Necessary and Sufficient Condition of Consensus for Affine Multi-Agent Cooperative Systems under Time-Varying Directed Networks

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***Abstract*—This work studies the necessary and sufficient condition for a class of affine multi-agent system under time varying directed network. A notion called $\infty-$*adjoin graph* of time-varying network, whose properties will be used as the criterion of system achieving consensus, is initiated. By a decomposition on $\infty-$*adjoin graph* of the time-varying directed network, the system may consist of several independent basic components and non-independent basic components of the system. Based on this composition and some assumptions on the system, we provided the necessary and sufficient condition for achieving consensus of systems. This work extends and modifies the results given in our previous paper[15].**

I. Introduction

The study of consensus for multi-agent systems can be traced back to Degroot[1](1974). Later, Tsitsiklis[2](1984) and Tsitsiklis, Bertsekas, Athans[3](1986) and others studied the similar model of Degroot's under time-varying and random communication network. Jadbabaie, Lin and Morse[4](2003) examined discrete multi-agent cooperative systems under time-varying symmetrical communication network. Moreau[5](2005) examined discrete-time multi-agent cooperative systems under time-varying asymmetric communication networks. Tahbaz-Salehi and Jadbabaie[6](2008) studied also about Degroot model under some time-varying communication network. Based on probability limit approach, [6] gave a necessary and sufficient conditions that the states of all agents reach consensus in stochastic process meaning.

Fax and Murray[7](2001) studied general consensus problems for continuous, first-order linear multi-agent cooperative system in networks. Olfati-Saber and Murray[8](2004), Lin, Francis and Maggiore[9](2005), Jin and Zheng[10](2009) studied the same problems under the communication networks, which are time-invariant directed networks. [8] gave the first important theoretical result, showing that if the communication network strongly connected, then the states of all agents reach consensus. [9] gave the necessary and sufficient conditions for the states of all agents reaching consensus. [10] studied the system, whose agents are with general dynamical properties, and pointed out that for such systems if they achieve consensus or not mainly depends on the topological structure of networks.

Jin and Zheng[11](2011) extended the linear models to affine nonlinear models and described some general characterization of the behavior for first-order affine multi-agent cooperative systems under time-invariant directed communication networks

Moreau[12](2004) studied first-order continuous multi-agent cooperative systems under time-varying directed communication networks. He gave a sufficient conditions that the states of all agents reach consensus. Cao, Zheng and Zhou[13](2008),[14](2011) studied Moreau model in the case that the communication networks are symmetric, and gave the necessary and sufficient conditions that the states of all agents reach consensus.

Jin and Zheng[15](2011) studied first-order affine multi-agent cooperative systems under time-varying directed communication networks. In a different approach from [12], [15] gave sufficient conditions of consensus under variety of communication structures.

Following [15] this paper further studies the consensus problems of affine multi-agent cooperative systems under time-varying directed communication networks. We pay main attention on the discussion of sufficient condition for consensus, where we provide a comprehensive proof on it. For the necessary conditions we modify the previous results in [16] without of proof.

II. Affine Multi-Agent Cooperative Systems

$\mathbf{A}$ is a set of agents. $a_i \in \mathbf{A}(i=1,\cdots,n)$ is an agent, whose state is x_i, $x_i \in \mathbf{R}^m$. a_i's neighbors is denoted by set $N(t,a_i)$, which includes all agents acting a_i at time t.

The neighbor sets of all agents define a directed network $\mathbf{G}(t) =< \mathbf{A}, \mathcal{E}(t) >$, where

$$\mathcal{E}(t) = \{(a_i,a_j)_t \in \mathbf{A}\times\mathbf{A} | a_i \in \mathbf{A}, a_j \in N(t,a_i)\}$$

$(a_i,a_j)_t \in \mathcal{E}(t)$ is a directed edge from a_j to $a_i (a_j \to a_i)$ at time t.

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$\mathbf{G}(t)$ is the communication network describing the action relations between the agents in $\mathbf{A}$ at time t.

In this work multi-agent dynamical system under *time-varying communication networks* can be written as

$$\dot{x}_i = f_i(x_i, x_{i_1}, \cdots, x_{i_s}) \qquad a_i \in \mathbf{A} \tag{1}$$

where

$$\{a_{i_1}, \cdots, a_{i_s}\} = N(t, a_i)$$

is the neighbor set of a_i at time t.

If the actions from $a_{i_p}(p = 1, \cdots, s)$ to a_i are of superposition, (1) can be written as

$$\dot{x}_i = \sum_{a_j \in N(t,a_i)} f_{ij}(x_i, x_j) \qquad a_i \in \mathbf{A} \tag{2}$$

Galilean Relativity: *A dynamical system is of* ***Galilean Relativity****[18] if its behavior is independent of observation coordinate systems, I.e. observers using different, but equivalent observation coordinate systems, will obtain the same observation result of system behaviors.*

When we study the natural phenomena, such as birds, fish, etc,(refer to Vicsek model[17](1995)) or some artificial systems such as multi-vehicle systems, multi-robot systems, etc, we will find all of them are of *Galilean Relativity*.

Now we assume that system (2) is of *Galilean Relativity*. When one takes any isometry on the state space of system (2), which is an Euclidean Space, denoted by a transformation $\mathcal{T} : \mathbf{R}^m \to \mathbf{R}^m;\ x \longmapsto \tilde{x}$ satisfying

$$\|\mathcal{T}x - \mathcal{T}y\|_2 = \|\tilde{x} - \tilde{y}\|_2 = \|x - y\|_2 \tag{3}$$

where $\|\bullet\|_2$ is Euclidean norm, one has following definition.

Definition *1:* [11] A multi-agent system (2) on $\mathbf{R}^m$ is an *affine dynamic system* if for any *isometry* $\mathcal{T}$

$$\begin{aligned} \frac{d\tilde{x}_i}{dt} &= \mathcal{T}\dot{x}_i = \sum_{a_j \in N(t,a_i)} f_{ij}(\mathcal{T}x_i, \mathcal{T}x_j) \\ &= \sum_{a_j \in N(t,a_i)} f_{ij}(\tilde{x}_i, \tilde{x}_j) \end{aligned}$$

The following result was given in paper [11].

Proposition *1:* [11] The multi-agent dynamic system (2) is *affine* if and only if each $f_{ij}(x_i, x_j)$ takes form:

$$f_{ij}(x_i, x_j) = f_{ij}(x_i - x_j) = q_{ij}\left(\|x_i - x_j\|_2\right)\vec{\mathbf{r}}_{ij} \tag{4}$$

where $q_{ij}(\bullet) \in \mathbf{R}$ is a scalar, $\vec{\mathbf{r}}_{ij} := \frac{x_i - x_j}{\|x_i - x_j\|_2}$ is the unit vector from x_j to x_i.

Definition *2:* [11] An *affine multi-agent dynamic system*

$$\dot{x}_i = f_i(x) = \sum_{a_j \in N(t,a_i)} -g_{ij}\left(\|x_i - x_j\|_2\right)\vec{\mathbf{r}}_{ij} \tag{5}$$

is a *cooperative system* if

$$g_{ij}\left(\|x_i - x_j\|_2\right)\begin{cases} = 0 & x_i = x_j \\ > 0 & x_i \neq x_j \end{cases} \tag{6}$$

where $g_{ij}(\bullet) \in \mathbf{R}$ is a scalar, $\vec{\mathbf{r}}_{ij} = \frac{x_i - x_j}{\|x_i - x_j\|_2}$ is the unit vector from x_j to x_i. g_{ij} is the *acting strength function* on a_i by a_j.

Assumption 1: The system (5) is smooth, i.e. the *action strength functions* $\{g_{ij}(\cdot);\ i, j \in \underline{n}\} \subset \mathbf{C}^1$

$x \in \mathbf{R}^m$, denote $x^{(k)}$ are k-th components of $x(k = 1, \cdots, m)$.

Definition *3:*

$$\dot{x}_i^{(k)} = \sum_{a_j \in N(t,a_i)} -\kappa_{ij}^k \frac{x_i^{(k)} - x_j^{(k)}}{\left|x_i^k - x_j^k\right|} \quad a_i \in \mathbf{A} \tag{7}$$

is the k-th *projection system* of *affine system* (5) where

$$\kappa_{ij}^k = g_{ij}\left(\|x_i - x_j\|\right)\frac{\left|x_i^{(k)} - x_j^{(k)}\right|}{\|x_i - x_j\|} \tag{8}$$

The following result was given in paper [11].

Proposition *2:* [11] The *projection systems* of *multi-agent cooperative system* (5) are *multi-agent cooperative systems*, i.e. κ_{ij}^k satisfy

$$\kappa_{ij}^k\begin{cases} = 0 & \left|x_i^{(k)} - x_j^{(k)}\right| = 0 \\ > 0 & \left|x_i^{(k)} - x_j^{(k)}\right| > 0 \end{cases}$$

III. Topology of Directed Graphs

$\mathbf{G} =< \mathbf{A}, \mathcal{E} >$ is a directed graph.

Definition *4: Connected relation* on $\mathbf{G}$

$\mathcal{W} \subset \mathbf{A} \times \mathbf{A}$ is a relation on $\mathbf{G}$ if

(1) $a_i \in \mathbf{A}$, then $(a_i, a_i) \in \mathcal{W}$;

(2) $(a_i, a_j) \in \mathcal{E}$, then $(a_i, a_j) \in \mathcal{W}$;

(3) $(a_i, a_k) \in \mathcal{W}$ and $(a_k, a_j) \in \mathcal{W}$, then $(a_i, a_j) \in \mathcal{W}$.

If $(a_i, a_j) \in \mathcal{W}$, then there are paths from a_j to a_i subject that $a_j \neq a_i$

One denotes $(a_i, a_j) \in \mathcal{W}$ by $\mathcal{W}(a_i, a_j)$.

Definition *5: Interconnected relation* on $\mathbf{G}$

$\mathcal{WW} \subset \mathbf{A} \times \mathbf{A}$ is a relation on $\mathbf{G}$ defined as follows. A pair $(a_i, a_j) \in \mathcal{WW}$ if both $\mathcal{W}(a_i, a_j)$ and $\mathcal{W}(a_j, a_i)$ hold. One denote $(a_i, a_j) \in \mathcal{WW}$ by $\mathcal{WW}(a_i, a_j)$.

If $\mathcal{WW}(a_i, a_j)$ exists, then we say that a_j interconnects to a_i subject that $a_j \neq a_i$.

It is easy to verify that $\mathcal{WW}$ is an equivalence relation on $\mathbf{A}$.

Define

$$\mathbf{A}_{a_i} = \{a_j | \mathcal{WW}(a_i, a_j)\}$$

which is called an interconnected equivalent on $\mathbf{A}$.

Let $\tilde{\mathbf{A}}$ be a partition of $\mathbf{A}$ and write

$$\widetilde{\mathbf{A}} = \{\mathbf{A}_1, \cdots, \mathbf{A}_s\} \tag{9}$$

Definition *6:* A class $\mathbf{A}_p(p \in \{1, \cdots, s\})$ of $\tilde{\mathbf{A}}$ is called a *basic set* of $\mathbf{A}$ if $\mathbf{A}_p$ is an interconnected equivalence class of $\mathbf{A}$.

Let (9) be the set of *basic sets* of $\mathbf{A}$. According to the graph theory ones have the following definition[19].

Definition *7:*

(1) $\mathbf{A}_p$ is a *basic set* of $\mathbf{A}$, the induced subgraph

$$\mathbf{G}_p =< \mathbf{A}_p, \mathcal{E} \cap (\mathbf{A}_p \times \mathbf{A}_p) >$$

is called a *maximum strongly connected subgraph* of $\mathbf{G}$ or a *strong component* of $\mathbf{G}$.

(2) If there is only one *basic set* $\mathbf{A} \in \tilde{\mathbf{A}}$, then the graph $\mathbf{G}$ is *strongly connected*.

Definition *8: Precursor closure*

(1) $a_i \in \mathbf{A}$, the *precursor closure* of a_i is defined as

$$P(a_i) = \{ a_j \in \mathbf{A} | \, \mathcal{W}(a_i, a_j) \} \, .$$

(2) $\mathbf{A}' \subset \mathbf{A}$, the *precursor closure* of $\mathbf{A}'$ is defined as

$$P(\mathbf{A}') = \bigcup_{a_i \in \mathbf{A}'} P(a_i) \; .$$

Definition *9:*

(1) $\mathbf{A}' \subset \mathbf{A}$ is an *independent subset* of $\mathbf{A}$ if $P(\mathbf{A}') \subset \mathbf{A}'$.

(2) $\mathbf{A}_p \in \tilde{\mathbf{A}}(p = 1, \cdots, s)$ is a *basic set* of $\mathbf{A}$. If $P(\mathbf{A}_p) = \mathbf{A}_p$, then $\mathbf{A}_p$ is called an *independent basic set* of $\mathbf{A}$, otherwise is called a *non-independent basic set* of $\mathbf{A}$.

(3) If $\mathbf{A}_p$ is an *independent basic set* of $\mathbf{A}$, then $\mathbf{G}_p$ is called an *independent strong component* of $\mathbf{G}$, otherwise is called a *non-independent strong component* of $\mathbf{G}$.

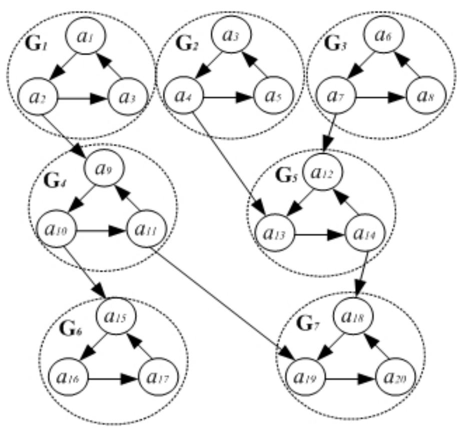

Fig. 1

In Fig.1 $\mathbf{G_1}, \mathbf{G_2}, \mathbf{G_3}$ are *independent components* of $\mathbf{G}$; The others are *non-independent*. For $\mathbf{A_6}$ it has $P(\mathbf{A_6}) = \{\mathbf{A_1}, \mathbf{A_4}, \mathbf{A_6}\}$. Also one has $P(\mathbf{A_1}) = \mathbf{A_1}$.

By definition, it is easy to verify the following proposition.

Proposition *3:* The following statements are equivalent.

(1) There is $a_j \in \mathbf{A}$ such that for any $a_i \in \mathbf{A}$ and $a_i \neq a_j$ a_j connects to a_i;

(2) There is only one *independent component* $\mathbf{G}_p =< \mathbf{A}_p, \mathcal{E} >$ in $\mathbf{G}$ and $a_j \in \mathbf{A}_p$.

In other worlds, there are spanning trees in $\mathbf{G}$ if $\mathbf{G}$ has the property (1) in above proposition. In this case a_j may be called a candidate root of spanning trees on $\mathbf{G}$. It is easy to verify that a_j is a candidate root if and only if there is only one *independent component* $\mathbf{G}_p =< \mathbf{A}_p, \mathcal{E}_p >$ in $\mathbf{G}$ and $a_j \in \mathbf{A}_p$.

$\mathbf{G}$ is *strongly connected* then there is only one *independent component* $\mathbf{G}$ in $\mathbf{G}$.

IV. Adjoin Graph of Time-Varying Network

In our work it is assumed that the *time-varying graph* $\mathbf{G}(t)$ be described by discrete-time series, i.e. there exist a sequence of time-invariant networks

$$\mathbf{G}_0, \cdots, \mathbf{G}_k, \cdots \tag{10}$$

such that when $t \in \mathbf{I}_k = [t_k, t_{k+1})$, $\mathbf{G}(t) = \mathbf{G}_k =< \mathbf{A}, \mathcal{E}_k >$. $\mathbf{I}_k = [t_k, t_{k+1}); k \geq 1$ are called *time-invariant intervals* of $\mathbf{G}(t)$.

Let

$$\tau_k = t_{k+1} - t_k$$

be the length of *time-invariant interval* $\mathbf{I}_k$, and define

$$\mathbf{I}_{(a_i, a_j)} := \{ \mathbf{I}_k = [t_k, t_{k+1}) | \, (a_i, a_j) \in \mathcal{E}_k; k \geq 1 \}$$

to be the set of *time-invariant intervals* $\mathbf{I}_k$ of $(a_i, a_j)_t \in \mathcal{E}(t)$.

$$l_{ij} = \sum_{\mathbf{I}_k \in \mathbf{I}_{(a_i, a_j)}} \tau_k = \sum_{\mathbf{I}_k \in \mathbf{I}_{(a_i, a_j)}} t_{k+1} - t_k$$

is called the *total connecting time-length* of $(a_i, a_j)_t$ on $\mathbf{G}(t); t \geq 0$.

The *total connecting time-lengths* of each edge of $\mathbf{G}(t)$ can be defined by an infinite integral of *adjacency matrix* of $\mathbf{G}(t)$.

Let

$$\mathbf{L}(t) = \begin{pmatrix} l_{11}(t) & \cdots & l_{1n}(t) \\ \vdots & \ddots & \vdots \\ l_{n1}(t) & \cdots & l_{nn}(t) \end{pmatrix}$$

be the *adjacency matrix* of $\mathbf{G}(t)$ where $l_{ij}(t) = 0$ for $(a_i, a_j)_t \notin \mathcal{E}(t)$ and $l_{ij}(t) = 1$ for $(a_i, a_j)_t \in \mathcal{E}(t)$.

Write

$$\begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} = \int_0^\infty \begin{pmatrix} l_{11}(t) & \cdots & l_{1n}(t) \\ \vdots & \ddots & \vdots \\ l_{n1}(t) & \cdots & l_{nn}(t) \end{pmatrix} dt$$

Then we can define an *adjoin matrix* of $\mathbf{L}(t)$ and denoted by $\mathbf{L}_\infty = [l'_{ij}]_{n \times n}$ such that

$$l'_{ij} = \begin{cases} 0 & l_{ij} < \infty \\ 1 & l_{ij} = \infty \end{cases}$$

Accordingly, we define the edge's set $\mathcal{E}_\infty$ of $\mathbf{G}_\infty$ such that $(a_i, a_j) \in \mathcal{E}_\infty$ if $l'_{ij} = 1$.

Definition *10:* [13] $\mathbf{G}_\infty =< \mathbf{A}, \mathcal{E}_\infty >$ is known as ∞-*adjoin graph* of $\mathbf{G}(t)$.

V. Lyapunov Stability Analysis

Under Assumption 1 we discuss the Lyapunov stability of system (5), which is a *cooperative system*.

Definition *11:* A convex set $\Xi(t)$ in $\mathbf{R}^m$ is called the *minimum convex cover-set* of $\{x_i(t) | a_i \in \mathbf{A}\}$ if any other convex cover-set $\Theta(t)$ of $\{x_i(t) | a_i \in \mathbf{A}\}$ in $\mathbf{R}^m$ must satisfy

$$\{x_i(t) | a_i \in \mathbf{A}\} \subset \Xi(t) \subset \Theta(t)$$

Theorem 1 was given in [11] and [15] originally.

Theorem *1:* For any $t \geq 0$ and $\delta t > 0$ the *minimum convex cover* $\Xi(t)$ of $\{x_i(t) | a_i \in \mathbf{A}\}$ of *multi-agent cooperative system* (5) satisfies

$$\Xi(t + \delta t) \subset \Xi(t).$$

According to Theorem 1 the following results can be obtained immediately.

Corollary *1:* The *multi-agent cooperative system* (5) is Lyapunov stable.

By *Weierstrass* Theorem, one can get following.

Corollary *2:* The limit of *minimum convex cover*

$$\lim_{t\to\infty}\Xi(t)=\Xi(\infty)$$

of *multi-agent cooperative system* (5) does exist.

VI. Sufficient condition for consensus

First we discuss the sufficient condition for achieving consensus.

Definition *12:* The sequence (10) of $\mathbf{G}(t)$ is *steady* if (10) is a finite or infinite sequence and satisfies that

1) There exist $\tau_{\min}>0$ for any *time-invariant intervals* $\mathbf{I}_k=[t_k,t_{k+1})$,

$$\tau_k=t_{k+1}-t_k>\tau_{\min}$$

2) There exist τ_B for any $(a_i,a_j)\in\mathcal{E}_\infty$, (a_i,a_j) must appear in $\mathbf{G}(t)$ in any time interval $(t,t+\tau_B)$.

By Proposition 2, the *projection systems* of *multi-agent cooperative system* (5) are *multi-agent cooperative systems*. If g_{ij} are functions of $\mathbf{C}^1$, then $k_{ij}^k(k=1,\cdots,m)$ in *projection systems* (7) are functions of $\mathbf{C}^1$. Every *projection systems* of *multi-agent cooperative system* (5) achieve consensus then the system (5) achieve consensus. Therefore, we only need to consider the situation that $x_i\in\mathbf{R}^1$.

Let $X(t)=\{\,x_i(t)|\,a_i\in A\}$, then write

$$\Delta(t)=[\underline{\Delta}(t),\overline{\Delta}(t)]=[\min X(t),\max X(t)]$$

which is the *minimum convex cover* of $X(t)$. By Corollary 2,

$$[\Delta_1,\Delta_2]=\lim_{t\to\infty}\Delta(t)$$

does exist. This means that for any $\varepsilon>0$ there exist t_k as $t>t_k$,

$$x_i(t)\in(\Delta_1-\varepsilon,\Delta_2+\varepsilon)\qquad i=1,\cdots,n$$

In what follows, it is always assumed that Assumption 1 is satisfied by system (5).

Proposition *4:* For *multi-agent cooperative system* (5) there exists constant $\gamma>0$ such that

$$\dot{x}_i\le\gamma\left(\Delta_2+\varepsilon-x_i\right)\qquad\forall\ a_i\in\mathbf{A}$$

Proof: Consider a closed interval $[0,\Delta_2-\Delta_1+2\varepsilon]$. Let

$$\gamma_{ij}=\max_{y\in[0,\Delta_2-\Delta_1+2\varepsilon]}\frac{g_{ij}(y)}{y}$$

By (6) and Assumption 1, $\{\gamma_{ij}\}$ do exist and each $\gamma_{ij}>0$.

$$\begin{aligned}\dot{x}_i&=\sum_{a_j\in N(t,a_i)}-\tfrac{g_{ij}(\|x_i-x_j\|)}{\|x_i-x_j\|}\left(x_i-x_j\right)\\&=\sum_{a_j\in N(t,a_i)}\tfrac{g_{ij}(\|x_i-x_j\|)}{\|x_i-x_j\|}\left(x_j-x_i\right)\end{aligned}$$

Let

$$\gamma'=\max\{\gamma_{ij}\,|i,j\in\underline{n}\}\qquad x_i,x_j\in(\Delta_1-\varepsilon,\Delta_2+\varepsilon)$$

$$\begin{aligned}\dot{x}_i&\le\sum_{a_j\in N(t,a_i)}\gamma'\left((\Delta_2+\varepsilon)-x_i\right)\\&\le\sum_{j\ne i}\gamma'\left((\Delta_2+\varepsilon)-x_i\right)=(n-1)\gamma'\left((\Delta_2+\varepsilon)-x_i\right)\end{aligned}$$

Let $\gamma=(n-1)\gamma'$, $\dot{x}_i\le\gamma\left(\Delta_2+\varepsilon-x_i\right).$ □

Lemma *1:* For *multi-agent cooperative systems* (5) it holds that given any two very small positive real numbers ε and ε' if $x_i(t_k)\in(\Delta_2-\varepsilon',\Delta_2+\varepsilon)$, then for a given $\tau>0$, when $t\in(t_k-\tau,t_k)$, $x_i(t)\in(\Delta_2-\varepsilon'',\Delta_2+\varepsilon)$ where $\varepsilon''>0$ is equivalently infinitesimal to ε and ε'.

Proof: Assume that $x_i(t_k-\tau)<\Delta_2-\varepsilon'$ and a_i tend to $x_i(t_k)=\Delta_2-\varepsilon'$. By proposition 4,

$$\dot{x}_i<\gamma\left((\Delta_2+\varepsilon)-x_i\right)\tag{11}$$

By (11), $x_i(t_k-\tau)>(\Delta_2+\varepsilon)-\left(\varepsilon+\varepsilon'\right)e^{\gamma\tau}$.

$$x_i(t_k-\tau)\in\left(\Delta_2+\varepsilon-\left(\varepsilon+\varepsilon'\right)e^{\gamma\tau},\Delta_2+\varepsilon\right)$$

As $\tau,\gamma>0$, $\left(\varepsilon+\varepsilon'\right)e^{\gamma\tau}-\varepsilon>0$. Let $\varepsilon''=\left(\varepsilon+\varepsilon'\right)e^{\gamma\tau}-\varepsilon$,

$$x_i(t_k-\tau)\in(\Delta_2-\varepsilon'',\Delta_2+\varepsilon)\qquad\square$$

Lemma *2:* Assume that the sequence (10) of $\mathbf{G}(t)$ is *steady* and $(a_i,a_j)\in\mathcal{E}_\infty$. If the *multi-agent cooperative system* (5) satisfies that, when $t\in(t_k-\tau_B,t_k]$,

$$x_i(t)\in(\Delta_2-\varepsilon'',\Delta_2+\varepsilon)$$

then there exist $t_k'\in(t_k-\tau_B,t_k]$

$$x_j(t_k')\in(\Delta_2-\varepsilon''',\Delta_2+\varepsilon)$$

where ε''' is equivalently infinitesimal to ε and ε''.

Proof: If the sequence (10) of $\mathbf{G}(t)$ is *steady*, then there exist $(t_k'-\tau_{\min},t_k')\subset(t_k-\tau_B,t_k)$ such that $(a_i,a_j)_t\in\mathcal{E}(t)$ when $t\in(t_k'-\tau_{\min},t_k')$. Assume that

$$Max\{x_j(t);t\in(t_k'-\tau_{\min},t_k')\}=\Delta_2-\sigma<\Delta_2-\varepsilon''$$

i.e.

$$x_j(t)\le\Delta_2-\sigma<\Delta_2-\varepsilon'';\ t\in(t_k'-\tau_{\min},t_k')$$

and the σ is independent of ε and ε''. Let

$$V_i^L=\min_{y\in(\Delta_1-\varepsilon,\Delta_2-\sigma],z\in(\Delta_2-\varepsilon'',\Delta_2+\varepsilon)}\{g_{ij}\left(\|y-z\|\right)\}$$

By (6), when $\sigma>\varepsilon''$

$$V_i^L>0$$

and one can verify that $V_i^L\to 0$ only if $\sigma-\varepsilon''\to 0$. Let

$$V_i^R=\max_{y\in(\Delta_2-\varepsilon'',\Delta_2+\varepsilon),z\in[y,\Delta_2+\varepsilon)}\{g_{ij}\,\|y-z\|\}$$

By (6), $g_{ij}(0)$=0. By Assumption 1, one writes

$$V_i^R=\varepsilon_v$$

where $\varepsilon_v > 0$ is equivalently infinitesimal to $\varepsilon, \varepsilon''$.

$$\dot{x}_i = \sum_{a_j \in N(t,a_i)} -g_{ij}\left(\|x_i - x_j\|\right)\frac{x_i - x_j}{\|x_i - x_j\|} = \dot{x}_i^L + \dot{x}_i^R$$

where

$$\begin{aligned}\dot{x}_i^L &= \sum_{a_j \in N(t,a_i), x_j < x_i} -g_{ij}\left(\|x_i - x_j\|\right)\tfrac{x_i - x_j}{\|x_i - x_j\|} \\ &= -\sum_{a_j \in N(t,a_i), x_j < x_i} g_{ij}\left(\|x_i - x_j\|\right) < 0 \\ \dot{x}_i^R &= \sum_{a_s \in N(t,a_i), x_s \geq x_i} -g_{ij}\left(\|x_i - x_s\|\right)\tfrac{x_i - x_s}{\|x_i - x_s\|} \\ &= \sum_{a_s \in N(t,a_i), x_s \geq x_i} g_{ij}\left(\|x_i - x_s\|\right) \geq 0\end{aligned}$$

When $t \in (t_k' - \tau_{\min}, t_k')$,

$$\dot{x}_i^L \leq -g_{ij}\left(\|x_i - x_j\|\right) \leq -V_i^L$$

One can further verify that

$$\dot{x}_i = \dot{x}_i^L + \dot{x}_i^R \leq -V_i^L + (n-2)V_i^R$$

or write

$$\dot{x}_i = \dot{x}_i^L + \dot{x}_i^R \leq -V_i^L + (n-2)\varepsilon_v \qquad (12)$$

As $\sigma > \varepsilon''$, the first term of the right side of (12), $-V_i^L$, is negative constant and interdependent of ε and ε''. Now one has

$$\int_{t_k' - \tau_{\min}}^{t_k'} \dot{x}_i(t)dt \leq -V_i^L \tau_{\min} + (n-2)\varepsilon_v \tau_{\min}$$

Furthermore, one gets

$$|x_i(t_k') - x_i(t_k' - \tau_{\min})| \geq V_i^L \tau_{\min} - (n-2)\varepsilon_v \tau_{\min}$$

However, when ε is small enough, there is δ such that

$$(V_i^L - (n-2)\varepsilon_v)\tau_{min} > \delta > 0$$

But, according to given condition

$$\left|x_i(t'_{k-\tau_{\min}}) - x_i(t_k')\right| \leq \varepsilon'' + \varepsilon$$

The right side of above inequality could be arbitrarily small. It is a contradiction. That implies the σ must dependent of ε'' and satisfying that $\varepsilon'' - \sigma \to 0$ when $\varepsilon'' \to 0$, which verify the existence of the ε''' as the Lemma claimed. □

A sufficient condition for achieving consensus is given in the following Theorem.

Theorem *2: Multi-agent cooperative system* (5) achieve consensus if

(1) The sequence (10) of $\mathbf{G}(t)$ is *steady*; and

(2) there is only one *independent strong component* in ∞-*adjoin graph* $\mathbf{G}_\infty$ of $\mathbf{G}(t)$.

Proof: We prove this theorem in two steps.

(I) Assume for the system $\mathbf{G}_\infty$ is *strongly connected*. Let

$$[\Delta_1, \Delta_2] = \lim_{t\to\infty} \Delta(t); \quad \Delta_2 > \Delta_1$$

For any $\varepsilon > 0$ there exist t_s such that

$$x_i(t) \in (\Delta_1 - \varepsilon, \Delta_2 + \varepsilon) \quad \forall a_i \in \mathbf{A},\ t > t_s$$

Furthermore, for any $t > t_s$ there exist $a_p, a_q \in \mathbf{A}$ such that

$$x_p(t) \in (\Delta_1 - \varepsilon, \Delta_1] \qquad x_q(t) \in [\Delta_2, \Delta_2 + \varepsilon)$$

Suppose there is $t_{k_0} = t_s + n\tau_B$ such that $x_{i_0}(t_{k_0}) \in [\Delta_2, \Delta_2 + \varepsilon)$. By Lemma 1, when $t \in (t_{k_0} - \tau_B, t_{k_0})$,

$$x_{i_0}(t) > \Delta_2 - \varepsilon_0(\varepsilon)$$

where $\varepsilon_0(\varepsilon)$ is equivalently infinitesimal to ε.

Let $(a_{i_0}, a_{i_1}) \in \mathcal{E}_\infty$. By Lemma 2, there exist $t_{k_1} \in (t_{k_0} - \tau_B, t_{k_0})$

$$x_{i_1}(t_{k_1}) \in (\Delta_2 - \varepsilon'(\varepsilon), \Delta_2 + \varepsilon)$$

and $\varepsilon'(\varepsilon)$ is equivalently infinitesimal to ε. By Lemma 1,

$$x_{i_1}(t) > \Delta_2 - \varepsilon_1(\varepsilon)$$

for $t \in (t_{k_1} - \tau_B, t_{k_1})$ and $\varepsilon_1(\varepsilon)$ is equivalently infinitesimal to ε.

Repeating previous procedure, one finds a serious of time instants such that

$$\begin{cases} t_{k_0} = t_s + n\tau_B \\ t_{k_1} \in (t_{k_0} - \tau_B, t_{k_0}) \\ \cdots \\ t_{k_p} \in (t_{k_{p-1}} - \tau_B, t_{k_{p-1}}) \end{cases}$$

where $p \leq n$ is the longest length of paths among agents, such that

$$x_{i_p}(t) > \Delta_2 - \varepsilon_p(\varepsilon)$$

when $t \in (t_{k_p} - \tau_B, t_{k_p})$ and $\varepsilon_0(\varepsilon), \cdots, \varepsilon_p(\varepsilon), \cdots$ are equivalently infinitesimal to ε and all $(a_{i_0}, a_{i_1}), \cdots, (a_{i_{p-1}}, a_{i_p}), \cdots$ are in $\mathcal{E}_\infty$.

Let

$$t_{k_0} - \tau_0 = t_{k_1} - \tau_1 = \cdots = t_{k_p} - \tau_B$$

By Lemma 1, accordingly we have serious of estimations such that

$$\begin{cases} x_{i_0}(t) > \Delta_2 - \varepsilon_0'(\varepsilon) \\ x_{i_1}(t) > \Delta_2 - \varepsilon_1'(\varepsilon) \\ \cdots \\ x_{i_p}(t) > \Delta_2 - \varepsilon_p'(\varepsilon) \end{cases}$$

when, respectively,

$$\begin{cases} t \in (t_{k_0} - \tau_0, t_{k_0}) = (t_{k_p} - \tau_B, t_{k_0}) \\ t \in (t_{k_1} - \tau_1, t_{k_1}) = (t_{k_p} - \tau_B, t_{k_1}) \\ \cdots \\ t \in (t_{k_p} - \tau_B, t_{k_p}) \end{cases}$$

Let

$$\tilde{\varepsilon}(\varepsilon) = \max\left\{\varepsilon_0'(\varepsilon), \varepsilon_1'(\varepsilon), \cdots, \varepsilon_p'(\varepsilon)\right\}$$

where the $\tilde{\varepsilon}(\varepsilon)$ is equivalently infinitesimal to ε. Thus, it is possible to choose ε small enough so that $\tilde{\varepsilon}(\varepsilon) \ll \Delta_2 - \Delta_1$. Then, for all $s = 0, 1, \cdots, p$,

$$x_{i_s}(t) > \Delta_2 - \tilde{\varepsilon}(\varepsilon)$$

when $t \in (t_{k_p} - \tau_B, t_{k_p})$.

As $\mathbf{G}_\infty$ is *strongly connected*, for any $a_i \neq a_{i_0}$ there is a path from a_i to a_{i_0} in $\mathbf{G}_\infty$. According to the previous discussion, there exist a time interval $(t' - \tau_B, t') \subset (t_s, t_s + n\tau_B)$ and the $\tilde{\varepsilon}(\varepsilon)$ such that

$$x_i(t) > \Delta_2 - \tilde{\varepsilon}(\varepsilon) > \Delta_1$$

when $t \in (t' - \tau_B, t')$. It implies that there is moment t such that

$$\underline{\Delta}_1(t) > \Delta_2 - \tilde{\varepsilon}(\varepsilon) > \Delta_1$$

It is a contradiction as, according to Theorem 1, $\underline{\Delta}_1(t)$ is monotonously increasing and its limit is Δ_1.

(II) There is one *independent strong component* in $\mathbf{G}_\infty$.

By (I), all agents in *independent strong component*

$$\mathbf{G}_\infty^1 =< \mathbf{A}_1, \mathcal{E}_\infty^1 >$$

of $\mathbf{G}_\infty$ will tend to a consistent state $\delta \in [\Delta_1, \Delta_2]$. Let

$$[\Delta_1, \Delta_2] = \lim_{t \to \infty} \Delta(t)$$

and

$$\delta \neq \Delta_2$$

For any $\varepsilon > 0$ there exist t_r as $t > t_r$

$$x_j(t) \in (\delta - \varepsilon, \delta - \varepsilon) \qquad a_j \in \mathbf{A}_1$$

For any $\varepsilon > 0$ there exist a_{i_0}

$$x_{i_0}(t) \in [\Delta_2, \Delta_2 + \varepsilon)$$

Let $t_s = \max\{t_r, t_s\}$, then $t > t_s$

$$a_{i_0} \in \mathbf{A} - \mathbf{A}_1$$

But by Proposition 3 there are paths from a_j to a_{i_0}. According to the previous discussion, there exist a time interval

$$(t' - \tau_B, t') \subset (t_s, t_s + n\tau_B)$$

and $\varepsilon(\varepsilon)$ which is equivalently infinitesimal of ε

$$x_j(t) > \Delta_2 - \varepsilon(\varepsilon) \qquad a_j \in \mathbf{A}_1$$

as $t \in (t' - \tau_B, t')$. This is a contradiction to $\delta \neq \Delta_2$. Similarly, one can show that $\delta \neq \Delta_1$ is not true. Therefor

$$\Delta_1 = \delta = \Delta_2 \qquad \square$$

VII. Necessary and sufficient condition for consensus

Under all assumptions given in this paper one has following.

Theorem *3:* [16] *Multi-agent cooperative system* (5) achieves consensus only if there is one *independent strong component* in its *∞-adjoin graph* $\mathbf{G}_\infty$ of $\mathbf{G}(t)$.

The proof of Theorem 3 is given in paper[16]. We do not provide the proof in this paper to save space.

Now we give the main result in this paper. By Theorem 2 and Theorem 3 one has

Theorem *4:* Assume the sequence (10) of $\mathbf{G}(t)$ is *steady*. *Multi-agent cooperative system* (5) achieves consensus if and only if there is only one *independent strong component* in its *∞-adjoin graph* $\mathbf{G}_\infty$ of $\mathbf{G}(t)$.

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