A Simple Approach to Nonlinear State Feedback Design

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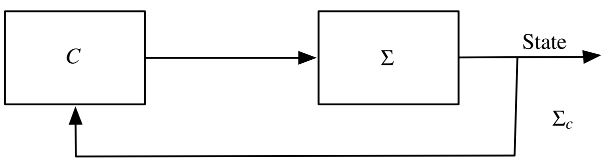
***Abstract*— Global state feedback controllers that asymptotically and robustly stabilize a nonlinear system are derived from the solution of inequalities obtained directly from the controlled system's equation.**

I. INTRODUCTION

We consider the design of state feedback controllers that drive a nonlinear system Σ from given initial conditions into a specified *target domain* in state space. It turns out that such feedback controllers can be derived from the solution of inequalities obtained directly from quantities given in the differential equation of the controlled system Σ. Furthermore, such feedback controllers exist if and only if these inequalities have a solution. When the target domain is a tight neighborhood of the origin, the technique yields asymptotic stabilization.

The control configuration is described in the figure below, where Σ is the controlled system, C is a state feedback controller, and Σ_c denotes the closed loop system.



The control objective is as follows.

Problem 1: Let Σ be an input/state system with the set X_0 of potential initial conditions, and let D_0 be an open domain in state space serving as the target domain. Find necessary and sufficient conditions for the existence of a state feedback controller C that takes Σ from every initial condition in X_0 into D_0 in finite time. If such a controller exists, provide a method for its design. □

State feedback controllers that solve Problem 1 are derived in section III from the solution of a set of inequalities. The inequalities are obtained directly from quantities given in the differential equation of Σ. Furthermore, whenever solvable, Problem 1 can be solved by *static* state feedback controllers – controllers that are described by a feedback function rather than by a differential equation. The controllers are robust: they can tolerate small implementation errors as well as small errors in the model of Σ. When the target domain D_0 is a tight neighborhood of the origin, these controllers yield asymptotic stabilization of Σ (section VI).

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Explicitly, the controlled system is described by

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ t \geq 0, \\ x(0) = x_0, \end{array} \tag{1}$$

where $f : R^n \times R^m \to R^n$ is a continuous function, $x(t) \in R^n$ and $u(t) \in R^m$ are the state and the input of Σ at the time t, and x_0 is the initial state.

In section III we show that, if a solution of Problem 1 exists, then it can be chosen as a static state feedback controller described by a state feedback function $\varphi : R^n \to R^m$. The input $u(t)$ of Σ is then given by $u(t) = \varphi(x(t))$, and the closed loop system Σ_φ is

$$\Sigma_\varphi : \begin{array}{l} \dot{x}(t) = f(x(t), \varphi(x(t))),\ t \geq 0, \\ x(0) = x_0. \end{array} \tag{2}$$

In sections II and III we show that an appropriate state feedback function φ can be calculated from the solution of a set of inequalities derived directly from the function f given in the differential equation (1) of the controlled system Σ. Furthermore, Problem 1 has a solution if and only if this set of inequalities has a solution.

We provide now a simplified (and somewhat inaccurate) summary of the process that leads to the solution of Problem 1 in sections II and III. At a state $x \in R^n$, let $U_1(x)$ be the set of all input values $u \in R^m$ for which the vector $f(x,u)$ points from x to the target domain D_0. Let D_1 be the set of all states $x \in R^n$ at which the set $U_1(x)$ is not empty. The set D_1 is derived by solving an inequality induced by the function f of (1).

At each point $x \in D_1$, choose a value $u(x) \in U_1(x)$ and define the state feedback function $\varphi(x) := u(x)$. Then, by (2), the path derivative $\dot{x}(t) = f(x(t), \varphi(x(t)))$ of Σ_φ is directed toward D_0 at every point $x(t) \in D_1$. Consequently, the state $x(t)$ of Σ_φ moves toward D_0 as time progresses.

Having built the set D_1, consider the difference set $D_2' := R^n \setminus D_1$ of the remaining states. At a state $x \in D_2'$, denote by $U_2(x)$ the set of all input values $u \in R^m$ for which the vector $f(x,u)$ points from x to a point of the set D_1. Let D_2 be the set of all points $x \in D_2'$ at which $U_2(x)$ is not empty. As before, at each point $x \in D_2$, choose a value $u(x) \in U_2(x)$ and define the state feedback function $\varphi(x) := u(x)$. The set $U_2(x)$ and the domain D_2 are calculated by solving an inequality based on the function f given in (1). Then, the derivative $\dot{x}(t) = f(x(t), \varphi(x(t)))$ points toward D_1 at all points $x(t)$ of D_2, and the state trajectory $x(t)$ takes Σ_φ from every point of D_2 toward D_1. Once a point of D_1 is reached, the values of the state feedback function φ previously defined on D_1 take Σ_φ to the target domain D_0. The resulting state feedback

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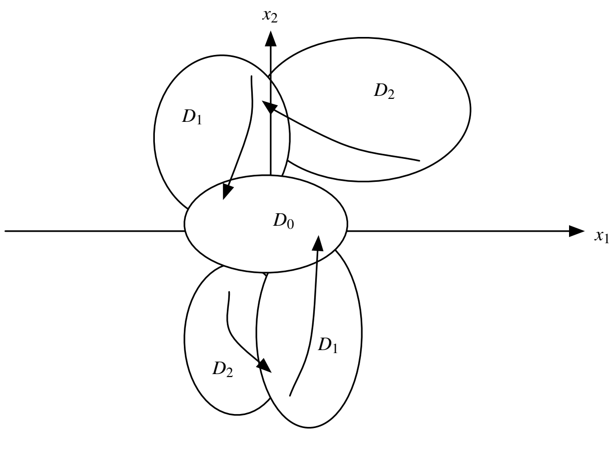
function φ takes Σ_φ to the target domain D_0 from all points of the union $D_1 \cup D_2$.

Continuing in this way, we build a sequence of domains $D_1, D_2, \ldots \subseteq R^n$: having derived the domains $D_1, D_2, \ldots, D_i$ for an integer $i \geq 1$, consider the difference set

$$D'_{i+1} := R^n \setminus \left(\bigcup_{j=0,\ldots,i} D_j \right).$$

At a state $x \in D'_{i+1}$, let $U_{i+1}(x)$ be the set of all input values $u \in R^m$ for which the vector $f(x,u)$ points from x to a point of D_i. Denote by D_{i+1} the set of all points $x \in D'_{i+1}$ at which $U_{i+1}(x)$ is not empty. The domain D_{i+1} is obtained from the solution of an inequality based on the given function f. At each point $x \in D_{i+1}$, choose a value $u(x) \in U_{i+1}(x)$ and define the state feedback function $\varphi(x) := u(x)$. Then, by (2), the path $x(t)$ of Σ_φ points toward D_i at all points of D_{i+1}; hence, Σ_φ moves from every point of D_{i+1} toward D_i. Once a point of D_i is reached, previously defined values of φ on D_i take Σ_φ into D_{i-1}. From there, previously defined values of φ take Σ_φ into D_{i-2}, and so on, until Σ_φ reaches the target domain D_0.

Schematically, the progression of Σ_φ toward the target domain D_0 can be described as in the figure below. Note that D_i may not be a connected set.



The resulting state feedback function φ takes Σ into D_0 in finite time from any point of

$$S(D_0) := \bigcup_{i \geq 0} D_i.$$

In section III we show that this is an exclusive feature of the set $S(D_0)$: there is no state feedback controller, not static nor dynamic, that can take Σ into D_0 in finite time from a state outside $S(D_0)$. Thus, a state feedback controller solving Problem 1 exists if and only if

$$X_0 \subseteq S(D_0).$$

When choosing the target domain D_0 as a tight neighborhood of the origin, our discussion leads to state feedback controllers that asymptotically stabilize Σ, as discussed in section VI. This results in a simple approach to global stabilization of nonlinear system by state feedback. The critical step is the solution of a set of inequalities based on the function f given in the differential equation (1) of Σ.

The fact that Problem 1 can be solved by a static state feedback controller does not imply that dynamic state feedback controllers are insignificant, since they offer broader capabilities of assigning the dynamical behavior of the closed loop system Σ_c. The static state feedback controllers derived here can be utilized to obtain a fraction representation of Σ; using such a fraction representation, one can derive dynamical state feedback controllers that assign desirable dynamics to the closed loop system ([5], [6], [7], [8]).

Alternative approaches to the control of nonlinear systems can be found in [11], [12], [5], [6], [18], [3], [17], [7], [2], [14], [16], [19], [15], [8], [1], [4], [13] and [9], in the references cited in these publications, and elsewhere.

This note is organized as follows. Section II presents basic concepts and notation. The derivation of state feedback controllers that solve Problem 1 is discussed in Section III. The issue of robustness is examined in Section IV, and Section V demonstrates the proposed technique with a detailed example. Robust asymptotic stabilization is studied in section VI, and section VII consists of a few concluding remarks.

II. Preliminaries

A. Notation

In practice, systems usually have bounds on the maximal input amplitude they can tolerate. To accommodate such bounds, we adopt the following assumption, where $|u| := \sqrt{u_1^2 + u_2^2 + \cdots + u_m^2}$ is the Euclidian norm of a vector $u = (u_1, u_2, \ldots, u_m) \in R^m$.

Assumption 1: The controlled system Σ permits only input signals u of magnitude $|u| \leq M$, where $M > 0$ is a specified real number.

Under Assumption 1, the solution $x(t)$ of (1) is a continuous function of time.

We denote by $B(s,\rho)$ the open ball of center $s \in R^n$ and radius $\rho > 0$, namely, $B(s,\rho) := \{x \in R^n : |x - s| < \rho\}$.

With a non-zero vector $z \in R^n$ we associate a *unit vector* $\hat{z}$ in the direction of z

$$\hat{z} := \begin{cases} z/|z| & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

where $\hat{z} = 0$ when $z = 0$. Then, $\hat{}$ is a continuous function at nonzero arguments.

The straight line segment that connects two distinct points $y, z \in R^n$ is

$$\ell(y,z) := \{\alpha(z - y) + y : \alpha \in [0,1]\}. \tag{3}$$

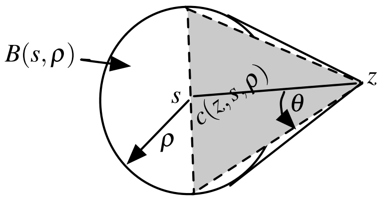
We build the following n-dimensional body.

Definition 1: Given two points $z, s \in R^n$ and a real number $\rho > 0$, the *ball-cone* $\chi(z,s,\rho)$ consists of all straight line segments that start in the open ball $B(s,\rho)$ and end at z:

$$\begin{aligned}\chi(z,s,\rho) &:= \bigcup_{y \in B(s,\rho)} \ell(y,z) \\ &= \{x \in R^n : x = \alpha(z - y) + y, \alpha \in [0,1], y \in B(s,\rho)\}.\end{aligned}$$

Here, z is the *apex* of $\chi(z,s,\rho)$ and s is the *base center*. $\square$
A ball-cone, pictured in the figure below, is akin to a cone, except that it has a ball instead of a cone's 'flat' base. The shaded area in the figure is a right circular cone $c(z,s,\rho)$ with vertex z, base center s, and radius ρ. The angle θ between the generator and the axis of $c(z,s,\rho)$ is called the *opening angle* of the ball-cone $\chi(z,s,\rho)$. It satisfies

$$\rho = |s-z|\tan\theta. \tag{4}$$



A ball-cone $\chi(z,s,\rho)$

B. Directional Error

Suppose it is necessary to take the system Σ of (1) from a state z to a state $s \neq z$ along the straight line segment $\ell(z,s)$ of (3). For this to happen, the derivative $\dot{x} = f(x,u)$ must point in the direction from z to s at all points $x \in \ell(z,s)$. In other words, at every state $x \in \ell(z,s)$, there must be an input value $u(x)$ for which $f(x,u(x))$ points in the direction of the unit vector $\widehat{(s-z)}$. As every point x of $\ell(z,s)$ is characterized by the triplet z,s,α of (3), we can write $u(z,s,\alpha)$ instead of $u(x)$. Then, Σ can be driven along $\ell(z,s)$ if and only if there are inputs $u(z,s,\alpha) \in R^m$ such that

$$\hat{f}(\alpha(z-s)+s,u(z,s,\alpha)) = \widehat{(s-z)} \text{ for all } \alpha \in [0,1]. \tag{5}$$

To be robustly implementable, (5) must be modified into a form that allows for small errors. To this end, let $\varepsilon > 0$ be a real number. Allowing a *directional error* of ε, we replace (5) by the requirement that there be an input function $u(z,s,\alpha) \in R^m$ satisfying

$$\left|\hat{f}(\alpha(z-s)+s,u(z,s,\alpha)) - \widehat{(s-z)}\right| < \varepsilon \tag{6}$$

for all $\alpha \in [0,1]$. The expression "an error of ε" refers to all errors of magnitude not exceeding ε.

Clearly, (6) must be valid at all states through which Σ might pass on its way. Due to the directional error, the motion of Σ may not be confined to the straight line segment $\ell(z,s)$. To determine the states through which Σ may pass, let $\theta(\varepsilon) \geq 0$ be the supremal angle between the two unit vectors $\hat{f}(\alpha(z-s)+s,u(z,s,\alpha))$ and $\widehat{(s-z)}$ consistent with a directional error of ε, namely, the angle between $\hat{f}(\alpha(z-s)+s,u(z,s,\alpha))$ and $\widehat{(s-z)}$ when $\left|\hat{f}(\alpha(z-s)+s,u(z,s,\alpha)) - \widehat{(s-z)}\right| = \varepsilon$.

Now, a ball-cone $\Gamma(z,s,\varepsilon)$ with opening angle $\theta(\varepsilon)$ and base center s has, by (4), the base radius

$$\rho(z,s,\varepsilon) = |z-s|\tan\theta(\varepsilon). \tag{7}$$

It is given by

$$\Gamma(z,s,\varepsilon) = \bigcup_{y\in B(s,\rho(z,s,\varepsilon))} \ell(z,y)$$
$$= \{x \in R^n : x = \alpha(z-y)+y, \alpha \in [0,1], y \in B(s,\rho(z,s,\varepsilon))\},$$

We show below that condition (6) must be valid within the entire ball-cone $\Gamma(z,s,\varepsilon)$ in order to be meaningful.

C. Ball-cones

In this subsection, we lay the foundation for proving the following fact: with a directional error of ε, a system that starts at a state z and moves toward s, stays within the closure $\overline{\Gamma}(z,s,\varepsilon)$ of the ball-cone $\Gamma(z,s,\varepsilon)$ until reaching the vicinity of s.

Proposition 1: Let $z,s \in R^n$ be two distinct points, and let $\varepsilon > 0$ be a directional error for which the opening angle of the ball-cone $\Gamma(z,s,\varepsilon)$ satisfies $\theta(\varepsilon) < \pi/4$. Then, $\overline{\Gamma}(z',s,\varepsilon) \subseteq \overline{\Gamma}(z,s,\varepsilon)$ for all $z' \in \overline{\Gamma}(z,s,\varepsilon)$.

Proof: (sketch) A point $z' \in \overline{\Gamma}(z,s,\varepsilon)$ is of the form $z' = \beta(z-y)+y$ for some $\beta \in [0,1]$ and $y \in \overline{B}(s,\rho(z,s,\varepsilon))$. The ball-cone with vertex at z' has the base radius $\rho(z',s,\varepsilon) = |z'-s|\tan\theta(\varepsilon)$ according to (7), and is given by

$$\overline{\Gamma}(z',s,\varepsilon) := \left\{ x \in R^n \,\middle|\, \begin{array}{l} x = \alpha(z'-y')+y', \\ \alpha \in [0.1], y' \in \overline{B}(s,\rho(z',s,\varepsilon)) \end{array} \right\}$$
$$= \left\{ x \in R^n \,\middle|\, \begin{array}{l} x = \alpha(\beta(z-y)+y-y')+y', \\ \alpha,\beta \in [0.1], y \in \overline{B}(\rho(z,s,\varepsilon), \\ y' \in \overline{B}(\rho(z',s,\varepsilon) \end{array} \right\}. \tag{8}$$

We need to show that the point

$$x = \alpha(\beta(z-y)+y-y')+y' \tag{9}$$

of (8) is in $\overline{\Gamma}(z,s,\varepsilon)$ for all $\alpha,\beta \in [0,1]$. To this end, rewrite (9) as

$$x = \alpha\beta(z-\eta)+\eta, \tag{10}$$

where

$$\eta = y - \frac{1-\alpha}{1-\alpha\beta}(y-y') \text{ for all } \alpha,\beta \in [0,1] \text{ satisfying } \alpha\beta \neq 1.$$

Denoting $\gamma(\alpha,\beta) := \frac{1-\alpha}{1-\alpha\beta}$, an examination shows that $\gamma(\alpha,\beta) \in [0,1]$ for all $\alpha,\beta \in [0,1], \alpha\beta \neq 1$. Therefore, η is a point on the straight line segment connecting y and y'. As $\theta(\varepsilon) < \pi/4$ by the proposition's assumption, $\rho(z',s,\varepsilon) < \rho(z,s,\varepsilon)$. Noting that the closed balls $\overline{B}(s,\rho(z,s,\varepsilon))$ and $\overline{B}(s,\rho(z',s,\varepsilon))$ are concentric with center at s, that $y \in \overline{B}(s,\rho(z,s,\varepsilon))$, and that $y' \in \overline{B}(s,\rho(z',s,\varepsilon))$, it follows by convexity that the straight line segment connecting y and y' is in $\overline{B}(s,\rho(z,s,\varepsilon))$; hence, so is η. By (10), this implies that $x \in \overline{\Gamma}(z,s,\varepsilon)$. $\blacksquare$

Considering that directional errors are usually small and that Proposition 1 is important to our discussion, we impose the following.

Assumption 2: The directional error satisfies $\theta(\varepsilon) < \pi/4$.

Notation 1: For a real number $\beta > 0$, the symbol $0(\beta)$ represents the set of all functions $\omega : R^n \to R^n$ for which $\lim_{\beta\to 0} |\omega(\beta)|/\beta = 0$. $\square$

The next statement is a technical refinement of Proposition 1 proved in [10]. It will help us show that, upon moving from z to s with a directional error of ε, the system stays within the ball-cone $\overline{\Gamma}(z,s,\varepsilon)$ until reaching a vicinity of s.

Lemma 1: Let $x, x', s, z \in R^n$ be points for which $x' - x = \beta\widehat{a} + \mu(x',x)$, where $\beta > 0$ is a real number, $\widehat{a}$ is a unit vector satisfying $|\widehat{a} - \widehat{(s-x)}| < \varepsilon$, and $\mu(x',x) \in 0(\beta)$. If $x \in \overline{\Gamma}(z,s,\varepsilon)$, then also $x' \in \overline{\Gamma}(z,s,\varepsilon)$ for sufficiently small $\beta > 0$. $\square$

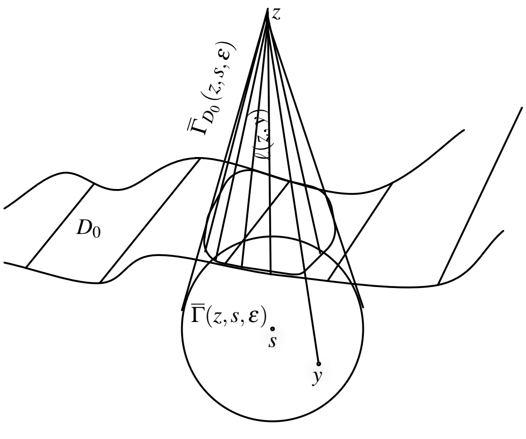
D. Interception

Let $\Delta(s,\varepsilon)$ be the set of all states at which the trajectory of Σ can be pointed in the direction of a state $s \in R^n$ with a directional error of ε. Incorporating Assumption 1, we have

$$\Delta(s,\varepsilon) = \left\{ x \in R^n \setminus s \;\middle|\; \begin{array}{l} |\hat{f}(x,u(x)) - \widehat{(s-x)}| < \varepsilon \\ \text{for some } u(x) \in R^m \\ \text{satisfying } |u(x)| \leq M \end{array} \right\}. \quad (11)$$

Problem 1 requires a feedback controller to take Σ from an initial state $x_0 = z$ into a target domain D_0. The path of Σ from z to D_0 is unpredictable due to directional errors, but Proposition 2 below shows that the closed loop system remains within the closed ball-cone $\overline{\Gamma}(z,s,\varepsilon)$. Therefore, if every path through $\overline{\Gamma}(z,s,\varepsilon)$ meets the target domain D_0, then D_0 will be reached, regardless of uncertainties. This motivates the following notion.

Definition 2: An open domain $D_0 \subseteq R^n$ *intercepts* the ball-cone $\Gamma(z,s,\varepsilon)$ if $\ell(z,y) \cap D_0 \neq \varnothing$ for all $y \in \overline{B}(s,\rho(z,s,\varepsilon))$. $\square$



Interception means that every ray from the apex z within $\overline{\Gamma}(z,s,\varepsilon)$ meets D_0, as depicted above. Note that we are interested in the "upper" part of the ball-cone – the part between the apex z and the set D_0; after that, D_0 is reached.

Definition 3: Let D_0 be an open subset of R^n that intercepts the ball-cone $\Gamma(z,s,\varepsilon)$, and denote by $\check{D}_0 := \overline{D}_0 \setminus D_0$ the boundary of D_0. The the *restriction* $\overline{\Gamma}_{D_0}(z,s,\varepsilon)$ is the set

$$\overline{\Gamma}_{D_0}(z,s,\varepsilon) := \begin{cases} \bigcup \ell(y,z) \;\middle|\; \begin{array}{l} y \in \overline{\Gamma}(z,s,\varepsilon) \cap \check{D}_0 \\ \text{and } \ell(y,z) \cap D_0 = \varnothing \end{array} & \text{if } z \notin D_0, \\ \varnothing & \text{if } z \in D_0. \end{cases}$$

When $z \notin D_0$, the restriction $\overline{\Gamma}_{D_0}(z,s,\varepsilon)$ consists of all points of $\overline{\Gamma}(z,s,\varepsilon)$ that are between the apex z and the target domain D_0, including z and the 'upper' boundary of D_0. Being a subset of $\overline{\Gamma}(z,s,\varepsilon)$, it is a bounded set. A closer examination shows that it is also a closed set (see [10] for details), and the following is true.

Lemma 2: Let $D_0 \subseteq R^n$ be an open set that intercepts the ball-cone $\Gamma(z,s,\varepsilon)$. Then, the restriction $\overline{\Gamma}_{D_0}(z,s,\varepsilon)$ is a compact set. $\square$

III. State Feedback Functions

A. State Feedback and Target Domains

The states from which the controlled system can be driven into the target domain along a path resembling a straight line are characterized as follows.

Proposition 2: Let Σ be a system described by (1), where the function f is continuous. Let $z, s \in R^n$ be a pair of points, let $\varepsilon > 0$ be a real number, let $\Delta(s,\varepsilon)$ be given by (11), and let $D_0 \subseteq R^n$ be an open domain that intercepts $\Gamma(z,s,\varepsilon)$. If $\overline{\Gamma}_{D_0}(z,s,\varepsilon) \subseteq \Delta(s,\varepsilon)$, then there is a state feedback function φ with directional error of ε that takes Σ from z into D_0 in finite time.

Proof: (sketch) Let $x(t)$ be the state of Σ at time t, and let $x(0) := z$ be the initial state. As $x(0) \in \overline{\Gamma}_{D_0}(z,s,\varepsilon) \subseteq \Delta(s,\varepsilon)$ by assumption, there is an input value $u(x(0)) \in R^m$ for which $|\hat{f}(x(0),u(x(0))) - \widehat{(s-x(0))}| < \varepsilon$. Now, let $\tau \geq 0$ be a real number for which the following is true for all $t \in [0,\tau]$: $x(t) \in \overline{\Gamma}_{D_0}(z,s,\varepsilon)$, and there is a state feedback function $u(x(t))$ that drives Σ with a directional error of ε toward s. Define

$$T := \sup \tau. \quad (12)$$

By (1), we can write for a real number $\delta > 0$ that

$$x(t+\delta) = x(t) + f(x(t),u(x(t)))\,\delta + 0(\delta). \quad (13)$$

By (12), there are times $t_1, t_2, ... \in [0,T]$ converging to T such that $x(t_i) \in \overline{\Gamma}_{D_0}(z,s,\varepsilon)$ for all $i = 1,2,...$ As $\overline{\Gamma}_{D_0}(z,s,\varepsilon)$ is compact by Lemma 2, the sequence $\{x(t_i)\}_{i=1}^{\infty}$ has a convergent subsequence $\{x(t_{i_k})\}_{k=1}^{\infty}$ and $\lim_{k\to\infty} x(t_{i_k}) \in \overline{\Gamma}_{D_0}(z,s,\varepsilon)$. As $x(t)$ is the solution of (1) with bounded input (Assumption 1), $x(t)$ is a continuous function of time; hence, $\lim_{k\to\infty} t_{i_k} = T$ implies $\lim_{k\to\infty} x(t_{i_k}) = x(T)$, so that $x(T) \in \overline{\Gamma}_{D_0}(z,s,\varepsilon)$. Since $\overline{\Gamma}_{D_0}(z,s,\varepsilon) \subseteq \Delta(s,\varepsilon)$ by assumption, $x(T) \in \Delta(s,\varepsilon)$. Thus, there is an input value $u(x(T)) \in R^m$ at which $|\hat{f}(x(T),u(T)) - \widehat{(s-x(T))}| < \varepsilon$. Using (13) with $t = T$ and Lemma 1, it follows there is a real number $\delta_1 > 0$ such that $x(T+\delta) \in \overline{\Gamma}_{D_0}(z,s,\varepsilon)$ for all $0 < \delta < \delta_1$, contradicting the supremality of T. Thus, $x(T+\delta) \in D_0$. $\blacksquare$

B. Expansion Sets

By Proposition 2, the system Σ can be driven into the target domain D_0 by a state feedback function with directional error of ε from any point of the set

$$E^1_f(D_0,\varepsilon) := \left\{ z \in R^n \;\middle|\; \begin{array}{l} D_0 \text{ intercepts } \Gamma(z,s,\varepsilon) \text{ for some} \\ s \in R^n \text{ and } \overline{\Gamma}_{D_0}(z,s,\varepsilon) \subseteq \Delta(s,\varepsilon). \end{array} \right\}$$

Definition 4: $E_f^1(D_0, \varepsilon)$ is the *expansion set* of D_0 relative to f with directional error of ε. □

By Definitions 2 and 3,

$$D_0 \subseteq E_f^1(D_0, \varepsilon). \tag{14}$$

Proposition 2 can now be restated as follows.

Proposition 3: Let Σ be a system described by (1) with a continuous function f. Let $\varepsilon > 0$ be a real number, let $D_0 \subseteq R^n$ be an open domain, and let $E_f^1(D_0, \varepsilon)$ be the expansion set. Then, there is a state feedback function φ with directional error of ε that takes Σ from every initial state $z \in E_f^1(D_0, \varepsilon)$ into D_0 in finite time. □

The fact that the target domain is an open set together with the continuity of the function f of (1) implies the following (see [10] for detailed proof).

Lemma 3: Let Σ be a system described by (1) with a continuous function f, and let D_0 be an open domain in R^n. Then, the expansion set $E_f^1(D_0, \varepsilon)$ is an open set for every $\varepsilon > 0$. □

C. More Expansion Sets

Based on Definition 4, we build a sequence of sets $E_f^0(D_0, \varepsilon), E_f^1(D_0, \varepsilon), E_f^2(D_0, \varepsilon), \ldots$, where, for $i = 0, 1, 2, \ldots$,

$$E_f^{i+1}(D_0, \varepsilon) := E_f^1(E_f^i(D_0, \varepsilon), \varepsilon), E_f^0(D_0, \varepsilon) := D_0.$$

By (14), we have $E_f^i(D_0) \subseteq E_f^{i+1}(D_0), i = 0, 1, 2, \ldots$ In these terms, Proposition 3 becomes

Proposition 4: Let $\varepsilon > 0$ be a real number, and let D_0 be an open domain in R^n. There is a static state feedback controller with directional error of ε that drives Σ from every state $z \in E_f^{i+1}(D_0, \varepsilon)$ into $E_f^i(D_0, \varepsilon)$, $i = 0, 1, \ldots$ □

Similarly, Lemma 3 yields

Lemma 4: Let Σ be a system described by the differential equation (1) with a continuous function f, and let D_0 be an open domain in R^n. Then, the expansion set $E_f^i(D_0, \varepsilon)$ is an open set for all $\varepsilon > 0$ and all $i = 1, 2, \ldots$ □

We have arrived at the main notion of our discussion.

Definition 5: Let D_0 be an open domain in R^n, let $f : R^n \times R^m \to R^n$ be a continuous function, and let $\varepsilon > 0$ be a real number. The *expansion* $E_f(D_0, \varepsilon)$ of D_0 with respect to f and ε is $E_f(D_0, \varepsilon) := \cup_{i \geq 0} E_f^i(D_0, \varepsilon)$. □

The following is a main result.

Theorem 1: Let Σ be a system described by the differential equation (1) with a continuous function f, and let D_0 be an open domain in R^n. Then, (i) and (ii) are equivalent.

(i) There is a state feedback function with directional error of ε that drives Σ from a state $z \in R^n$ into D_0 in finite time.

(ii) $z \in E_f(D_0, \varepsilon)$.

Furthermore,

(iii) For a state $z \notin E_f(D_0, \varepsilon)$, there is no state feedback controller – not static nor dynamic – that drives Σ from z into D_0 in finite time with a directional error of ε.

Proof: *(sketch)* Consider a point $z \in E_f(D_0, \varepsilon)$. By (5), there is a first integer i such that $z \in E_f^i(D_0, \varepsilon)$. By Proposition 4, there then is a state feedback function φ with a directional error of ε that drives Σ from z to a point $z_1 \in E_f^{i-1}(D_0, \varepsilon)$ in finite time. Similarly, the state feedback function φ can be extended to take Σ from z_1 to a point $z_2 \in E_f^{i-2}(D_0, \varepsilon)$ in finite time, and so on, until Σ reaches a point $z_i \in D_0$, and it follows that *(ii)* implies *(i)*. For proofs that *(i)* implies *(ii)* and that *(iii)* is valid, see [10]. ■

In [10] it is shown that the feedback function of Theorem 1 can be selected to be piecewise continuous.

Theorem 1 provides a simple and effective method for calculating state feedback functions that drive a given system Σ into a target domain with a directional error of ε: at each state x of $E_f^i(D_0, \varepsilon)$, choose a state feedback function φ for which the vector $f(x, \varphi(x))$ points to a point of $E_f^{i-1}(D_0, \varepsilon)$. Such a value of φ is obtained by solving an inequality based on the function f – the function given in the differential equation of the controlled system Σ. An example is provided in Section V.

IV. Robust Control

A *robust implementation* of a state feedback controller is an implementation with an unspecified nonzero directional error. Robust implementation depend on the next notion.

Definition 6: Let $f : R^n \times R^m \to R^n$ be a continuous function, and let D_0 be an open domain in R^n. The *super extension set* of D_0 with respect to f is $E_f(D_0) := \cup_{\varepsilon > 0} E_f(D_0, \varepsilon)$. □

A slight reflection yields the following consequence of Theorem 1 (see [10] for details).

Theorem 2: Let Σ be a system described by the differential equation (1) with a continuous function f. Let D_0 be an open domain in R^n, and let $E_f(D_0)$ be the super expansion set. Then, (i) and (ii) are equivalent.

(i) There is a robust implementation of a static state feedback controller that drives Σ from a state $z \in R^n$ into D_0 in finite time.

(ii) $z \in E_f(D_0)$.

Furthermore,

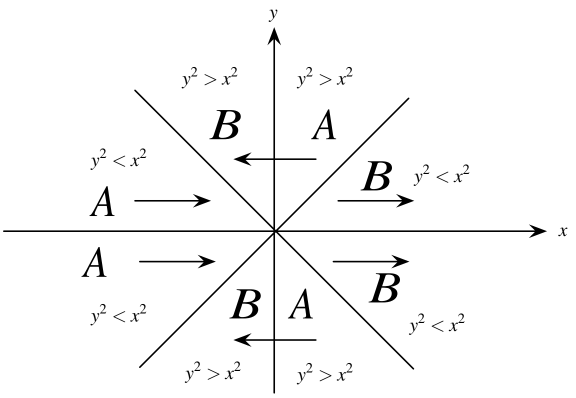
(iii) For a state $z \notin E_f(D_0)$, there is no robust implementation of a state feedback controller – not a static nor a dynamic controller – that takes Σ from z into D_0 in finite time. □

V. Example

Example 1: Consider the system

$$\Sigma : \begin{array}{l} \dot{x} = x^2 - y^2 \\ \dot{y} = u \end{array} = f(x, y, u),$$

with the target domain $D_0 = B(0, 0.1)$. For the sake of simplicity, we ignore here the input magnitude bound M; it can be readily incorporated. Note that when $x^2 > y^2$, we have $\dot{x} > 0$, so the state moves generally to the right; when $x^2 < y^2$, we have $\dot{x} < 0$, and the state moves generally toward the left, as indicated by the arrows in the figure below. As $\dot{y} = u$, the vertical 'tilt' of the state's trajectory can be assigned by selecting u. Thus, f can be pointed toward the origin by selecting an appropriate u only in the domains marked A in the figure, so that $E_f^1(D_0) = \{(x, y) : y^2 > x^2 \text{ and } x > 0; \text{ or } y^2 < x^2 \text{ and } x < 0\}$.



A slight reflection shows that, in the remaining parts of the plane, f can be pointed toward $E_f^1(D_0)$ from every point by selecting an appropriate u, so that $E_f^2(D_0) = R^2 \setminus E_f^1(D_0)$. The domains that form $E_f^2(D_0)$ are marked B in the figure above. According to Theorem 2, there is a state feedback function φ that takes Σ to a close vicinity of the origin from every bounded domain in state space.

To obtain a state feedback function φ, define the domains

$$A' := \left\{ (x,y) \,\middle|\, \begin{array}{l} x^2 - y^2 < 0, x > 0, |y/x| < 100; \text{ or} \\ x^2 - y^2 > 0, x < 0, |y/x| < 100; \end{array} \right\}$$

and $B' := R^2 \setminus A'$. Then, an examination shows that the following feedback function assigns directions to $f(x,y,\varphi(x,y))$ that point to the origin from within A', and point to A' from within B':

$$\varphi(x,y) := \begin{cases} (x^2 - y^2)y/x \text{ if } (x,y) \in A'; \\ (5(x^2 - y^2) + 1) \text{ if } (x,y) \notin A' \text{ and } y \geq 0, x > 0; \\ (5(x^2 - y^2) - 1) \text{ if } (x,y) \notin A' \text{ and } y > 0, x < 0; \\ (5(y^2 - x^2) - 1) \text{ if } (x,y) \notin A' \text{ and } y \leq 0, x > 0; \\ (5(y^2 - x^2) + 1) \text{ if } (x,y) \notin A' \text{ and } y < 0, x < 0. \end{cases}$$

VI. Asymptotic Stabilization

In this section, we seek a state feedback function φ for which the state of Σ_φ approaches the origin asymptotically as $t \to \infty$. We assume that Σ has a stationary point at the origin, i.e., that $f(0,0) = 0$. To find such a state feedback function φ, we proceed in two steps:
(i) Use the technique of Theorem 2 to find a state feedback function φ_1 that brings Σ from the initial state into a close vicinity $V = B(0,\rho)$ of the origin, where $\rho > 0$ is 'small'.
(ii) Use the linear approximation of Σ at the origin

$$\dot{x}(t) = \frac{\partial f(0,0)}{\partial x} x(t) + \frac{\partial f(0,0)}{\partial u} u(t) \tag{15}$$

to derive a linear state feedback function φ_2 that takes Σ asymptotically to the origin from within V.

Patching φ_1 and φ_2 together into one function φ yields a state feedback function that drives Σ asymptotically from an initial state x_0 to the origin, yielding asymptotic stabilization. This leads to the following statement.

Theorem 3: Let Σ be a system described by the differential equation (1), where the function f is twice continuously differentiable and $f(0,0) = 0$. Assume that the linear approximation (15) of Σ at the origin forms a stabilizable linear system, and let $X_0 \subseteq R^n$ be the set of all potential initial states of Σ. Then, there is a real number $\rho^* > 0$ for which the following two statements are equivalent.
(i) Σ is robustly and asymptotically stabilizable over the domain X_0 of initial states.
(ii) $X_0 \subseteq E_f(B(0,\rho^*))$. □
A computation of ρ^* is provided in [10].

VII. Conclusion

A general framework was developed for the design of nonlinear state feedback controllers. The main step of this framework involves the solution of a set of inequalities based on the function f given in the differential equation (1) of the controlled system Σ. As the example of section V demonstrates, the calculation of stabilizing state feedback controllers is relatively simple in this framework.

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