A New Method Based on the Polytopic Linear Differential Inclusion for the Nonlinear Filter

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***Abstract*—This paper describes a new nonlinear filter for the nonlinear system, motivated by the the deficiencies of the complexity and large calculation number in the general nonlinear filter. The new filter is performed in three stages: First, the predicted state quantities of the nonlinear system are obtained by the prediction equation of the EKF. Then, the estimation error system is represented via an uncertain polytopic linear model, on the bias of which, the rectification equations with constant coefficients for the predicted errors are designed, without the need to evaluate the Jacobian matrixes on line. Finally, the state estimates are given through updating the predictions by the rectified quantities. The main novelty of the paper is the application of the Polytopic Linear Differential Inclusion in the nonlinear system, leading to the simplified design of the nonlinear filter and the improved real time performance of the new filter than the EKF, though the accuracy is a little decline. Its effectiveness is demonstrated by using the statistics result of the calculation number for the filters and an example of application in the attitude estimation system.**

I. INTRODUCTION

Approximate nonlinear filters are generally proposed for the nonlinear system since the optimal solution to the nonlinear filtering problem is infinite dimensional [1], and the most widely used approximate nonlinear filter is the extended Kalman filter (EKF). The EKF is iterative and easy to implement. However, it simply approximates the nonlinear functions with the first order Taylor series evaluated at the current optimal estimated value, resulting in the poor accuracy for some practical applications [2,3], which derives the development for several new alternatives to the standard EKF, such as higher order truncated EKF [4], the modified gain EKF [5], wighted EKF [6], EKF based on the neural network [7]. These methods have improved the accuracy of the EKF in some extent, but they have not overcome the shortage of the EKF in essence since they retain the basic structure of the EKF, and the computational cost of the filters have increased at the same time. Besides, the Jacobian matrixes are also required to evaluate online, the calculation of which can be a very cumbersome and error-prone process, especially for the complicated or the high dimensional systems [3,8].

In general, it is easier to approximate a probability distribution rather than to approximate an arbitrary nonlinear function or transformation [9]. The Unscented Kalman filter (UKF) is proposed following this intuition. Compared with the EKF, the UKF has some excellent properties: (1) The UKF approximates the probability density distribution of the nonlinear function by the unscented transformation, avoiding introducing the model error during the linearization process in the EKF; (2) It predicts the mean and covariance accurately up to the third order [4,9], which in the EKF is just to the first order; (3) It's not necessary to evaluate the Jacobian matrixes in the UKF and the calculation number doesn't increase too much either [10]. But the EKF is the better choice than the UKF when consider both the calculation numbers and the stability of the filters [11,12].

Another famous nonlinear filter is the particle filter [13], which is optimal in theory and applicable for the non-Gaussian noise. There is an evident flaw in the particle filter, which is the particle degeneracy problem. Many improved algorithms have been done for the particle filter, for example, regularized particle filter [14], Gaussian particle filter [15], Rao-Blackwellized particle filter [16] and so on. Nevertheless, that the calculation numbers are large and the real time capability and stability of the filters are pool remain exiting in the particle filters.

The differential inclusion theory represents the nonlinear system by a linear differential inclusion (LDI) model and the original nonlinear system is the son system of the linear differential inclusion system (LDIS). Though it introduces some conservativeness in the system model, the linear property applies a new method for the nonlinear filter design, which can be much easier than designing the filter for the nonlinear system directly. In this paper, a new nonlinear filter is proposed based on the LDI theory. The nonlinear system is represented by the uncertain polytopic linear differential inclusion (PLDI) model with the existent condition for describing the general nonlinear system via a LDI model given in [17], on the basis of which the novel nonlinear filter is designed.

The paper is organized as follows: section II and Section III present the new nonlinear filters for the continuous and the discrete nonlinear system respectively. In each section, the estimation error system of the nonlinear filter is described by a PLDI model first. Then, the rectification equations for the state predicted errors are designed based on the robust H_2 filtering method. The statistic calculation number of the new discrete nonlinear filter and comparison between the new filter and the EKF are also shown in the section III. Section IV demonstrates an attitude estimation example and compares the results of this method with that of the EKF.

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II. New Filter For the Continues Nonlinear System

Consider the following continuous nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + w, \\ y &= h(x) + v, \end{aligned} \tag{1}$$

where, $x \in R^n$ is the system state vector, $y \in R^m$ is the observation vector, $f(\cdot)$ and $h(\cdot)$ are nonlinear functions assumed to be continuously differentiable, $w \in R^n$ and $v \in R^m$ are the process noise and the measurement noise respectively. It is assumed that the noises w and v are zero mean white uncorrelated noises.

The state predicted estimates $\hat{x}$ for system (1) satisfy

$$\dot{\hat{x}} = f(x). \tag{2}$$

And the measurement of the system (1) can be predicted as

$$\hat{y} = h(\hat{x}). \tag{3}$$

Define the state estimation errors and the measurement prediction errors as

$$\Delta x = x - \hat{x}, \Delta y = y - \hat{y}.$$

The estimation error system can be expressed as

$$\begin{aligned} \Delta\dot{x} &= f(x) - f(\hat{x}) + Bn_g, \\ \Delta y &= h(x) - h(\hat{x}) + Dn_g, \end{aligned} \tag{4}$$

where,

$$B = \begin{bmatrix} I_{n\times n} & 0_{n\times m} \end{bmatrix}, D = \begin{bmatrix} 0_{m\times n} & I_{m\times m} \end{bmatrix}, n_g = \begin{bmatrix} w \\ v \end{bmatrix}.$$

Lemma 1 [17]. Consider the nonlinear system (1), for all x, w, v and t, if there exists a matrix

$$F = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial h}{\partial x} \end{bmatrix} \in \Omega,$$

where $\Omega \subseteq R^{(n+m)\times n}$, then the estimation error system (4) can be represented via the following LDI model

$$\begin{bmatrix} \Delta\dot{x} \\ \Delta y \end{bmatrix} \in Co\Omega\Delta x + \begin{bmatrix} B \\ D \end{bmatrix} n_g, \tag{5}$$

where, $Co\Omega$ denotes a convex set generated by the set Ω.

Theorem 1. If the estimation error system (4) can be described by the model (5) and Ω is a real compact set, there exists an uncertain polytopic linear system (6) including the estimation error system (4).

$$\begin{aligned} \Delta\dot{x} &= A\Delta x + Bn_g, \\ \Delta y &= C\Delta x + Dn_g, \end{aligned} \tag{6}$$

where, $(A, C) = \sum_{i=1}^{l} \lambda_i(A_i, C_i)$, $\sum_{i=1}^{l} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$.

Proof: In general, a convex set can be approximated by an endo-polytope for arbitrary precision [18], so there is always an uncertain polytopic linear system which can be used to approximate the LDI system (5), that means the estimation error system (4) can be always represented by an uncertain polytopic linear model.

Denote

$$\phi(x) = \begin{bmatrix} f(x) \\ h(x) \end{bmatrix},$$

it's obvious that $\phi(x) : R^n \to R^{(m+n)}$ is a continuously differentiable function. The matrix F can be expressed as

$$F = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial h}{\partial x} \end{bmatrix} = \phi(x) \in \Omega.$$

According to the mean-value theorem [17]: if $\frac{\partial \phi}{\partial x} \in \Omega$, for any x and $\hat{x}$, there exists $\zeta \in \begin{bmatrix} min(x, \hat{x}), max(x, \hat{x}) \end{bmatrix}$ satisfies

$$\phi(x) - \phi(\hat{x}) = \frac{\partial \phi}{\partial x}(\zeta)\Delta x. \tag{7}$$

As $F(\zeta) = \frac{\partial \phi}{\partial x}(\zeta) \in \Omega$ and Ω is a real compact set, each element of $F(\zeta)$ has the maximum and minimum. For the nonlinear system (1) with single state and none observation, that is $n = 1, m = 0$, $F(\zeta)$ can be written as

$$F(\zeta) = \frac{F(\zeta) - F_{min}}{F_{max} - F_{min}} F_{max} + \frac{F_{max} - F(\zeta)}{F_{max} - F_{min}} F_{min}, \tag{8}$$

where, F_{max} and F_{min} are the maximum and minimum of $F(\zeta)$ respectively.

Let

$$\begin{aligned} \lambda_1 &= \frac{F(\zeta) - F_{min}}{F_{max} - F_{min}}, F_1 = F_{max}, \\ \lambda_2 &= \frac{F_{max} - F(\zeta)}{F_{max} - F_{min}}, F_2 = F_{min}, \end{aligned} \tag{9}$$

then, (8) can be rewritten as

$$F(\zeta)\Delta x = (\lambda_1 F_1 + \lambda_2 F_2)\Delta x,$$

where the parameters λ_1 and λ_2 satisfy $0 \leq \lambda_1 \leq 1$, $0 \leq \lambda_1 \leq 1$, $\lambda_1 + \lambda_2 = 1$. The estimation error system (4) is equivalent to the uncertain polytopic linear system (6), if λ_1 and λ_2 are valued as (9), otherwise, the system (4) is included in the system (6).

For the nonlinear system (1) with n-dimension state vector and m-dimension observation vector, the matrix $F(\zeta)$ can be written as

$$F(\zeta) = \frac{\partial \phi}{\partial x} = \sum_{i=1}^{m+n} \sum_{j=1}^{n} F(i, j) F_{ij} \tag{10}$$

where, the matrix F_{ij} is the matrix with the element in roll i and column j valued 1 and the others all valued 0. According to (8), it is easy to represent each element of the above equation on the right by a polytope with two vertexes. Then it can immediately obtain the uncertain polytopic linear system model (6) including the estimation error system (4) by the polytope overlay algorism. ■

Lemma 2 [19]. Consider the uncertain polytopic linear system (6), a filter with the following form that achieves a suboptimal guaranteed filtering error covariance bound exist,

$$\begin{aligned} \Delta\dot{\hat{x}}_m &= A_F \Delta\hat{x}_m + B_F \Delta y, \\ \Delta\hat{x} &= C_F \Delta\hat{x}_m \end{aligned} \tag{11}$$

if there exists a solution to the following LMI:

$$\begin{aligned} & mintrace(Z) \\ s.t. & \begin{bmatrix} -G_{11}-G_{11}^T & -G_2-G_{21}^T & \Psi_{1i} & S_A+P_{12i}-F_{21}^T & G_{11}B_i+S_BD_i \\ * & -G_2-G_2^T & \Psi_{2i} & S_A+P_{22i}-\alpha_2G_2^T & G_{21}B_i+S_BD_i \\ * & * & \Psi_{3i} & \Psi_{4i} & F_{11}B_i+\alpha_1S_BD_i \\ * & * & * & \alpha_2S_A+\alpha_2S_A^T & F_{21}B_i+\alpha_2S_BD_i \\ * & * & * & * & -I \end{bmatrix} < 0, \\ & \begin{bmatrix} Z & I & -S_C \\ * & P_{11i} & P_{12i} \\ * & * & P_{22i} \end{bmatrix} > 0, i=1,2,\cdots,l \end{aligned} \tag{12}$$

where, α_1 and α_2 are fixed parameters,

$$\begin{aligned} \Psi_{1i} &= G_{11}A_i+S_BC_i+P_{11i}-F_{11}^T, \\ \Psi_{2i} &= G_{21}A_i+S_BC_i+P_{12i}^T-\alpha_1G_2^T, \\ \Psi_{3i} &= F_{11}A_i+\alpha_1S_BC_i+A_i^TF_{11}^T+\alpha_1C_i^T, \\ \Psi_{4i} &= \alpha_1S_A+A_i^TF_{21}^T+\alpha_2C_i^TS_B^T. \end{aligned}$$

The suboptimal filter is given by the following equations:

$$A_F=G_2^{-1}S_A, B_F=G_2^{-1}S_B, C_F=S_C. \tag{13}$$

Theorem 2. Consider the nonlinear system (1), if $F \in \Omega$ and Ω is the compact subset of the real set $R^{(n+m)\times n}$, a constant gain filter that gives a suboptimal guaranteed filtering error covariance bound can be derived from the following form:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x})+C_F\Delta\hat{x}, \\ \Delta\dot{\hat{x}} &= A_F\Delta\hat{x}+B_F[h(x)-h(\hat{x})]. \end{aligned} \tag{14}$$

where, A_F, B_F, C_F are the filter coefficients to be determined, which can be gained by (13).

Proof: According to the theorem 1 and the conditions given in the theorem 3, the estimation error system (4) can be represented by the uncertain polytopic linear system (6). Since the system (6) is the father system of the estimation error system (4),based on the Lemma 2, one can be easy to conclude that the rectification quantities for the predicted state errors can be determined by (11), and the filtering error covariance is suboptimal for the estimation error system (4). Following the above conclusion, it is evident to get the theorem 2. ■

III. New Filter for the Discrete Nonlinear System

The result of Theorems 1 can be easily extended to the discrete nonlinear systems. Consider the following discrete nonlinear system

$$\begin{aligned} x_k &= f(x_{k-1})+Bn_{k-1}, \\ y_{k-1} &= h(x_{k-1})+Dn_{k-1}, \end{aligned} \tag{15}$$

where, $x_k \in R^n$ is the system state vector, $y_{k-1} \in R^m$ is the observation vector, $f(\cdot)$ and $h(\cdot)$ are nonlinear functions assumed to be continuously differentiable, $n_{k-1} \in R^p$ is the noise assumed to be zero mean white uncorrelated noise. Note that for the case when the process noise and the measurement noise are different, say B_1w_{k-1} and D_1v_{k-1}, one can simply put $B = \begin{bmatrix} B_1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & D_1 \end{bmatrix}$ and let $n_{k-1} = \begin{bmatrix} w_{k-1}^T & v_{k-1}^T \end{bmatrix}^T$ in the system (15).

Let $\hat{x}_k$ be the predicted state estimates at the moment k, denote the state estimation errors and the measurement prediction errors are

$$\Delta x_k = x_k-\hat{x}_k, \Delta y_{k-1} = y_{k-1}-\hat{y}_{k-1}.$$

The estimation error model can be expressed as

$$\begin{aligned} \Delta x_k &= f(x_{k-1})-f(\hat{x}_{k-1})+Bn_{k-1}, \\ \Delta y_{k-1} &= h(x_{k-1})-h(\hat{x}_{k-1})+Dn_{k-1}, \end{aligned} \tag{16}$$

Theorem 3. There exists an uncertain polytopic linear system with the following form involved the estimation error system (16), if $x_k \in \Omega_1$ and $y_{k-1} \in \Omega_2$, Ω_1 and Ω_2 are compact sets.

$$\begin{aligned} \Delta x_k &= A\Delta x_{k-1}+Bn_{k-1}, \\ \Delta y_{k-1} &= C\Delta x_{k-1}+Dn_{k-1}. \end{aligned} \tag{17}$$

Proof: Since $x_k \in \Omega_1$ and $y_{k-1} \in \Omega_2$, Ω_1 and Ω_2 are compact sets and the functions $f(x_{k-1})$ and $h(x_{k-1})$ are continuously differentiable, the partial derivative of the functions $f(\cdot)$ and $h(\cdot)$ are both compact sets. Similar to the continuous time case, following the same line as in the proof of theorem 1, it is easy to obtain the theorem 3 above. ■

Lemma 3 [19]. Consider the system (17), a filter of the form (18) that achieves a suboptimal guaranteed filtering error covariance bound can be derived from the optimization (19).

$$\begin{aligned} \Delta\hat{x}_{m\ k+1} &= A_F\Delta\hat{x}_{m\ k}+B_F\Delta y_k, \\ \Delta\hat{x}_k &= C_F\Delta\hat{x}_{m\ k}+D_F\Delta y_k, \end{aligned} \tag{18}$$

$$\begin{aligned} & mintrace(Z) \\ s.t. & \begin{bmatrix} G_{11}+G_{11}^T-P_{11i} & G_2+G_{21}^T-P_{12i} & \psi_{1i} & S_A-F_{21}^T & G_{11}B_i+S_BD_i \\ * & G_2+G_2^T-P_{22i} & \psi_{2i} & S_A-\alpha_2G_2^T & G_{21}B_i+S_BD_i \\ * & * & \psi_{3i} & \psi_{4i} & -F_{11}B_i-\alpha_1S_BD_i \\ * & * & * & P_{22i}-\alpha_2S_A-\alpha_2S_A^T & -F_{21}B_i-\alpha_2S_BD_i \\ * & * & * & * & I \end{bmatrix} > 0, \\ & \begin{bmatrix} Z & I-S_DC_i & -S_C & -S_DC_i \\ * & P_{11i} & P_{12i} & 0 \\ * & * & P_{22i} & 0 \\ * & * & * & I \end{bmatrix} > 0, i=1,2,\cdots,l \end{aligned} \tag{19}$$

where, α_1 and α_2 are fixed parameters,

$$\begin{aligned} \psi_{1i} &= G_{11}A_i+S_BC_i-F_{11}^T, \\ \psi_{2i} &= G_{21}A_i+S_BC_i-\alpha_1G_2^T, \\ \psi_{3i} &= P_{11i}-F_{11}A_i-\alpha_1S_BC_i-A_i^TF_{11}^T-\alpha_1C_i^TS_B^T, \\ \psi_{4i} &= P_{12i}-\alpha_1S_A-A_i^TF_{21}^T-\alpha_2C_i^TS_B^T. \end{aligned}$$

TABLE I: The statistics of the calculation numbers for the two filters

operation	the new filter	EKF
addition	$2n^2 - 2n + 2mn$	$2mn^2 - 2mn + 2m^2n + 2n^3 - n^2 - n$
multiplication[a]	$2n^2 + 2mn$	$2n^3 + 2mn^2 + 2m^2n + mn$

[a] The calculation number for the matrix inversion in the EKF isn't figured out in the result shown in the Table 1, since it is difficult to make statistics.

The suboptimal filter is given by the following equations

$$A_F = G_2^{-1} S_A, B_F = G_2^{-1} S_B, C_F = S_C, D_F = S_D. \quad (20)$$

The filter designed for the father system is also suitable for the son system. Following this intuition and similar to the continuous case, it is evident to get the following result under the condition that the uncertain polytopic system (17) includes the estimation error system (16).

Theorem 4. Consider the system (15), if $x_k \in \Omega_1$, $y_{k-1} \in \Omega_2$, Ω_1 and Ω_2 are compact sets, a constant gain filter that gives a suboptimal guaranteed filtering error covariance bound can be derived from the following equations:
state predictions:

$$\hat{x}_{k/k-1} = f(\hat{x}_{k-1}), \quad (21)$$

predictions for the rectification:

$$\Delta\hat{x}_{m\ k+1} = A_F \Delta\hat{x}_{m\ k} + B_F[y_k - h(\hat{x}_{k/k-1})], \quad (22)$$

rectification quantities:

$$\Delta\hat{x}_k = C_F \Delta\hat{x}_{m\ k} + D_F[y_k - h(\hat{x}_{k/k-1})], \quad (23)$$

state estimates:

$$\hat{x}_k = \hat{x}_{k/k-1} + \Delta\hat{x}_k, \quad (24)$$

where, A_F , B_F , C_F , D_F are the filter coefficients to be determined, which can be gained by the equation (20).

The real-time performance of the nonlinear filter is an important impact on the decision whether the filter is practical. And The nonlinear filters are generally hardly to satisfy the real-time performance required in the nonlinear estimation, since the algorithms are complex and the calculations numbers are large. In order to compare the real-time performance between the new filter and the EKF, the statistics of the calculation number for the two filters is shown in the Table 1, taking the discrete nonlinear filter for example. Since the state prediction equations are same in the two filters, only the calculation numbers of the rectification in each step are shown in the Table 1. The parameters n and m represent the dimensions of the system state vector and the observation vector respectively.

According to the statistics, it is obvious to find out that the multiplication calculation number in the EKF is much more than that in the new filter. And the addition calculation number in the EKF is equal to that in the new filter, if and if only $m = n = 1$, but the multiplication calculation number is 3 step more in the EKF meanwhile. Since the multiplication is much more complex than the addition, that the calculation number of the EKF is much more than the new filter is evident under any case. Therefore it can be easy to conclude that the real-time performance of the new filter is much better than the EKF.

IV. Example Application

The spacecraft attitude estimation system is chosen for the applications and comparison of the new filter and the EKF, for it has significant nonlinearities and has been extensively in the literature.

The dynamics equation for the system is

$$J\dot{\omega} + [\omega\times]J\omega = T_c + T_d, \quad (25)$$

where, J is the inertial matrix of the spacecraft, ω is the angular velocity vector, T_c and T_d are the control and disturb torques respectively, $[\omega\times]$ is the skew symmetric matrix generated from the angular velocity vector.

The modified Rodrigues parameter (MRP) kinematic equation is given by

$$\dot{\sigma} = M(\sigma)\omega, \quad (26)$$

where, σ is the modified Rodrigues parameter vector, $M(\sigma)$ is expressed as

$$M(\sigma) = \frac{1}{4}[(1 - \sigma^T\sigma)I_3 + 2[\sigma\times] + 2\sigma\sigma^T]. \quad (27)$$

The star tracker is used in most practical applications for the attitude estimation, for it is the most accurate. It's assumed that the sensor coordinate system coincides with the spacecraft body reference coordinate system. The observation model for the sensor is written as

$$S_b = R(\sigma)S_i + \Delta S, \quad (28)$$

where, S_b is the sensor observation vector, S_i is the unite stellar direction vector, ΔS is the measurement noise assumed to be white noise, $R(\sigma)$ is the attitude matrix obtained from the MRP according to the relation

$$R(\sigma) = I_3 - \frac{4(1 - \sigma^T\sigma)}{1 + \sigma^T\sigma}[\sigma\times] + \frac{8}{1 + \sigma^T\sigma}[\sigma\times]^2.$$

The spacecraft attitude state is given by the angular velocity vector and the attitude MRP

$$x = \begin{bmatrix} \omega^T & \sigma^T \end{bmatrix}^T. \quad (29)$$

The predicted attitude state estimates satisfy

$$J\dot{\hat{\omega}} + [\hat{\omega}\times]J\hat{\omega} = \hat{T}_c, \quad (30)$$

$$\dot{\hat{\sigma}} = M(\hat{\sigma})\hat{\omega}. \quad (31)$$

Denote the attitude state error vectors Δx are

$$\Delta x = \begin{bmatrix} \Delta\sigma^T & \Delta\omega^T \end{bmatrix}^T, \Delta\sigma = \sigma \otimes \hat{\sigma}^{-1}, \Delta\omega = \omega - \hat{\omega}, \quad (32)$$

where, $\hat{\sigma}^{-1}$ is the inversion of $\hat{\sigma}$ expressed as $\hat{\sigma}^{-1} = -\hat{\sigma}$. $\otimes$ represents the multiplication for the MRP expressed as

$$\sigma_a \otimes \sigma_b = \frac{(1-\sigma_b^T\sigma_b)\sigma_a + (1-\sigma_a^T\sigma_a)\sigma_b - 2\sigma_a \times \sigma_b}{1 + (\sigma_a^T\sigma_a)(\sigma_b^T\sigma_b) - 2\sigma_a^T\sigma_b} \quad (33)$$

The dynamics equations of attitude estimation error system can be expressed as the following form by (25), (26), (30) and (31) with the method of Taylor series expansion:

$$\Delta\dot{x} = \begin{bmatrix} -[\hat{\omega}\times] & \frac{1}{4}I_3 \\ 0_3 & F_\omega \end{bmatrix} \Delta x + O_1(\Delta x) + \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix}, \quad (34)$$

where,

$$F_\omega = -J^{-1}[\hat{\times}]J + J^{-1}[(J\hat{\omega})\times], n_\omega = J^{-1}(T_c + T_d - \hat{T}_c),$$

$O_1(\Delta x)$ are the higher order terms of the Taylor series.

Define the predicted observation errors are

$$\Delta S_b = S_b - \hat{S}_b, \quad (35)$$

where, $\hat{S}_b$ is the predicted observation vector. And the observation equation of the attitude estimation error system is

$$\Delta S_b = \begin{bmatrix} [4(R(\hat{\sigma})S_i)\times] & 0_3 \end{bmatrix} \Delta x + O_2(\Delta x) + \Delta S, \quad (36)$$

where, $O_2(\Delta x)$ are the higher order terms of the Taylor series.

According to the Theorem 1 and Theorem 3, the attitude estimation error system can be represented via an uncertain polytopic linear model even if for the discretizing attitude estimation error system, since the attitude state is of bounded variation on the finite interval in the practial engineering. Then, the attitude estimator can be derived from the following equations based on the Theorem 4.

attitude predictions:

$$\hat{\omega}_{k,k-1} = \hat{\omega}_{k-1} + J^{-1}\{\hat{T}_{c,k-1} - [\hat{\omega}_{k-1}\times]J\hat{\omega}_{k-1}\} \cdot \Delta T, \quad (37)$$

$$\hat{\sigma}_{k,k-1} = \hat{\sigma}_{k-1} + M(\hat{\sigma}_{k-1})\hat{\omega}_{k-1} \cdot \Delta T, \quad (38)$$

attitude estimates:

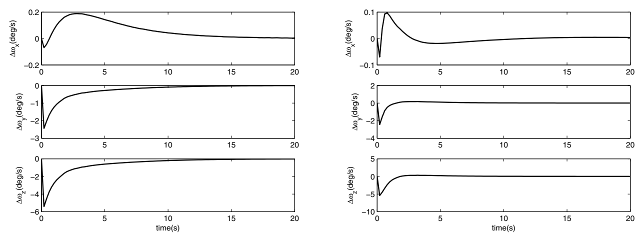
$$\hat{\omega}_k = \hat{\omega}_{k,k-1} + \Delta\hat{\omega}_k, \quad (39)$$

$$\hat{\sigma}_k = \Delta\hat{\sigma}_k \otimes \hat{\sigma}_{k,k-1}. \quad (40)$$

where, the rectification quantities for the predicted attitude $\Delta\hat{x}_k = \begin{bmatrix} \Delta\hat{\omega}_k^T & \Delta\hat{\sigma}_k^T \end{bmatrix}^T$ are evaluated by (22) and (23), and ΔT is the sample time for the discretization.

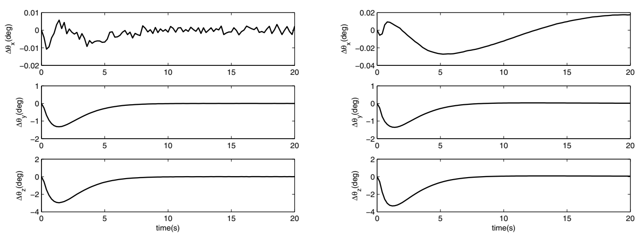
The initial true the angular velocity vector of the attitude determination system is $\begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix}^T$ rad/s, the initial true 3-1-2 Euler angles are $\begin{bmatrix} 10 \ 20 \ 50 \end{bmatrix}^T$ deg. The standard covariance of the star senor measurement noise and the disturb torques are $5''$ and 10^{-3} N·m. The initial attitude estimates are valued as zero, and the initial covariance is $10^{-8}I_6$. Both filters are implemented in the discrete time, and the sample time for the discretization is 0.2 s.

The initial attitude estimation errors of the angular velocity vector and the attitude MRP for the two filters are shown in Fig.1 and Fig. 2 respectively. The estimation errors of the attitude states in the new filter both converge over the simulate time quickly and the initial attitude estimation errors in the new filter are nearly the same with those in the EKF, implying that the attitude estimation in the new filter is right.



(a) the new filter (b) the EKF

Fig. 1: The initial estimation errors of the angular velocity

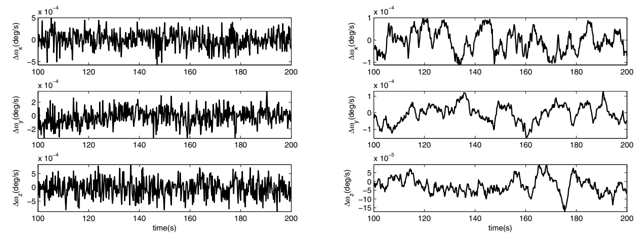


(a) the new filter (b) the EKF

Fig. 2: The initial estimation errors of the attitude MRP

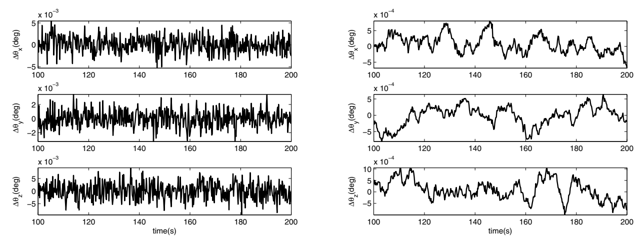
The steady attitude estimation errors of the angular velocity and the attitude MRP for the two filters are shown in Fig.3 and Fig. 4 respectively, where the attitude MRP is converted into the attitude 3-1-2 Euler angles to make the simulation results more intuitive. The steady attitude estimation errors in the new filter are unbiased, and the order of the magnitude for the steady attitude estimation errors are acceptable, indicating that the new filter works well for the spacecraft attitude estimation. Choose the maximum absolute steady estimation errors as the performance evaluation index for the two filters, the estimation accuracy for the triaxial angular velocities in the new filter are 5.707×10^{-4} deg/s, 3.643×10^{-4} deg/s and 8.853×10^{-4} deg/s, which in the EKF are 1.117×10^{-4} deg/s, 1.538×10^{-4} deg/s and 1.721×10^{-4} deg/s respectively. The estimation accuracy for the 3-1-2 Euler angles in the new filter are 0.0054 deg, 0.0034 deg and 0.0094 deg, which in the EKF are 0.0008 deg, 0.0008 deg and 0.0010 deg respectively. The magnitude order of the attitude estimation precision for thetriaxial angular velocities and the 3-1-2 Euler angles in the new filter are 10^{-4} deg/s and 0.001 deg, satisfying the requirement for the general attitude estimation, though the attitude estimation precision of the new filter is a little decline than that of the EKF.

According to the statistics results of the calculation number for the two filters shown in the Table 1, the calculation numbers of the addition in the new filter and the EKF are 132 and 1182 in each step of the attitude estimates, and the



(a) the new filter (b) the EKF

Fig. 3: The steady estimation errors of the angular velocity



(a) the new filter (b) the EKF

Fig. 4: The steady estimation errors of the 3-1-2 Euler angles

calculation numbers of the multiplication are 144 and 1332 respectively. It's evident that the calculation number of the new filter is much less than that of the EKF, implying that the real-time performance of the new filter is much better than the EKF in the application to the spacecraft attitude estimation. Thus, it can be concluded that the new filter should be preferred over the EKF in virtually nonlinear estimation application with low accuracy demand.

V. Conclusion

This paper develops a novel nonlinear filter for the general nonlinear system, in order to overcome the deficiencies of the general nonlinear filter with large number and complex calculation. Rather than approximate the Taylor series to an arbitrary oder, the new filter represents the estimation error system via a PLDI model, converting the nonlinear filtering design problem into the uncertain linear filtering design problem, which helps simplify the nonlinear filtering design. And the coefficients of the rectification equations in the new filter are constant without the need to evaluate the Jacobian matrixes, resulting in less calculation number and easier implementation. The calculation number statistics for the new filter and the EKF demonstrates that the new filter performs a better real-time property, though the example of the application to the attitude estimation shows the accuracy of the filter is a little decline than the EKF. Colligate its performance and implementation advantages, it can be easy to get conclusion that the new filter should be preferred over other nonlinear filters in virtually nonlinear estimation applications with lower accuracy demand.

Acknowledgment

This work was supported by Major State Basic Research Development Program under grant 2012CB720000. The authors would like to thank the Assistant Professor Zhongkui Li of the Automatic department in Peking University and Doctor Zhe Gao in Key laboratory for Intelligent Control & Decision of Complex Systems, their great expertise and the constructive criticism are gratefully acknowledged.

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