The Active Disturbance Rejection Control for Nonlinear Systems Using Time-Varying-Gain

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***Abstract*—The active disturbance rejection control, as a new control strategy in dealing with the large uncertainties, has been developed rapidly in the last two decades. Basically, the active disturbance rejection control is composed of three main parts: the differential tracking; the extended state observer; and the extended observer-based feedback control. In these three parts, the extended state observer plays a crucial role toward the active disturbance rejection control. The most of the extended state observers are based on the constant high gain parameter tuning which results inherently in the peaking problem near the initial time, and at most the attenuation effect for the uncertainty. In this paper, a time-varying-gain extended state observer is proposed for a class of nonlinear systems, which is shown to reject completely the disturbance and to avoid effectively the peaking phenomena by the proper choice of the gain function. The convergence of the extended state observer for the open-loop system is independently proved. The convergence for the closed-loop system which is based on the extended state observer feedback is also presented. Examples and numerical simulations are used to illustrate the convergence and the peaking diminution.**

I. Introduction

It is generally believed that "if there is no uncertainty in the system, the control, or the environment, feedback control is largely unnecessary"([2]). Basically, there are two kinds of control strategies in dealing with the uncertainties. One is using the high gain in feedback-loop to attenuate or reject the disturbance (system uncertainty and external disturbance) such as the well-known sliding mode control, internal model principle, and the high gain observer based feedback control ([17]). Another is to estimate the uncertainty first and cancel the effect of the uncertainty in the feedback-loop such as the external principle ([20]). However, both internal and external model methods require the dynamic of the unknown disturbance. A remarkable estimation/cacellation control strategy in dealing with the completely unknown uncertainties that come from both the internal and external was initiated by Han in [14], and later in [12], [13]. This new technology is later called the active disturbance rejection control (ADRC) ([15]). The main idea of the ADRC is that the "total disturbance" which consists of the unknown part of the system dynamics and control, and the external disturbance can be estimated online through the extended state observer and then be canceled in the feedback-loop.

In the past two decades, the ADRC has been successfully applied in many engineering control problems, for example see [6], [22], [24], just list a few.

On the other hand, although many successful engineering applications have been made, the theoretical research lags, however, behind the applications. The convergence of the linear ADRC with constant high gain is first proved in [16], [23]. Very recently, we give a convergence proof for nonlinear ADRC with constant high gain parameter for a class of SISO system ([8]), and MIMO nonlinear systems ([9]).

Let us briefly outline the main steps in the ADRC designed in [12], [15]. Consider the following nonlinear system with un-modeled system dynamics and external disturbance

$$\begin{cases} \dot{x}(t) = A_n x(t) + B_n[f(t, x(t)) + w(t) + u(t)], \\ y(t) = x_1(t), \end{cases} \tag{1.1}$$

where

$$A_n = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \; B_n^\top = C_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{1.2}$$

I_{n-1} is the $(n-1)$-th identity matrix, $x = (x_1, x_2, \ldots, x_n)^\top$ is the system state, u the input (control), y the output (measurement), f a possibly unknown system function, and w the uncertain external disturbance. $f + w$ is called the (unknown) "total disturbance". The control purpose of the ADRC is to design the output feedback control so that the system state can track the reference signal v and its derivatives by estimating/calellating, in real time, the total disturbance. It covers output regulation and the stabilization as its special cases.

The first part towards the ADRC is the tracking differentiator (TD). Han himself proposed a noise-tolerance TD to recover the derivatives of the reference signal v ([14], [7], [11]) which is actually an independent topic in control theory.

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For other works on this aspect, we refer to [3], [19] and the references therein.

The second step toward the ADRC is to design the "extended state observer" (ESO) to estimate not only the state of the system but also the uncertainties by the output. The ESO is an extension of the traditional state observer, which is again another independent topic in control theory. Now we give a brief introduction of the ESO because it plays a crucial role in design of the ADRC. The more details can be found in [5], [10], [12], [15]. In [12], Han proposed the following ESO:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) - \alpha_1 g_1(\hat{x}_1(t) - y(t)), \\ \qquad \vdots \\ \dot{\hat{x}}_n(t) = \hat{x}_{n+1}(t) - \alpha_n g_n(\hat{x}_1(t) - y(t)) + u(t), \\ \dot{\hat{x}}_{n+1}(t) = -\alpha_{n+1} g_{n+1}(\hat{x}_1(t) - y(t)), \end{cases} \tag{1.3}$$

for system (1.1). The main idea of the ESO is to choose some suitable nonlinear functions g_i, and tuning gain parameters α_i so that the first n states of ESO (1.3) converge to the corresponding states of system (1.1), and the $(n+1)$-th state of ESO (1.3) converges to the total disturbance $f+w$. However, in Han's original papers, there is no general principle to choose these functions and parameters. The first step is put forward in [5] where the constant high gain is introduced in the linear ESO.

In [10], we propose a high gain nonlinear ESO that covers the linear ESO as its special case. The big problem for the high gain ESO is the inherent peaking value problem near the initial time due to the small tuning value of ε. The saturation function approach is used in [4] to overcome the peaking problem in the feedback-loop by the linear high gain ESO under the assumption that the initial value bound is supposed to be known, but the ESO part which is significant independently as indicated, is not touched.

In this paper, we consider a more complicated nonlinear system of the following (see [4]):

$$\begin{cases} \dot{x}(t) = A_n x(t) + B_n[f(t, x(t), \zeta(t), w(t)) \\ \qquad\quad + b(t)u(t)], \\ \dot{\zeta}(t) = f_0(t, x(t), \zeta(t), w(t)), \end{cases} \tag{1.4}$$

where $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^m$ are system states, A_n, B_n are defined as (1.2), $f, f_0 \in C^1(\mathbb{R}^{n+m+2}, \mathbb{R})$ are possibly unknown nonlinear functions, $u \in \mathbb{R}$ is the input (control), and $y = C_n x = x_1$ is the output (measurement), $b \in \mathbb{C}^1(\mathbb{R}, \mathbb{R})$ contains some uncertainty with nominal value b_0.

We design the following nonlinear time-varying gain ESO for system (1.4) as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + \frac{1}{r^{n-1}(t)} g_1(r^n(t)(y(t) - \hat{x}_1(t))), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) + \dfrac{1}{r^{n-2}(t)} g_2(r^n(t)(y(t) - \hat{x}_1(t))), \\ \qquad \vdots \\ \dot{\hat{x}}_n(t) = \hat{x}_{n+1}(t) + g_n(r^n(t)(y(t) - \hat{x}_1(t))) \\ \qquad\quad + b_0 u(t), \\ \dot{\hat{x}}_{n+1}(t) = r(t) g_{n+1}(r^n(t)(y(t) - \hat{x}_1(t))), \end{cases} \tag{1.5}$$

to estimate the states $x_1, x_2, \ldots x_n$ and total disturbance

$$x_{n+1}(t) \triangleq f(t, x(t), \zeta(t), w(t)) + [b(t) - b_0]u(t), \tag{1.6}$$

which is also called the extended state, where r is the time-varying gain to be increased gradually.

The last step of the ADRC is to design an ESO-based output feedback control to achieve the reference signal tracking in cancelation of the total disturbance by the state of the ESO.

It should be pointed out that the appropriate choice of the nonlinear functions g_i in (1.5) would improve the accuracy and reduce the peaking value under the same gain parameter compared with the linear ones ([10]). Moreover, the suitable time varying gain can reduce dramatically the peaking value of both the ESO and the ADRC. This nonlinear ESO and time varying high gain nature are the main contributions of the present paper.

II. Convergence of the closed-loop system

Let us recall the whole process of the ADRC for (1.4). The first part of the ADRC is the TD which is designed to estimate the derivatives of a given signal v ([11]). The TD part is relatively independent of other two parts of the ADRC, we do not couple TD in the closed-loop but instead, using directly z_i in the feedback-loop.

The second part of the ADRC is to design the ESO to estimate both state and the total disturbance of system (1.4). The ESO used in the closed-loop of this paper is (1.5). However, since the separation principle is not valid automatically for nonlinear systems, for the convergence of the ESO based output feedback-loop, we require different conditions for the ESO that will be stated later.

Now suppose that we have got the estimations for both the state and the total disturbance, we then use estimation/cancellation strategy to design the ESO based output feedback control as follows:

$$\begin{aligned} u = \tfrac{1}{b_0}\big(& u_0\big(\hat{x}_1(t) - z_1(t), \ldots, \ \hat{x}_n(t) - z_n(t)\big) \\ & + z_{n+1} - \hat{x}_{n+1}\big), \end{aligned} \tag{2.1}$$

where $\hat{x}_{n+1}$ is used to cancel the total disturbance x_{n+1} and u_0 is the nominal control. The objective of the control is to make the error $(x_1(t)-z_1(t), x_2(t)-z_2(t)\ldots, x_n(t)-z_n(t))$ converge to 0 as time goes to infinity in the prescribed way. Precisely,

$$x_i(t) - z_i(t) \approx y^{(i-1)}(t), \ i = 1, 2, \cdots, n,$$

where y_i satisfies

$$y^{(n)}(t) = u_0\left(y(t), y'(t), \ldots, y^{(n-1)}(t)\right). \tag{2.2}$$

It is seen that different to the high gain approach in dealing with the uncertainty (see, e.g., ([21, Lemma 2.2 and 2.4]), we do not need the high gain in the feedback loop.

We point out that the scheme of the ADRC can dealt with at least two kinds of control problems: stabilization ($v \equiv 0$) and the output regulation.

Now we are in a position to give some Assumptions for proving the convergence of closed-loop of system (1.4) with its ESO (1.5) under the output feedback (2.1).

The following Assumption (AC1) is about the unknown functions f and f_0.

Assumption (AC1). There exists positive constants M_1, K_0, K_1, K_2 such that $\sup_{t\in[0,\infty)}(|w(t)| + \dot{w}(t)| + |b(t)| + |\dot{b}(t)|) < M_1$; and for any $t \in [0,\infty),\ x \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^m,\ w \in \mathbb{R}$,

$$\sum_{i=1}^{m}\left|\frac{\partial f}{\partial \zeta_i}(t,x,\zeta,w)\right| + \sum_{i=1}^{n}\left|\frac{\partial f}{\partial x_i}(t,x,\zeta,w)\right| \le K_0 + \varpi(w),$$
$$\left|\frac{\partial f}{\partial t}(t,x,\zeta,w)\right| + \left|\frac{\partial f}{\partial w}(t,x,\zeta,w)\right| + \|f_0(t,x,\zeta,w)\| \le K_1 + K_2\|x\| + \varpi(w).$$

For the nonlinear functions in the ESO (1.5), we need the following Assumption (AC2) that is a little bit more restrictive than Assumption (A4) for the open-loop system.

Assumption (AC2). There exist radially unbounded, positive definite functions $\mathcal{V} \in C^1(\mathbb{R}^{n+1},\ \bar{\mathbb{R}}^+)$, $\mathcal{W} \in C(\mathbb{R}^{n+1}, \bar{\mathbb{R}}^+)$ such that

- $\displaystyle\sum_{i=1}^{n}(x_{i+1} - g_i(x_1))\frac{\partial \mathcal{V}}{\partial x_i} - g_{n+1}(x_1)\frac{\partial \mathcal{V}}{\partial x_{n+1}} \le -\mathcal{W}(x)$;
- $\displaystyle\left|g_{n+1}(x_1)\frac{\partial \mathcal{V}}{\partial x_{n+1}}(x)\right| < N_0\mathcal{W}(x)$, $\displaystyle\|x\|^2 + \sum_{i=1}^{n}\left|x_i\frac{\partial \mathcal{V}}{\partial x_i}(x)\right| + \sum_{i=1}^{n}\left|g_i(x_1)\frac{\partial \mathcal{V}}{\partial x_i}(x)\right| + \left|\frac{\partial \mathcal{V}}{\partial x_{n+1}}(x)\right|^{\theta} + \|x\|\left|\frac{\partial \mathcal{V}}{\partial x_{n+1}}(x)\right| + \left|\frac{\partial \mathcal{V}}{\partial x_{n+1}}(x)\right|^2 \le N_1\mathcal{W}(x)$ for all $x \in \mathbb{R}^{n+1}$ with some $N_0,\ N_1 > 0$ and $\theta > 1$.

Assumption (AC3) below is on the function u_0 in feedback control (2.1).

Assumption (AC3). All partial derivatives of u_0 are globally bounded by L, and there exist continuous, radially unbounded positive definite functions $V \in C^1(\mathbb{R}^n, \bar{\mathbb{R}}^+),\ W \in C(\mathbb{R}^n, \bar{\mathbb{R}}^+)$ such that

$$\sum_{i=1}^{n-1} x_{i+1}\frac{\partial V}{\partial x_i}(x) + u_0(x)\frac{\partial V}{\partial x_n}(x) \le -W(x),$$
$$\|x\|^2 + \left|\frac{\partial V}{\partial x_n}(x)\right|^2 \le N_2 W(x),\ \forall\, x \in \mathbb{R}^n.$$

Since $\mathcal{V},\ \mathcal{W},\ V, W$ are continuous, radially unbounded and positive definite functions in Assumptions (AC2) and (AC3), it follows from lemma 4.3 of [18, p.145] that there exist continuous wedge functions $\kappa_i,\ \varkappa_i (i = 1,2,3,4)$ such that for any $x \in \mathbb{R}^{n+1}, y \in \mathbb{R}^n$

$$\begin{array}{l} \kappa_1(\|x\|) \le \mathcal{V}(x) \le \kappa_2(\|x\|), \\ \kappa_3(\|x\|) \le \mathcal{W}(x) \le \kappa_4(\|x\|), \\ \varkappa_1(\|y\|) \le V(y) \le \varkappa_2(\|y\|), \\ \varkappa_3(\|y\|) \le \mathcal{W}(y) \le \varkappa_4(\|y\|). \end{array} \tag{2.3}$$

Assumption (AC4) below is on these wedge functions.

Assumption (AC4). For any $x \ge 0$, $\kappa_4(x) \le N_3\varkappa_3(x)$ for some $N_3 > 0$.

The following Assumption (AC5) is on the gain function $r(t)$:

Assumption (AC5). $r \in C^1(\bar{\mathbb{R}}^+, \bar{\mathbb{R}}^+),\ r(t), r'(t) > 0$, $\lim_{t\to+\infty} r(t) = +\infty$, and there exists a constant $M > 0$ such that $\overline{\lim}_{t\to+\infty}\frac{\dot{r}(t)}{r(t)} \le M$.

The real situation in numerical experiment and engineering application is that the time varying gain $r(t)$ is small in the initial time to avoid the peaking value, and then increase rapidly to a certain acceptable large constant r_0 such that the approximation error in the given area in a relative short time.

By this idea, we modify the gain r in (1.5) as follows:

$$r(t) = \begin{cases} e^{at}, & 0 \le t < \frac{1}{a}\ln r_0, \\ r_0, & t \ge \frac{1}{a}\ln r_0, \end{cases} \tag{2.4}$$

where r_0 is a large number so that the errors between the solutions of (1.5) and (1.4) are in the prescribed scale. Under this gain parameter, Assumption (A4) is replaced by Assumption (A4*) below.

Theorem 2.1: Assume Assumptions (AC1)-(AC4), and $N_0\sup_{t\in[0,\infty)}|(b(t) - b_0)/b_0| < 1$. Then $j = 1,2,\ldots,n$.

(i) If the gain parameter satisfies Assumption (AC5), then

$$\lim_{t\to\infty}|x_i(t) - \hat{x}_i(t)| = 0,\ \lim_{t\to\infty}|x_j(t) - z_j(t)| = 0, \tag{2.5}$$

(ii) If the gain parameter satisfies (2.4), then for any given $\sigma > 0$, there exists constant $r^* > 0$ such that for all $r_0 > r^*$,

$$|x_i(t) - \hat{x}_i(t)| < \sigma,\ |x_j(t) - z_j(t)| < \sigma,\ t > t_0,$$

where t_0 is a r_0-dependent constant, $i = 1,2,\ldots,n+1$.

Proof. Let

$$\begin{array}{l} \eta_i(t) = r^{n+1-i}(t)\left(x_i(t) - \hat{x}_i(t)\right), i = 1,\ldots,n+1, \\ \mu_j(t) = x_j(t) - z_j(t),\ j = 1,\ldots,n. \end{array} \tag{2.6}$$

According to the different choices of r, the proof is divided into two parts.

Part I. The gain parameter r satisfies Assumption (A3). In this case, the error equation can be written as

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \dot{\eta}_1(t) = r(t)(\eta_2(t) - g_1(\eta_1(t))) + \frac{n\dot{r}(t)}{r(t)}\eta_1(t), \\ \qquad\vdots \\ \dot{\eta}_n(t) = r(t)(\eta_{n+1}(t) - g_n(\eta_1(t))) \\ \qquad + \frac{\dot{r}(t)}{r(t)}\eta_n(t), \\ \dot{\eta}_{n+1}(t) = -r(t)g_{n+1}(\eta_1(t)) + \dot{x}_{n+1}(t), \end{array}\right. \\ \left\{\begin{array}{l} \dot{\mu}(t)(t) = A_n\mu(t) + B_n[u_0(\hat{x}_1(t) - z_1(t), \\ \ldots, \hat{x}_n(t) - z_n(t)) + \eta_{n+1}(t)], \\ \mu(t) = (\mu_1(t), \mu_2(t), \ldots, \mu_n(t))^\top. \end{array}\right. \end{array}\right. \tag{2.7}$$

We first need to estimate the derivative of the total disturbance.

$$
\begin{aligned}
&\dot{x}_{n+1}(t)=\frac{d}{dt}[f(t,x(t),\zeta(t),w(t))\\
&+(b(t)-b_0)u(t)]\\
&=\frac{\partial f}{\partial t}(t,x(t),\zeta(t),w(t))\\
&+\sum_{i=1}^{n-1}x_{i+1}(t)\frac{\partial f}{\partial x_i}(t,x(t),\zeta(t),w(t))\\
&+[u_0(\hat{x}_1(t)-z_1(t),\ldots,\hat{x}_n(t)-z_n(t))\\
&+z_{n+1}(t)+\eta_{n+1}(t)]\frac{\partial f}{\partial x_{n+1}}(t,x(t),\zeta(t),w(t))\\
&+f_0(t,x(t),\zeta(t),w(t))\cdot\frac{\partial f}{\partial \zeta}(t,x(t),\zeta(t),w(t))\\
&+\dot{w}(t)\frac{\partial f}{\partial \zeta}(t,x(t),\zeta(t),w(t))\\
&+\frac{\dot{b}(t)}{b_0}\big(u_0\big(\hat{x}_1(t)-z_1(t),\ \ldots,\ \hat{x}_n(t)-z_n(t)\big)\\
&+z_{n+1}-\hat{x}_{n+1}\big)\\
&+\left|\frac{b(t)-b_0}{b_0}\right|\Bigg[\sum_{i=1}^{n}\Bigg(\hat{x}_{i+1}(t)+\frac{1}{r^{n-i}(t)}g_i(\eta_1(t))\\
&-z_{i+1}\Bigg)\times\frac{\partial u_0}{\partial y_i}(\hat{x}_1(t)-z_1(t),\ \ldots,\hat{x}_n(t)-z_n(t))\\
&+\dot{z}_{n+1}(t)-r(t)g_{n+1}(\eta_1(t))\Bigg].
\end{aligned}
\tag{2.8}
$$

By Assumptions (A2), (AC1)-(AC3), there exists a $B>0$ such that

$$
\begin{aligned}
&|\dot{x}_{n+1}(t)|\le B\Big(1+\|\eta(t)\|+\|\mu(t)\|\\
&+\sum_{i=1}^{n}|g_i(\eta_1(t))|\Big)\\
&+\left|\frac{b(t)-b_0}{b_0}\right|r(t)|g_{n+1}(\eta_1(t))|.
\end{aligned}
\tag{2.9}
$$

Define Lyapunov functions $\mathfrak{V},\mathfrak{W}$: $\mathbb{R}^{2n+1}\to\bar{\mathbb{R}}^{+}$ as follows:

$$
\mathfrak{V}(x,y)=\mathcal{V}(x)+V(y),\ \mathfrak{W}(x,y)=\mathcal{W}(x)+W(y). \tag{2.10}
$$

Finding the derivative of Lyapunov $\mathfrak{V}$ along the solution of (2.7) yields

$$
\begin{aligned}
&\frac{d\mathfrak{V}(\eta(t),\mu(t))}{dt}\bigg|_{(2.7)}\\
&=\sum_{i=1}^{n+1}\dot{\eta}_i(t)\frac{\partial\mathcal{V}}{\partial\eta_i}(\eta(t))+\sum_{i=1}^{n}\dot{\mu}_i(t)\frac{\partial\mathcal{V}}{\partial\mu_i}(\mu(t))\\
&=r(t)\Bigg(\sum_{i=1}^{n}\big(\eta_{i+1}(t)-g_i(\eta_1(t))\big)\frac{\partial\mathcal{V}}{\partial\eta_i}(\eta(t))\\
&-g_{n+1}(\eta_1(t))\frac{\partial\mathcal{V}}{\partial\eta_{n+1}}(\eta(t))\Bigg)\\
&+\sum_{i=1}^{n}\frac{(n+1-i)\dot{r}(t)\eta_i(t)}{r(t)}\frac{\partial\mathcal{V}}{\partial\eta_i}(\eta(t))\\
&+\dot{x}_{n+1}(t)\frac{\partial\mathcal{V}}{\partial\eta_{n+1}}(\eta(t))\\
&+\sum_{i=1}^{n-1}\mu_{i+1}(t)\frac{\partial V}{\partial\mu_i}(\mu(t))+[u_0(\hat{x}_1(t)-z_1(t),\ldots,\\
&\hat{x}_n(t)-z_n(t))+\eta_{n+1}(t)]\frac{\partial V}{\partial\mu_{n+1}}(\mu(t)).
\end{aligned}
\tag{2.11}
$$

From Assumption (A3), we may assume that there exists $t_1>0$ such that for any $t>t_1$ $\dot{r}(t)/r(t)<2M$ and

$$
\begin{aligned}
r(t)>\max\Bigg\{&\frac{2N_1N_2\left[B+(L+1)\sqrt{n+1}\right]}{\Delta},\\
&\frac{2\left[2n(n+1)M+(n^2+n+2)B\right]N_1}{\Delta}\Bigg\},
\end{aligned}
$$

where

$$
\Delta=1-N_0\sup_{t\in[0,\infty)}|(b(t)-b_0)/b_0|,
$$

by which and Assumptions (AC2)-(AC4), we have, for all $t>t_1$, that

$$
\begin{aligned}
&\frac{d\mathfrak{V}(\eta(t),\mu(t))}{dt}\bigg|_{(2.7)}\\
&\le-\Delta r(t)\mathcal{W}(\eta(t))+B\left|\frac{\partial\mathcal{V}}{\partial\eta_{n+1}}(\eta(t))\right|\\
&+\frac{\Delta r(t)}{4}\mathcal{W}(\eta(t))-W(\mu(t))\\
&+\sqrt{\frac{\Delta r(t)}{2}}\sqrt{\mathcal{W}(\eta(t))}\sqrt{W(\mu(t))}\\
&-\tfrac{\Delta}{2}r(t)\mathcal{W}(\eta(t))+B\left|\frac{\partial\mathcal{V}}{\partial\eta_{n+1}}(\eta(t))\right|-\tfrac{1}{2}W(\mu(t)).
\end{aligned}
\tag{2.12}
$$

The remainder of the proof is split into the following three steps.

Step 1. The boundedness of $\|(\eta(t),\mu(t))\|$.

Let

$$
\begin{aligned}
R=\max\Bigg\{&\kappa_3^{-1}\left(\left(\tfrac{2BN_1^{1/\theta}}{\Delta r(t_1)}\right)^{\theta/(\theta-1)}\right),\\
&\varkappa_3^{-1}\left((2B)^{\theta/(\theta-1)}(N_1N_3)^{1/(\theta-1)}\right)\Bigg\}.
\end{aligned}
$$

If $\|(\eta(t),\mu(t))\|>2R$, then there are two cases. When $\|\eta(t)\|>R$ and $t>t_1$, it has

$$
\begin{aligned}
&\frac{d\mathfrak{V}(\eta(t),\mu(t))}{dt}\bigg|_{(2.7)}\\
&\le\mathcal{W}^{1/\theta}(\eta(t))\\
&\quad\times\left(-\tfrac{\Delta}{2}r(t)\kappa_3^{(\theta-1)/\theta}(\|\eta(t)\|)+BN_1^{1/\theta}\right)\\
&<0.
\end{aligned}
\tag{2.13}
$$

When $\|\eta(t)\|<R$, then $\|\mu(t)\|>R$, and hence

$$
\begin{aligned}
&\frac{d\mathfrak{V}(\eta(t),\mu(t))}{dt}\bigg|_{(2.7)}\\
&\le B(N_1N_3)^{1/\theta}\varkappa_3^{1/\theta}(R)-\tfrac{1}{2}\varkappa_3(R)<0,\ t>t_1.
\end{aligned}
\tag{2.14}
$$

Let

$$
\mathcal{A}=\left\{(x,y)\in\mathbb{R}^{2n+1}:\ \mathfrak{V}(x,y)\le C\triangleq\max_{\|(x,y)\|\le 2R}\mathfrak{V}(x,y)\right\}.
$$

From continuity and radially unboundedness of $\mathfrak{V}$, we known that $\mathcal{A}\subset\mathbb{R}^{2n+1}$ is bounded. If $(\eta(t),\mu(t))\in\mathcal{A}^C$ then $\|(\eta(t),\mu(t))\|>2R$. Combining the two cases above, we get

$$
\frac{d\mathfrak{V}(\eta(t),\mu(t))}{dt}\bigg|_{(2.7)}<0,\ (\eta(t),\mu(t))\in\mathcal{A}^C.
$$

So there exists a $t_2 > t_1$ such that $(\eta(t), \mu(t))$ lies in bounded set $\mathcal{A}$ for any $t > t_2$, which means that $(\eta(t), \mu(t))$ is uniformly bounded for $t > t_2$.

Step 2. The convergence of $\eta(t)$ to 0.

Since $(\eta(t), \mu(t))$ is uniformly bounded for $t > t_2$, it follows from the continuity of g_i, $\dfrac{\partial \mathcal{V}}{\partial x}$ and (2.9) that there exists a $B_1 > 0$ such that for every $t > t_2$

$$\begin{array}{l}\left[B\left(1+\|\eta(t)\|+\|\mu(t)\|+\sum_{i=1}^{n}|g_i(\eta_1(t))|\right)\right.\\ \left.+\left|\dfrac{b(t)-b_0}{b_0}\right| r(t)|g_{n+1}(\eta_1(t))|\right]\left|\dfrac{\partial \mathcal{V}}{\partial \eta_{n+1}}(\eta(t))\right|\\ +\sum_{i=1}^{n}\dfrac{(n+1-i)\dot{r}(t)\eta_i(t)}{r(t)}\left|\dfrac{\partial \mathcal{V}}{\partial \eta_i}(\eta(t))\right|\\ \le B_1+N_0\sup_{t\in[0,\infty)}|(b(t)-b_0)/b_0|r(t)\mathcal{W}(\eta(t)).\end{array}$$

Finding the derivative of $\mathcal{V}(\eta(t))$ along the solution of (2.7) gives

$$\left.\frac{\mathcal{V}(\eta(t))}{dt}\right|_{(2.7)} \le -\Delta r(t)\mathcal{W}(\eta(t))+B_1,\ t>t_2. \tag{2.15}$$

For any given $\sigma > 0$, find $t_{1\sigma} > t_2$ so that $r(t) > \frac{2B_1}{\Delta\kappa_3(\kappa_2^{-1}(\sigma))}$ for all $t \ge t_{1\sigma}$. If $\mathcal{V}(\eta(t)) > \sigma$, then $\mathcal{W}(\eta(t)) \ge \kappa_3(\|\eta(t)\|) \ge \kappa_3(\kappa_2^{-1}(\mathcal{V}(\eta(t)))) \ge \kappa_3(\kappa_2^{-1}(\sigma))$. It follows that

$$\left.\frac{\mathcal{V}(\eta(t))}{dt}\right|_{(2.7)} \le -B_1 < 0. \tag{2.16}$$

So there exists a $t_{\sigma_2} > t_{\sigma_1}$ such that $\mathcal{V}(\eta(t)) \le \sigma$ for all $t > t_{\sigma 2}$. This shows that $\mathcal{V}(\eta(t)) \to 0$ as $t \to \infty$. Since $\|\eta(t)\| \le \kappa_1^{-1}(\mathcal{V}(\eta(t)))$, we finally get $\lim_{t\to\infty}\eta(t) = 0$.

Step 3. The convergence of $\mu(t) \to 0$.

From Step 1, $\mu(t)$ is uniformly bounded for $t > t_2$. This together with the continuity of $\frac{\partial V}{\partial x_n}$ shows that there exist a constant $C_1 > 0$ such that $(L+1)\sqrt{n+1}\left|\frac{\partial V}{\partial \mu_{n+1}}(\mu(t))\right| < C_1$. Hence the derivative of V along the solution of (2.7) satisfies

$$\left.\frac{dV(\mu(t))}{dt}\right|_{(2.7)} \le -W(\mu(t))+C_1\|\eta(t)\|. \tag{2.17}$$

For any $\sigma > 0$, by Step 2, one can find a $t_{\sigma 3} > t_2$ such that $\|\eta(t)\| < \frac{1}{2C_1}\varkappa_3(\varkappa_2^{-1}(\sigma))$ for all $t > t_{\sigma 3}$. It then follows that for $V(\mu(t)) > \sigma$ with $t > t_{\sigma 3}$,

$$\begin{array}{rl}\left.\dfrac{dV(\mu(t))}{dt}\right|_{(2.7)} & \le -W(\mu(t))+C_1\|\eta(t)\|\\ & < \varkappa_3(\varkappa_2^{-1}(V(\mu(t))))+C_1\|\eta(t)\|\\ & \le -\varkappa_3(\varkappa_2^{-1}(\sigma))+C_1\|\eta(t)\|\\ & \le -\frac{1}{2}\varkappa_3(\varkappa_2^{-1}(\sigma))<0.\end{array} \tag{2.18}$$

So, one can find a $t_{\sigma 4} > t_{\sigma 3}$ such that $V(\mu(t)) < \sigma$ for $t > t_{\sigma 4}$. This together with $\|\mu(t)\| \le \varkappa_1^{-1}(V(\mu(t)))$ gives $\lim_{t\to\infty}\mu(t) = 0$. This accomplishes the first part (i).

Part II. $r(t)$ satisfies (2.4)**.** In this case, the error should be separated in different intervals $\left[0, \frac{1}{a}\ln r_0\right)$ and $\left[\frac{1}{a}\ln r_0, \infty\right)$ respectively. In $\left[0, \frac{1}{a}\ln r_0\right)$, the error equation is the same as (2.7), while in $\left[\frac{1}{a}\ln r_0, \infty\right)$, the error satisfies

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\dot{\eta}_1(t)=r(t)(\eta_2(t)-g_1(\eta_1(t))),\\ \qquad\vdots\\ \dot{\eta}_n(t)=r(t)(\eta_{n+1}(t)-g_n(\eta_1(t))),\\ \dot{\eta}_{n+1}(t)=-r(t)g_{n+1}(\eta_1(t))+\dot{x}_{n+1}(t),\end{array}\right.\\ \left\{\begin{array}{l}\dot{\mu}(t)(t)=A_n\mu(t)+B_n[u_0(\hat{x}_1(t)-z_1(t),\\ \ldots,\hat{x}_n(t)-z_n(t))+\eta_{n+1}(t)],\\ \mu(t)=(\mu_1(t),\mu_2(t),\ldots,\mu_n(t))^\top,\end{array}\right.\end{array}\right. \tag{2.19}$$

which is similar to (2.7) but is a little bit simpler. So the convergence in this case is similar with the first part. The details are omitted. The proof is complete. ∎

To end this section, we present some numerical simulations for illustration.

Example 2.1: Consider the following control system:

$$\left\{\begin{array}{l}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=f(t,x_1(t),x_2(t),\zeta(t),w(t))+b(t)u(t).\\ \dot{\zeta}(t)=f_0(t,x_1(t),x_2(t),\zeta(t),w(t)),\\ y(t)=x_1(t),\end{array}\right. \tag{2.20}$$

where f, f_0 are unknown system functions, w is the external disturbance, control parameter b is also unknown but its nominal value b_0 is given, and y is the output. The control purpose is to design the output feedback control so that y tracks reference signal v.

In the numerical simulation, we set

$$\begin{array}{l}f_0(t,x_1,x_2,\zeta,w)\\ \quad=a_1(t)x_1+\sin x_2+\cos\zeta+w,\\ f(t,x_1,x_2,\zeta,w)\\ \quad=a_2(t)x_2+\sin(x_1+x_2)+\zeta+w,\\ a_1(t)=1+\sin t,\ a_2(t)=1+\cos t,\\ b(t)=10+0.1\sin t,\ b_0=10,\\ w(t)=1+\cos t+\sin 2t,\ v(t)=\sin t,\end{array} \tag{2.21}$$

It is easily to verify that Assumption (AC1) is satisfied.

In order to approximate states x_1, x_2 and extended state $x_3 \triangleq f(t,x_1,x_2,\zeta,w)+(b-b_0)u$ of system (2.1) by output $y = x_1$, the ESO is designed as (1.5) with $n = 2$, $g_1(x) = 6x+\phi(x)$, $g_2(x) = 11x$, $g_3(x) = 6x$, $x \in \mathbb{R}$, where

$$\phi(x)=\left\{\begin{array}{ll}-\frac{1}{4\pi}, & x\in(-\infty,-\pi/2),\\ \frac{\sin x}{4\pi}, & x\in(-\pi/2,\pi/2),\\ \frac{1}{4\pi}, & x\in(\pi/2,\infty).\end{array}\right. \tag{2.22}$$

It is easy to verify that functions g_i satisfies Assumption (AC2) with

$$\begin{array}{l}\mathcal{V}(y)=1.7y_1^2+0.7y_2^2+1.5333y_3^2-y_1y_2-1.4y_1y_3-y_2y_3,\\ \mathcal{W}(y)=\frac{1}{2}\left(y_1^2+y_2^2+y_3^2\right),\\ \kappa_1(x)=0.09x,\ \kappa_2(x)=2.33x^2,\\ \kappa_3(x)=\kappa_4(x)=\frac{1}{2}x^2,\ x\ge 0.\end{array}$$

The gain parameter is chosen as (2.4) with $a = 5$, $r_0 = 200$.

For simplicity, we use directly $z_1(t) = \sin t$, $z_2(t) = \cos t$, $z_3(t) = -\sin t$ as the target states. In the feedback control (2.1), u_0 is designed as

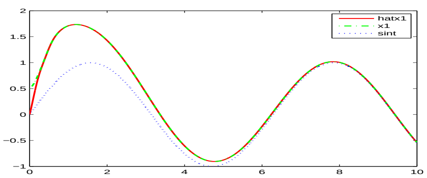
$$u_0(x_1,x_2)=-2x_1-4x_2-\phi(x_1). \tag{2.23}$$

Define the Lyapunov functions V, W and wedge functions $\varkappa_i, i = 1, 2, 3, 4$ in Assumption (AC3) as

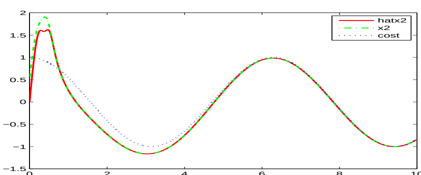
$$\begin{aligned} &V(x_1, x_2) = 1.375x_1^2 + 0.1875x_2^2 + 0.5x_1x_2, \\ &W(x_1, x_2) = 0.5x_1^2 + 0.5x_2^2, \\ &\varkappa_1(y) = 0.13y^2, \ \varkappa_2(y) = 1.43y^2, \\ &\varkappa_3(y) = \kappa_4(y) = 0.5y^2. \end{aligned} \tag{2.24}$$

One can also easily to verify that Assumptions (AC3) and (AC4) are satisfied.

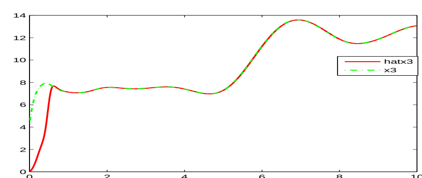
The numerical results are plotted in Figure 1. From Figure 1,



(a) x_1(green), $\hat{x}_1$(red), $\sin t$(blue)



(b) x_2(green), $\hat{x}_2$(red), $\cos t$(blue)



(c) x_3(green), $\hat{x}_3$(red)

Fig. 1. Numerical results of (2.20) for f to be chosen as (2.21).

we can see that: a) $\hat{x}_i (i = 1, 2, 3)$ of (1.5) converge to x_1, x_2 of system (2.1) and its total disturbance $x_3 \triangleq f(t, x_1, x_2, \zeta, w) + (b(t) - b_0)u(t)$ in a relative short time. The most remarkable fact is that the peaking value phenomena near the initial time is almost vanished; b) Under the feedback control, the output x_1 and its derivative x_2 tracks reference signal $\sin t$ and its derivative very satisfactorily without peaking problem.

III. Concluding Remarks

In this paper, we propose an extended state observer (ESO) and the active disturbance rejection control (ADRC) with time-varying gain parameter for a quite general class of nonlinear systems with large uncertainties from the dynamics, control, and the external disturbance. It is shown that the ESO is convergent for the open-loop system, which can be considered as an independent work. The closed-loop system under the ESO based output feedback control is also convergent, which is the separation principle for this special control strategy. In other words, the asymptotic stability for both the ESO and ADRC are achieved by rejecting the uncertainty. The most advantage of the control strategy lies in its estimation/cancellation strategy and hence the control energy can be significantly reduced as reported in control practice ([25]). In addition, by the proper choice of the time varying gain function, the peaking problems can be eliminated effectively. Some numerical simulations illustrate the effect of the control.

References

[1] A. Adams and John J.F. Fournier, *Sobolev Spaces*, 2nd edition, Academic press, 2003.
[2] R. Brockett, New issues in the mathematics of control, in: *"Mathematics Unlimited-2001 and Beyond"*, Springer-Verlag, Berlin, 2001, 189-219.
[3] A.M. Dabroom and H.K. Khalil, Discrete-time implementation of high-gain observers for numerical differentiation, *Internat. J. Control*, 12(1997), 1523-15371.
[4] L.B. Freidovich and H.K. Khalil, Performance recovery of feedback-linearization-based designs, *IEEE Trans. Automat. Control*, 53(2008), 2324-2334.
[5] Z.Q. Gao, Scaling and bandwith-parameterization based controller tuning, *American Control Conference*, 2003, 4989-4996.
[6] F.J. Goforth and Z.Q. Gao, An active disturbance rejection control solution for hysteresis compensation, *American Control Conference*, 2008, 2202-2208.
[7] B.Z. Guo, J.Q. Han and F.B. Xi, Linear tracking-differentiator and application to online estimation of the frequency of a sinusoidal signal with random noise perturbation, *Internat. J. Systems Sci.*, 33(2002), 351-358.
[8] B.Z. Guo and Z.L. Zhao, On convergence of nonlinear active disturbance rejection for SISO systems, *The 24th Chinese Control and Decision Conference*, 2012, 3524-3529.
[9] B.Z.Guo and Z.L. Zhao, On convergence of the nonlinear active disturbance rejection control for MIMO Systems, *SIAM J. Control Optim.*, to appear.
[10] B.Z. Guo and Z.L. Zhao, On the convergence of an extended state observer for nonlinear systems with uncertainty, *Systems Control Lett.*, 60(2011), 420-430.
[11] B.Z. Guo and Z.L. Zhao, On convergence of tracking differentiator, *Internat. J. Control*, 84(2011), 693-701
[12] J.Q. Han, A class of extended state observers for uncertain systems, *Control & Decision*, 10(1)(1995), 85-88 (in Chinese).
[13] J.Q. Han, The active disturbance rejection control and its applications, *Control & Decision*, 13(1)(1998), 19-23 (in Chinese).
[14] J.Q. Han, Control theory: Model approach or control approach, *J. Systems Sci. Math. Sci.*, 9(4)(1989), 328-335 (in Chinese).
[15] J.Q. Han, From PID to active disturbance rejection control, *IEEE Trans. Ind. Electron.*, 56(2009), 900-906.
[16] Y. Huang, W.C. Xue and X.X. Yang, Active disturbance rejection control: Methodology, theoretical analysis and applications, *29th Chinese Control Conference*, 2010, 6113-6120.
[17] H.K. Khalil, High-gain observers in nonlinear feedback control, *International Conference on Control, Automation and Systems*, Seoul, Korea, 2008, 47-57.
[18] H.K. Khalil, *Nonlinear Systems* , New Jersey, Prentice Hall, 2002.
[19] A. Levant, Robust exact differentiation via sliding mode technique, *Automatica*, 34(1998), 379-384.
[20] A. Medvedev and G. Hillerstrom, An extertnal model control system, *Control Theory Adv. Tech.*, 10(1995), 1643-1665.
[21] A. Teel and L. Praly, Tools for semiglobal stabilization by partial state and output feedback, *SIAM J. Control Optim*, 33 (1995), 1443-1488.
[22] D. Wu, Design and analysis of precision active disturbance rejection control for noncircular turning process, *IEEE Trans. Ind. Electron.*, 56(2009), 2746-2753.
[23] Q. Zheng, L.Gao and Z. Q. Gao, On stability analysis of active disturbance rejection control for nonlinear time-varying plants with unknow dynamics, *IEEE Conference on Decision and Control*, 2007, 3501-3506.
[24] Q. Zheng, L. Gong, D.H. Lee and Z.Q. Gao, Active disturbance rejection control for MEMS gyroscopes, *American Control Conference*, 2008, 4425-4430.
[25] Q. Zheng and Z. Gao, An energy saving, factory-validated disturbance decoupling control design for extrusion processes, *The 10th World Congress on Intelligent Control and Automation*, 2012.