On a trolley-like problem in the presence of a nonlinear friction and a bounded fuel expenditure

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***Abstract*—We consider a problem of maximization of the distance traveled by a material point in the presence of a nonlinear friction under a bounded thrust and fuel expenditure. Using the maximum principle we obtain the form of optimal control and establish conditions under which it contains a singular subarc. This problem seems to be the simplest one having a mechanical sense in which singular subarcs appear in a nontrivial way.**

INTRODUCTION

We consider the following optimal control problem:

$$\begin{cases} \dot{s} = x, & s(0) = 0, & s(T) \to \max, \\ \dot{x} = u - \varphi(x), & x(0) = 0, & x(t) \text{ is free} \\ \dot{m} = -u, & m(0) = m_0, & m(T) \geq m_T, \\ u \in [0, 1]. \end{cases} \tag{1}$$

Here $s(t)$ and $x(t)$ are one-dimensional position and velocity of a vehicle, $m(t)$ describes the total mass of vehicle's body and fuel, $u(t)$ is the rate of fuel expenditure, $\varphi(x)$ is a twice smooth function describing the "friction" (media resistance) depending on the velocity. We assume that $\varphi(0) = 0$, $\varphi'(0) \geq 0$, and $\varphi''(x) > 0$ for all $x > 0$. This object can be considered as a material point moving along a horizontal track and being forced by a nonnegative thrust. Our aim is to maximize the distance passed by the object in a given time T under a fuel limitation. Here $m_T \in (0, m_0)$ is the mass of "empty" vehicle without fuel.

This problem can also be considered as a simplification of the Goddard problem [1] in the following two aspects: first, in the equation for acceleration we do not take into account the change of the mass of the object; second, the object moves in a horizontal, not in a vertical direction, which mathematically means that if the speed is zero and the thrust is not applied, the speed keeps zero value on.

It is well-known that a typical feature of optimal trajectories in the Goddard problem is the presence of singular arcs. However, in the original Goddard problem it is hardly possible to investigate optimality conditions analytically, and so, even quantitative properties of optimal trajectories are obtained by using numerical calculations (see, e.g. [12]-[13]).

Problem (1) includes rather simple equations, which allow us to determine the form of optimal trajectories analytically. At the same time, these trajectories still may contain singular subarcs for some typical forms of the friction function φ.

I. MAXIMUM PRINCIPLE FOR PROBLEM (1)

Let $s(t), x(t), m(t), u(t),\ t \in [0, T]$ be an optimal process. According to the Pontryagin Maximum Principle (MP), there exist constants $(\alpha_0, \alpha, \beta_s, \beta_x, \beta_m)$, not all identically zero, and Lipschitz functions $\psi_s(t)$, $\psi_x(t), \psi_m(t)$, that generate *the endpoint Lagrange function*

$$l = -\alpha_0 s(T) - \alpha(m(T) - m_T) + \\ + \beta_s s(0) + \beta_x x(0) + \beta_m (m(0) - m_0) \tag{2}$$

and *the Pontryagin function*

$$H(s, x, m, u) = \psi_s\, x + \psi_x\, (u - \varphi(x)) - \ \psi_m\, u, \tag{3}$$

such that the following conditions are satisfied:

(a) nonnegativity condition: $\alpha_0 \geq 0,\ \ \alpha \geq 0$,

(b) nontriviality condition:

$$(\alpha_0, \alpha, \beta_s, \beta_x, \beta_m) \neq (0, 0, 0, 0, 0),$$

(c) complementarity slackness condition:

$$\alpha(m(T) - m_T) = 0, \tag{4}$$

(d) costate (adjoint) equations

$$\begin{cases} -\dot{\psi}_s = H_s = 0, \\ -\dot{\psi}_x = H_x = \psi_s - \psi_x \varphi'(x), \\ -\dot{\psi}_m = H_m = 0. \end{cases} \tag{5}$$

(e) transversality conditions:

$$\begin{cases} \psi_s(0) = \beta_s, & \psi_s(T) = \alpha_0, \\ \psi_x(0) = \beta_x, & \psi_x(T) = 0, \\ \psi_m(0) = \beta_m, & \psi_m(T) = \alpha. \end{cases} \tag{6}$$

(f) the "energy conservation law": $H(s, x, m, u) \equiv const$,

(g) and the maximality condition: for almost all t,

$$\max_{0\le u'\le 1} H(s(t),x(t),m(t),u') = \\ = H(s(t),x(t),m(t),u(t)). \quad (7)$$

According to (5)-(6), in order to simplify further computations we set $\psi_x \equiv \alpha_0$, $\psi_m \equiv \alpha$ and write $\psi(t)$ instead of $\psi_x(t)$.Then the maximality condition (7) gives us optimal control in the form

$$u(t) \in \mathrm{Sign}^+(\psi-\alpha), \quad (8)$$

where the set-valued function

$$\mathrm{Sign}^+(z) = \begin{cases} \{1\}, & z>0, \\ [0,1], & z=0, \\ \{0\}, & z<0, \end{cases}$$

and the costate$\psi(t)$ is determined by the equation

$$\begin{cases} \dot{\psi} = -\alpha_0 + \psi_x \varphi'(x), \\ \psi(T) = 0. \end{cases} \quad (9)$$

Recall that by definition $\Delta m = m_0 - m_T > 0$. If $\Delta m \ge T$ then the optimal control is obviously $u \equiv 1$. So, in further considerations we assume that

$$0 < \Delta m < T. \quad (10)$$

II. Analysis of the maximum principle

Consider first the abnormal case $\alpha_0 = 0$. Then equation (9) for $\psi(t)$ restricts to a homogeneous one, and condition $\psi(T) = 0$ yields $\psi(t) \equiv 0$. Hence $\beta_x = 0$ (see (6)) and nontriviality condition gives us $\alpha > 0$. Then (8) yields $u(t) \equiv 0$, and from equations (1) we have $m(t) = const = m_0$, which contradicts complementarity slackness condition (4). Hence the normal case $\alpha_0 > 0$ is realised and we may take $\alpha_0 = 1$.

Proposition 2.1: $\psi(t) > 0$ for all $t < T$.

Proof: According to (9), $\dot{\psi}(T) = -1$. Since $\dot{\psi}$ is continuous, $\psi(t) > 0$ in a left neighborhood of T. Suppose there exists $t' < T$ such that $\psi(t') = 0$ and $\psi(t) > 0$ on (t',T). From (9) we have $\dot{\psi}(t') = -1$ again, which contradicts the previous inequality. ■

Proposition 2.2: $\alpha > 0$.

Proof: Suppose that $\alpha = 0$. Then from (8) we have $u \equiv 1$ for a.a. t. Hence $\Delta m = T$, which contradicts (10). ■

From the last proposition and (8) it follows that there exists $t_2 < T$ such that $u = 0$ for a.a. $t \in (t_2,T)$. Moreover, since $\alpha > 0$, condition (4) gives $m(T) = m_T$ and hence

$$\int_0^T u\,dt \;=\; \Delta m > 0. \quad (11)$$

Note that if $\varphi(x)$ is linear (i.e. only assumptions $\varphi(0) = 0$ and $\varphi'(0) \ge 0$ are hold) the analysis of PMP is more simple. Let us consider the case of $\varphi(x) = \gamma x$, where $\gamma > 0$. Then (9) determines $\psi(t) = \left(1 - e^{\gamma(t-T)}\right)/\gamma$ which is positive on $[0,T]$ and decreases monotonically from $\psi(0) > 0$ to 0. In this case the optimality condition (8) leads to that optimal control always has a bang-bang form $u = (1,0)$ on $((0,\Delta m),(\Delta m,T))$.

Consider the set $M_0 = \{t : \psi(t) = \alpha\}$. Obviously, M_0 is closed. Moreover, it is not empty (otherwise $\psi < 0$ on $(0,T)$, hence $u \equiv 0$, which contradicts (11)).

Proposition 2.3: M_0 is a connected set.

Proof: Suppose the opposite. Then there exists an interval $\omega = (t',t'')$ such that $\psi(t') = \psi(t'') = \alpha$ and either i) $\psi(t) < \alpha$ on ω, or ii) $\psi(t) > \alpha$ on ω.

Consider the case i). Since $\dot{\psi}(t') \le 0$ and $\dot{\psi}(t'') \ge 0$, from (9) it follows that $\varphi'(x(t')) \le \varphi'(x(t''))$ and hence $x(t') \le x(t'')$ by the strict monotonicity of φ'. But $u \equiv 0$ along ω, so $x(t') > x(t'')$ by (1), a contradiction.

Case ii) is analysed similarly. ■

Thus, M_0 is either a point or a non-zero segment $\Delta = [t_1,t_2]$.

Let us consider the last case. Here $\psi(t) \equiv \alpha$ along Δ, and maximality condition doesn't allow to find the optimal control directly. This is a singular subarc of optimal trajectory. Differentiating the above equality, we obtain $\dot{\psi} = -1 + \alpha\varphi'(x) \equiv 0$ on Δ. Since φ' strictly increases, this yields $x(t) = const$ and hence $\dot{x} = u - \varphi(x) = 0$, which gives us the form of optimal control along Δ:

$$u_{sing}(t) = \varphi(x(t)). \quad (12)$$

Proposition 2.4: $M_0 \subset (0,T)$, i.e. $t = 0$ and $t = T$ do not belong to M_0.

Proof: Since $\psi(T) = 0 < \alpha$, $T \notin M_0$. So, we just need to show that $0 \notin M_0$. Taking into account that M_0 is a segment $[t_1,t_2]$, suppose first that $M_0 = \{0\}$. Then $\psi < \alpha$ on $(0,T)$, which yields $u \equiv 0$ for a.a. $t < T$ and contradicts (11). Now suppose $M_0 = [0,t_2]$, where $0 < t_2 < T$. Then along $[0,t_2]$ we have $x = const = x(0) = 0$, $u = \varphi(x) = 0$, hence $u \equiv 0$ along the whole $[0,T]$, which contradicts (11). ■

Thus, M_0 is a segment $[t_1,t_2]$ (with possible $t_1 = t_2$) lying inside $(0,T)$.

Proposition 2.5: $\psi(t) > \alpha$ along $(0,t_1)$.

Proof: Suppose this is wrong. Then since $\psi(t) \ne \alpha$ on $(0,t_1)$, we have $\psi < \alpha$, which yields $u \equiv 0$ on $(0,t_1)$. Hence, $x \equiv 0$ and $\varphi(x) \equiv 0$. Since $\dot{\psi}(t_1) = 0$, from (9) we have $\varphi'(0) = 1/\alpha$. Thus, on $(0,t_1)$ we have $\dot{\psi} = -1 + \psi/\alpha < 0$ since $\psi < \alpha$. Hence, $\psi < \alpha$ and decreases on $(0,t_1)$, so $\psi(t_1) < \alpha$, which contradicts the relation $t_1 \in M_0$. ■

Thus, $\psi(t)$ is a nonincreasing function. Moreover, it has the following form. First, it decreases from $\psi(0) > \alpha$ to α. Then, the following two cases may be realized:

i) ψ crosses the value α at time t_1 and then decreases from α to $\psi(T) = 0$. For convenience, we use the notation $u = (u_1,u_2,...)$ on $\Delta_1,\Delta_2,...$, where $\Delta_1,\Delta_2,...$ are some intervals, if $u(t) = u_1$ on Δ_1, $u(t) = u_2$ on Δ_2, etc. Here, $u = (1,0)$ on $((0,t_1),\,(t_1,T))$ is a bang-bang control.

ii)ψ goes to value α at time t_1, stays at this level along $[t_1,t_2]$ and then decreases from α to 0. Here we have a bang-singular-bang control $u = (1,\varphi(x(t_1)),0)$ on $((0,t_1),\,(t_1,t_2),\,(t_2,T))$ with a singular subarc (t_1,t_2). Note that, for a given starting point t_1 of singular subarc, the corresponding end point is defined as

$$t_2 = t_1 + \frac{m_0 - t_1 - m_T}{\varphi(x(t_1))}. \tag{13}$$

This can be easily obtained from equations (1). Obviously, since $u \equiv 1$ on $(0, t_1)$, we have $m(t_1) = m_0 - t_1$. In view of $u \equiv 0$ on $[t_2, T]$, we have $m(t_2) = m_T$. On (t_1, t_2) we have $u = \varphi(x(t_1))$, which leads to $m(t_1) - m(t_2) = \varphi(x(t_1))(t_2 - t_1)$. The last relation implies (13).

Let us establish conditions under which each of these cases is realised.

III. Necessary and sufficient conditions for optimal control to be of the bang-bang form

The bang-bang form of optimal control is equivalent to that $u = 1$ for all $0 \le t < \gamma = \Delta m$ and $u = 0$ for all $\gamma < t < T$. Let us construct the pair of functions $x(t), \psi(t)$ on $[0, \gamma)$, denoting it as $x^l(t),\ \psi^l(t)$, and independently on $[\gamma, T]$, denoting it as $x^r(t),\ \psi^r(t)$. According to (1) and (9), each pair should satisfy, respectively, the equations

$$\begin{cases} \dot\psi^l(t) = -1 + \psi^l(t)\varphi'(x^l), \\ \dot x^l(t) = 1 - \varphi(x^l), \\ x^l(0) = 0. \end{cases}$$

$$\begin{cases} \dot\psi^r(t) = -1 + \psi^r(t)\varphi'(x^r), \\ \dot x^r(t) = -\varphi(x^r), \\ \psi^r(T) = 0. \end{cases}$$

By setting $\psi(0) = q$, where $q > 0$ (according to Proposition 2.1) is an unknown parameter, we can determine the functions $x^l(t),\ \psi^l(t)$ from the following initial value problem:

$$\begin{cases} \dot\psi^l(t) = -1 + \psi^l(t)\varphi'(x^l), \\ \dot x^l(t) = 1 - \varphi(x^l), \\ \psi^l(0) = q, \\ x^l(0) = 0. \end{cases} \tag{14}$$

Note that (see Proposition 2.1) $\psi^l(\gamma) > 0$. Since $x(t),\ \psi(t)$ are continuous, the following relations should hold at time γ:

$$x^l(\gamma) = x^r(\gamma), \qquad \psi^l(\gamma) = \psi^r(\gamma).$$

Moreover, since ψ does not increase, the following condition should hold: $\dot\psi^l(\gamma) \le 0$. In view of (9) the last one can be rewritten in the form $\psi^l(\gamma)\,\varphi'\left(x^l(\gamma)\right) \le 1$. This is a restriction on the parameter q. Further we can find $x^r(t),\ \psi^r(t)$ from the following initial value problem:

$$\begin{cases} \dot\psi^r(t) = -1 + \psi^r(t)\varphi'(x^r), \\ \dot x^r(t) = -\varphi(x^r), \\ \psi^r(\gamma) = \psi^l(\gamma), \\ x^r(\gamma) = x^l(\gamma). \end{cases} \tag{15}$$

The last one determines $\psi^r(T) = \psi(T, q)$ which should be equal to 0 according to PMP. Thus, we can formulate a necessary and sufficient condition for theoptimal control to be of the bang-bang form as the following

Theorem 3.1: Optimal control in problem (1) has the bang-bang form if and only if there exists such $q > 0$ that the pairs of functions $x^l(t),\ \psi^l(t)$ and $x^r(t),\ \psi^r(t)$ determined by the equations (14) and (15) respectively satisfy the following relations:

$$\psi^l(\gamma) > 0, \tag{16}$$

$$\psi^l(\gamma)\,\varphi'\left(x^l(\gamma)\right) \le 1. \tag{17}$$

$$\psi^r(T) = 0. \tag{18}$$

Conditions (16)–(17) determine an interval of feasible q. Condition (18) determines the interval of those T that the corresponding optimal trajectory has the bang-bang form. Examples of application of Theorem 3.1 are presented in section VI below.

IV. Geometrical arguments for the existence of singular arc

In this section we obtain a sufficient condition of the presence of singular arc along optimal trajectory using geometrical considerations. Since $s(t) = \int_0^t x(\tau)d\tau$, our optimal control problem (1) is equivalent to find such a bang-singular switching time moment $t_1 \in [0, \Delta m]$ that the square under the graph of corresponding $x(t)$ is maximal.

Take a small $\varepsilon > 0$ and compare squares under the graphs of bang-bang trajectory $\hat x(t)$ with $\hat u = (1, 0)$ on $((0, \gamma),\ (\gamma, T))$ corresponding to line OBF, and bang-singular-bang trajectory $x(t)$ with $u = (1,\ \varphi(\hat x(\gamma - \varepsilon),\ 0)$ on $((0, \gamma - \varepsilon),\ (\gamma - \varepsilon,\ \gamma + \delta(\varepsilon)),\ (\gamma + \delta(\varepsilon),\ T))$ corresponding to line $OADE$ in Fig 1.

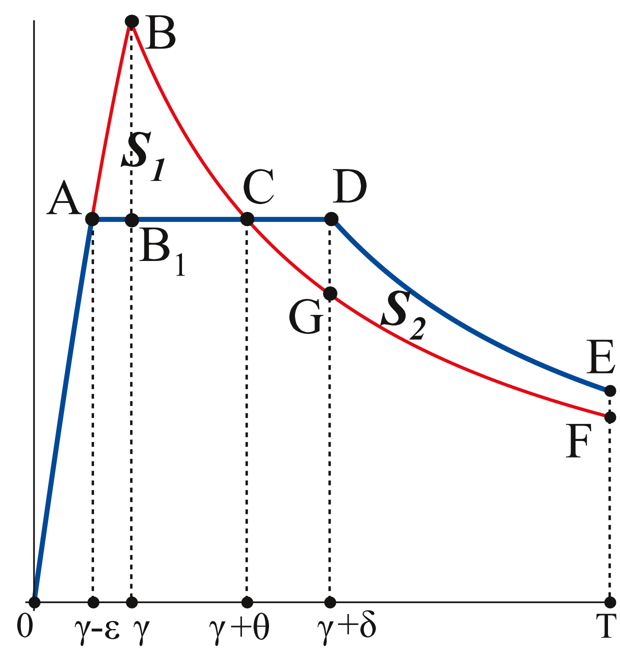

Fig. 1. Trajectory OBF corresponds to bang-bang control; trajectory $OADE$corresponds to bang-singular-bang control.

From geometrical considerations it is easily seen that, if we obtain $S_2(\varepsilon) > S_1$, where S_1 is the square of ABC and $S_2(\varepsilon)$ is the square of $CDEF$, then the bang-bang trajectory is not optimal.

To simplify further formulas, define $x_\gamma = \hat{x}(\gamma)$, $\varphi_\gamma = \varphi(x_\gamma)$ and $p_\gamma = 1 - \varphi_\gamma$. One can show that, up to terms of order ε^2,

$$\begin{aligned} S_1 &= \frac{p_\gamma}{2\varphi_\gamma}\varepsilon^2, \qquad \delta(\varepsilon) = \frac{p_\gamma}{\varphi_\gamma}\left(\varepsilon + \frac{\varphi'_\gamma}{\varphi_\gamma}\varepsilon^2\right), \\ |DG| &= x(\gamma+\delta) - \hat{x}(\gamma+\delta) = \Delta x = \frac{p_\gamma \varphi'_\gamma}{2\varphi_\gamma}\varepsilon^2. \end{aligned} \tag{19}$$

The last relation gives that

$$S_{GCD} \le \frac{1}{2}|CD|\cdot|DG| = o(\varepsilon^2).$$

Hence,

$$S_2(\varepsilon) = S_{DGFE} + o(\varepsilon^2) = \int_{\gamma+\delta}^{T} (\hat{x}(\tau) - x(\tau))\, d\tau + o(\varepsilon^2).$$

On the interval on $[\gamma+\delta, T]$, we have

$$\begin{cases} \dot{\hat{x}} = -\varphi(\hat{x}), \\ \hat{x}(\gamma+\delta) = x(\gamma-\varepsilon) - \Delta x, \end{cases} \qquad \begin{cases} \dot{x} = -\varphi(x), \\ x(\gamma+\delta) = x(\gamma-\varepsilon), \end{cases}$$

whence $\bar{x}(t) = x(t) - \hat{x}(t)$ satisfies (up to terms of order ε^2)

$$\dot{\bar{x}} = -\varphi'(\hat{x}(t))\,\bar{x}, \qquad \bar{x}(\gamma+\delta) = \Delta x,$$

and now our aim is to find $S_2(\varepsilon) = \int_{\gamma+\delta}^{T} \bar{x}(\tau)d\tau$. We will use the following well-known property of linear ODE systems.

Lemma 4.1: Let $\bar{x}(t)$ be defined by the equation $\dot{\bar{x}} = A(t)\bar{x}$, where $A(t)$ is an integrable function. Then the linear functional

$$\bar{J} = l\bar{x}(T) + \int_{t_0}^{T} c(t)\bar{x}(t)dt,$$

where $c(t)$ is an integrable function and l is a real number, can be represented in the form $\bar{J} = -\bar{\psi}(t_0)\bar{x}(t_0)$, where the function $\bar{\psi}(t)$ is determined by the equation

$$\dot{\bar{\psi}} = -\bar{\psi}A(t) + c(t), \qquad \bar{\psi}(T) = -l.$$

Proof: We can write $\bar{J} = l\bar{x}(T) + \int_{t_0}^{T}\left(c\bar{x} + \bar{\psi}\left(\dot{\bar{x}} - A\bar{x}\right)\right)dt$. Integrating $\bar{\psi}\dot{\bar{x}}$ by parts we get

$$\int_{t_0}^{T} \bar{\psi}\dot{\bar{x}}dt = \bar{\psi}(T)\bar{x}(T) - \bar{\psi}(t_0)\bar{x}(t_0) - \int_{t_0}^{T}\dot{\bar{\psi}}\bar{x}dt,$$

so we obtain

$$\begin{aligned} \bar{J} &= \left(l + \bar{\psi}(T)\right)\bar{x}(T) - \bar{\psi}(t_0)\bar{x}(t_0) + \\ &+ \int_{t_0}^{T}\left(c - \left(\dot{\bar{\psi}} + \bar{\psi}A\right)\right)\bar{x}dt = -\bar{\psi}(t_0)\bar{x}(t_0). \end{aligned}$$

■

(Note that Lemma 4.1 is valid also in the case where $\bar{x}$ is a vector function.)

In our case $A(t) = -\varphi'(\hat{x}(t))$, $l = 0$, $c(t) \equiv 1$, $\bar{\psi}(t) = -\psi(t)$, so $S_2(\varepsilon) = \psi(\gamma)\frac{p_\gamma\varphi'_\gamma}{2\varphi_\gamma}\varepsilon^2$, and condition $S_2(\varepsilon) > S_1$ in view of (19) is equivalent to

$$\psi(\gamma + \delta(\varepsilon))\,\varphi'(x(\gamma+\delta(\varepsilon))) > 1.$$

The existence of such $\varepsilon > 0$ that the last inequality takes place is guaranteed if $\psi(\gamma)\varphi'(x_\gamma) > 1$, which allows us to formulate a sufficient condition for the existence of singular arc.

Theorem 4.1: Let $\hat{x}(t)$ be a bang-bang trajectory (may be not optimal one) with a switching time $\gamma = \Delta m$, and let $\psi(t)$ be determined according to (9) on $[\gamma, T]$. If

$$\psi(\gamma)\,\varphi'(\hat{x}(\gamma)) > 1,$$

then the optimal trajectory in problem (1) contains a singular subarc.

This theorem, obtained by geometrical arguments, is in accordance with the above Theorem 3.1.

V. Algorithm to find a singular subinterval

Suppose that for a given T the optimal control in (1) is of bang-singular-bang form. The algorithm described allows one to find the start and end times of the singular subarc.

Given $\Delta m = m_0 - m_T$, a function $\varphi(x)$, and accuracy $\varepsilon > 0$, define $\hat{t} = \Delta m$, and a function $\hat{x}(t)$ on $[0, \hat{t}]$ from the equation $\dot{x} = 1 - \varphi(x)$, $x(0) = 0$. Taking $k = 0$ and $t_1^0 \in (0, \hat{t})$, go to step (1) of the following algorithm.

(1) Compute $x_1^k = \hat{x}(t_1^k)$ and set $t_2^k = t_1^k + \left(m_0 - t_1^k - m_T\right)/\varphi(x_1^k)$.
(2) If $t_2^k \ge T$, increase t_1^k and go to step (1). Else, go to step (3).
(3) Find $\psi^k(T)$ from the following IVP on $[t_2^k, T]$:

$$\begin{cases} \dot{\psi}^k(t) = -1 + \psi^k(t)\varphi'(x(t)), \\ \psi^k(t_2^k) = \alpha^k = 1/\varphi'(x_1^k), \\ \dot{x}(t) = -\varphi(x(t)), \\ x(t_2^k) = x_1^k. \end{cases} \tag{20}$$

(4) (a) If $|\psi^k(T)| \le \varepsilon$, finish the computations with the resulting $t_1^* = t_1^k$.
  (b) If $\psi^k(T) > \varepsilon$, take $t_1^{k+1} > t_1^k$ and go to step (1).
  (c) If $\psi^k(T) < -\varepsilon$, take $t_1^{k+1} < t_1^k$ and go to step (1).

We obtain that the singular subarc of optimal trajectory begins at t_1^* and ends at $t_2^* = t_1^* + (m_0 - m_T - t_1^*)/\varphi(\hat{x}(t_1^*))$.

The described algorithm uses two important properties of $\psi^k(t)$ defined by (20), which are stated in the following two propositions.

Proposition 5.1: $\psi^k(t) < \alpha^k$ and $\dot{\psi}^k(t) < 0$ on (t_2^k, T).

Proof: Dropping the index k, we obtain that $\ddot{\psi} = -\psi\varphi''\varphi + \varphi'(-1 + \psi\varphi')$ is a continuous function. Since $\dot{\psi}(t_2) = 0$, we have $\ddot{\psi}(t_2) < 0$, so $\ddot{\psi} < 0$ in a right neighborhood of t_2. Hence, in this neighborhood, $\dot{\psi} < 0$, so $\psi < \alpha$, and then, in view of (20) and since x decreases, these inequalities hold on the whole interval $(t_2, T]$. ■

Proposition 5.2: $\psi^k(T)$ decreases if the starting time t_1^k of singular subarc increases.

Proof: Dropping again the index k and taking $t_1' < t_1$, we get $x_1' = \hat{x}(t_1') > x_1 = \hat{x}(t_1)$ and so, $\alpha' < \alpha$. Then $t_2' < t_2$ and, as was shown in Sec. 6, $x'(t_2) < x(t_2)$. This implies that $x'(t) < x(t)$ on $[t_2, T]$.

Since by Proposition 5.1 $\psi'(t)$ decreases on $[t_2', T]$, we have $\psi'(t_2) < \psi'(t_2') = \alpha' < \alpha$, whereas $\psi(t_2) = \alpha$. Taking into account that $\varphi'(x'(t)) < \varphi(x(t))$ on $[t_2, T]$, we get that $\psi'(t) < \psi(t)$ on this interval. ■

Propositions 5.1 and 5.2 allow one to use the above algorithm to find the start and end points of singular subarc.

VI. Numerical Experiments

In this section we consider the case of simple $\varphi(x) = x^2/2$, $m_0 = 1$ and $m_T = 0.1$. Thus, we have $\gamma = \Delta m = 0.9$.

1) Find the “thersold value” T^* by the algorithm in the end of Sec. 5. To do this, note that according to (14), $\psi^l(\gamma, q)$ depends linearly on the initial value $\psi^l(0, q) = q$:

$$\psi^l(t, q) = \left(q - \int_0^t F(z)dz\right)/F(t),$$

where

$$F(z) = e^{-\int_0^z \varphi'(x^l)d\tau},$$

which allows us to find q according of step (i) of the algorithm:

$$q = \int_0^\gamma F(z)dz + \frac{F(\gamma)}{\varphi'(x^l(\gamma))} = 1.654917.$$

Further computations give $T^* = 3.414422$.

2) Consider $T = 3 < T^*$. Applying Theorem 3.1 we find that there exists a unique $p = 1.577724$ which satisfies (16)-(18). Hence, the optimal control is of bang-bang form: $u = 1$ on $(0, \gamma)$ and $u = 0$ on $(\gamma, T.)$

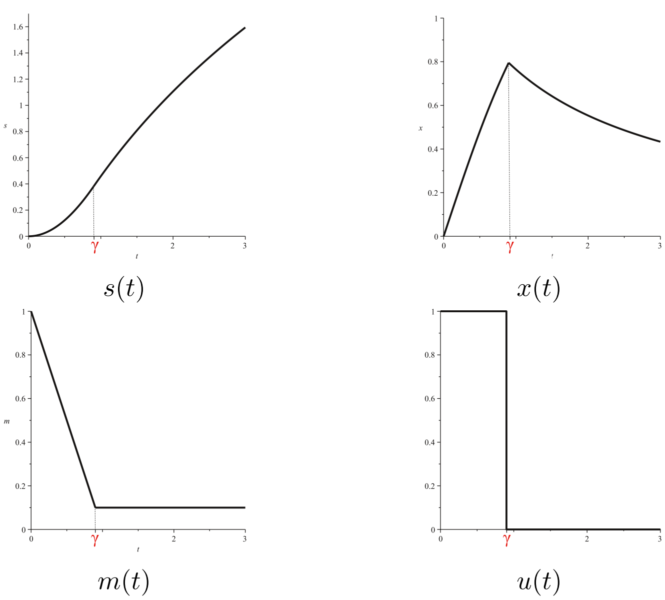

Fig. 2. Graphs of the optimal trajectory and control for $T = 3$.

3) Consider $T = 4 > T^*$. Here we find that only $p = 1.744562$ satisfies (16) and (18). But this contradicts (17), so the optimal control is of bang-singular-bang form. Applying the algorithm in Sec. 7, we obtain that the singular subarc starts at $t_1^* = 0.7912296$ and ends at $t_2^* = 1.2133897$.

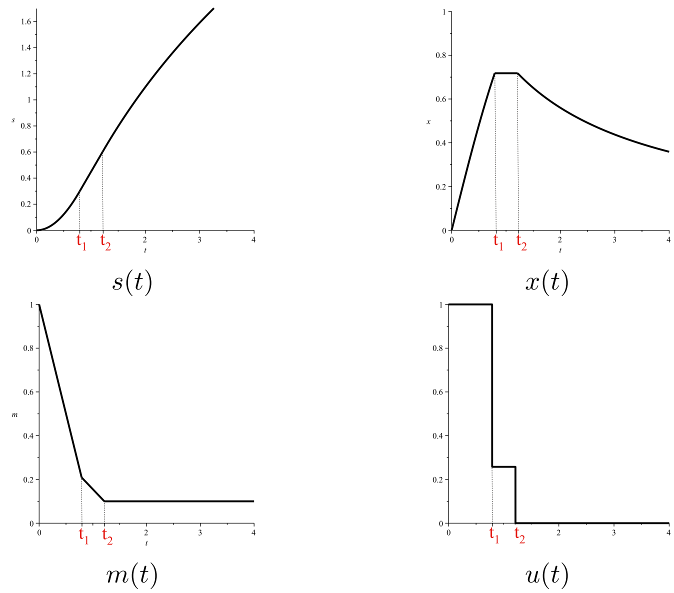

Fig. 3. Graphs of the optimal trajectory and control for $T = 4$

4) Consider $T = 5 > T^*$. Here we find that only

$$p = 1.860953$$

satisfies (16) and (18). But this contradicts (17), so the optimal control is of bang-singular-bang form. Applying the algorithm in Sec. 7, we obtain that the singular subarc starts at $t_1^* = 0.6746996$ and ends at $t_2^* = 1.818053$.

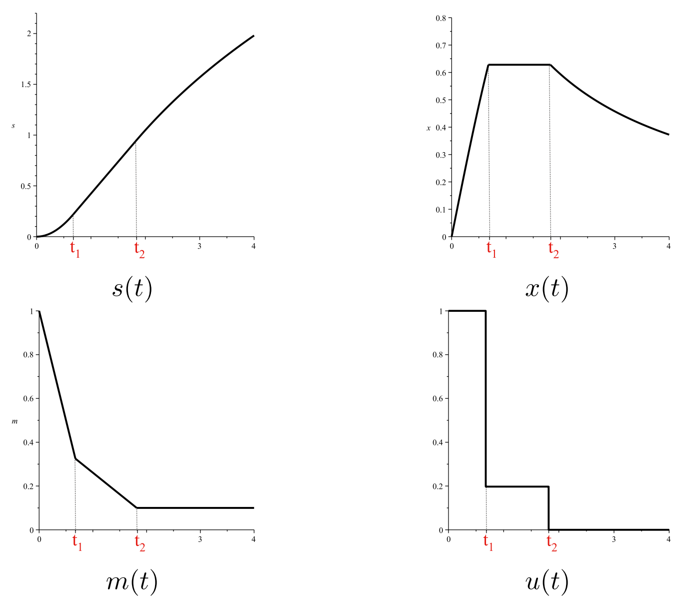

Fig. 4. Graphs of the optimal trajectory and control for $T = 5$.

5) Consider $T = 6 > T^*$. Here we find that only

$$p = 1.946773$$

satisfies (16) and (18). But this contradicts (17), so the optimal control is of bang-singular-bang form. Applying the algorithm in Sec. 7, we obtain that the singular subarc starts at $t_1^* = 0.600071$ and ends at $t_2^* = 2.469399$.

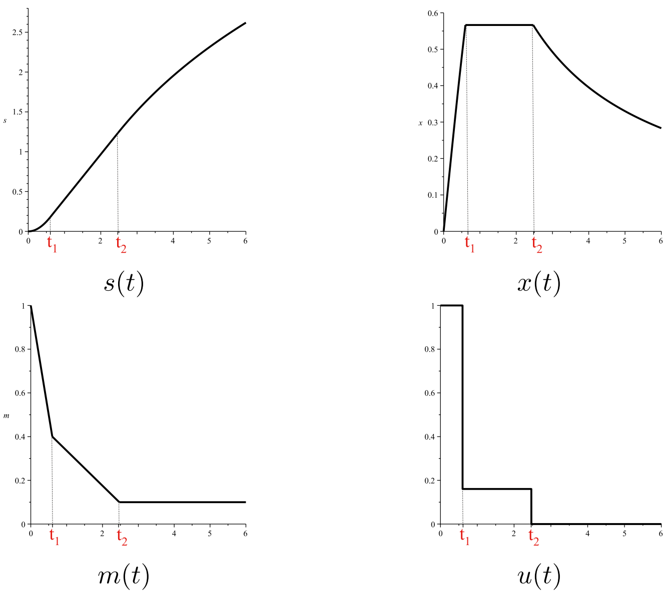

Fig. 5. Graphs of the optimal trajectory and control for $T = 6$.

In experiments provided above, all computations were performed on a computer with Pentium(R) Dual-Core CPU 2.2 GHz, 2 GB RAM under 32-bit Windows 7 Professional (SP1) operating system. For the numerical solution of the BVP (20), the C$^{\#}$-written class library partly described in [15] was used.

Conclusion

For a simplified Goddard problem the form of optimal control is obtained from the Pontryagin Maximum Principle. analytically. Necessary and sufficient conditions of bang-bang form of this control are formulated (Theorem 3.1) together with geometrical form of sufficient conditions of existence of singular subarc (Theorem 4.1). Iterative algorithm to find bounds of singular arc is suggested. Numerical experiments show that there exists a "threshold value" T^* such that for all $T > T^*$ optimal trajectory in problem (1) contains singular subarc. A method to determine T^* is also presented.

Further investigations may contain the following.

Firts of all, it is interesting to verify the possibilities of the MP analytical investigation in case of a complication of the considered model (for example, obtained by introducing $m(t)$ expenditure in the equation for $x(t)$).

The second interesting question is how to modify the algorithm described above to make it feasible for numerical investigation of the "multi-staged" variant of the considered problem. More precisely, it is interesting to obtain the conditions under which the optimal trajectory contains singular subarcs on different "inter-separation" time ranges. In the described case of $\varphi(x) = x^2/2$, it may be easily verified that the optimal trajectory may contains a singular subarc only for a times greater than the time of first stage separation.

References

[1] Robert H. Goddard, *A method on reacing extreme altitudes*, Smitsonian Miscellaneous Collection, vol. 71, No. 2, 1919, 82 pages

[2] L. Pontryagin, V. Boltyanski, R. Gamkrelidze, and E. Michtchenko, *The Mathematical Theory of Optimal Processes*, Wiley Interscience, New York, 1962.

[3] Boris Garfinkel, *A solution of the Goddard problem*, Ballistic research laboratories report No. 1194, 1963, 31 pages

[4] H. Munick, *Goddard problem with bounded thrust*, AIAA Journal, vol. 3, 1965, pp. 12831285.

[5] A. E. Bryson and Y.-C. Ho, *Applied optimal control*, Hemisphere Publishing, New-York, 1975.

[6] Panagiotis Tsiotras and Henry J. Kelleyt, *Goddard Problem with Constrained Time of Flight*, J. of Guidance, Control and Dynamics, vol. 15, No. 2, March-April 1992, pp. 289–296.

[7] H. Seywald and E.M. Cliff, *Goddard problem in presence of a dynamic pressure limit*, Journal of Guidance, Control, and Dynamics, vol. 16, No. 4, 1993, pp. 776–781.

[8] E. Hairer and G. Wanner, *Solving ordinary differential equations, II*, Springer-Verlag, Berlin, second edition, 1996

[9] L.F. Shampine, J. Kierzenka, and M.W. Reichelt, *Solving boundary value problems for ordinary differential equations in Matlab with bvp4c*, http://www.mathworks.com/bvp tutorial, 2000.

[10] B. Bonnard and M. Chyba, *Singular trajectories and their role in control theory*, vol. 40 of Math matiques and Applications, Springer-Verlag, Berlin, 2003

[11] Knut Graiche, Nicolas Petit, *Solving the Goddard problem with thrust and dynamic pressure constraints using saturation functions*, Proceedings of the 17th World Congress The International Federation of Automatic Control Seoul, Korea, July 6-11, 2008, pp. 14301–14306.

[12] J.F. Bonnans, P. Martinon, E. Trelat, *Singular arcs in the generalized Goddard's Problem*, J. Optimization Theory and Applications, vol. 139, No. 2, 2008, pp. 439–461.

[13] P. Martinon, F. Bonnans, J. Laurent-Varin, E. Trelat, *Numerical study of optimal trajectories with singular arcs for an Ariane 5 launcher*,J. Guidance, Control, and Dynamics vol. 32, No. 1, 2009, pp. 51-55.

[14] C.Ponssard, K.Graichen, N.Petit, J.Laurent-Varin, *Ascent optimization for a heavy space launcher*, Proceedings of the European Control Conference 2009, Budapest, Hungary, August 23–26, 2009, pp. 3033–3038.

[15] I.Samylovskiy, *Implementation and use of .NET and Silverlight class library for the numerical investigation of dynamical control systems*, Proceedings of the ICMC Summer Meeting in Differential Equations 2013 Chapter, Sao Carlos, Sao Paulo, Brasil, February 4–7, 2013, p. 73.