High-order Consensus in High-order Multi-agent Systems

Yu-Ping Tian and Ya Zhang
School of Automation, Southeast University, Nanjing, 210096, China
Email: yptian@seu.edu.cn

***Abstract*—This paper studies the high-order consensus problem for high-order multi-agent systems with heterogeneous dynamics and diverse communication delays. A necessary and sufficient condition is given for the existence of high-order consensus solution to heterogeneous multi-agent systems. The obtained condition shows that for systems with diverse communication delays, high-order consensus does not require each self-delay of agent is equal to the corresponding communication delay. A matching condition for self-delays and communication delays is derived.**

I. Introduction

Consensus problems have been extensively studied for first- or second-order integrator multi-agent systems (MASs) [4], [6], [7]. Recently, increasing interest has been observed in the study of MASs with general linear time-invariant (LTI) dynamical agents. This is because in many practical applications agents cannot be simplified as mass points and more complex dynamics should be considered. In most existing references on this problem, however, the agents are assumed to be identical [11], [8]. In this case the existence of consensus solutions are automatically guaranteed. For a system consisting of heterogeneous dynamical agents, a question to be answered is whether there exists a consensus solution to such a system. This paper will show that the existence of high-order consensus solution to heterogeneous MASs is related not only to the connectivity of interconnection graph of MASs but also to the dynamics of agents and communication channels. In particular, this paper studies the delay effect on the existence of consensus solutions, which is closely related to the design of consensus protocols. For systems with communication delays, each agent gets delayed state information from its neighbors and thus only delayed information can be used in consensus protocols. Note that when a constant-consensus is reached, states of all the agents in the system share the same value at any time. So, in many consensus protocols the delayed states of neighbors obtained by each agent are compared with its own current state instead of delayed state. This kind of consensus protocol has been studied for fixed or even switched graphs by using different analysis methods [12], [1], [9]. For second-order or high-order consensus problem, consensus state is no longer a constant value. To cope with this problem, currently existing consensus protocols introduce self-delays for agents, which are assumed to be exactly equal to the corresponding communication delays [3]. In practice, however, communication delays can be only estimated approximately. Therefore, a high-order consensus protocol which does not need exact delay values is of great importance for practical application of the consensus theory.

In this paper we investigate the high-order consensus problem for heterogeneous multi-agent systems with unknown communication delays. Differing from the existing references which consider identical agents, this paper gives up the state-space model and Kronecker product technique but uses the input-output model. Using frequency-domain analysis method we show that high-order consensus may exist in systems with heterogeneous agents. A necessary and sufficient condition is given for the existence of high-order consensus solution to heterogeneous multi-agent systems. By this condition it is shown that for systems with diverse communication delays, high-order consensus does not require each self-delay of agent is equal to the corresponding communication delay. A matching condition for self-delays and communication delays is derived.

Notions used in this paper are quite standard. A weighted digraph $G = (V, E, A)$ is a triplet of a set of vertices $V = \{v_1, ..., v_n\}$, a set of edges $E \subseteq V \times V$ and a weighted adjacency matrix $A = [a_{ij}] \in R^{n\times n}$. We assume that the adjacency elements associated with the edges of the digraph are positive, i.e., $a_{ij} > 0 \Leftrightarrow e_{ij} \in E$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in V : (v_i, v_j) \in E\}$. The Laplacian matrix of $G(V, E, A)$ is defined as $L = D - A$, where $D = \mathrm{diag}\{\sum_{j=1}^{n} a_{ij}, \quad i \in \overline{1, n}\}$ is the degree matrix of G. For any two integers n_1 and n_2 such that $n_1 \leq n_2$, denote by $\overline{n_1, n_2}$ the set of all the integers in the interval $[n_1, n_2]$. Denote by $\mathbf{1}_n$ the vector $[1, \cdots, 1]^T$ of n dimension, and $\mathrm{span}(\mathbf{1}_n) = \{\alpha \in R^{n\times 1} : \alpha = a\mathbf{1}_n, a \in R\}$. For a matrix $M \in C^{n\times n}$, $\rho(M)$ stands for the spectral radius of M.

II. High-order consensus

Suppose the interconnection topology of the system is described by a digraph $G = (V, E, A)$ with $|V| = n$. We assume that the interconnection topology of the system is a connected undirected graph or a digraph containing a globally reachable node. Then, the Laplacian matrix L has a simple eigenvalue 0, i.e., $\det(L) = 0$ and $\mathrm{rank}(L) = n-1$. Moreover, the definition of L implies that $L \cdot \mathbf{1}_n = 0$.

Let the model of the ith agent ($i \in \overline{1, n}$) be given by the following transfer function

$$G_i(s) = \frac{Y_i(s)}{U_i(s)}$$

978-1-4673-5769-2/13/$31.00 ©2013 IEEE

$$= \frac{\mathrm{e}^{-T_i s}}{s^{\nu}(c_{im_i}s^{m_i-\nu} + c_{im_i-1}s^{m_i-\nu-1} + \cdots + c_{i\nu})}, \quad (1)$$

where $Y_i(s) \in C$ and $U_i(s) \in C$ denote the Laplace transformation of the output and input, respectively, of the ith agent;T_i is the input delay; ν and m_i are positive integers satisfying $m_i \geq \nu$; $c_{ik} \in R$ are system parameters. For any non-negative integer k, denotes by $y_i^{(k)}(t)$ the kth order derivative of the output $y_i(t)$ of the ith agent.

Definition 1 Multi-agent system (1) is said to reach *the rth order asymptotic consensus* if

$$\lim_{t\to\infty} |y_i(t) - y_j(t)| = 0, \ \forall i, j \in \overline{1,n}, \quad (2)$$

$$\lim_{t\to\infty} |y_i^{(k)}(t)| = 0, \ \forall i \in \overline{1,n}, \ \forall k > r \quad (3)$$

for system solutions from any admissible initial conditions, and

$$\lim_{t\to\infty} |y_i^{(k)}(t)| \neq 0, \ \forall i \in \overline{1,n}, \ \forall k \in \overline{0,r} \quad (4)$$

for system solutions from some admissible initial conditions.

Note that when $r = 0$, the above definition implies that

$$\lim_{t\to\infty} y_i(t) = c, \ \forall i \in \overline{1,n},$$

where $c \in R$ is a constant. In this case we say the system achieves a *constant consensus*. So, constant consensus can be regarded as *the zeroth order consensus* by Definition 1. We also note by this definition the constant c can not identically be zero for all initial conditions, i.e., it excludes the case of *trivial consensus* which actually implies each agent is asymptotically stabilized.

Let τ_{ij} be the communication delay from agent j to agent i. Then, at time instance t the information got by agent i from agent j is $y_j^{(k)}(t - \tau_{ij})$ instead of $y_j^{(k)}(t)$. Let the consensus protocol be

$$u_i(t) = \kappa_i \sum_{k=0}^{r} b_{ik} \left(\sum_{j\in N_i} a_{ij}(y_j^{(k)}(t-\tau_{ij}) - y_i^{(k)}(t-\tau'_{ij})) \right) - \kappa_i \sum_{k=r+1}^{m_i} b_{ik} y_i^{(k)}(t) \quad (5)$$

where $\kappa_i > 0$, $b_{ik} \in R$ are some control parameters, τ'_{ij} denotes the estimation of communication delay τ_{ij} used by agent i. Sometimes $\tau'_{ij}, j \in N_i$ are also referred to as self-delays of agent i. We will show that the high-order consensus can be achieved even though the self-delays are not equal to the communication delays.

III. CONSENSUS CONDITION

It is easy to get the closed-loop form of system (1) with protocol (5) as

$$\sum_{k=\nu}^{m_i} c_{ik} y_i^{(k)}(t) =$$

$$\kappa_i \sum_{k=0}^{r} b_{ik} \sum_{j\in N_i} a_{ij}(y_j^{(k)}(t - T_i - \tau_{ij}) - y_i^{(k)}(t - T_i - \tau'_{ij}))$$

$$-\kappa_i \sum_{k=r+1}^{m_i} b_{ik} y_i^{(k)}(t), \quad i \in \overline{1,n}. \quad (6)$$

Taking the Laplace transform under zero initial condition for the above equation yields

$$\sum_{k=\nu}^{m_i} c_{ik} s^k Y_i(s) =$$

$$\kappa_i \mathrm{e}^{-T_i s} \sum_{k=0}^{r} b_{ik} s^k \sum_{j\in N_i} a_{ij}(Y_j(s)\mathrm{e}^{-\tau_{ij}s} - Y_i(s)\mathrm{e}^{-\tau'_{ij}s})$$

$$-\kappa_i \sum_{k=r+1}^{m_i} b_{ik} s^k Y_i(s), \quad i \in \overline{1,n}. \quad (7)$$

Let

$$Y(s) = [Y_1(s), \cdots, Y_n(s)]^{\mathrm{T}}, \quad (8)$$

$$c_i(s) = \sum_{k=\nu}^{m_i} c_{ik} s^{k-r-1} + \sum_{k=r+1}^{m_i} \kappa_i b_{ik} s^{k-r-1}, \quad (9)$$

$$b_i(s) = \sum_{k=0}^{r} b_{ik} s^k, \quad (10)$$

$$l_{ij}(s) = \begin{cases} -a_{ij}\mathrm{e}^{-\tau_{ij}s}, & i \neq j \\ \sum_{j=1}^{n} a_{ij}\mathrm{e}^{-\tau'_{ij}s}, & i = j \end{cases} \quad (11)$$

$$(12)$$

and

$$H(s) = \mathrm{diag}\left\{ \frac{b_i(s)}{c_i(s)}\mathrm{e}^{-T_i s}, \ i \in \overline{1,n} \right\}, \quad (13)$$

$$K = \mathrm{diag}\{\kappa_i, \ i \in \overline{1,n}\}, \quad (14)$$

$$L(s) = \{l_{ij}(s)\}. \quad (15)$$

Then, the closed-loop system can be shown by Fig. 1. The frequency-domain model of the closed-loop system is given by

$$Y(s) = -\frac{1}{s^{r+1}} H(s)KL(s)Y(s), \quad (16)$$

and the return difference equation of the system is given by

$$I + \frac{1}{s^{r+1}} H(s)KL(s) = 0.$$

Direct use of the equation

$$\det\left(I + \frac{1}{s^{r+1}} H(s)KL(s) \right) = 0 \quad (17)$$

may ignore cancelation between the poles of the closed-loop system and the poles of the open-loop system at $s = 0$. To avoid these cancelations we consider the following equation

$$\det(s^{r+1}I + H(s)KL(s)) = 0 \quad (18)$$

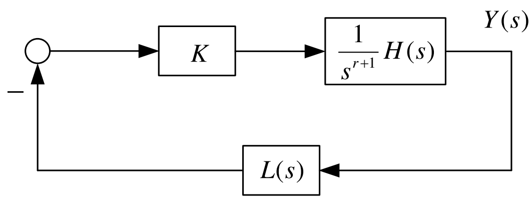

Fig. 1. Interconnected system.

as the characteristic equation of the system. Obviously,

$$\det(I+\frac{1}{s^{r+1}}H(s)KL(s)) = \det(\frac{1}{s^{r+1}}I)\det(s^{r+1}I+H(s)KL(s)). \qquad (19)$$

Therefore, all the none-zero solutions of (18) are contained in the solutions of (19), and vice versa.

The following proposition gives an obvious condition for justifying if the open-loop poles at $s=0$ enter into the set of the closed-loop poles.

Proposition 1 Suppose $H(s)KL(s)$ is analytic at $s=0$. If $\mathrm{rank}[H(s)KL(s)] = n-1$, for all $s \in C$ at which $H(s)KL(s)$ is analytic, then

$$\det(s^{r+1}I+H(s)KL(s)) = s^{r+1}g(s),$$

where $g(s)$ has neither poles nor zeros at $s=0$.

Proof. The proposition can be easily proved by converting $H(s)KL(s)$ into a diagonal matrix through a similarity transformation. $\diamondsuit$

Proposition 1 shows that $s=0$ is a repeated pole with multiplicity $r+1$ of the closed-loop system. This also implies that the solution of the closed-loop system can be expressed as

$$y(t) = \sum_{i=0}^{r} C_i t^i + \sum_{k=1}^{M} P_k(t)\mathrm{e}^{\lambda_k t}, \qquad (20)$$

where $C_i \in R^n$ are constant vectors, $P_i(t) \in R^n$ are vectors whose elements are polynomials of t, λ_k are zeros of $g(s)$, and M is finite or infinite positive integer. Furthermore, if $\mathrm{Re}\lambda_i < 0$, we have $y(t) \to \sum_{i=0}^{r} C_i t^i$ as $t \to \infty$. We call $y(t) = \sum_{i=0}^{r} C_i t^i$ as the steady state of the system.

Lemma 1 Assume that $\det(s^{r+1}I+H(s)KL(s)) = s^{r+1}g(s)$, where $g(s)$ has neither poles nor zeros $s=0$. Then, for any $s \neq 0$, $g(s)=0$ if and only if

$$\det(I+\frac{1}{s^{r+1}}H(s)KL(s)) = 0. \qquad (21)$$

Proof. The lemma is obvious due to the equation (19). $\diamondsuit$

By definition of semi-stability [10], a linear time-invariant system is said to be steady semi-stable if all its characteristic roots are inside the LHP or at the origin of the complex plane. By Lemma 1, to verify the steady semi-stability of the system, we just need to check if all the zeros of (21) have negative real parts. Note that the generalized Nyquist stability criterion can be used for this purpose [2], [10].

Denote $\hat{L}(s) = H(s)KL(s)$ for convenience, and denote by $\hat{L}^{(k)}(s)$ the kth order derivative of $\hat{L}(s)$.

Theorem 1 Assume the multi-agent system (1) with consensus protocol (5) is steady semi-stable and the transfer function $\hat{L}(s)$ is analytic in a neighborhood of $s=0$. Then, the rth consensus is reached at the steady state if $\hat{L}^{(k)}(0)\cdot \mathbf{1}_n = 0$, for all $k \in \overline{0,r}$, and $\mathrm{rank}[\hat{L}(s)] = n-1$ for all $s \in C$ at which $\hat{L}(s)$ is analytic.

Proof. By Proposition 1 we know that the system has zeros at $s=0$ with multiplicity $r+1$, and the system solution can be expressed by (20). With the assumption of the steady semi-stability, from (20) it is easy to see that

$$y^{(j)}(t) \to 0,\ \forall j \geq r+1 \qquad (22)$$

and

$$y(t) \to \sum_{k=0}^{r} C_k t^k \qquad (23)$$

when $t \to \infty$. So, conditions (3) and (4) in Definition II are already satisfied. We just need to check (2).

Let s be in the neighborhood $\mathcal{D}$ of $s=0$ in which $\hat{L}(s)$ is analytic. By Taylor formula for functions of a complex variable we have

$$\hat{L}(s) = L_1(s) + L_2(s)s^{r+1}, \qquad (24)$$

where

$$\begin{aligned} L_1(s) &= \hat{L}(0)+\hat{L}^{(1)}(0)s+\cdots+\frac{1}{r!}\hat{L}^{(r)}(0)s^r, && (25)\\ L_2(s) &= \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \\ &\quad +\frac{1}{(r+2)!}\hat{L}^{(r+2)}(0)s+\cdots. && (26)\end{aligned}$$

Substituting (24) into (16) yields

$$s^{r+1}Y(s) = -(\hat{L}(0)+\hat{L}^{(1)}(0)s+\cdots+\frac{1}{r!}\hat{L}^{(r)}(0)s^r+\cdots)Y(s).$$

Note that the above equation hold in the neighborhood $\mathcal{D}$ of $s=0$. When the system is steady semi-stable, $Y(s)$ is analytic for $\mathrm{Re}s > 0$. Then, by taking the inverse Laplace transformation of the equation we have

$$\begin{aligned} -y^{(r+1)}(t) &= \hat{L}(0)y(t)+\hat{L}^{(1)}(0)y^{(1)}(t)+\cdots \\ &\quad +\frac{1}{r!}\hat{L}^{(r)}(0)y^{(r)}(t) \\ &\quad +\frac{1}{(r+1)!}\hat{L}^{(r+1)}(0)y^{(r+1)}(t)+\cdots \quad (27)\end{aligned}$$

for $t \to \infty$. Applying (22) to the above equation we get

$$\hat{L}(0)y(t) + \hat{L}^{(1)}(0)y^{(1)}(t) + \cdots + \frac{1}{r!}\hat{L}^{(r)}(0)y^{(r)}(t) \to 0 \quad (28)$$

when $t \to \infty$.

Differentiating (23) r times yields

$$\begin{array}{rcl} \dot{y}(t) & \to & \sum_{k=1}^{r} kC_k t^{k-1}, \\ \ddot{y}(t) & \to & \sum_{k=2}^{r} k(k-1)C_k t^{k-2}, \\ & \vdots & \\ y^{(r)}(t) & \to & r!C_r. \end{array} \quad (29)$$

Substituting (23) and (29) into (28) yields

$$E_r t^r + E_{r-1} t^{r-1} + \cdots + E_1 t + E_0 \to 0, \quad (30)$$

where

$$\begin{array}{rcl} E_r & = & \hat{L}(0)C_r, \\ E_{r-1} & = & \hat{L}(0)C_{r-1} + r\hat{L}^{(1)}(0)C_r, \\ & \vdots & \\ E_{r-j} & = & \hat{L}(0)C_{r-j} + C_{r-j+1}^{1}\hat{L}^{(1)}(0)C_{r-j+1} \\ & & +C_{r-j+2}^{2}\hat{L}^{(2)}(0)C_{r-j+2} \\ & & + \cdots + C_r^j \hat{L}^{(j)}(0)C_r, \\ & \vdots & \\ E_0 & = & \hat{L}(0)C_0 + \hat{L}^{(1)}(0)C_1 + \hat{L}^{(2)}(0)C_2 \\ & & + \cdots + \hat{L}^{(r)}(0)C_r, \\ C_r^j & = & \frac{r!}{(r-j)!j!}. \end{array} \quad (31)$$

Equation (30) holds if and only if $E_{r-j} = 0, \forall j \in \overline{0,r}$.

From $E_r = 0$ we get $C_r \in \text{span}(\mathbf{1}_n)$ because $\text{rank}[\hat{L}(0)] = n-1$ and $\hat{L}(0)\mathbf{1}_n = 0$. Using the result $C_r \in \text{span}(\mathbf{1}_n)$ and the assumption $\hat{L}^{(1)}(0)\mathbf{1}_n = 0$, from $E_{r-1} = 0$ it follows that $\hat{L}(0)C_{r-1} = 0$ which implies $C_{r-1} \in \text{span}(\mathbf{1}_n)$. Conducting this procedure to the end, i.e., $E_0 = 0$, we get $C_{r-j} \in \text{span}(\mathbf{1}_n), \forall j \in \overline{0,r}$. Thus, we have $y(t) \in \text{span}(\mathbf{1}_n)$ when $t \to \infty$, which, by (23), also implies that $y^{(k)}(t) \in \text{span}(\mathbf{1}_n), \forall k \in \overline{1,r}$. $\diamondsuit$

IV. Existence of high-order consensus solutions

Theorem 1 gives some sufficient consensus condition for the system. Here we try to find a necessary and sufficient condition of the existence of high-order consensus solutions. First, we show that the condition given by Proposition 1 can be further weakened as follows.

Proposition 2 Suppose $\hat{L}(s)$ is analytic in a neighborhood of $s = 0$. Then,

$$\det(s^{r+1}I + \hat{L}(s)) = s^{r+1}g(s)$$

with $g(s)$ having neither poles nor zeros at $s = 0$, if and only if $\text{rank}[\hat{L}(0)] = n-1$, and there exists a nonzero constant vector $\alpha \in \text{span}(\mathbf{1}_n)$ such that $\hat{L}^{(k)}(0) \cdot \alpha = 0, \forall k \in \overline{0,r}$, and $\alpha + \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \cdot \alpha \notin \text{span}(\hat{L}(0))$.

Proof. Omitted due to page limitation. For details see [10]. $\diamondsuit$

Remark. When $\text{rank}[\hat{L}(0)] = n-1$, a sufficient condition for $\mathbf{1}_n + \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \cdot \mathbf{1}_n \notin \text{span}(\hat{L}(0))$ is $\mathbf{1}_n + \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \cdot \mathbf{1}_n = c\mathbf{1}_n$, where c is a nonzero constant. And a more sufficient condition for is $\hat{L}^{(r+1)}(0) \cdot \mathbf{1}_n = 0$.

When $s = 0$ is a repeated pole with multiplicity $r+1$ of the closed-loop system, the solution of the closed-loop system can be expressed by equation (20).

Now, we are ready to present the following theorem.

Theorem 2 Assume the multi-agent system (1) with consensus protocol (5) is steady semi-stable and the transfer function $\hat{L}(s) = H(s)KL(s)$ is analytic in a neighborhood of $s = 0$. Then, the rth consensus is reached at the steady state if and only if $\hat{L}^{(k)}(0) \cdot \mathbf{1} = 0$, $\forall k \in \overline{0,r}$, $\text{rank}[\hat{L}(0)] = n-1$, and $\mathbf{1}_n + \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \cdot \mathbf{1}_n \notin \text{span}(\hat{L}(0))$.

Proof. (*Sufficiency*) Suppose $\hat{L}^{(k)}(0) \cdot \mathbf{1} = 0$, $\forall k \in \overline{0,r}$, $\text{rank}[\hat{L}(0)] = n-1$, and $\mathbf{1}_n + \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \cdot \mathbf{1}_n \notin \text{span}(\hat{L}(0))$. Then, by Proposition IV we know that the system has zeros at $s = 0$ with multiplicity $r+1$, and the system solution can be expressed by (20). The rest of the sufficiency part of the this proof is as the same as given in the proof of Theorem 1.

(*Necessity*) Suppose the system reaches the consensus defined by (2), (3) and (4). Note that (3), (4) and the steady semi-stability assumption imply that the system has zeros at $s = 0$ of multiplicity $r+1$. So, by Proposition 2, the necessity is obvious. $\diamondsuit$

V. Constant consensus

For the case of constant consensus, i.e., the case when $r = 0$, without loss of generality, we let $b_{i0} = 1$ for all $i \in \overline{1,n}$. Then, the protocol (5) reduces to the following form

$$u_i(t) = \kappa_i \left(\sum_{j \in N_i} a_{ij}(y_j(t-\tau_{ij}) - y_i(t-\tau'_{ij})) \right) - \kappa_i \sum_{k=1}^{m_i} b_{ik} y_i^{(k)}(t), \quad (32)$$

the closed-loop system equation becomes

$$\sum_{k=\nu}^{m_i} c_{ik} s^k Y_i(s) = \kappa_i \mathrm{e}^{-T_i s} \sum_{j \in N_i} a_{ij}(Y_j(s)\mathrm{e}^{-\tau_{ij}s} - Y_i(s)\mathrm{e}^{-\tau'_{ij}s}) - \kappa_i \sum_{k=1}^{m_i} b_{ik} s^k Y_i(s),\ i \in \overline{1,n},$$

and $H(s)$ defined in (13) takes the form

$$H(s) = \text{diag}\left\{ h_i(s),\ i \in \overline{1,n} \right\},$$

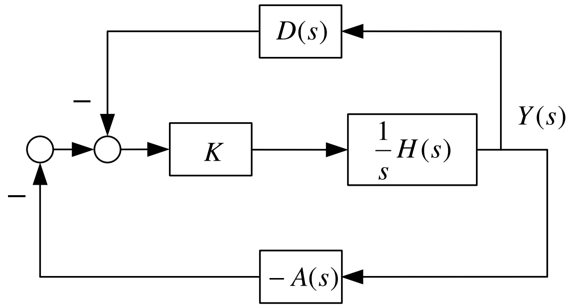

Fig. 2. Equivalent diagram for constant consensus.

where

$$h_i(s) = \frac{\mathrm{e}^{-T_i s}}{\sum_{k=\nu}^{m_i} c_{ik}s^{k-1} + \kappa_i \sum_{k=1}^{m_i} b_{ik}s^{k-1}}. \quad (33)$$

Let us apply Theorem 2 to the case of constant consensus. For this case, Theorem 2 requires: (1) the marginal stability of the closed-loop system, (2) $\mathrm{rank}[\hat{L}(0)]$=$n-1$ and $\hat{L}(0)\cdot\mathbf{1} = 0$. Since $\kappa_i \neq 0$, requirement (2) implies that $\mathrm{rank}[L(0)] = n-1$ and $L(0)\cdot\mathbf{1} = 0$. It is well known that this is equivalent to the connectivity condition for the interconnection graph, i.e., the digraph has a globally reachable node.

To check the marginal stability of the closed-loop system, we may make some loop transformation on the system. Denote by $\tilde{L}(s)$ the open-loop transfer function matrix after the loop transformation. Then, based on the extended Spectral Radius Theorem for steady semi-stability [10], we get the following result immediately.

Theorem 3 Consider the multi-agent system (1) with the protocol (32). Assume the interconnection digraph has a globally reachable node, then, the system achieves a constant consensus, if

$$\rho(\tilde{L}(\mathrm{j}\omega)) < 1,\ \forall\omega \in R, \omega \neq 0; \quad (34)$$

and

$$\rho(\tilde{L}(\mathrm{j}\omega)) = 1,\ \text{and}\ \det(I + \tilde{L}(\mathrm{j}\omega)) = 0 \text{ for } \omega = 0. \quad (35)$$

Since $L(s)$ defined in (15) can be rewritten as

$$\begin{aligned} L(s) &= D(s) - A(s) \\ &= \mathrm{diag}\{d_i(s),\ i \in \overline{1,n}\} - \{a_{ij}(s)\}, \end{aligned}$$

where

$$d_i(s) = \sum_{j=1}^{n} a_{ij}\mathrm{e}^{-\tau'_{ij}s}.$$

and

$$a_{ij}(s) = \begin{cases} a_{ij}\mathrm{e}^{-\tau_{ij}s}, & i \neq j, \\ 0, & i = j, \end{cases}$$

the system diagram shown by Fig. 1 can be equivalently transformed as shown by Fig. 2. In this case we have

$$\tilde{L}(s) = \mathrm{diag}\{g_i(s),\ i \in \overline{1,n}\}(-A(s)), \quad (36)$$

where

$$g_i(s) = \frac{\kappa_i h_i(s)}{s + \kappa_i h_i(s)\sum_{j=1}^{n} a_{ij}\mathrm{e}^{-\tau'_{ij}s}}. \quad (37)$$

Straightforward calculation shows that

$$\rho(\tilde{L}(\mathrm{j}\omega)) = 1,\ \text{and}\ \det(I + \tilde{L}(\mathrm{j}\omega)) = 0 \text{ for } \omega = 0.$$

So, the requirement (35) of Theorem 3 is satisfied. Therefore, under the assumption that the interconnection digraph has a globally reachable node, a sufficient condition for achieving a constant consensus is

$$\rho(\mathrm{diag}\{g_i(\mathrm{j}\omega),\ i \in \overline{1,n}\}(-A(\mathrm{j}\omega)) < 1,\ \forall\omega \in R, \omega \neq 0. \quad (38)$$

By using Gershgorin's disc lemma, a more conservative but scalable condition is

$$|g_i(\mathrm{j}\omega)| < (\sum_{j=1}^{n} a_{ij})^{-1},\ \forall\omega \in R, \omega \neq 0, \forall i \in \overline{1,n}. \quad (39)$$

This condition was first given by [5].

Similarly, from the Small Gain Theorem for semi-stability and Positive Realness Theorem [10] we can also get the following two theorems for the constant consensus problem.

Theorem 4 Consider the multi-agent system (1) with the protocol (32). Assume the interconnection digraph has a globally reachable node, then, the system achieves a constant consensus, if

$$\|\tilde{L}(\mathrm{j}\omega)\| < 1,\ \forall\omega \in R, \omega \neq 0; \quad (40)$$

and

$$\|\tilde{L}(\mathrm{j}\omega)\| = 1,\ \text{and}\ \det(I + \tilde{L}(\mathrm{j}\omega)) = 0,\ \text{for } \omega = 0, \quad (41)$$

where $\|\hat{G}\|$ denotes any matrix norm satisfying $\|AB\| \leq \|A\|\cdot\|B\|$.

Theorem 5 Consider the multi-agent system (1) with the protocol (32). Assume the interconnection digraph has a globally reachable node, then, the system achieves a constant consensus, if

$$1 + \mathrm{Re}(\lambda_i(\tilde{L}(\mathrm{j}\omega))) > 0,\ \forall\omega \in R, \omega \neq 0, \forall i \in \overline{1,n}; \quad (42)$$

and

$$1 + \mathrm{Re}(\lambda_i(\tilde{L}(\mathrm{j}\omega))) = 0,\ \text{for } \omega = 0, i \in \overline{1,n}. \quad (43)$$

VI. CONSENSUS IN IDEAL NETWORKS

Before applying Theorem 2 to ideal networks, i.e., networks with zero communication delays and constant channel dynamics, let us review the following fact from graph theory.

Let G be a digraph having at least one globally reachable node. If we choose only one globally reachable node, say v_j, of G, and cut off all the edges from v_j, then, obviously v_j is still a globally reachable node of G. But, if we do this operation for a node which is not globally reachable in G, or simultaneously do this operation for two or more globally reachable nodes, then, G has no globally reachable node any more. This is equivalently to say, if $L \in R^{n\times n}$ is a Laplacian of a digraph having at leat one globally reachable node, then $\operatorname{rank}[\operatorname{diag}\{b_1, \cdots, b_n\}L] = n-1$ if and only if $b_i \neq 0, \forall i \in \overline{1,n}$, or $b_i \neq 0, \forall i \in \overline{1,n}\backslash j, b_j = 0$ and v_j is a globally reachable node of G, where $\overline{1,n}\backslash j$ denotes the set of integers from 1 to n excluding j.

Now, let us consider a multi-agent systems based on an ideal network. Suppose the topology digraph G contains at least one globally reachable node. Then, we have $\operatorname{rank}[L] = n-1$ and $L \cdot \mathbf{1}_n = 0$. In this case we have $\tau_{ij} = \tau'_{ij} = 0$ and $L(s) = L$. From $L \cdot \mathbf{1}_n = 0$ it is easy to get $\hat{L}^{(k)}(0) \cdot \mathbf{1}_n = 0$, $\forall k \in \overline{0,r}$ and $\mathbf{1}_n + \frac{1}{(r+1)!}\hat{L}^{(r+1)}(0) \cdot \mathbf{1}_n = \mathbf{1}_n \notin \operatorname{span}(\hat{L}(0))$. Finally, by denoting $g_i(s) = \frac{b_i(s)}{c_i(s)}$, we know that $\operatorname{rank}[\hat{L}(0)] = \operatorname{rank}[L] = n-1$ if and only if $g_i(0) \neq 0$, or $g_i(0) \neq 0, \forall i \in \overline{1,n}\backslash j, g_j(0) = 0$ and v_j is a globally reachable node of digraph G. Thus, we get the following result as a corollary of Theorem 2.

Theorem 6 In an ideal network with zero communication delays and constant channel dynamics, the rth consensus will be achieved for a steady semi-stable system at its steady state if and only if the topology graph contains at least one globally reachable node, and the agents's dynamics and the protocol satisfy $g_i(0) \neq 0$, or $g_i(0) \neq 0, \forall i \in \overline{1,n}\backslash j, g_j(0) = 0$ and v_j is a globally reachable node of digraph G.

VII. INTEGRATOR-CHAIN SYSTEMS WITH DIVERSE COMMUNICATION DELAYS

Consider the multi-agent system, whose agents are chains of integrators, i.e., $m_i = \nu$, and $h_i(s) = 1$. In this case $g_i(s) = b_i(s)$, and thus we have

$$\begin{aligned}
\hat{L}(s) &= \operatorname{diag}\{\kappa_i b_i(s)\}L(s), \\
\hat{L}^{(1)}(s) &= \operatorname{diag}\{\kappa_i b_i(s)\}L^{(1)}(s) + \operatorname{diag}\{\kappa_i b_i^{(1)}(s)\}L(s), \\
&\cdots \\
\hat{L}^{(\nu-1)}(s) &= \operatorname{diag}\{\kappa_i b_i(s)\}L^{(\nu-1)}(s) + \cdots.
\end{aligned}$$

Since $\kappa_i > 0$ and $b_{i0} \neq 0$, we have $\operatorname{rank}[\hat{L}(0)] = \operatorname{rank}[L(0)] = n-1$ when the topology graph contains a globally reachable node. Now, we check the condition $\hat{L}^{(k)}(0)\cdot\mathbf{1} = 0$, $\forall k \in \overline{0,(\nu-1)}$. For the second order consensus ($\nu = 2$), this condition implies that

$$\sum_{j=1}^{n} a_{ij}\tau_{ij} = \sum_{j=1}^{n} a_{ij}\tau'_{ij}. \tag{44}$$

And generally, the consensus condition for the νth order consensus can be obtained as

$$\begin{cases} \sum_{j=1}^{n} a_{ij}\tau_{ij} = \sum_{j=1}^{n} a_{ij}\tau'_{ij}, \\ \cdots \\ \sum_{j=1}^{n} a_{ij}\tau_{ij}^{\nu-1} = \sum_{j=1}^{n} a_{ij}(\tau'_{ij})^{\nu-1}. \end{cases} \tag{45}$$

Equation (44) or (45) uncovers a very interesting fact that the second-order or high-order consensus does not necessarily require $\tau'_{ij} = \tau_{ij}$. We call condition (45) as the *matching condition* for self-delays.

ACKNOWLEDGMENT

The authors would like to thank the National Natural Science Foundation of China (under grants 61273110, 60974041 and 60934006, 61105113) for financial supports for this work.

REFERENCES

[1] M. Cao, A. S. Morse, & B. D. O. Anderson, Reaching an agreement using delayed information, *IEEE Conference on Decision and Control*, San Diego, CA, USA, pp.3375–3380, 2006.

[2] C. A. Desoer and Y.-T. Wang, On the generalized Nyquist Stability Criterion. *IEEE Transactions on Automatic Control*, 25(2), 187–196, 1980 .

[3] J. Hu and Y. Hong, Leader-following coordination of multi-agent systems with coupling time delays, *Physica A*, 374, 853–863, 2007.

[4] A. Jadbabaie, J. Lin, and A. S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Transactions on Automatic Control*, 48, 988–1001, 2003.

[5] D. Lee and M. W. Spong, Agreement with non-uniform information delays. *Proceedings of the American Control Conference*, Minneapolis, Minnesota, USA, pp.756–761, 2006

[6] L. Moreau, Stability of multiagent systems with time-dependent communication links, *IEEE Transaction on Automatic Control*, 50, 169–182, 2005.

[7] W. Ren and R. W. Beard, Consensus seeking in multi-agent systems under dynamically changing interaction topologies, *IEEE Transactions on Automatic Control*, 50, 655–661, 2005.

[8] J. H. Seo, H. Shim, and J. Back, Consensus of high-order linear systems using dynamic output feedback compensator: low gain approach, *Automatica*, 45, 2659–2664, 2009.

[9] Y.-P. Tian and C.-L. Liu, Consensus of multi-agent systems with diverse input and communication delays. *IEEE Trans. on Auto. Control, 53*, 2122–2128, 2008.

[10] Y.-P. Tian, Frequency-domain Analysis and Design of Destributed Control Systems, John Wiley & Sons, Singapore, 2012.

[11] J. H. Wang, D. Z. Cheng, and X. M. Hu, Consensus of multi-agent linear dynamic systems, *Asian Journal of Control*, 10, 144–155, 2008.

[12] W. Wang and J. J. E. Slotine, Contraction analysis of time-delayed communications and group cooperation, *IEEE Transaction on Automatic Control*, 51, 712–717, 2006.