_1)\dot{e}(s)\mathrm{d}s \\ &- \int_{t-\tau_2}^{t} \dot{e}^T(s)(I_N \otimes \mathcal{R}_2)\dot{e}(s)\mathrm{d}s \end{aligned}$$

From the equality $-a^T\Omega a + (a - b)^T\Omega(a - b) - b^T\Omega b = -2a^T\Omega b$, one has

$$\begin{aligned} &- (d_1 - 1)e^T(t)(I_N \otimes \mathcal{Q}_1)e(t) \\ &+ (d_1 - 1)e^T(t - \tau_1(t))(I_N \otimes \mathcal{Q}_1)e(t - \tau_1(t)) \\ &- (d_1 - 1)\varphi^T(t)(I_N \otimes \mathcal{Q}_1)\varphi(t) \\ =& - 2(d_1 - 1)e^T(t)(I_N \otimes \mathcal{Q}_1)\varphi(t) \end{aligned}$$

and

$$\begin{aligned} &- (d_2 - 1)e^T(t)(I_N \otimes \mathcal{Q}_2)e(t) \\ &+ (d_2 - 1)e^T(t - \tau_2(t))(I_N \otimes \mathcal{Q}_2)e(t - \tau_2(t)) \\ &- (d_2 - 1)\psi^T(t)(I_N \otimes \mathcal{Q}_2)\psi(t) \\ =& - 2(d_2 - 1)e^T(t)(I_N \otimes \mathcal{Q}_2)\psi(t). \end{aligned}$$

Together with

$$\begin{aligned} &- \int_{t-\tau_1}^{t} \dot{e}^T(s)(I_N \otimes \mathcal{R}_1)\dot{e}(s)\mathrm{d}s \le - \int_{t-\tau_1(t)}^{t} \dot{e}^T(s) \\ &\times (I_N \otimes \mathcal{R}_1)\dot{e}(s)\mathrm{d}s \le -\tau_1^{-1}\varphi^T(t)(I_N \otimes \mathcal{R}_1)\varphi(t) \end{aligned}$$

and

$$\begin{aligned} &- \int_{t-\tau_2}^{t} \dot{e}^T(s)(I_N \otimes \mathcal{R}_2)\dot{e}(s)\mathrm{d}s \le - \int_{t-\tau_2(t)}^{t} \dot{e}^T(s) \\ &\times (I_N \otimes \mathcal{R}_2)\dot{e}(s)\mathrm{d}s \le -\tau_2^{-1}\psi^T(t)(I_N \otimes \mathcal{R}_2)\psi(t), \end{aligned}$$

one has

$$\dot{V}(t) \le \zeta^T(t)\Phi_1\zeta(t) + \sum_{j=1}^{2} \zeta^T(t)\Phi_2^{(j)}\zeta(t)$$

where $\zeta(t) = [e^T(t), \varphi^T(t), \psi^T(t)]^T$, Φ_1 is the submatrix of the first three rows and columns of the matrix Φ, $\Phi_2^{(j)} = \tau_j\tilde{\mathcal{H}}^T(I_N \otimes \mathcal{R}_j)\tilde{H}$ and $\tilde{\mathcal{H}} = [\mathcal{H}, \gamma_1 L_a \otimes H_1, -\gamma_2 L_b \otimes H_2]$. Using Schur complement Lemma, it follows that $\Phi < 0$ infers that $\Phi_1 + \Phi_2^{(1)} + \Phi_2^{(2)} < 0$ which assures the validity of the conclusion. This completes the proof. ■

Theorem 4: For the system (1) under the protocol (7), suppose the state communication topology $\mathcal{G}_a$ is connected and undirected and the protocol communication topology $\mathcal{G}_b$ is complete. For any given system structure (A, B, C), coupling strengths γ_1, γ_2, and suitable feedback gains K, F, the protocol (7) solves the consensus problem if there exist $\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_j, \tilde{\mathcal{R}}_j > 0$ $(j = 1, 2)$ such that:

$$\tilde{\Phi}_i = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} & E^T\tilde{\mathcal{R}}_1 & E^T\tilde{\mathcal{R}}_2 \\ * & \tilde{\Phi}_{22} & 0 & \tilde{\Phi}_{24} & (\gamma_1\lambda_{i+1}(L_a)H_1)^T\tilde{\mathcal{R}}_2 \\ * & * & \tilde{\Phi}_{33} & \tilde{\Phi}_{34} & -(\gamma_2 N H_2)^T\tilde{\mathcal{R}}_2 \\ * & * & * & -\tau_1^{-1}\tilde{\mathcal{R}}_1 & 0 \\ * & * & * & * & -\tau_2^{-1}\tilde{\mathcal{R}}_2 \end{bmatrix} < 0,$$

for $i = 1, 2, \ldots, N-1$, where

$$\begin{aligned} \tilde{\Phi}_{11} &= (\mathcal{A}^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}\mathcal{A}) - \gamma_1\lambda_{i+1}(L_a)(H_1^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}H_1) \\ &\quad + \gamma_2 N(H_2^T\tilde{\mathcal{P}} + \tilde{\mathcal{P}}H_2) + \sum_{j=1}^{2} d_j\tilde{\mathcal{Q}}_j, \\ \tilde{\Phi}_{12} &= \gamma_1\lambda_{i+1}(L_a)\tilde{\mathcal{P}}H_1 - (d_1 - 1)\tilde{\mathcal{Q}}_1, \\ \tilde{\Phi}_{13} &= -\gamma_2 N\tilde{\mathcal{P}}H_2 - (d_2 - 1)\tilde{\mathcal{Q}}_2, \\ \tilde{\Phi}_{22} &= (d_1 - 1)\tilde{\mathcal{Q}}_1 - \tau_1^{-1}\tilde{\mathcal{R}}_1, \\ \tilde{\Phi}_{24} &= (\gamma_1\lambda_{i+1}(L_a)H_1)^T\tilde{\mathcal{R}}_1, \\ \tilde{\Phi}_{33} &= (d_2 - 1)\tilde{\mathcal{Q}}_2 - \tau_2^{-1}\tilde{\mathcal{R}}_2, \\ \tilde{\Phi}_{34} &= -(\gamma_2 N H_2)^T\tilde{\mathcal{R}}_1, \\ E &= \mathcal{A} - \gamma_1\lambda_{i+1}(L_a)H_1 + \gamma_2 N H_2. \end{aligned}$$

Proof: Similar to the proof of the Theorem 2, by making the same transformation and denoting $\varphi(t) = \theta(t) - \theta(t - \tau_1(t)), \psi(t) = \theta(t) - \theta(t - \tau_2(t))$, the asymptotical stability of the system (9) is equivalent to the following model:

$$\begin{aligned} \dot{\theta}(t) = &(I_{N-1} \otimes \mathcal{A} - \gamma_1\Lambda \otimes H_1 + \gamma_2 N I_{N-1} \otimes H_2)\,\theta(t) \\ &+ \gamma_1(\Lambda \otimes H_1)\varphi(t) - \gamma_2 N(I_{N-1} \otimes H_2)\psi(t) \quad (11) \end{aligned}$$

Constructing the Lyapunov functional $V(t) = V_1(t) + V_2(t) + V_3(t)$ with

$$\begin{aligned} V_1(t) &= \theta^T(t)(I_{N-1} \otimes \tilde{\mathcal{P}})\theta(t), \\ V_2(t) &= \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} \theta^T(s)(I_{N-1} \otimes \tilde{\mathcal{Q}}_j)\theta(s)\mathrm{d}s, \\ V_3(t) &= \sum_{j=1}^{2} \int_{t-\tau_j}^{t} (s - t + \tau_j)\dot{\theta}^T(s)(I_{N-1} \otimes \tilde{\mathcal{R}}_j)\dot{\theta}(s)\mathrm{d}s, \end{aligned}$$

by similar analysis and techniques used in Theorem 3, one has

$$\dot{V}(t) \le \sum_{i=1}^{N-1} \zeta_i(t)^T\tilde{\Phi}_1^i\zeta_i(t) + \sum_{i=1}^{N-1}\sum_{j=1}^{2} \zeta_i^T(t)\tilde{\Phi}_2^{(ij)}\zeta_i(t)$$

where $\zeta_i = [e_i^T, \varphi_i^T, \psi_i^T]^T$, $\tilde{\Phi}_1^i, \tilde{\Phi}_2^{(ij)}$ are corresponding matrices which are similar to Theorem 3. Using Schur complement Lemma, it follows that $\tilde{\Phi}_i < 0$ infers that $\tilde{\Phi}_1^i + \tilde{\Phi}_2^{(i1)} + \tilde{\Phi}_2^{(i2)} < 0$ which assures the validity of the conclusion. This completes the proof. ■

Remark 3: When graphs $\mathcal{G}_a$ and $\mathcal{G}_b$ are both undirected (special balanced digraphs). If we assume the condition $L_aL_b = L_bL_a$ hold, then one can obtain similar conclusions as given in Theorem 2 and 4. Such assumptions imposed here can simplify the verified conditions especially when the number of nodes N and the dimension n of the agents in the network are very large.

C. Consensus Protocol' Parameters Selecting

• Since (A, B) is stabilizable, there exists a symmetric positive definite solution $D_1 > 0$ for the following Riccati equation:

$$A^TD_1 + D_1A - D_1BB^TD_1 = 0$$

and choose the matrix $K = -B^TD_1$.

• Since (A, C) is detectable, there exists a symmetric positive definite solution $D_2 > 0$ for the following Riccati equation:

$$AD_2 + D_2A^T - D_2C^TCD_2 = 0$$

and choose the matrix $F = -D_2C^T$.

• The coupling strengths can be chosen as follows:

$$\gamma_1 \ge \frac{1}{\min\limits_{i=2,\ldots,N}\{\mathrm{Re}(\lambda_i)\}}, \qquad \gamma_2 \ge \frac{1}{\min\limits_{i=2,\ldots,N}\{\mathrm{Re}(\mu_i)\}}$$

where $\lambda_i, \mu_i (i =, 2\ldots, N)$ are the nonzero eigenvalues of L^a and L^b respectively.

IV. Illustrative Example

In this section, some examples are given to illustrate the theoretical results obtained in the previous sections.

Example 1: When the state communication topology $\mathcal{G}_a$ and the protocol communication topology $\mathcal{G}_b$ are different balanced digraphs, we consider the following illustrative example.

Consider a group of 8 linear multi-agent systems, the state communication topology and the protocol communication topology are given in Fig. 1 and Fig. 2 respectively, the agents' dynamics are given by the system (1) with

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

It can be verified that (A, B, C) is stabilizable and observable. By solving the Riccati equation, the feedback gain matrix K and F of the protocol (2) are given as $K = [-0.4249, -0.2568, -0.5392, -1.2064]$ and $F = [-0.5671, -0.4549, -1.2218, -0.7122]^T$. Select the coupling strengths associate with topologies in Fig. 1 and Fig. 2 as $\gamma_1 = 0.5, \gamma_2 = 1.39$. The initial values of the augmented system (3) are given randomly. By solving the matrix inequality in Theorem 1, one can get the solution $\mathcal{P}$. Under such cases,

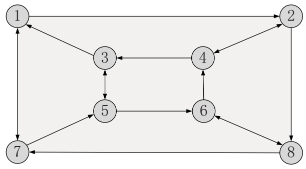

Fig. 1: Balanced state communication topology.

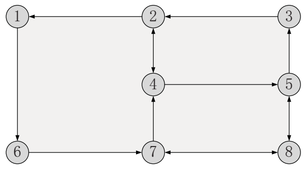

Fig. 2: Balanced protocol communication topology.

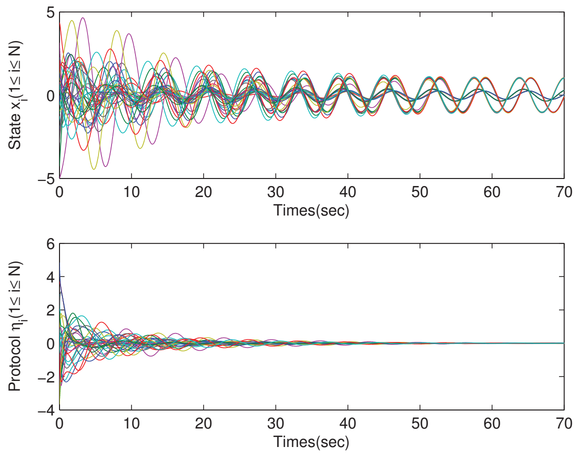

Fig. 3: States and Protocols convergence under the protocol (2).

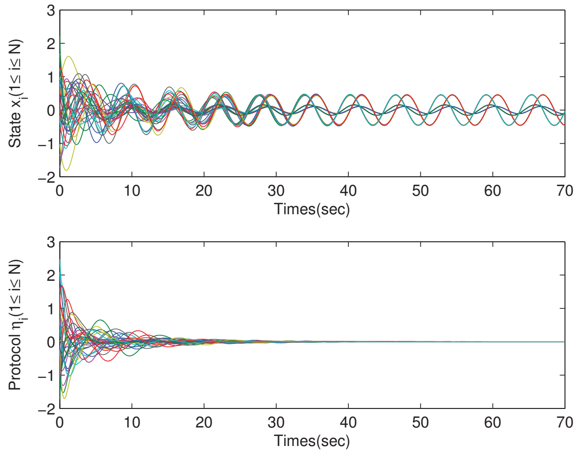

Fig. 4: States and protocols convergence under the protocol (7).

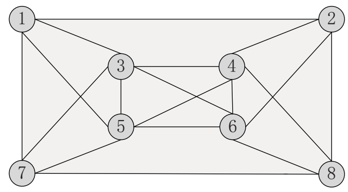

Fig. 5: Undirected state communication topology.

the state trajectories and the protocol trajectories are shown in Fig. 3.

Next, we consider the protocol (7) (still under the topologies given in Fig. 1 and Fig. 2) to analyze the convergence of the whole network. The time-varying delays are given as $\tau_1(t) = 0.04\sin^2(0.6t)$, $\tau_2(t) = 0.05\cos^2(0.5t)$ with $\tau_1 = 0.04, \tau_2 = 0.05, d_1 = 0.024, d_2 = 0.025$. The initial functions for the system (8) are given randomly on the interval $t \in [-0.5, 0]$. By solving the corresponding matrix inequality in Theorem 3, one can get the solution $\mathcal{P}$, $\mathcal{Q}_j$, $\mathcal{R}_j$. Under such cases, the state trajectories and the protocol trajectories are shown in Fig. 4.

Example 2: When the state communication topology is undirected (given in Fig.5) and the protocol communication topology is complete, we employ the above example as a special case to check Theorem 2. The initial values and the feedback gain matrices K, F are the same as in Example 1. The coupling strengths associating with the topology in Fig. 5 and the complete graph are taken as $\gamma_1 = 0.3618$, $\gamma_2 = 0.125$. By solving the matrix inequalities in Theorem 2, we can get $\tilde{\mathcal{P}}$. Due to the space limitation, it was omitted here. The state trajectories and the protocol trajectories are shown in Fig. 6.

Next, we consider the protocol (7) (still under the topology in Fig. 5) to analyze the convergence of the whole network. The time-varying delays are given as $\tau_1(t) = 0.04\sin^2(0.6t)$, $\tau_2(t) = 0.05\cos^2(0.5t)$ with $\tau_1 = 0.04, \tau_2 = 0.05, d_1 = 0.024, d_2 = 0.025$. By solving the matrix inequalities in Theorem 4, we get $\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_j, \tilde{\mathcal{R}}_j > 0$ $(j = 1, 2)$. Under such cases, the state trajectories and the protocol trajectories are shown in Fig. 7.

Remark 4: From the simulation results, we can see that when the protocol communication topology $\mathcal{G}_b$ turns into a complete graph which means all the agents in the network know what there neighbors will do to reach an agreement, the convergence speed will be much faster than the previous cases. Also from the evaluation of the protocol states, we know that $\lim_{t\to\infty} \eta_i(t) = 0$, which means $\lim_{t\to\infty} u_i(t) =$

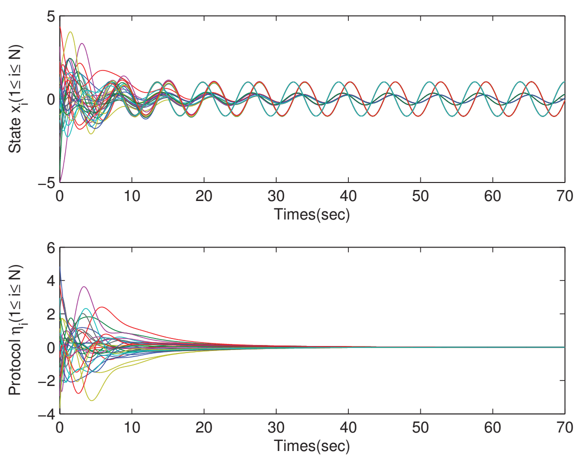

Fig. 6: States and protocols convergence under the protocol (2).

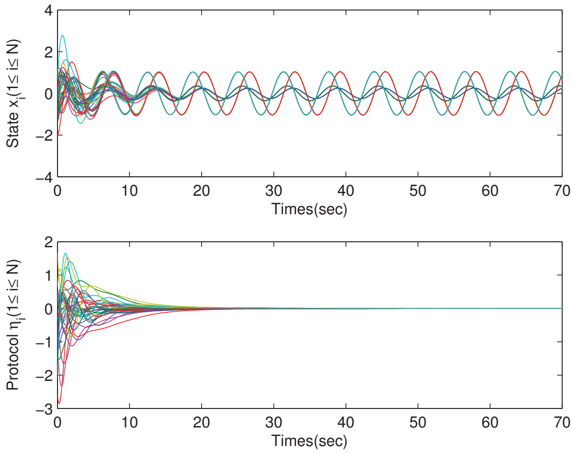

Fig. 7: States and protocols convergence under the protocol (7).

$\lim_{t\to\infty} K\eta_i(t) = 0$, i.e., when the whole network reaches the consensus, the external controllers do not need to work any more.

V. Conclusions

In this paper, we have investigated the dynamic average consensus problem of the networked linear multi-agent systems by a dynamic observed protocol, where the consensus protocol is constructed by double communication topologies, namely the agents' state communication topology $\mathcal{G}_a$ and the agents' protocol communication topology $\mathcal{G}_b$. Considering with (without) communication delays, we have established some consensus criteria which are easy to be verified.

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