Cluster Consensus of Boolean Multi-Agent Systems

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***Abstract*—Consensus of the multi-agent systems (MASs) is ubiquitous in nature. Over the past decade, consensus of the MASs has received an increasing attention from various disciplines. Inspired by the cluster consensus of the discrete-time MAS, this paper aims at exploring the cluster consensus of the Boolean MAS and applies it to modeling the synchronous flashing of fireflies. Based on the graph theory and the Boolean matrix analysis, two cluster consensus criteria are established for the Boolean MASs with fixed and switching topologies. Furthermore, numerical simulations are also given to validate the effectiveness of these proposed criteria.**

I. INTRODUCTION

Synchronization, which refers to that several organisms simultaneously repeat the same activity at regular intervals of time ([1]), is a very common animal group behavior in nature. In fact, it involves two distinct factors: synchronism and rhythm. In particular, there is a kind of activities which involves two states, such as the flashing of fireflies, the croak of tree frogs, and the chirping of the crickets, cicadas and katydids, all of which includes two states, flashing on or off, chirping or non-chirping. All of them could be synchronized by responding to those of their neighbors ([1], [2], [3]).

For the convenience of observing and recording, researches on synchronous behaviors of animals usually focus on the flashing of fireflies ([1], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]). Biologists confirm experimentally that the rhythm of the recurring flashing is under the control of a neural center which is influenced by the flashing of neighboring fireflies, that is, each firefly modifies its own rhythm by interacting with others, which leads to synchronization eventually. There are various synchronization of firefly flashing. Synchronous flashing occurs both in stationary ([4]) and flying ([5]) groups of fireflies. Apart from the global synchronization of flashing, the phenomena of clustered synchronization, which means that different groups of fireflies develop distinguishable rhythms and result in locally synchronous flashing, have also been observed ([9]).

Recently, the synchronous and rhythmic flashing of fireflies has been paid a renewed attention among many researchers. Many models have been constructed to study the synchronous flashing of fireflies. In [14], Kim proposed a biological model utilizing spiking neuron to explain the synchronous flashing of fireflies. In [15], a system of coupled oscillators was applied to modeling and analyzing synchronous flashing of fireflies. In [16], Liu *et al.* constructed a model of networked oscillators that incorporated several important ingredients of many real-life systems to study the synchronous flashing of fireflies. However, none of these models could explain all phenomena, such as the global and clustered synchronization, or the synchronous flashing of stationary and flying fireflies. Moreover, biological models mainly depend on the experimental results without sufficient theoretical support, while mathematical models are mainly based on the theory of coupled oscillators, which is difficult to analyze for nonlinear cases.

In [17], Chen *et al.* investigated cluster consensus of the discrete-time Multi-agent system. A multi-agent system (MAS) is a system composed of multiple interacting autonomous agents. Consensus of a MAS means a network of interacting agents achieving a common agreement by information exchange, and cluster consensus refers that a MAS consisting of multiple subgroups achieves consensus in each subgroup. In this article, we extend the cluster consensus of the MAS in [17] to the Boolean MAS, and use it to model the synchronous flashing of fireflies. The Boolean MAS could be used to model the synchronous flashing of fireflies for the following reasons:

1) A firefly corresponds to an agent in the Boolean MAS, and the flashing behavior of each firefly could be characterized as a two-state behavior, i.e., the light on or off of firefly's flashing correspond to the state value one or zero, respectively.
2) The flashing rhythm is controlled by the interactions with neighboring fireflies, which can be represented by the arcs connecting the vertices with its neighbors in a digraph.
3) A firefly has a neural delay from the brain to the light organ, which can be explained as a discrete-time MAS, i.e., each firefly's state at time t is updated according to a rule depending on the states of its neighbors at time $t-1$.
4) Phenomena of clustered synchronization of the flashing corresponds to the cluster consensus of the MASs, while synchronous flashing occurred in stationary and flying congregations corresponds to the time-variant and time-invariant MAS.

This article is organized as follows, in Section II, several

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necessary preliminaries on the graph theory and the Boolean matrix are introduced. The fundamental problem is formulated in Section III. In Section IV, based on the graph theory and the Boolean matrix analysis, two cluster consensus criteria for the Boolean MASs with fixed and switching topologies are obtained. In Section V, numerical simulations are given to verify the effectiveness of the proposed cluster consensus criteria. Finally, some concluding remarks are made in Section VI.

II. Preliminaries

A. Preliminaries on Graph Theory and Boolean Matrix

A graph is defined as $\mathcal{G} = \{V, \mathcal{E}\}$ comprising a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and a set of (unordered or ordered) pairs of vertices $\mathcal{E} \subseteq V \times V$ representing the relationships of vertices. The graph is called undirected if the pairs of vertices are unordered, and each unordered pair of vertices in $\mathcal{E}$ is called an edge. The graph is called directed (or digraph) if the pairs of vertices are ordered, and each ordered pair of vertices in $\mathcal{E}$ is called an arc. For an edge or arc (v_i, v_j), v_i and v_j are called the endpoints. Especially, if (v_i, v_j) is an arc, v_i is called the initial vertex and v_j the terminal vertex. A loop is an arc with the same initial and terminal vertex.

In a graph, a walk is a sequence of vertices $\{v_{i_s}\}_{s=1}^{k}$ satisfying $\left(v_{i_s}, v_{i_{s+1}}\right) \in \mathcal{E}$ for $1 \leq s \leq k-1$. The length of the walk is defined as the number of the edges (or arcs) contained in the walk. The walk is closed if $v_{i_1} = v_{i_k}$. A walk with distinct vertices is called a path. A cycle is a path with $v_{i_1} = v_{i_k}$. In particular, for a digraph, we refer to the directed walk, closed directed walk, directed path and directed cycle. A digraph is called strongly connected if there exists a directed path connecting any two vertices in the digraph.

Given k graphs $\mathcal{G}_i = \{V, \mathcal{E}_i\}, i = 1, 2, \ldots, k$ with the same set of vertices, the union of these graphs is defined by $\bigcup_{i=1}^{k} \mathcal{G}_i = \{V, \bigcup_{i=1}^{k} \mathcal{E}_i\}$. A sequence of digraphs $\{\mathcal{G}_k\}_{k=0}^{\infty}$ is called jointly connected if there exists an h and a subsequence $\{k_i\}_{i=1}^{\infty}$ of natural numbers such that $1 \leq k_{i+1} - k_i \leq h$ and each $\bigcup_{k=k_i}^{k_{i+1}-1} \mathcal{G}_k$ is strongly connected.

A matrix $A = \{a_{ij}\}_{i,j=1}^{n}$ is called nonnegative if all of its entries are nonnegative, denoted by $A \geq 0$. A matrix A is called positive if all of its entries are positive, denoted by $A > 0$. A matrix A is called reducible if either (i) $n = 1$ and $A = 0$, or (ii) $n \geq 2$ and there exists a permutation matrix P such that $P^T A P$ is in block upper triangular form. A matrix A is called irreducible if it is not reducible. A nonnegative matrix A is called primitive if it is irreducible and has exactly one eigenvalue of maximum modulus. For a matrix A, its corresponding digraph $\mathcal{G}(A) = \{V, \mathcal{E}\}$ is constructed by $(v_j, v_i) \in \mathcal{E}$ if and only if $a_{ij} \neq 0$. The set of neighbors of a vertex v_i is defined by $N_{v_i} = \{v_j \in V \mid (v_j, v_i) \in \mathcal{E}\}$.

A Boolean (or logical) matrix, which can be used to represent a binary relation between a pair of finite sets, is a matrix with entries from the Boolean domain $\{0, 1\}$. For a Boolean matrix A, its corresponding digraph $\mathcal{G}(A) = \{V, \mathcal{E}\}$ is constructed by $(v_j, v_i) \in \mathcal{E}$ if and only if $a_{ij} = 1$. $\left(A^l\right)_{ij} = 1$ means there exists a directed walk of length l from the vertex v_j to the vertex v_i. The operations of the Boolean matrices inherit the logical operations. Specifically, suppose $A = \{a_{ij}\}_{i,j=1}^{n}$, $B = \{b_{ij}\}_{i,j=1}^{n}$ are two Boolean matrices, and $x = (x_1, x_2, \ldots, x_n)^T$, then $(A+B)_{ij} = a_{ij} \vee b_{ij}$, $(AB)_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$, $(Ax)_i = \bigvee_{k=1}^{n} (a_{ik} \wedge x_k)$, where the operations of $\vee$ and $\wedge$ are shown in Table. I.

TABLE I
Logical operations

$\vee$	1	0
1	1	1
0	1	0

$\wedge$	1	0
1	1	0
0	0	0

Furthermore, denote $\mathcal{G}(A) = \{V, \mathcal{E}_A\}$, $\mathcal{G}(B) = \{V, \mathcal{E}_B\}$ and $\mathcal{G}(AB) = \{V, \mathcal{E}_{AB}\}$ as the corresponding digraphs of matrices A, B and AB, then $(AB)_{ij} = 1$ if and only if there exists a k such that $a_{ik} = b_{kj} = 1$, which can also be illustrated in the view of graph as follows, $(v_j, v_i) \in \mathcal{E}_{AB}$ if and only if there exists $v_k \in V$ satisfying $(v_j, v_k) \in \mathcal{E}_B$ and $(v_k, v_i) \in \mathcal{E}_A$.

B. Propositions and Canonical Form of Matrix

Before the fundamental problem is presented, some necessary propositions are following.

Proposition 2.1: ([23], pp. 516) A matrix A is primitive if and only if there exists an integer $m > 0$ such that A^m is positive.

Particularly, a primitive Boolean matrix refers to that A^m is a matrix with all of its entries being ones for some integer m.

Proposition 2.2: ([24], pp. 55) A matrix A is irreducible if and only if its corresponding digraph $\mathcal{G}(A) = \{V, \mathcal{E}\}$ is strongly connected.

Given a strongly connected digraph $\mathcal{G} = \{V, \mathcal{E}\}$, let $\{w_i^k\}_{k=1}^{\infty}$ denote all the closed directed walks containing the vertex v_i, and $\{l_i^k\}_{k=1}^{\infty}$ denote the corresponding lengths of them. The period of v_i is defined by $d_i = \gcd\{l_i^k\}_{k=1}^{\infty}$, where "gcd" means the greatest common divisor. The index of imprimitivity of a strongly connected digraph $\mathcal{G}$ is defined as $d = \gcd\{l^k\}_{k=1}^{\infty}$, where $\{l^k\}_{k=1}^{\infty}$ denotes the lengths of all the closed directed walks $\{w^k\}_{k=1}^{\infty}$ of $\mathcal{G}$.

Proposition 2.3: For a strongly connected digraph $\mathcal{G} = \{V, \mathcal{E}\}$, d_i denotes the period of vertex v_i for $1 \leq i \leq n$, and d denotes the index of imprimitivity of $\mathcal{G}$, then

1) $d_1 = d_2 = \cdots = d_n = d$.
2) For any two vertices $v_i, v_j \in V$, let l and l' be the lengths of any two directed walks from the vertex v_i to the vertex v_j, then $l \equiv l' \pmod{d}$.
3) The set of vertices V can be partitioned into d parts $\{V_r\}_{r=1}^{d}$ satisfying $V = \bigcup_{r=1}^{d} V_r$ and $V_i \cap V_j = \emptyset$ for any $i \neq j$. Furthermore, l_{ij} denotes any directed walk starting from a vertex in V_i and ending at a vertex in V_j, then $l_{ij} \equiv j - i \pmod{d}$.

Proof: Since 1) and 2) are given in [25], pp. 102, we only prove 3). For any given vertex $v_{k_0} \in V$, and any other vertex

$v_i \in V$, let l_i be the length of a directed walk from v_{k_0} to v_i, for $r = 1, 2, \ldots d$, define $V_r = \{v_i \in V \mid l_i \equiv r-1 \pmod{d}\}$, whose validation can be guaranteed by 2), then $V = \bigcup_{r=1}^{d} V_r$ and $V_i \cap V_j = \emptyset$ for any $i \neq j$ as a result of 2).

Take $v_{i_0} \in V_i$ and $v_{j_0} \in V_j$, let l_{ij} be the length of a directed walk from v_{i_0} to v_{j_0}, and l_{ki} be the length of a directed walk from v_{k_0} to v_{i_0}. Thus, there exists a directed walk from v_{k_0} to v_{j_0} with the length $l_{ki} + l_{ij}$. By the construction of V_r, $l_{ki} \equiv i-1 \pmod{d}$, $l_{ki} + l_{ij} \equiv j-1 \pmod{d}$. Therefore, $l_{ij} \equiv j - i \pmod{d}$. ■

Remark 2.1: A definition is put forward as a result of 1): the period of a strongly connected digraph $\mathcal{G} = \{V, \mathcal{E}\}$ is defined as the period of any vertex. What's more, a cluster factorization algorithm is given by 3). It should be noticed that the construction of d parts $\{V_r\}_{r=1}^{d}$ in 3) is independent of the selection of the vertex v_{k_0}. In fact, for any vertex $v'_{k_0} \in V$ ($v'_{k_0} \neq v_{k_0}$), assume $v'_{k_0} \in V_k$ with $k \in \{1, 2, \ldots, d\}$, suppose $v_{j_1}, v_{j_2} \in V_r$ with $r \in \{1, 2, \ldots, d\}$, and let l_1, l_2 be the lengths of two directed walks from v'_{k_0} to v_{j_1}, v_{j_2}, respectively. Then by 3), $l_1 \equiv r - k \pmod{d}$, $l_2 \equiv r - k \pmod{d}$. Thus, $l_1 \equiv l_2 \pmod{d}$, which implies that the construction of $\{V_r\}_{r=1}^{d}$ is independent of the selection of the vertex v_{k_0}.

Let A be an irreducible matrix, i.e., the corresponding digraph $\mathcal{G}(A) = \{V, \mathcal{E}\}$ is strongly connected, and d be the period of $\mathcal{G}(A)$. Partition the set of vertices V into d pairwise disjoint sets $\{V_r\}_{r=1}^{d}$ according to the cluster factorization algorithm in Proposition 2.3). Rearrange the indices of matrix A by the order of $V_1, V_2, \ldots, V_d$, obtaining a matrix $\tilde{A}$. More detailed procedure is illustrated in [17], which is omitted here. The procedure introduced above can also be illustrated as follows. Suppose P is a permutation matrix, the matrix $\tilde{A} = P^T A P$ can be obtained with the following algebraic form,

$$\tilde{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_d \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{d-1} & 0 \end{pmatrix}, \tag{1}$$

where A_i is a $|V_{i+1}| \times |V_i|$ matrix for $1 \leq i \leq d-1$, and A_d is a $|V_1| \times |V_d|$ matrix. Here, $\tilde{A}$ is called the canonical form of A.

III. Problem Formulation

Consider a MAS consisting of n autonomous agents where $x_i(t)$ denotes the state of agent i at time t for $i = 1, 2, \ldots, n$ and $t \geq 1$, the dynamics of agent i is described as follows,

$$x_i(t+1) = \bigvee_{j=1}^{n} \left(a_{ij}(t) \wedge x_j(t)\right), \tag{2}$$

where $a_{ij}(t), x_i(t) \in \{0, 1\}$ with $i, j = 1, 2, \ldots, n$. It is assumed that all the operations of the Boolean matrices and vectors are the logical operations in this article.

A MAS with the form (2) is called a Boolean MAS if all the values in the MAS are either 0 or 1 and all the operations (except $t+1$) of the MAS are logical. The Boolean MAS (2) can be rewritten in the matrix form,

$$x(t+1) = A(t)x(t), \tag{3}$$

with $A(t) = \{a_{ij}(t)\}_{i,j=1}^{n}$ is a Boolean matrix, and $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$.

The graph theory is one of the most common and effective methods to study the MASs. A Boolean MAS (3) can be represented by $A(t)$'s corresponding digraph $\mathcal{G}(A) = \{V, \mathcal{E}\}$ with $V = \{v_1, v_2, \ldots, v_n\}$, where v_i represents agent i, and the arc (v_i, v_j) represents the information exchange from agent i to agent j.

Definition 3.1: For the Boolean MAS (3), suppose $\mathcal{G}(A) = \{V, \mathcal{E}\}$ with $V = \{v_1, v_2, \ldots, v_n\}$, if there exists a t_0 and k pairwise disjoint sets $\{V_r\}_{r=1}^{k}$ with $\bigcup_{r=1}^{k} V_r = V$ such that

$$x_i(t) = x_j(t),\ \forall v_i, v_j \in V_r,\ t \geq t_0,$$

then the Boolean MAS (2) achieves cluster consensus. Particularly, if $k = 1$, then

$$x_i(t) = x_j(t),\ \forall v_i, v_j \in V,\ t \geq t_0,$$

the Boolean MAS (2) achieves consensus.

Necessarily, some assumptions of $A(t)$ are proposed here.

Assumption 3.1: $\mathcal{G}(A(t))$ is fixed and strongly connected.

Assumption 3.2: Each $\mathcal{G}(A(t))$ is strongly connected and has a period of d, and $|V| = md$. There exists a permutation matrix P such that $\tilde{A}(t) = P^T A(t) P$ with the following form,

$$\tilde{A}(t) = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_d(t) \\ A_1(t) & 0 & \cdots & 0 & 0 \\ 0 & A_2(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{d-1}(t) & 0 \end{pmatrix}, \tag{4}$$

where each $A_i(t)$ is an $m \times m$ Boolean matrix with positive diagonal for $1 \leq i \leq d$ and $t \geq 1$.

Assumption 3.3: The sequence $\{\mathcal{G}\left(A_{k_i(t)}(t)\right)\}_{t=1}^{\infty}$ is jointly connected for $1 \leq i \leq d$, where $1 \leq k_i(t) \leq d$ and $k_i(t) \equiv t + i - 1 \pmod{d}$.

IV. Main Results

A. Boolean MAS with Fixed Topology

Lemma 4.1: ([25], pp. 102) The Boolean matrix A is primitive if and only if the corresponding digraph $\mathcal{G}(A)$ is strongly connected and the index of imprimitivity of $\mathcal{G}(A)$ is $d = 1$.

Lemma 4.2: Suppose the Boolean matrix A is irreducible, the index of imprimitivity of its corresponding digraph $\mathcal{G}(A)$ satisfies $d > 1$, then,

1) ([17]) A is similar to some Boolean matrix $\tilde{A}$ with the algebraic form (1), where each A_i has no zero row nor zero column.
2) ([25], pp. 103) $\tilde{A}^d = \mathrm{diag}\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_d\}$, where $\tilde{A}_i = A_{i_1} A_{i_2} \cdots A_{i_d}$ is a $|V_i| \times |V_i|$ primitive matrix, and $i_r \equiv i + d - r \pmod{d}$ with $i_r, i, r \in \{1, 2, \ldots, d\}$.

Theorem 4.1: If Assumption 3.1 holds for the Boolean MAS (2), then the Boolean MAS can reach cluster consensus with the clusters being partitioned as in Proposition 2.3. Moreover, the number of the clusters is equal to the period of $\mathcal{G}(A)$.

Proof: Since $\mathcal{G}(A(t))$ is fixed and strongly connected by Assumption 3.1, $A(t) = A$ and $\mathcal{G}(A)$ is irreducible. Thus, the Boolean matrix A is irreducible according to Proposition 2.2. Let $\tilde{A} = P^T AP$ be the canonical form (1) of A. By Lemma 4.2, $\tilde{A}^d = \text{diag}\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_d\}$ where $\tilde{A}_i$ is primitive for $1 \le i \le d$. Hence, there exists a k_0 such that,

$$\tilde{A}^{kd} = \begin{pmatrix} \mathbf{1}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_d \end{pmatrix}, \ \forall k \ge k_0,$$

where $\mathbf{1}_i$ is the $|V_i| \times |V_i|$ matrix with all entries being ones, $1 \le i \le d$.

Denote $P^T x(1) = \left(\tilde{x}_1^T, \tilde{x}_2^T, \ldots, \tilde{x}_d^T\right)^T$, where $\tilde{x}_i$ is a $|V_i| \times 1$ vector with $1 \le i \le d$. For any $1 \le i \le d$ and $k \ge k_0$,

$$\begin{aligned} & A^{kd+i} x(1) \\ = \ & P\tilde{A}^{kd+i} P^T x(1) \\ = \ & P\tilde{A}^{kd+i} \left(\tilde{x}_1^T, \tilde{x}_2^T, \ldots, \tilde{x}_d^T\right)^T \\ = \ & P\left(\gamma_{d-i+1}\mathbf{1}_{V_1}^T, \gamma_{d-i+2}\mathbf{1}_{V_1}^T, \ldots, \gamma_{d-i+d}\mathbf{1}_{V_1}^T\right)^T. \end{aligned}$$

Here $\mathbf{1}_{V_i}$ is the $|V_i| \times 1$ vector with all entries being ones, and $\gamma_i = \mathbf{1}_{V_i}^T \tilde{x}_i$ with $i = 1, 2, \ldots, d$. It should be emphasized that the subscript i of γ_i is in the sense of modular d. Thus the proof is completed. ∎

Remark 4.1: The proof above shows that $\gamma_d \to \gamma_{d-1} \to \cdots \to \gamma_1 \to \gamma_d$ can be viewed as the consensus trajectory of each agent. It corresponds to the static consensus value in the traditional consensus problem. Thus, for any $1 \le i \le n$, there exists some $\tau_i \in \{1, 2, \ldots, d\}$ and k_0 such that

$$x_i(kd + \tau_i - s) = \gamma_s, \ \forall 1 \le s \le d, \ \forall k \ge k_0.$$

B. Boolean MAS with Switching Topology

Theorem 4.1 deals with the cluster consensus of the Boolean MAS with fixed topology. For the Boolean MAS with switching topology, cluster consensus can also be achieved under appropriate assumptions. Similarly, we propose some lemmas before the final Theorem.

Lemma 4.3: Suppose $A = \{a_{ij}\}_{i,j=1}^n$ and $B = \{b_{ij}\}_{i,j=1}^n$ are two Boolean matrices with positive diagonal, then

1) $AB \ge A + B \ge A$.
2) If A and B are irreducible, then $AB = A$ holds if and only if $A > 0$.
3) Denote $f(A)$ as the number of ones in A, then $f(AB) \ge f(A)$. Furthermore, if A and B are irreducible, then $f(AB) = f(A)$ holds if and only if $A > 0$.

Proof: Since $a_{ii} = b_{jj} = 1$ for all $1 \le i, j \le n$, either $a_{ij} = 1$ or $b_{ij} = 1$ can guarantee $(AB)_{ij} = 1$, which implies $AB \ge A + B \ge A$. Particularly, $AB = A$ if $A > 0$.

Next, we will show that $AB = A$ implies $A > 0$. Suppose the corresponding digraphs of the Boolean matrices A, B and AB are $\mathcal{G}(A) = \{V, \mathcal{E}_A\}$, $\mathcal{G}(B) = \{V, \mathcal{E}_B\}$, and $\mathcal{G}(AB) = \{V, \mathcal{E}_{AB}\}$. $AB = A$ and $AB \ge A + B$ imply that $A \ge B$, which is equivalent to $\mathcal{E}_A \supseteq \mathcal{E}_B$. Claim that $A > 0$. Otherwise, there exists an a_{ij} such that $a_{ij} = 0$, i.e., $(v_j, v_i) \notin \mathcal{E}_A$. Notice that B is irreducible, i.e., $\mathcal{G}(B)$ is strongly connected, thus there exists a directed path $v_j v_{k_1} v_{k_2} \cdots v_{k_l} v_i$ in $\mathcal{G}(B)$ from v_j to v_i. Thereby, $\left(v_{k_{l-1}}, v_{k_l}\right) \in \mathcal{E}_B$ and $(v_{k_l}, v_i) \in \mathcal{E}_B \subseteq \mathcal{E}_A$, which further implies that $\left(v_{k_{l-1}}, v_i\right) \in \mathcal{E}_{AB} = \mathcal{E}_A$. Similarly, $(v_{k_s}, v_i) \in \mathcal{E}_A$ holds for $s = 1, \ldots, l-2$. Therefore, $(v_j, v_{k_1}) \in \mathcal{E}_B$ and $(v_{k_1}, v_i) \in \mathcal{E}_A$ implies $(v_j, v_i) \in \mathcal{E}_{AB} = \mathcal{E}_A$, i.e., $a_{ij} = 1$, which is a contradiction.

3) can be easily derived from 1) and 2). ∎

Remark 4.2: 1) in Lemma 4.3 can be illustrated as follows. Suppose $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are two digraphs with loop on each vertex, then $\mathcal{G}(AB) \supseteq (\mathcal{G}(A) \cup \mathcal{G}(B))$.

Lemma 4.4: Suppose D_i is an $m \times m$ irreducible Boolean matrix with positive diagonal for $1 \le i \le r$ and $r \ge m^2$, then $D_r D_{r-1} \cdots D_1 > 0$.

Proof: According to 1) in Lemma 4.3, it is obvious that the product of two irreducible Boolean matrices are still irreducible. By 1) and 3) in Lemma 4.3, for any $1 \le k \le r-1$,

$$\begin{aligned} D_r D_{r-1} \cdots D_k &\ge D_r D_{r-1} \cdots D_{k+1}, \\ f(D_r D_{r-1} \cdots D_k) &\ge f(D_r D_{r-1} \cdots D_{k+1}). \end{aligned}$$

We claim that there exists a k_0 $(1 \le k_0 \le r-1)$ satisfying

$$f(D_r D_{r-1} \cdots D_{k_0}) = f(D_r D_{r-1} \cdots D_{k_0+1}). \tag{5}$$

Otherwise, $f(D_r D_{r-1} \cdots D_k) \ge f(D_r D_{r-1} \cdots D_{k+1}) + 1$ for any $1 \le k \le r-1$, thus,

$$\begin{aligned} & f(D_r D_{r-1} \cdots D_1) \ge f(D_r D_{r-1} \cdots D_2) + 1 \ge \cdots \\ & \ge f(D_r) + r - 1 \ge m + 1 + r - 1 > r \ge m^2, \end{aligned}$$

which contradicts to the inequality $f(D_r D_{r-1} \cdots D_1) \le m^2$.

By (5) and 3) in Lemma 4.3, $D_r D_{r-1} \cdots D_{k_0+1} > 0$. Thus, $D_r D_{r-1} \cdots D_1 \ge D_r D_{r-1} \cdots D_{k_0+1} > 0$. ∎

Theorem 4.2: If Assumption 3.2-3.3 hold for the Boolean MAS (2), then the Boolean MAS can achieve cluster consensus.

Proof: By (4),

$$\begin{aligned} \prod_{t=1}^{kd} \tilde{A}(t) &= \tilde{A}(kd)\tilde{A}(kd-1)\cdots\tilde{A}(1) \\ &= \text{diag}\{\tilde{A}_1(k), \tilde{A}_2(k), \ldots, \tilde{A}_d(k)\}, \end{aligned}$$

where $\tilde{A}_i(k) = A_{k_i(kd)}(kd) A_{k_i(kd-1)}(kd-1) \cdots A_{k_i(1)}(1)$.

Next, we will show that there exists a k_0 such that $\tilde{A}_i(k) > 0$ for any $1 \le i \le d$, $k \ge k_0$. According to Assumption 3.3, $\{\mathcal{G}\left(A_{k_i(t)}(t)\right)\}_{t=1}^{\infty}$ is jointly connected, i.e., there exists an h and a subsequence $\{t_s^i\}_{s=1}^{\infty}$ such that $1 \le t_{s+1}^i - t_s^i \le h$ and each $\bigcup_{t=t_s^i}^{t_{s+1}^i - 1} \mathcal{G}\left(A_{k_i(t)}(t)\right)$ is strongly connected.

By Remark 4.2,

$$\mathcal{G}\left(\prod_{t=t_s^i}^{t_{s+1}^i-1} A_{k_i(t)}(t)\right) \supseteq \bigcup_{t=t_s^i}^{t_{s+1}^i-1} \mathcal{G}\left(A_{k_i(t)}(t)\right),\ \forall s \geq 1.$$

Therefore, each $\mathcal{G}\left(\prod_{t=t_s^i}^{t_{s+1}^i-1} A_{k_i(t)}(t)\right)$ is strongly connected. And since $A_i(t)$ is positive diagonal as in Assumption 3.2, $\prod_{t=t_s^i}^{t_{s+1}^i-1} A_{k_i(t)}(t)$ is irreducible with positive diagonal. Denote $D_{i,s} = \prod_{t=t_s^i}^{t_{s+1}^i-1} A_{k_i(t)}(t)$. Without loss of generality, let $t_1^i = 1$. For any r with $r \geq m^2$, there exists a c_i such that $c_i d \geq t_{r+1}^i - 1$. Thus,

$$\tilde{A}_i(k) \geq \tilde{A}_i(c_i) \geq D_{i,r} D_{i,r-1} \cdots D_{i,1},\ \forall k \geq c_i.$$

Since $D_{i,j}$ is an $m \times m$ irreducible Boolean matrix with positive diagonal for $1 \leq j \leq r$, and $r \geq m^2$, by Lemma 4.4,

$$\tilde{A}_i(k) \geq D_{i,r} D_{i,r-1} \cdots D_{i,1} > 0,\ \forall k \geq c_i.$$

Denote $c_0 = \max\{c_i \mid 1 \leq i \leq d\}$, therefore,

$$\tilde{A}_i(k) = \hat{\mathbf{1}}_m > 0,\ \forall k \geq c_0,\ 1 \leq i \leq d,$$

where $\hat{\mathbf{1}}_m$ is the $m \times m$ matrix with all entries being ones. The rest of the proof is similar to that of Theorem 4.1 and thus omitted here. ■

V. Numerical Simulations

Consider a Boolean MAS with fixed topology. The corresponding digraph of A is shown in Fig. 1. According to the cluster factorization algorithm in 3) of Proposition 2.3, the vertices of the digraph in Fig. 1 are classified into two clusters, which can be shown by the two colors in Fig. 1, corresponding to two clusters of agents in the Boolean MAS. Fig. 2 shows the cluster consensus of the Boolean MAS with fixed topology. Here black and white represent $x_i = 0$ and $x_i = 1$, respectively. The agents in the left group from left to right and top to bottom are $v_1, v_3, v_5, v_7, v_9, v_{11}, v_{12}, v_{14}, v_{18}, v_{19}$, and the right group, $v_2, v_4, v_6, v_8, v_{10}, v_{13}, v_{15}, v_{16}, v_{17}, v_{20}$. What's more, it can also be used to illustrate the flashing of fireflies, with black and white representing light off and on, respectively.

Moreover, for the case of the Boolean MAS with switching topology, the Boolean matrix $A(t)$ is given by,

$$A(t) = \begin{cases} A_{\sigma(t)}, & t = 1, 3, 5, \cdots \\ A_{\sigma(t-1)}, & t = 2, 4, 6, \cdots. \end{cases}$$

Here, $\sigma(t) \in \{1, 2\}$ is a random variable with each value taking a probability of 0.5. In this article, the values of $\sigma(t)$ are generated by Matlab: $\sigma(1) = 2, \sigma(3) = 2, \sigma(5) = 1, \sigma(7) = 2, \sigma(9) = 2, \sigma(11) = 1, \sigma(13) = 1$. The corresponding digraphs of A_1 and A_2 are shown in Fig. 3. Similarly, the two colors in Fig. 3 represent the same two clusters of the two digraphs. Fig. 4 shows the cluster consensus of the Boolean MAS with switching topology, where the agents in the left group from left to right and top to bottom are $v_1, v_2, v_4, v_5, v_9, v_{12}, v_{15}, v_{16}, v_{19}, v_{20}$, and the right group, $v_3, v_6, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}, v_{17}, v_{18}$.

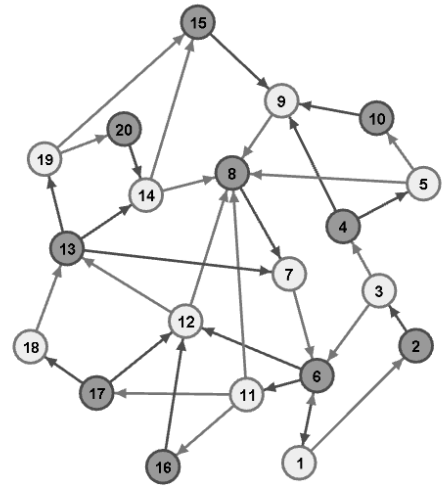

Fig. 1. Corresponding digraph of A for the case of fixed topology

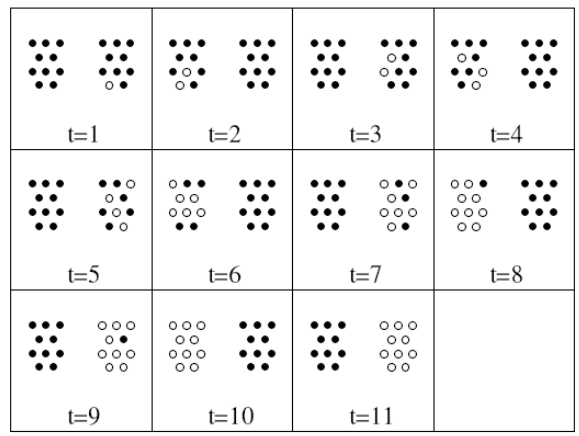

Fig. 2. Cluster consensus with fixed topology

VI. Conclusion

This article investigates the cluster consensus of the Boolean MAS, and uses it to model the synchronous flashing of fireflies. By the graph theory and the Boolean matrix analysis, two cluster consensus criteria are proposed for the Boolean MASs with fixed and switching topologies. Moreover, numerical simulations are also given to verify the effectiveness of the criteria. The cluster consensus of the Boolean MAS in this article is a periodical dynamic consensus, rather than a static consensus, which means once the Boolean MAS achieves cluster consensus, the states of the agents within a subgroup will be identical and switch periodically. So far, the periodical phenomenon of synchronous flashing fireflies has rarely been reported in the related observational researches. Thus, the periodical phenomenon showed in our model has some biological significance, which might be verified by further observation. Further, the results presented in this article may potentially lead to more practical applications of the Boolean MASs.

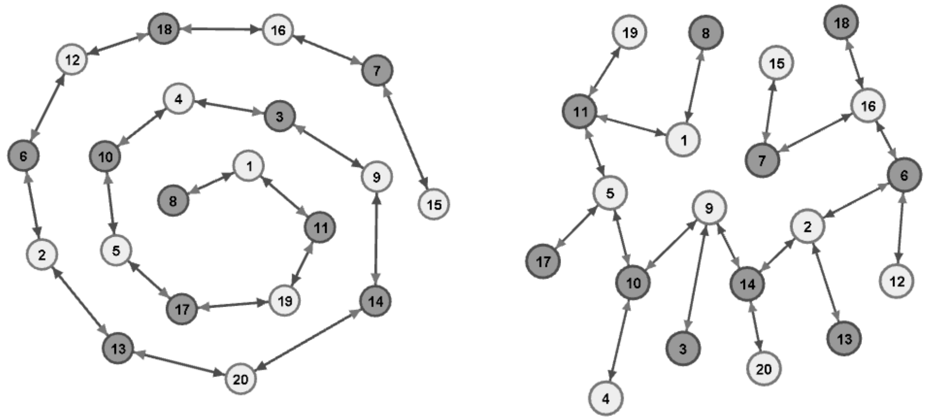

Fig. 3. Corresponding digraphs of A_1 and A_2 for the case of switching topology

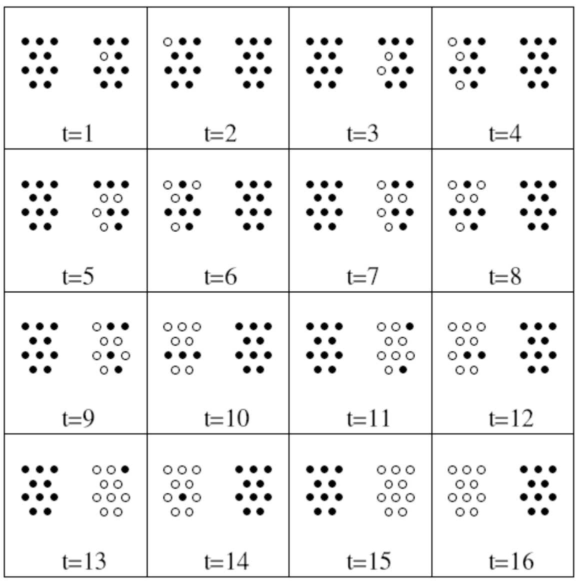

Fig. 4. Cluster consensus with switching topology

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