State Estimation for Genetic Regulatory Networks with Time-varying Delay Using Stochastic Sampled-data

Tae H. Lee, M.J. Park, O.M. Kwon, Ju H. Park*, S.M. Lee

***Abstract*—This paper considers genetic regulatory networks with time-varying delay. By construction of a suitable Lyapunov-Krasovskii functional and utilization of stochastic sampled-data, a delay-dependent state estimation for the concerned systems is established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. One numerical example is given to illustrate the effectiveness of the proposed method.**

***Keywords*—State estimator, Genetic regulatory networks, Time-varying delay, Stochastic sampled-data.**

I. INTRODUCTION

During the last few years, the state estimation problem is being on the rise in various dynamic systems. Because the states in the systems, particularly, large scale systems, are not completely available in the system outputs in practice applications, the state estimation for the systems is important both in theory and in practice. Since the availability of the states of the system is a prerequisite in the field of control engineering, especially, in state feedback control, various approaches to state estimation criteria for the systems have been investigated. Moreover, it is well known that time-delay often occurs due to the finite speed of information processing in the implementation of dynamic systems, and causes undesirable dynamic behaviors such as performance degradation and instability of the systems. Therefore it is worth to consider both above mentioned issues and it has been studied in the literature [1].

On the other hand, genetic regulatory networks (GRNs) are a nonlinear dynamical system which describes the highly complex interaction between mRNAs and proteins. So, the GRNs have received considerable attentions in research area of biological and biomedical sciences due to their extensive applications in networks of gene of high connectivity and complexity [2]. There are two types of GRNs model, that is the Boolean model and the differential equation model. For reasons of continuous values of the gene regulation systems, the differential equation model of GRNs will be considered in this paper.

In this regard, in [3]-[4], the state estimation problem for the GRNs with time-delay were considered. Lv *et al.* [3] had studied the robust distributed state estimation problem for delayed GRNs with SUM logic and multiple sensors based on the Lyapunov functional method and the stochastic analysis technique. By resorting to the Lyapunov functional method and some stochastic analysis tools, the robust state estimation problem was addressed for the GRNs with parameters uncertainties and stochastic disturbance in [4]. However, to the best of our knowledge, no related results have been established for state estimation of GRNs using sampled-data.

Since [5] and [6] introduced a concept that discontinuous sampled control inputs treat time-varying delayed continuous signals, the sampled-data control scheme ([5], [6]) has been applied to many dynamic systems although applied actual control signals are discontinuous. In addition, the necessity of the controller with varying sampling interval has strongly come to the fore because of its usefulness in practical system. For example, in networked control systems, if a constant sampling period is adopted, the sampling period should be large enough to avoid network congestion when the network is occupied by the most users, so network bandwidth cannot be sufficiently used when the network is idle. More recently, stochastically varying sampling intervals are considered, which is said to be further extended scheme to the case of time-varying sampling intervals. In [7], mean square stability of networked control system with stochastically varying network induced delay is studied. In [8], an $\mathcal{H}_\infty$ control for sampled-data control system with probabilistic sampling is also investigated.

Motivated by this mentioned above, in this paper, a state estimation problem for GRNs with time-varying delays will be studied by using stochastic sampled-data. By construction of a suitable Lyapunov-Krasovskii functional and utilization of reciprocally convex approach, a state estimation condition for GRNs with time-varying delays and sampled-data is derived in terms of LMIs which can be solved efficiently by use of standard convex optimization algorithms such as interior-point methods [9]. Moreover, we propose a discontinuous Lyapunov functional approach in order to use the full information of sawtooth structure characteristic of sampling input delays. Finally, one numerical example is included to show the effectiveness of the proposed method.

Notations: $\Phi(i,j)$ denotes the ith row, jth column element(or block matrix) of matrix Γ. $diag\{\cdots\}$ denotes diagonal matrix. $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$, respectively, mean the expectation of the

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stochastic variable x and the expectation of the stochastic variable x conditional on the stochastic variable y. $Pr\{\alpha\}$ means the occurrence probability of the event α.

II. Problem formulation

The functional differential equation model of GRN is described as

$$\begin{cases} \dot{m}_i(t) = a_i m_i(t) + h_i(p_1(t-\tau_1(t),\ldots,p_N(t-\tau_1(t)))) \\ \dot{p}_i(t) = c_i p_i(t) + d_i m_i(t-\tau_2(t)), \quad i=1,\ldots,N, \end{cases} \tag{1}$$

where $m_i(t)$, $p_i(t) \in \mathbb{R}$ are concentrations of mRNA and protein of the ith node, a_i and c_i are degradation rates of mRNA and protein respectively, d_i is the translation rate, $\tau_1(t)$ and $\tau_2(t)$ are time-varying delays, $h_i(\cdot)$ represents the feedback regulation of the protein, which is usually of the Michaelis-Menten or Hill form.

In this paper, we take $h_i(p_1(t-\tau_1(t)),\ldots,p_N(t-\tau_1(t))) = \sum_{j=1}^{N} h_{ij}(p_j(t-\tau_i(t)))$, which is called SUM logic [10]. If transcription factor j is an activator of the gene i, then

$$h_{ij}(p_j(t)) = b_{ij}\frac{(p_j(t)/\beta)^H}{1+(p_j(t)/\beta)^H}$$

and if transcription factor j is a repressor of the gene j, then

$$\begin{aligned} h_{ij}(p_j(t)) &= b_{ij}\frac{1}{1+(p_j(t)/\beta)^H} \\ &= b_{ij}\left(1-\frac{(p_j(t)/\beta)^H}{1+(p_j(t)/\beta)^H}\right), \end{aligned}$$

where H is the Hill coefficient, β is a positive constant, b_{ij} is the dimensionless transcriptional rate of the transcriptional factor j to i.

Then the system (1) can be rewritten as

$$\begin{cases} \dot{m}_i(t) = a_i m_i(t) + \sum_{j=1}^{N} w_{ij} g_j(p_j(t-\tau_1(t))) + B_i \\ \dot{p}_i(t) = c_i p_i(t) + d_i m_i(t-\tau_2(t)), \quad i=1,\ldots,N, \end{cases} \tag{2}$$

where $g_j(x) = \frac{(x/\beta)^H}{1+(x/\beta)^H}$, $B_i = \sum_{j\in I_i} b_{ij}$, I_i is the set of all the repressor of gene i and $W=(w_{ij}) \in \mathbb{R}^{n\times n}$ is defined as

$$w_{ij} = \begin{cases} b_{ij} \text{ if transcription factor } j \text{ is} \\ \quad \text{an activator of the gene } i \\ -b_{ij} \text{ if transcription factor } j \text{ is} \\ \quad \text{an repressor of the gene } i \\ 0 \text{ if there is no link from gene } j \text{ to } i \end{cases}$$

The system (2) can be rewritten in vector-matrix form:

$$\begin{cases} \dot{m}(t) = Am(t) + Wg(p(t-\tau_1(t))) + B \\ \dot{p}(t) = Cp(t) + Dm(t-\tau_2(t)) \end{cases}, \tag{3}$$

where $m(t) = [m_1(t),\ldots,m_N(t)]^T$, $p(t) = [p_1(t),\ldots, p_N(t)]^T$, $A = diag\{a_1,\ldots,a_N\}$, $C = diag\{c_1,\ldots,c_N\}$, $D = \{d_1,\ldots,d_N\}$, $B = [B_1,\ldots,B_N]^T$, $g(P(t-\tau_1(t))) = [g_1(p_1(t-\tau_1(t))),\ldots,g_N(p_N(t-\tau_1(t)))]^T$.

Suppose that $m^* = [m_1^*,\ldots,m_N^*]^T$ and $p^* = [p_1^*,\ldots, p_N^*]^T$ are equilibrium points of the system (3). By using state transformation $x(t) = m(t) - m^*$ and $y(t) = p(t) - p^*$, one can shift the equilibrium points m^* and p^* into the origin. The transformed model is as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Wf(y(t-\tau_1(t))) \\ \dot{y}(t) = Cy(t) + Dx(t-\tau_2(t)) \end{cases}, \tag{4}$$

where $x(t) = [x_1(t),\ldots,x_N(t)]^T$ with $x_i(t) = m_i(t) - m_i^*$, $y(t) = [y_1(t),\ldots,y_N(t)]^T$ with $y_i(t) = p_i(t) - p_i^*$, $f(y(t)) = [f_1(y_1(t)),\ldots,f_N(y_N(t))]^T$ with $f_i(y_i(t)) = g_i(y_i(t)+p^*) - g_i(p^*)$, respectively.

It should be noted that the feedback regulation of the protein, $g_i(t)$, is monotonically increasing function with saturation such that

$$0 \le \frac{g_i(a)-g_i(b)}{a-b} \le l_i, \quad \forall a \ne b \in \mathbb{R},$$

where l_i is a positive constant. And the time-varying delays $\tau_1(t)$ and $\tau_2(t)$ are continuous functions satisfying $0 \le \tau_1(t) \le \sigma_1$, $\dot{\tau}_1(t) \le \mu_1$, $0 \le \tau_2(t) \le \sigma_2$, $\dot{\tau}_2(t) \le \mu_2$ where σ_i and μ_i $(i=1,2)$ are positive constant values.

In real systems, especially, large-scale systems or high-order complex systems, only partial information about the network components is available. Therefore, in order to obtain the true state of the GRN (concentrations of the mRNA and protein), one would need to estimate them from available measurements.

Suppose that the network measurements are $z_x(t) = Mx(t)$, $z_y(t) = Ny(t)$ where $z_x(t), z_y(t) \in \mathbb{R}^k$ are the actual measurement outputs and M and N are known constant matrices with appropriate dimension.

The purpose of this paper is to present an efficient estimation algorithm to observe the concentrations of mRNA and protein from the available network output. For this end, the following full-order observer using sampled-data is proposed:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Wf(\hat{y}(t-\tau_1(t))) + u_1(t), \\ \dot{\hat{y}}(t) = C\hat{y}(t) + D\hat{x}(t-\tau_2(t)) + u_2(t), \end{cases} \tag{5}$$

where $\hat{x}(t)$ and $\hat{y}(t) \in \mathbb{R}^n$ are the estimation of the state $x(t)$ and $y(t)$, respectively, and $u_1(t)$ and $u_2(t) \in \mathbb{R}^n$ are the control input which will be designed later.

Define the error vectors by $e_x(t) = x(t) - \hat{x}(t)$ and $e_y(t) = y(t) - \hat{y}(t)$. Then the error dynamical system is expressed from (4) and (5):

$$\begin{cases} \dot{e_x}(t) = Ae_x(t) + W\bar{f}(t-\tau_1(t)) - u_1(t), \\ \dot{e_y}(t) = Ce_y(t) + De_x(t-\tau_2(t)) - u_2(t), \end{cases} \tag{6}$$

where $\bar{f}(t) = f(y(t)) - f(\hat{y}(t))$.

It should be noted that nowadays, most of controllers are the digital controller or networked to the system. These control systems can be modeled by sampled-data control systems. So the sampled-data control approach is eligible to receive much attention. Therefore, in this paper, the following controller using sampled-data with stochastic sampling is considered:

$$\begin{cases} u_1(t) = K_1(z_x(t_k) - M\hat{x}(t_k)) = K_1 M e_x(t_k), \\ u_2(t) = K_2(z_y(t_k) - N\hat{y}(t_k)) = K_2 N e_y(t_k), \end{cases} \tag{7}$$
$$t_k \le t < t_{k+1}, \quad k=0,1,2,\ldots$$

where K_1 and K_2 are the observer gain to be determined later, t_k is the updating instant time of the Zero-Order-Hold (ZOH) and define the sampling interval $h = t_{k+1} - t_k$.

In this paper, we consider that the sampling interval h is randomly change in finite set $[h_1, \ldots, h_m]$. For simplicity, we assume that the number of sampling intervals is two such that $0 < h_1 < h_2$, where the probabilities of the occurrence of each sampling interval are

$$Pr\{h = h_1\} = \beta, \;\; Pr\{h = h_2\} = 1 - \beta,$$

where $\beta \in [0\ 1]$ is a known constant.

In order to design the controller using sampled-data with stochastic sampling, the concept of the time-varying delayed control input which is proposed in [5]-[6], is adopted in this paper. Thus, by defining $d(t) = t - t_k, \;\; t_k \le t < t_{k+1}$, the controller (7) can be represented as following:

$$\begin{cases} u_1(t) = K_1 M e_x(t_k) = K_1 M e_x(t - d(t)), \\ u_2(t) = K_2 N e_y(t_k) = K_2 N e_y(t - d(t)). \end{cases} \tag{8}$$

Now, we define the stochastic variables $\alpha(t)$ and $\beta(t)$ such that

$$\alpha(t) = \begin{cases} 1 \; 0 \le d(t) < h_1 \\ 0 \; h_1 \le d(t) < h_2 \end{cases}, \;\; \beta(t) = \begin{cases} 1 \; h = h_1 \\ 0 \; h = h_2 \end{cases}$$

with following probability:

$$Pr\{\alpha(t) = 1\} = \beta + \frac{h_1}{h_2}(1 - \beta), Pr\{\beta(t) = 1\} = \beta. \tag{9}$$

Thus the system (6) can be expressed as

$$\begin{cases} \dot{e_x}(t) = A e_x(t) + W \bar{f}(t - \tau_1(t)) \\ \qquad -\big(\alpha(t) K_1 M e_x(t - d_1(t)) \\ \qquad +(1 - \alpha(t)) K_1 M e_x(t - d_2(t))\big), \\ \dot{e_y}(t) = C e_y(t) + D e_x(t - \tau_2(t)) \\ \qquad -\big(\alpha(t) K_2 N e_y(t - d_1(t)) \\ \qquad +(1 - \alpha(t)) K_2 N e_y(t - d_2(t))\big), \\ t_k \le t < t_{k+1}, \end{cases} \tag{10}$$

where $0 \le d_1(t) < h_1$ and $h_1 \le d_2(t) < h_2$.

Definition 1. [7] The error system (10) is said to be mean square stable if for any $\varepsilon > 0$, there is a $\rho(\varepsilon) > 0$ such that $\mathbb{E}\{\|e(t)\|^2\} < \varepsilon, \;\; t > 0$ when $\mathbb{E}\{\|e(0)\|^2\} < \rho(\varepsilon)$. In addition, if $\lim_{t\to\infty} \mathbb{E}\{\|e(t)\|^2\} = 0$, for any initial conditions, then the error system (10) is said to be globally mean square asymptotically stable.

Remark 1. The time-varying delays $d_1(t)$ and $d_2(t)$ in Eq. (10) are independent on the stochastic interval. So, by introducing a stochastic variable $\alpha(t)$, we can remodel system (6) to (10) which is general time-varying delay system.

Remark 2. To extend the case that the number of sampling period of the time-varying delay $d(t)$ is m, is more worthful. So, in order to apply this case to considered system, m sampling period is defined as $0 < h_1 < \ldots < h_m$ and the probability of occurrence is calculated as [8]:

$$\begin{aligned} &Pr\{h = h_i\} = \beta_i, \\ &Pr\{h_{i-1} \le d(t) < h_i\} = \sum_{j=i}^{m} \beta_j \frac{d_i - d_{i-1}}{d_j} = \alpha_i, \end{aligned} \tag{11}$$

where $i = 1, \ldots, m$ and $h_0 = 0$. Note that $\sum_{i=1}^{m} \alpha_i = 1$ and $\sum_{i=1}^{m} \beta_i = 1$.

III. Main results

In this section, a design problem of the state estimator for GRNs with time-varying delays using the stochastic sampled-data will be investigated via a discontinuous Lyapunov functional approach.

The following is a main result of this paper.

Theorem 1. For given positive constants β, μ_i, γ_i, h_i $(i = 1, 2)$ and a diagonal matrix $L = diag\{l_1, \ldots, l_n\}$, the error system (10) is globally mean square asymptotically stable, if there exist positive-definite matrices Q_i, Y_i, $P = diag\{P_1, P_2\}$, $Z_i = diag\{Z_{i1}, Z_{i2}\}$, $U_i = diag\{U_{i1}, U_{i2}\}$, $S_i = diag\{S_{i1}, S_{i2}\}$ $(i = 1, 2)$, a positive diagonal matrix Λ, and any matrices $G_i, H_i, X_i = diag\{X_{i1}, X_{i2}\}$ $(i = 1, 2)$ satisfying the following LMIs :

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \star & \Phi_3 \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} U_i & X_i \\ \star & U_i \end{bmatrix} > 0, \;\; \forall i = 1, 2, \tag{13}$$

where

$$\Phi_1 = \begin{bmatrix} \Gamma_1 & 0 & 0 & 0 & 0 & 0 & G_1 W \\ \star & -(1-\mu)Q_2 & 0 & D^T G_2^T & 0 & 0 & 0 \\ \star & \star & -Y_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \Gamma_2 & 0 & 0 & 0 \\ \star & \star & \star & \star & \Gamma_3 & 0 & 0 \\ \star & \star & \star & \star & \star & -Y_1 & 0 \\ \star & \star & \star & \star & \star & \star & -\Lambda \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} \Gamma_4 & \Gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_6 & 0 \\ \star & \Gamma_7 & \Gamma_8 & \alpha X_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \Gamma_9 & \Gamma_{10} & 0 & 0 & 0 & 0 & \Gamma_{11} & 0 \\ \star & \star & \star & -\alpha U_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 & \Gamma_{14} \\ \star & \star & \star & \star & \star & \Gamma_{15} & \Gamma_{16} & \alpha X_{22} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Gamma_{17} & \Gamma_{18} & 0 & \Gamma_{19} \\ \star & \star & \star & \star & \star & \star & \star & -\alpha U_{22} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \Gamma_{20} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \Gamma_{21} \end{bmatrix},$$

$$\begin{aligned} &\Phi_2(1,1) = \beta p S_{11} + (1-\beta) p \frac{h_1}{h_2} S_{21} + \alpha(U_{11} - X_{11} \\ &- H_1 M), \Phi_2(1,2) = \alpha X_{11}, \Phi_2(1,3) = -(1-\alpha) H_1 M, \\ &\Phi_2(1,9) = P_1 - G_1 + \gamma_1 A^T G_1^T, \; \Phi_2(2,10) = \gamma_2 D^T G_2^T, \\ &\Phi_2(4,5) = \beta p S_{12} + (1-\beta) p \frac{h_1}{h_2} S_{22} + \alpha(U_{12} - X_{12} \\ &- H_2 N), \Phi_2(4,6) = \alpha X_{12}, \Phi_2(4,7) = -(1-\alpha) H_2 N, \\ &\Phi_2(4,10) = P_2 - G_2 + \gamma_2 C^T G_2^T, \; \Phi_2(7,9) = \gamma_1 W^T G_1^T, \\ &\text{the rest entries of } \Phi_2 \text{ are } 0, \end{aligned}$$

$$\Gamma_1 = Q_2 + Y_2 + \alpha Z_{11} - \beta p S_{11} - (1-\beta)p\frac{h_1}{h_2}S_{21}$$
$$-\alpha U_{11} + G_1 A + A^T G_1^T, \Gamma_2 = Q_1 + Y_1 + \alpha Z_{12}$$
$$-\beta p S_{12} - (1-\beta)p\frac{h_1}{h_2}S_{22} - \alpha U_{12} + G_2 C + C^T G_2^T,$$
$$\Gamma_3 = -(1-\mu)Q_1 + \Lambda L, \Gamma_4 = -\beta p S_{11} - (1-\beta)p\frac{h_1}{h_2}S_{21}$$
$$+\alpha(-2U_{11} + X_{11}^T + X_{11}), \Gamma_5 = \alpha(U_{11} - X_{11}),$$
$$\Gamma_6 = -\gamma_1 \alpha M^T H_1^T, \Gamma_7 = -\alpha(Z_{11} + U_{11} + U_{21})$$
$$-(1-\beta)p\frac{h_2 - h_1}{h_2}S_{21}, \Gamma_8 = (1-\beta)p\frac{h_2 - h_1}{h_2}S_{21}$$
$$+\alpha(U_{21} - X_{21}), \Gamma_9 = -(1-\beta)p\frac{h_2 - h_1}{h_2}S_{21} + \alpha(-2U_{21}$$
$$+X_{21}^T + X_{21}), \Gamma_{10} = \alpha(U_{21} - X_{21}),$$
$$\Gamma_{11} = -\gamma_1(1-\alpha)M^T H_1^T, \Gamma_{12} = -\beta p S_{12}$$
$$-(1-\beta)p\frac{h_1}{h_2}S_{22} + \alpha(-2U_{12} + X_{12}^T + X_{12}),$$
$$\Gamma_{13} = \alpha(U_{12} - X_{12}), \Gamma_{14} = -\gamma_2 \alpha N^T H_2^T,$$
$$\Gamma_{15} = -\alpha(Z_{12} + U_{12} + U_{22}) - (1-\beta)p\frac{h_2 - h_1}{h_2}S_{22},$$
$$\Gamma_{16} = (1-\beta)p\frac{h_2 - h_1}{h_2}S_{22} + \alpha(U_{22} - X_{22}),$$
$$\Gamma_{17} = -(1-\beta)p\frac{h_2 - h_1}{h_2}S_{22} + \alpha(-2U_{22} + X_{22}^T + X_{22}),$$
$$\Gamma_{18} = \alpha(U_{22} - X_{22}), \Gamma_{19} = -\gamma_2(1-\alpha)N^T H_2^T,$$
$$\Gamma_{20} = \alpha h_1^2 U_{11} + (1-\alpha)H_2^2 U_{21} + \beta h_1^2 S_{11}$$
$$+(1-\beta)h_2^2 S_{21} - \gamma_1(G_1 + G_1^T),$$
$$\Gamma_{21} = \alpha h_1^2 U_{12} + (1-\alpha)H_2^2 U_{22} + \beta h_1^2 S_{12}$$
$$+(1-\beta)h_2^2 S_{22} - \gamma_2(G_2 + G_2^T), p = \frac{\pi^2}{4}.$$

Then, the desired observer gain matrices are given by $K_1 = G_1^{-1}H_1$ and $K_2 = G_2^{-1}H_2$.

Proof. Denote $\eta(t) = \left[e_x^T(t)\ e_y^T(t)\right]^T$ and consider the following discontinuous Lyapunov functional for error system (10) is

$$V(e_t) = V_1(e_t) + V_2(e_t) + V_3(e_t) + V_4(e_t), \qquad (14)$$
$$t \in [t_k,\ t_{k+1}),$$

where

$$V_1(e_t) = \eta^T(t)P\eta(t),$$
$$V_2(e_t) = \int_{t-\tau_1(t)}^{t} e_y^T(s)Q_1 e_y(s)ds + \int_{t-\sigma_1}^{t} e_y^T(s)Y_1 e_y(s)ds$$
$$+ \int_{t-\tau_2(t)}^{t} e_x^T(s)Q_2 e_x(s)ds + \int_{t-\sigma_2}^{t} e_x^T(s)Y_2 e_x(s)ds,$$
$$V_3(e_t) = \alpha\left(\int_{t-h_1}^{t}\eta^T(s)Z_1 e(s)ds\right.$$
$$\left.+h_1\int_{t-h_1}^{t}\int_{s}^{t}\dot\eta^T(u)U_1\dot\eta(u)duds\right)$$
$$+(1-\alpha)\left(\int_{t-h_2}^{t-h_1}\eta^T(s)Z_2 e(s)ds\right.$$
$$\left.+(h_2-h_1)\int_{t-h_2}^{t-h_1}\int_{s}^{t}\dot\eta^T(u)U_2\dot\eta(u)duds\right),$$
$$V_4(e_t) = \beta\left(h_1^2\int_{t_k}^{t}\dot\eta^T(s)S_1\dot\eta(s)ds\right.$$
$$\left.-\frac{\pi^2}{4}\int_{t_k}^{t}(\eta(s)-\eta(t_k))^T S_1(\eta(s)-\eta(t_k))ds\right)$$
$$+(1-\beta)\left(h_2^2\int_{t_k}^{t}\dot\eta^T(s)S_2\dot\eta(s)ds\right.$$
$$\left.-\frac{\pi^2}{4}\int_{t_k}^{t}(\eta(s)-\eta(t_k))^T S_2(\eta(s)-\eta(t_k))ds\right).$$

Here, $V_4(t)$ originate from [11] and makes full use of the sawtooth structure characteristic of sampling input delays. From the paper [12], it is easy to find that $V_4(t) \geq 0$. In addition, it is correct that $\lim_{t\to t_k^-} V(t) \geq V(t_k)$, because $V_4(t)$ will disappear at $t = t_k$.

Define the infinitesimal operator $\mathcal{L}$ of $V(e_t)$ is defined as follows:

$$\mathcal{L}V(e_t) = \lim_{h\to 0^+}\frac{1}{h}\{\mathbb{E}\{V(e_{t+h})|e_t\} - V(e_t)\}. \qquad (15)$$

Then from (14) and (15), we obtain

$$\mathbb{E}\{\mathcal{L}V_1(t)\} = \mathbb{E}\{2\eta^T(t)P\dot\eta(t)\}, \qquad (16)$$
$$\mathbb{E}\{\mathcal{L}V_2(t)\} = \mathbb{E}\Big\{e_y^T(t)Q_1 e_y(t) - (1-\mu_1)$$
$$\times e_y(t-\tau_1(t))Q_1 e_y(t-\tau_1(t)) + e_y^T(t)Y_1 e_y(t)$$
$$-e_y^T(t-\sigma_1)Y_1 e_y(t-\sigma_1) + e_x^T(t)Q_2 e_x(t)$$
$$-(1-\mu_2)e_x(t-\tau_2(t))Q_2 e_x(t-\tau_2(t))$$
$$+e_x^T(t)Y_2 e_x(t) - e_x^T(t-\sigma_2)Y_2 e_x(t-\sigma_2)\Big\}, \qquad (17)$$
$$\mathbb{E}\{\mathcal{L}V_3(t)\} = \mathbb{E}\Big\{\alpha\Big(\eta^T(t)Z_1\eta(t) - \eta^T(t-h_1)Z_1$$
$$\times\eta(t-h_1) + h_1^2\dot\eta^T(t)U_1\dot\eta(t) - h_1\int_{t-h_1}^{t}\dot\eta^T(s)U_1\dot\eta(s)ds\Big)$$
$$+(1-\alpha)\Big(\eta^T(t-h_1)Z_1\eta(t-h_1) - \eta^T(t-h_2)Z_1$$
$$\times\eta(t-h_2) + h_2^2\dot\eta^T(t)U_1\dot\eta(t) - (h_2-h_1)$$
$$\times\int_{t-h_2}^{t-h_1}\dot\eta^T(s)U_1\dot\eta(s)ds\Big)\Big\}, \qquad (18)$$
$$\mathbb{E}\{\mathcal{L}V_4(t)\} = \mathbb{E}\left\{\beta\left(h_1^2\dot\eta^T(t)S_1\dot\eta(t) - \frac{\pi^2}{4}\begin{bmatrix}\eta(t)\\ \eta(t-d(t))\end{bmatrix}\right.\right.$$
$$\times\begin{bmatrix}S_1 & -S_1\\ \star & S_1\end{bmatrix}\left.\begin{bmatrix}\eta(t)\\ \eta(t-d(t))\end{bmatrix}\right)$$
$$+(1-\beta)\left(h_2^2\dot\eta^T(t)S_2\dot\eta(t) - \frac{\pi^2}{4}\begin{bmatrix}\eta(t)\\ \eta(t-d(t))\end{bmatrix}\right.$$
$$\left.\left.\times\begin{bmatrix}S_2 & -S_2\\ \star & S_2\end{bmatrix}\begin{bmatrix}\eta(t)\\ \eta(t-d(t))\end{bmatrix}\right)\right\} \qquad (19)$$

It is noted that if a sampling interval $h = h_1$ then a sampling time-varying delay, $d(t)$, is in a interval $[0\ h_1]$ and if a sampling interval $h = h_2$, then $d(t)$ exists in two intervals such that $[0\ h_1]$ and $[h_1\ h_2]$ with probabilities $\frac{h_1}{h_2}$ and $\frac{h_2-h_1}{h2}$, respectively. Therefore in order to fully use the information

of a sawtooth structure delay, $d(t)$, let us modify Eq. (19) as follows:

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}V_4(t)\} = \mathbb{E}\left\{\beta\left(h_1^2\dot{\eta}^T(t)S_1\dot{\eta}(t)\right.\right.\\
&-\frac{\pi^2}{4}\begin{bmatrix}\eta(t)\\ \eta(t-d_1(t))\end{bmatrix}\begin{bmatrix}S_1 & -S_1\\ \star & S_1\end{bmatrix}\left.\begin{bmatrix}\eta(t)\\ \eta(t-d_1(t))\end{bmatrix}\right)\\
&+(1-\beta)\left(h_2^2\dot{\eta}^T(t)S_2\dot{\eta}(t) - \frac{h_1\pi^2}{4h_2}\begin{bmatrix}\eta(t)\\ \eta(t-d_1(t))\end{bmatrix}\right.\\
&\times\begin{bmatrix}S_2 & -S_2\\ \star & S_2\end{bmatrix}\begin{bmatrix}\eta(t)\\ \eta(t-d_1(t))\end{bmatrix} - \frac{(h_2-h_1)\pi^2}{4h_2}\\
&\left.\left.\times\begin{bmatrix}\eta(t-h_1)\\ \eta(t-d_2(t))\end{bmatrix}\begin{bmatrix}S_2 & -S_2\\ \star & S_2\end{bmatrix}\begin{bmatrix}\eta(t-h_1)\\ \eta(t-d_2(t))\end{bmatrix}\right)\right\}. \quad (20)
\end{aligned}$$

By using Jensen's inequality [13] and Theorem 1 in [14], the integral terms of the $\mathcal{L}V_3(t)$ can be bounded as

$$\begin{aligned}
&-\alpha h_1\int_{t-h_1}^{t}\dot{\eta}^T(t)U_1\dot{\eta}(s)ds\\
&\le -\alpha\begin{bmatrix}\delta_{11}(t)\\ \delta_{21}(t)\end{bmatrix}^T\begin{bmatrix}\frac{1}{1-\kappa_1}U_1 & 0\\ \star & \frac{1}{\kappa_1}U_1\end{bmatrix}\begin{bmatrix}\delta_{11}(t)\\ \delta_{21}(t)\end{bmatrix}\\
&\le -\alpha\begin{bmatrix}\delta_{11}(t)\\ \delta_{21}(t)\end{bmatrix}^T\begin{bmatrix}U_1 & X_1\\ \star & U_1\end{bmatrix}\begin{bmatrix}\delta_{11}(t)\\ \delta_{21}(t)\end{bmatrix}, \quad (21)\\
&-(1-\alpha)(h_2-h_1)\int_{t-h_2(t)}^{t-h_1}\dot{\eta}^T(t)U_2\dot{\eta}(s)ds\\
&\le -(1-\alpha)\begin{bmatrix}\delta_{12}(t)\\ \delta_{22}(t)\end{bmatrix}^T\begin{bmatrix}\frac{1}{1-\kappa_2}U_2 & 0\\ \star & \frac{1}{\kappa_2}U_2\end{bmatrix}\begin{bmatrix}\delta_{12}(t)\\ \delta_{22}(t)\end{bmatrix}\\
&\le -(1-\alpha)\begin{bmatrix}\delta_{12}(t)\\ \delta_{22}(t)\end{bmatrix}^T\begin{bmatrix}U_2 & X_2\\ \star & U_2\end{bmatrix}\begin{bmatrix}\delta_{12}(t)\\ \delta_{22}(t)\end{bmatrix}, \quad (22)
\end{aligned}$$

where $\delta_{11}(t) = \int_{t-d_1(t)}^{t}\dot{\eta}(s)ds$, $\delta_{21}(t) = \int_{t-h_1}^{t-d_1(t)}\dot{\eta}(s)ds$, $\delta_{12}(t) = \int_{t-d_2(t)}^{t-h1}\dot{\eta}(s)ds$, $\delta_{22}(t) = \int_{t-h_2}^{t-d_2(t)}\dot{\eta}(s)ds$, $\kappa_1 = (h_1-d_1(t))(h_1)^{-1}$, $\kappa_2 = (h_2-d_2(t))(h_2-h_1)^{-1}$.

From the property of the nonlinear function $g(\cdot)$, we can obtain the following equation:

$$\begin{aligned}
&\lambda\bar{f}(t-\tau_1(t))^T\bar{f}(t-\tau_1(t))\\
&\le \lambda e_y^T(t-\tau_1(t))Le_y(t-\tau_1(t)). \quad (23)
\end{aligned}$$

According to the error system (10), for any appropriately dimensioned matrices G_1 and G_2, the following equations hold:

$$\begin{aligned}
&\mathbb{E}\Big\{2\left[e_x^T(t)G_1+\gamma_1\dot{e}_x^T(t)G_1\right]\Big[-\dot{e}_x(t)+Ae_x(t)\\
&+W\bar{f}(t-\tau_1(t)) - \alpha K_1Me_x(t-d_1(t))\\
&-(1-\alpha)K_1Me_x(t-d_2(t))\Big]\Big\} = 0, \quad (24)\\
&\mathbb{E}\Big\{2\left[e_y^T(t)G_2+\gamma_2\dot{e}_y^T(t)G_2\right]\Big[-\dot{e}_y(t)+Ce_y(t)\\
&+De_x(t-\tau_2(t)) - \alpha K_2Ne_y(t-d_1(t))\\
&+(1-\alpha)K_2Ne_y(t-d_2(t))\Big]\Big\} = 0, \quad (25)
\end{aligned}$$

where we let $H_1 = G_1K_1$ and $H_2 = G_2K_2$.

Utilizing the relationship (16)-(18), (20)-(25), thus we have the following a new bound for $\mathbb{E}\{\mathcal{L}V(t)\}$:

$$\mathbb{E}\{\mathcal{L}V(t)\} \le \mathbb{E}\{\zeta^T(t)\Phi\zeta(t)\}, \quad (26)$$

where $\zeta(t) = \left[e_x^T(t)\ e_x^T(t-\tau_2(t))\ e_x^T(t-\sigma_2)\ e_y^T(t)\ e_y^T(t-\tau_1(t))\ e_y^T(t-\sigma_1)\ \bar{f}^T(t-\tau_1(t))\ e_{xm}^T(t)\ e_{ym}^T(t)\ \cdots\ \dot{e}_x^T(t)\ \dot{e}_y^T(t)\right]^T$ with $e_{xm}^T(t) = \left[e_x^T(t-d_1(t))\ e_x^T(t-h_1)\cdots e_x^T(t-d_2(t))\ e_x^T(t-h_2)\right]$, $e_{ym}^T(t) = \left[e_y^T(t-d_1(t))\ \cdots\ e_y^T(t-h_1)\ e_y^T(t-d_2(t))\ e_y^T(t-h_2)\right]$.

Therefor, if LMIs (12)-(13) hold, then the error system (10) is mean square stable by Definition 1. This completes the proof. ■

Remark 4. Note that Theorem 1 is independent from time-varying system delays, $\tau_1(t)$ and $\tau_2(t)$, because this paper focus on designing the state estimator using stochastic sampled-data. The proposed result in this paper can be easily extended to the delay dependent result by using existing works.

IV. Numerical example

In this section, in order to demonstrate the effectiveness of the proposed method, we present a numerical simulation result for the GRN model. Consider GRNs (4) with parameters: $A = diag\{-3,-3,-3\}$, $C = diag\{-2.5,-2.5,-2.5\}$, $D = diag\{0.8,0.8,0.8\}$, $g_i(p_i(t)) = \frac{p_i^2(t)}{1+p_i^2(t)}$, $N = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0.5\\ 0 & 1 & 1\end{bmatrix}$, $M = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 0.5 & 0.5 & 1\end{bmatrix}$, $W = \begin{bmatrix}0 & 0 & -2.5\\ -2.5 & 0 & 0\\ 0 & -2.5 & 0\end{bmatrix}$, where we can easily know $l_i = 0.65$. The time-varying system delays are $\tau_1(t) = 1+0.9\sin t$ and $\tau_2(t) = 1+0.5\cos t$, then $\mu_1 = 0.9$ and $\mu_2 = 0.5$.

By solving the conditions (12) and (13) in Theorem 1 with $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, $h_1 = 0.01$, $h_2 = 0.05$ and $\beta = 0.8$, the observer gains K_1 and K_2 are obtained by

$$\begin{aligned}
K_1 &= \begin{bmatrix}0.2930 & 3.6471 & -0\\ -0.6783 & 5.1041 & 0\\ 8.5585 & -8.7511 & -0\end{bmatrix},\\
K_2 &= \begin{bmatrix}2.3029 & 1.3267 & -2.9878\\ -2.9505 & 1.2891 & 2.3128\\ 1.2967 & -1.3177 & 1.3240\end{bmatrix}. \quad (27)
\end{aligned}$$

In Fig. 1, the state trajectories of system (4) are depicted. Under the calculated observer gains, K_1 and K_2, the simulation result of the controlled error signals and the sampled control inputs are presented in Fig. 2 and Fig. 3, respectively. As seen in Fig. 2, the trajectories of error signals are indeed well stabilized. It means that the designed estimator closely monitor the considered GRN system (4) by control inputs which are seen in Fig. 3. Finally, Fig. 4 displays the sampling interval parameter h which is stochastically changed.

V. Conclusions

In this paper, the state estimator for GRNs with time-varying delay has been designed by using sampled-data with stochastic sampling interval. The sampled-data control system has been remodeled to a delay system with stochastic variables by using input-delay approach. In order to use full information of sawtooth structure characteristic of the

sampling delay, the discontinuous Lyapunov functional has been proposed. Then, the criterion for designing the state estimator has been expressed in terms of LMIs. One numerical example has been illustrated to show the performance of the proposed method.

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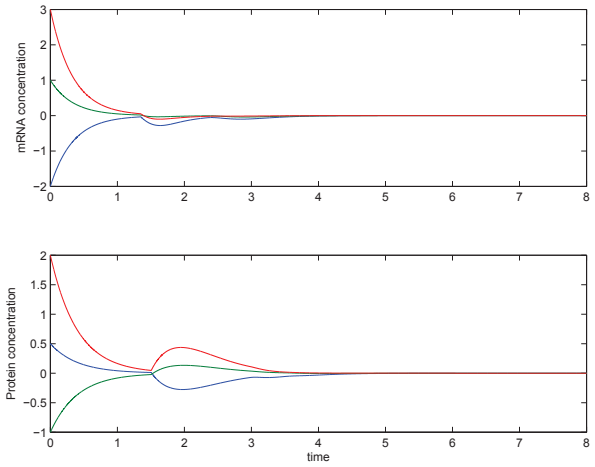

Fig. 1. The time trajectories of $m_i(t)$ and $p_i(t)$ $(i = 1, 2, 3)$

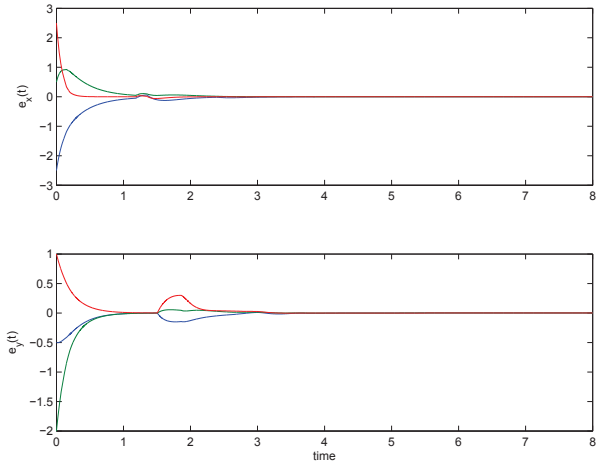

Fig. 2. The error signals between the system (4) and observer (5)

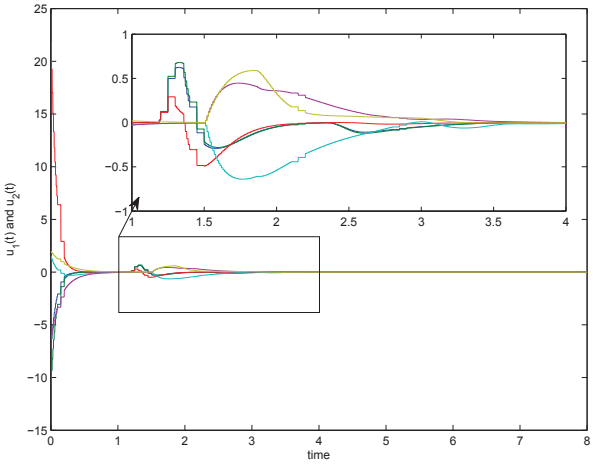

Fig. 3. The stochastic sampled-data control signals

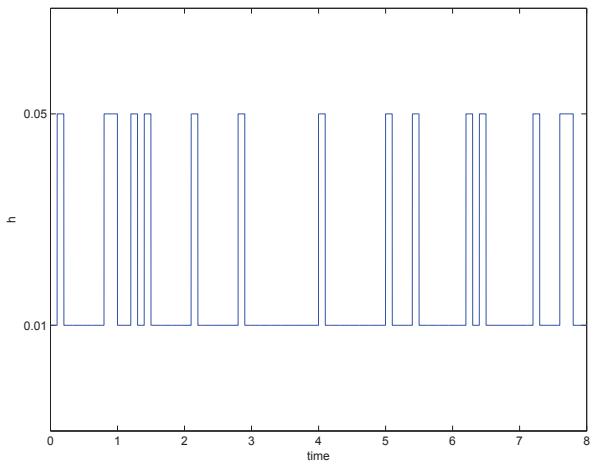

Fig. 4. The stochastic sampling interval