Hyperstability Analysis of Switched Systems Subject to Integral Popovian Constraints

M. De la Sen and S. Alonso-Quesada
Department of Electricity and Electronics
University of the Basque Country, UPV/EHU
48940-Leioa, Bizkaia, Spain
e-mails: manuel.delasen@ehu.es, santiago.alonso@ehu.es

A. Ibeas
Department of Telecommunications and Systems Engineering
Universitat Autònoma de Barcelona
08193-Bellaterra, Barcelona, Spain

Abstract—**This paper studies the asymptotic hyperstability of switched time-varying dynamic systems. The system is subject to switching actions among linear time-invariant parameterizations in the feed-forward loop for any feedback regulator controller. Moreover, such controllers can be also subject to switching through time while being within a class which satisfies a Popov's-type integral inequality. Asymptotic hyperstability is proven to be achievable under very generic switching laws if (i) at least one of the feed-forward parameterization possesses a strictly positive real transfer function, (ii) a minimum residence time interval is respected for each activation time interval of such a parameterization and (iii) a maximum allowable residence time interval is simultaneously maintained for all active parameterization which are not positive real, if any.**

Keywords—hyperstability; asymptotic hyperstability; switched dynamic systems; switching laws

I. INTRODUCTION

The problems of hyperstability and asymptotic hyperstability have received important attention in Control Theory because global closed-loop stability is achieved for a wide class of nonlinear devices under the only constraint that they satisfy Popov's-type integral inequalities [1-3]. It is needed that the linear feed-forward part of the system has a positive real transfer function for hyperstability and a strictly positive one for asymptotic hyperstability. In that way, global stability is achieved for a family of nonlinear controllers making the problem to be more independent of controller ageing or certain ranges of component dispersion along the fabrication process of the controller components. On the other hand, global stability of switched systems with several parameterizations has been investigated through exhaustive research work performed along the last years. In some cases, global asymptotic stabilization is achievable irrespective of the switching law. This property is typically guaranteed for linear switched systems when all the parameterizations possess a common Lyapunov function [4, 5]. In the general case, global stabilization of the switched system is achievable, depending on the switching time instants, provided that sufficiently large minimum time intervals are respected at certain stable active parameterizations [6-8].

This paper investigates conditions of asymptotic hyperstability of switched linear systems under regulation controls generated from nonlinear devices satisfying Popov's-type integral inequalities. *It is assumed that*: **(i)** a set of linear parameterizations in the feed-forward loop, subject to switching through time to select the active parameterization on a certain time interval; and **(ii)** a, in general, nonlinear controller function taking values in a set of at most the same number of nonlinear feedback devices being also subject to switching. *The following results* are found: **(i)** if all the feed-forward loop linear parameterizations are strictly positive real (then their matrices of dynamics being all Hurwitz) and, furthermore, the corresponding matrices of dynamics possess a common Lyapunov function then asymptotic hyperstability of the system is achieved unconditionally for any switching law for any nonlinear controller device satisfying a Popov's type integral inequality, and **(ii)** if there is at least one parameterization with strictly positive real transfer function then asymptotic hyperstability of the switched system might be achieved for switching laws satisfying generic constraints on the switching time instants. The remaining active parameterizations are not required to be either stable or with associate positive real transfer functions. Those constraints consist basically in respecting a minimum allowable residence time interval for a set of marked testing active parameterizations involving strictly positive real functions and, simultaneously, a maximum allowable residence time interval for the remaining active parameterizations.

II. SWITCHED CLOSED-LOOP SYSTEM

Consider the n-dimensional single-input single-output switched nonlinear feedback dynamic system:

$$\begin{aligned}&\dot{x}(t)=A_{\sigma(t)}(t)\,x(t)+b_{\sigma(t)}(t)\,u(t)\ \ ;\ \ x(0)=x_0\in\mathbf{R}^n\ ;\\ &y(t)=c^T_{\sigma(t)}x(t)+d_{\sigma(t)}u(t)\ \ ;\ \ u(t)=-\varphi_{\sigma_0(t)}\big(y(t),\,t\big),\end{aligned} \tag{1}$$

for $t\in\mathbf{R}_{0+}:=\mathbf{R}_+\cup\{0\}$, where $x\in\mathbf{R}^n$, $u(t)\in\mathbf{R}$ and $y(t)\in\mathbf{R}$ are, respectively, the state n-vector, the scalar input, which is a feedback regulation control, and the scalar output where:

(i) $\sigma:\mathbf{R}_{0+}\to\bar{p}$ and $\sigma_0:\mathbf{R}_{0+}\to\bar{p}_0$, $p_0\le p$, with $\bar{p}:=\{1,\,2,\,\dots,\,p\}$ and $\bar{p}_0:=\{1,\,2,\,\dots,\,p_0\}\subseteq\bar{p}$, for some given finite numbers $p,\,p_0\in\mathbf{N}$ of parameterizations of (1). The first one describes a switching law among such various constant parameterizations defined by the set of quadruples $\{(A_i, b_i, c_i, d_i) : i \in \overline{p}\}$ of (1) of elements whose orders are compatible with the corresponding signals dimensionalities.

(ii) The function $\sigma : \mathbf{R}_{0+} \to \overline{p}$ is defined as $\sigma(t) = j = j(t) = j(t_i)$ $\forall t \in [t_i, t_{i+1})$ for some integer $j \in \overline{p}$, each $t_i \in \{t_i\}$ and each integer $i \in \overline{N} \subseteq \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, where $STI = STI(\sigma) = \{t_i\}$ is the sequence of switching time instants generated from some given switching law $SL = SL(\sigma)$ subject to $t_{i+1} - t_i \geq T_r \geq 0$ $\forall i \in \mathbf{N}$. Such a function assigns at certain time intervals a particular parameterization of the feed-forward part of the system which is modified at the switching time instants.

(iii) The function $\sigma_0 : \mathbf{R}_{0+} \to \overline{p}_0$ is defined as $\sigma_0(t) = j_0 = j_0(t) = j_0(t_{i_0})$ $\forall t \in [t_{i_0}, t_{i_0+1})$ so that $STI_0 = STI_0(\sigma_0) = \{t_{i_0}\} \subseteq STI \equiv \{t_i\}$, with the switching constraint $t \in STI_0 \Rightarrow t \in STI$, is the sequence of switching time instants of the feedback nonlinear part of the dynamic system. This means that the number of parameterizations of the nonlinear feedback device function is at most that of the linear feed-forward one. Such a function assigns for each time interval a particular parameterization of the feedback nonlinear device of the system which is modified at the switching time instants.

(iv) The nonnegative real number $T_r = T_r(\sigma)$ is the minimum residence time interval at each parameterization and either $p = 1$ or $j(t_i) \neq j(t_{i+1})$ for all $t_i, t_{i+1} \in STI$. If $T_r = 0$ then the switching law is unconditional in the sense that switching is fully arbitrary. In the linear case, unconditional switching is possible in a stable way if all the parameterizations possess a common Lyapunov function. Otherwise, a minimum residence time $T_r > 0$ is required at each parameterization to guarantee the stabilization of the linear time-varying switched system if all of them are stable or at least one should be stable, subject to a minimum residence time when such a stable parameterization is active, which depends of the whole sequences and respective active time intervals at the rest of parameterizations.

(v) The set $\overline{N}$ is a denumerable (proper or improper) subset of $\boldsymbol{N}$ that, if finite, describes switching processes with a finite number of switches among the *p* distinct parameterizations.

(vi) The control input is generated as the nonlinear output-feedback function $u(t) = -\varphi_{\sigma_0(t)}(y(t), t)$, which is assumed to be piecewise continuous while satisfying the integral Popov's-type inequality:

$$\int_0^t \varphi_{\sigma_0(\tau)}(y(\tau), \tau)\, y(\tau)\, d\tau \geq -\gamma > -\infty, \qquad (2)$$

$\forall t \in \mathbf{R}_{0+}$ for some $\gamma \in \mathbf{R}_+$. Note that $\varphi : \overline{p}_0 \times \mathbf{R}^2 \to \mathbf{R}$ defined by $\varphi(y(t), t) = \varphi_{\sigma_0(t)}(y(t), t)$ for $\sigma_0(t) = j_0 = j_0(t) = j_0(t_{i_0})$ $\forall t \in [t_{i_0}, t_{i_0+1})$ for some integer $j_0 \in \overline{p}_0$ for each $t_{i_0} \in \{t_{i_0}\}$ is piecewise continuous within its definition domain for any switching law $\sigma_0 : \mathbf{R}_{0+} \to \overline{p}_0 \subset \overline{p}$ if $\varphi_{j_0} : \mathbf{R}^2 \to \mathbf{R}$ with $j_0 \in \overline{p}_0$ are all piecewise continuous. It is not required in principle that the nonlinear devices be distinct for each distinct parameterization in the feed-forward loop. The closed-loop system is displayed in Figure 1 below.

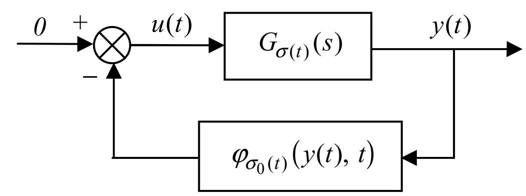

Fig. 1. Block diagram of the feedback nonlinear system.

Note that the switched feedback law $u(t) = -\varphi_{\sigma_0(t)}(y(t), t)$ together with (2) implies that (2) itself is equivalent to:

$$E(t) := \int_0^t y(\tau)\, u(\tau)\, d\tau \leq \gamma < \infty, \qquad (3)$$

$\forall t \in \mathbf{R}_{0+}$ and for some $\gamma \in \mathbf{R}_+$, where $E(t)$ is an input-output energy measure of the feed-forward linear part of the closed-loop system.

III. Switching Condition for Convergence to Zero of the Control Input

This section investigates parameterization switching sufficiency-type conditions for the input to the linear feed-forward loop to converge asymptotically to zero as time tends to infinity. The switching laws can involve parameterizations which are not strictly positive real being subject either a) to a finite number of switching actions, b) to appropriate alternating with strictly positive real ones subject to maximum allowable residence times or c) to saturation-vanishing conditions of the input to the feed-forward linear loop. The time-varying piecewise constant parameterization $(A_{\sigma(t)}, b_{\sigma(t)}, c_{\sigma(t)}, d_{\sigma(t)})$ of the switched system (1) changes of values at time instants in STI. It is well-known that in the absence of switching, i.e. if $p = 1$, the closed-loop system is said to be hyperstable if the linear transfer function $G(s) = c^T (sI - A)^{-1} b + d$ is positive real, i.e. it belongs to the set PR of positive real functions fulfilling $\mathrm{Re}\{G(s)\} \geq 0$ for $\mathrm{Re}\{s\} > 0$, and the feedback law satisfies (2). If $G(s)$ is strictly positive real, i.e. if it is in the set SPR of strictly positive real functions fulfilling $\mathrm{Re}\{G(s)\} > 0$ for $\mathrm{Re}\{s\} \geq 0$, then the closed-loop system is said to be asymptotically hyperstable. It is well-known that realizable positive real functions are either stable or critically stable of relative order either zero or one. Their critically stable poles, if any, are single and with non-negative associate residues. Strictly positive real transfer functions are, in particular, stable.

In order to simplify the formalism, we will refer indistinctly to positive realness and strict positive realness either for transfer functions or for their state-space realizations. The feedback system is said to be hyperstable (respectively, asymptotically hyperstable) if it is globally stable for any nonlinear output-feedback law satisfying Popov's inequality (2). For any given switching rule $\sigma : \mathbf{R}_{0+} \to \overline{p}$, let us consider the impulse response of the feed-forward linear block $g(t) := \mathbf{L}^{-1}\{G(s)\}$; i.e. the Laplace anti-transform of $G(s)$, denote the Fourier transform of a function $f(t)$ as $\mathbf{F}(f)$ and also define the subsequent auxiliary truncated input:

$$u_{\sigma_0(t_{i_0})}(t) = \begin{cases} u(t) & , \quad t \in \left[t_{i_0}, t_{i_0+1}\right) \\ 0 & , \quad t \in \left(-\infty, t_{i_0}\right) \cup \left[t_{i_0+1}, \infty\right) \end{cases}, \tag{4}$$

$\forall t_{i_0} \in STI_0$, for $i_0 = 0, 1, \ldots, q_0(t)-1$ $\forall t \in \mathbf{R}$ if the switching action never ends, and otherwise

$$u_{\sigma_0(t_{i_0})}(t) = \begin{cases} u(t) & , \quad t \in \left[t_{i_0}, t_{i_0+1}\right) \\ 0 & , \quad t \in \left(-\infty, t_{i_0}\right) \cup \left[t_{i_0+1}, \infty\right) \end{cases}, \tag{5}$$

$\forall t_{i_0} \in STI_0$, for $i_0 = 0, 1, \ldots, q_0(t)-1$ $\forall t\left(\le t_{q0}\right) \in \mathbf{R}$ and

$$u_{\sigma_0(t_{q0})}(t) = \begin{cases} u(t) & , \quad t \in \left[t_{q0}, \infty\right) \\ 0 & , \quad t \in \left(-\infty, t_{q0}\right) \end{cases}, \tag{6}$$

$\forall t\left(\ge t_{q0}\right) \in \mathbf{R}$ where $t_{q0(t)} = t_{q0} < \infty$ is the last switching time instant under the axiom $t_0 (=0) \in STI \cap STI_0$ for any $\sigma : \mathbf{R}_{0+} \to \overline{p}$, $\sigma_0 : \mathbf{R}_{0+} \to \overline{p}_0$, and:

$$\begin{aligned} q = q(t) &:= \max\left\{z \in N \cup \{0\} \;:\; t_{q(t)} (\in STI) \le t\right\}, \\ q_0 = q_0(t) &:= \max\left\{z \in N \cup \{0\} \;:\; t_{q0(t)} (\in STI_0) \le t\right\}, \end{aligned} \tag{7}$$

so that t_q and t_{q0} are, respectively, the last switching time instants of the feed-forward linear parameterization and of the nonlinear feedback device in the time interval $[0, t]$. In the same way, given $STI = \{t_i\}$, we define the impulse response as:

$$g_\sigma(t) = g_\sigma(t_i) \quad \forall t \in [t_i, t_{i+1}). \tag{8}$$

If the convolution and Parseval's theorems are jointly applied to (3) for zero initial conditions and extending the definition of the input on $\mathbf{R}$ with $u(t) = 0$ and $g_\sigma(t) = 0$ for $t < 0$, one gets:

$$\begin{aligned} E(t) &= \sum_{i=1}^{q(t)} \int_{t_{i-1}}^{t_i} y(\tau)\, u(\tau)\, d\tau + \int_{q(t)}^{t} y(\tau)\, u(\tau)\, d\tau \\ &= \sum_{i=1}^{q(t)} \left(\int_{t_{i-1}}^{t_i} \int_0^\tau g(\tau')\, u(\tau-\tau')\, u(\tau)\, d\tau'\, d\tau \right) \\ &\quad + \int_{t_{q(t)}}^{t} \int_0^\tau g(\tau')\, u(\tau-\tau')\, u(\tau)\, d\tau'\, d\tau \\ &= \sum_{i=1}^{q(t)} \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g_{\sigma(t_{i-1})}(\tau') u_{\sigma_0(t_{i-1})}(\tau-\tau') d\tau' \right) u_{\sigma_0(t_{i-1})}(\tau) d\tau \right] \\ &\quad + \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g_{\sigma(t_q)}(\tau')\, u_{\sigma_0(t_q)}(\tau-\tau')\, d\tau' \right) u_{\sigma_0(t_q)}(\tau)\, d\tau \\ &= \frac{1}{2\pi} \left(\sum_{i=1}^{q(t)} \left[\int_{-\infty}^{\infty} \mathbf{F}\left\{ \int_{-\infty}^{\infty} g_{\sigma(t_{i-1})}(\tau') u_{\sigma_0(t_{i-1})}(\tau-\tau') d\tau' \right\} \mathbf{F}\left\{u_{\sigma_0(t_{i-1})}(\tau)\right\} d\omega \right] \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \mathbf{F}\left\{ \int_{-\infty}^{\infty} g_{\sigma(t_{q(t)})}(\tau')\, u_{\sigma_0(t_{q(t)})}(\tau-\tau')\, d\tau' \right\} \mathbf{F}\left\{u_{\sigma_0(t_{q(t)})}(\tau)\right\} d\omega \right) \\ &= \frac{1}{2\pi} \left(\sum_{i=1}^{q(t)} \int_{-\infty}^{\infty} \mathrm{Re}\left\{G_{\sigma(t_{i-1})}(j\omega)\right\} \left|U_{\sigma_0(t_{i-1})}(j\omega)\right|^2 d\omega \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \mathrm{Re}\left\{G_{\sigma(t_{q(t)})}(j\omega)\right\} \left|U_{\sigma_0(t_{q(t)})}(j\omega)\right|^2 d\omega \right), \end{aligned} \tag{9}$$

$\forall t \in \mathbf{R}_{0+}$ since $\mathrm{Im}\{j\omega\} = -\mathrm{Im}\{-j\omega\}$ $\forall \omega \in \mathbf{R}$ where $j = \sqrt{-1}$ is the complex unit. Now, proceed to calculate a lower-bound of (9) by using again Parseval's theorem and the hodograph symmetry property $\mathrm{Re}\{j\omega\} = \mathrm{Re}\{-j\omega\}$ $\forall \omega \in \mathbf{R}$ to yield:

$$\begin{aligned} E(t) \ge \sum_{i=1}^{q(t)} &\left(\min_{\omega \in \mathbf{R}_{0+}} \left\{ \mathrm{Re}\left\{G_{\sigma(t_{i-1})}(j\omega)\right\} \right\} \int_{-\infty}^{\infty} u^2_{\sigma_0(t_{i-1})}(\tau)\, d\tau \right) \\ &+ \min_{\omega \in \mathbf{R}_{0+}} \left\{ \mathrm{Re}\left\{G_{\sigma(t_{q(t)})}(j\omega)\right\} \right\} \int_{-\infty}^{\infty} u^2_{\sigma_0(t_{q(t)})}(\tau)\, d\tau, \end{aligned} \tag{10}$$

$\forall t \in \mathbf{R}_{0+}$. Now, consider the sequence of switching time instants until time t of the given switching law $\sigma : \mathbf{R}_{0+} \to \overline{p}$, $STI(t) = STI \cap [0, t)$ $\forall t \in \mathbf{R}_{0+}$ and decompose it as the disjoint union $STI(t) = STI_p(t) \cup STI_n(t) \cup STI_z(t)$ $\forall t \in \mathbf{R}_{0+}$ as follows:

$$\begin{aligned} &STI_p(\sigma, t) = \\ &\quad STI_p(t) := \left\{ t_i \in STI(t) \;:\; \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\} \right\} > 0 \right\}, \\ &STI_n(\sigma, t) = STI_n(t) := \left\{ t_i \in STI(t) \;:\right. \\ &\quad \left. \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\} \right\} = -\max_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\} \right\} < 0 \right\} \textit{ and} \\ &STI_z(\sigma, t) = \\ &\quad STI_z(t) := \left\{ t_i \in STI(t) \;:\; \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\} \right\} = 0 \right\}. \end{aligned} \tag{11}$$

$STI_p = STI_p(\sigma)$, $STI_n = STI_n(\sigma)$ and $STI_z = STI_z(\sigma)$ are defined in the same way by including all the respective switching instants of the switching law $\sigma : \mathbf{R}_{0+} \to \overline{p}$, that is, the right-hand-side sets of the definitions in (11) modified for $t_i \in STI$. Note that the sequences in (11) are defined so that $t \in STI \Rightarrow t \notin STI(t)$ following the definition convention $STI(t) = STI \cap [0,\, t)$.

Lemma 1 (*non-negativity of the input-output energy*): Define the switching-dependent amount:

$$
\begin{aligned}
g_\sigma(t_i) &:= \sum_{t_k \in STI_p(t_i)} \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{ G_{\sigma(t_k)}(j\omega) \right\} \right\} \int_0^{T_{k-1}} u^2(t_{k-1}+\tau)\, d\tau \\
&- \sum_{t_k \in STI_n(t_i)} T_{k-1} \max_{\omega \in R_{0+}} \left\| \mathrm{Re}\left\{ G_{\sigma(t_k)}(j\omega) \right\} \right\| \max_{t_{k-1} \le \tau \le t_k} \left\{ u^2(\tau) \right\},
\end{aligned} \tag{12}
$$

$\forall t_i \in STI$, where $t_{k+1} = t_k + T_k \quad \forall t_k \in STI$ so that the $\sigma(t_k)$ parameterization of the feed-forward part of the system is active during a time interval T_k in-between two consecutive switching time instants. Then, the following properties hold:

(i) The input-output energy measure $E(t)$ is nonnegative for all time irrespective of the input $u: \mathbf{R}_{0+} \to \mathbf{R}$ if the switching law $\sigma: \mathbf{R}_{0+} \to \overline{p}$ satisfies any of the two conditions below:

(i.1) All the active parameterizations are positive real and

(i.2) Any active parameterization in an interval $[t_i,\, t_{i+1})$ which is not positive real is preceded by a strictly positive real one on $[t_{i-1},\, t_i)$ while subject to a maximum (being potentially finite or infinity) residence time interval satisfying the constraint:

$$
T_i = t_{i+1} - t_i < \frac{g_\sigma(t_i)}{\max_{\omega \in \mathbf{R}_{0+}} \left\| \mathrm{Re}\left\{ G_{\sigma(t_i)}(j\omega) \right\} \right\| \max_{0 \le \tau \le T_i} \left\{ u^2(t_i+\tau) \right\}}. \tag{13}
$$

(ii) A necessary condition to guarantee that

$$
\begin{aligned}
g_\sigma(t) &:= g_\sigma(t_i) + \mu(t_i) \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{ G_{\sigma(t_i)}(j\omega) \right\} \right\} \int_{t_i}^{t} u^2(\tau)\, d\tau \\
&- (1-\mu(t_i))(t-t_i) \max_{\omega \in R_{0+}} \left\| \mathrm{Re}\left\{ G_{\sigma(t_i)}(j\omega) \right\} \right\| \max_{t_i \le \tau < t} \left\{ u^2(\tau) \right\},
\end{aligned} \tag{14}
$$

$\forall t \in [t_i,\, t_{i+1})$ is nonnegative is that the first parameterization after an arbitrary finite time is positive real in order to guarantee the non-negativity for all time of the input-output energy measure where $\mu: R_{0+} \to \{0,\, 1\}$ is a binary indicator function of value $\mu(t) = \mu(t_i) = 1 \quad \forall t \in [t_i,\, t_{i+1})$ if $t_i \in STI_p \cup STI_z$ and $\mu(t) = \mu(t_i) = 0$ if $t_i \in STI_n$. A necessary condition to guarantee that (14) is nonnegative for all time, irrespective of the input, if the number of switches is finite is that the last active parameterization be positive real.

(iii) Assume that the first active parameterization after an arbitrary finite time is strictly positive real and that all non positive real parameterization, if any, satisfies the constraint of maximum residence time interval (13). Then, the input-output energy measure is positive for all $t > 0$.

Proof: Consider $g_\sigma(t)$ defined in (14). It turns out that $g_\sigma(0) = 0$. Also, $g_\sigma(t) \ge 0 \quad \forall t \in [0,\, t_1]$ with $t_1 \in STI$ if the first active parameterization $\sigma(0)$ in the time interval $[0,\, t_1)$ is positive real since

$$
\begin{aligned}
g_\sigma(t) &:= g_\sigma(t_0) + \mu(t_0) \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{ G_{\sigma(t_0)}(j\omega) \right\} \right\} \int_{t_0}^{t} u^2(\tau)\, d\tau \\
&= \mu(t_0) \min_{\omega \in R_{0+}} \left\{ \mathrm{Re}\left\{ G_{\sigma(t_0)}(j\omega) \right\} \right\} \int_{t_0}^{t} u^2(\tau)\, d\tau,
\end{aligned} \tag{15}
$$

$\forall t \in [t_0,\, t_1]$. Now, proceed by complete induction by assuming that $g_\sigma(\tau) \ge 0$ for $\tau \in [0,\, t_i]$ and any given $t_i \in STI$. Thus, $g_\sigma(t_i) \ge 0 \Rightarrow g_\sigma(t_i+\tau) \ge 0$ for any $\tau \in [0,\, T_i]$ by construction if:

(a) $t_i \in STI_p \cup STI_z$ so that $g_\sigma(t_i) \ge 0$ implies that $g_\sigma(t) \ge 0 \quad \forall t \in [0,\, t_{i+1})$, then $t_{i+1} \in STI$ may be any positive arbitrary time instant, or

(b) $t_i \in STI_p \cup STI_z$ so that $g_\sigma(t_i) = 0$ implies that $g_\sigma(t) \ge 0 \quad \forall t \in [0,\, t_{i+1})$ and then $t_{i+1} \in STI$ may be any positive arbitrary time instant, or

(c) $T_i = t_{i+1} - t_i < \dfrac{g_\sigma(t_i)}{\max_{\omega \in \mathbf{R}_{0+}} \left\| \mathrm{Re}\left\{ G_{\sigma(t_i)}(j\omega) \right\} \right\| \max_{0 \le \tau \le T_i} \left\{ u^2(t_i+\tau) \right\}}$ if $g_\sigma(t_i) \ge 0$ and $t_i \in STI_n$ which is subject to a maximum allowable guaranteed upper-bound except for the case of identically zero input on the current switching interval which allows an arbitrary next switching time instant $t_{i+1} \in STI_p \cup STI_z$. Thus, Property (i) follows for (i.1) from case (a) and it follows for (i.2) from cases (b)-(c).

Property (ii) is proven as follows. Note that $g_\sigma(t) \ge g_\sigma(t_0) = g_\sigma(0) > 0 \quad \forall t \in [t_0 = 0,\, t_1)$ for any $t_1 \in STI$ since the first active parameterization after an arbitrary finite time is strictly positive real. From the cases (a)-(b) of the proof of Property (i), it follows that

$$
\begin{aligned}
& g_\sigma(t) \ge g_\sigma(t_0) = g_\sigma(0) > 0 \quad \forall t \in [t_0 = 0,\, t_1) \Rightarrow \\
& \Rightarrow g_\sigma(t) \ge g_\sigma(t_1) \ge g_\sigma(t_0) = g_\sigma(0) > 0 \quad \forall t \in \left[t_0 = 0,\, t^*\right),
\end{aligned} \tag{16}
$$

with $t^* \in STI$ being the first switching time instant activating a positive real parameterization on $\left[t^*,\, t^{**}\right)$ with t^{**} being the time instant of the next activation of a positive real parameterization. From the case (c) of the proof of Properties (i) for (i.1) and (i.2), it follows that $g_\sigma(t^*) > 0 \Rightarrow g_\sigma(t) > 0 \quad \forall t \in \left[t_0 = 0,\, t^{**}\right)$. The subsequent time intervals of alternate activation of positive real and nonpositive real parameterizations are discussed in the same way leading to a complete induction proof of Property (iii). The two necessary conditions of Property (iii) follow directly by using simple contradiction arguments. ***

Lemma 2 (*uniform boundedness and nonnegativity of the input-output energy measure*): Assume that the switching law $\sigma: \mathbf{R}_{0+} \rightarrow \overline{p}$ satisfies Lemma 1 and, furthermore, the nonlinear feedback device satisfies:

$$\int_{t_i}^{t_i+\eta} \varphi_{\sigma_0(\tau)}\big(y(\tau), \tau\big)\, y(\tau)\; d\tau \geq -\gamma - \int_0^{t_i} \varphi_{\sigma_0(\tau)}\big(y(\tau), \tau\big)\, y(\tau)\; d\tau \;, (17)$$

$\forall t_i, t_{i+1} \in STI$, any $\eta \in [0, T_i)$ where $T_i = t_{i+1} - t_i$, and

$$\int_{t_q}^{t_q+\eta} \varphi_{\sigma_0(\tau)}\big(y(\tau), \tau\big)\, y(\tau)\; d\tau \geq -\gamma - \int_0^{t_q} \varphi_{\sigma_0(\tau)}\big(y(\tau), \tau\big)\, y(\tau)\; d\tau \;, (18)$$

for any $\eta \in [0, \infty)$ if $STI = \{t_0, t_1, \ldots, t_q\}$ with $q < \infty$. Then, $0 \leq E(t) \leq \gamma < \infty \;\; \forall t \in \mathbf{R}_{0+}$.

Proof: It follows directly from Lemma 1, the fact that (17) and (18) guarantee Popov's inequality (2) and the equivalence of (2) and (3). ***

Remark 1: Note that Lemma 1 is formulated for the feed-forward linear part irrespective of the feedback law. Note also that the sufficient-type conditions of Lemma 1 guarantee that the switching law $\sigma: \mathbf{R}_{0+} \rightarrow \overline{p}$ leads to a nonnegative input-output energy measure for all time under the necessary condition of fulfilment of such sufficient ones that the first active parameterization at the finite initial switching time instant t_0 be positive and that the last one be also positive real if there is a finite number of switchings (otherwise, sufficiency conditions fail in both cases). In this last case, the last active parameterization is required to be globally stable in order to guarantee the boundedness of the input-output energy for any admissible piecewise continuous input satisfying Popov's inequality (2) under Lemma 2 holds. ***

Remark 2: Any switching law $\sigma: \mathbf{R}_{0+} \rightarrow \overline{p}$ of the feed-forward loop involving only the activation of parameterizations with positive real transfer functions satisfies all the positivity conditions of Lemma 1. This result is a direct generalization of a well-known previous one for single parameterizations. ***

Lemma 3: Assume a switching law $\sigma: \mathbf{R}_{0+} \rightarrow \overline{p}$ of the feed-forward loop such that all $\varphi_j: \mathbf{R}^2 \rightarrow \mathbf{R}$ with $j \in \overline{p}_0$ are piecewise continuous for any active parameterizations while satisfying (17)-(18) guaranteeing the integral Popov's inequality (2). Then, the following properties hold:

(i) If all the active parameterizations are strictly positive real then the input is uniformly bounded for all time and, furthermore, $\exists \lim_{t\to\infty}\{u(t)\} = 0$.

(ii) Assume that the first active parameterization after an arbitrary finite time is strictly positive real and that all active non positive real parameterization, if any, is preceded by a strictly positive real one while satisfying the constraint of maximum allowable residence time interval (13). Then, the input to the feed-forward loop is uniformly bounded for all time and, furthermore, $\exists \lim_{t\to\infty}\{u(t)\} = 0$.

(iii) Assume that the switching law activates infinitely many times strictly positive real parameterizations and that first active parameterization after an arbitrary finite time is strictly positive real. Assume also that the system feed-forward loop of any active parameterization on $[t_i, t_{i+1})$ which is not positive real has no pole at $s = 0$ and satisfies the saturation-vanishing input constraint $|u(t)| \leq K e^{-\lambda t_i} \;\; \forall t \in [t_i, t_{i+1})$ with $t_i \in STI_n$ for some real constants $\lambda > 0$ and $K > 0$ subject to $\infty > \lambda > \max\left\{\lambda_0, \max_{t_i \in STI_n}\left\{\frac{\ln(T_i)}{2T_i}\right\}\right\}$ with $T_i = t_{i+1} - t_i$ for some prefixed $\lambda_0 \in \mathbf{R}_+$. Then, the input to the feed-forward loop is uniformly bounded for all time and, furthermore, $\exists \lim_{t\to\infty}\{u(t)\} = 0$.

Proof: Property (i) is proven as follows. It all the parameterizations are strictly positive real then $STI(t) = STI_p(t) \;\; \forall t \in \mathbf{R}_{0+}$ and one gets from (3), (12) and (15) that:

$$\begin{aligned} 0 &\leq \min_{t_k \in STI,\, \omega \in R_{0+}} \left\{\mathrm{Re}\left\{G_{\sigma(t_k)}(j\omega)\right\}\right\} \sum_{t_k \in STI(t_i)} \left(\int_{t_{k-1}}^{t_k} u^2(\tau)\; d\tau\right) \\ &\leq g_\sigma(t_i) := \sum_{t_k \in STI(t_i)} \min_{\omega \in R_{0+}} \left\{\mathrm{Re}\left\{G_{\sigma(t_k)}(j\omega)\right\}\right\} \left(\int_0^{T_{k-1}} u^2(t_{k-1} + \tau)\; d\tau\right) \quad (19) \\ &\leq \gamma < \infty \;, \end{aligned}$$

$\forall t_i \in STI$ when there is an infinite number of active parameterizations, or

$$\begin{aligned} 0 &\leq \sum_{t_i \in STI} \min_{\omega \in R_{0+}} \left\{\mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\}\right\} \left(\int_{t_i}^{t_{i+1}} u^2(\tau)\; d\tau\right) \\ &\leq \liminf_{t\to\infty} \{g_\sigma(t)\} \leq \limsup_{t\to\infty} \{g_\sigma(t)\} \leq \gamma < \infty \;, \end{aligned} \quad (20)$$

$\forall t_i \in STI$ and

$$\begin{aligned} 0 &\leq g_\sigma(t) := g_\sigma(t_q) \\ &\quad + \min_{\omega \in R_{0+}} \left\{\mathrm{Re}\left\{G_{\sigma(t_q)}(j\omega)\right\}\right\} \left(\int_{t_q}^{t} u^2(\tau)\; d\tau\right) \leq \gamma < \infty, \end{aligned} \quad (21)$$

$\forall t \geq t_q\, (\in STI)$ if $STI \cap (t_q, \infty) = \varnothing$ when there is a finite number of active parameterizations. Note that all parameterizations are finite for all $s\in\mathbf{C}$ since strictly positive real transfer functions cannot possess critical poles, then they are integrators-free. First, assume that $\min_{\omega\in\mathbf{R}_{0+}}\left\{\mathrm{Re}\left\{G_{\sigma(t)}\right\}\right\}\ge d>0$, i.e. the relative order of the strictly positive real transfer function is zero. Note that the above amounts are, furthermore, strictly bounded from below by zero if the input u: $\mathbf{R}_{0+}\to\mathbf{R}$ is piecewise continuous and non-identically zero. Furthermore, if the switching action never ends then $\mathrm{card}(STI)=\infty$ and $t_i(\in STI)\to\infty$. Thus, $\lim_{t_i(\in STI)\to\infty}\left\{\int_{t_i}^{t_i+\tau}u^2(\tau)\,d\tau\right\}=0\ \ \forall\tau\in[0,\,T_i)$ and since the input is piecewise continuous and the sequence of integrals $\left\{\int_{t_i}^{t_i+\tau}u^2(\tau)\,d\tau\right\}$ has infinitely many elements, it follows that $\exists\lim_{t\to\infty}\{u(t)\}=0$ since $\min_{t_i\in STI,\,\omega\in\mathbf{R}_{0+}}\left\{\mathrm{Re}\left\{G_{\sigma(t_i)}\right\}\right\}>0$. If switching ends in a finite time instant $t_q\in STI$ for some $q\in\mathbf{N}_0$ then $\mathrm{card}(STI)=q+1<\infty$ and there is no switching for $t>t_q$ so that $\lim_{t\to\infty}\left\{\int_{t_q}^{t}u^2(\tau)\,d\tau\right\}=0$ and, again, $\exists\lim_{t\to\infty}\{u(t)\}=0$ since $\int_{t_q}^{t}u^2(\tau)\,d\tau$ is a strictly increasing function of time for any $0\le t_q<\infty$ and any nonzero input, then contradicting (21), unless $\exists\lim_{t\to\infty}\{u(t)\}=0$.

Next, assume that $\min_{\omega(<\infty)\in\mathbf{R}_{0+}}\left\{\mathrm{Re}\left\{G_{\sigma(t)}\right\}\right\}>0$ and $\lim_{\omega\to\infty}\left\{\mathrm{Re}\left\{G_{\sigma(t)}\right\}\right\}=0$, i.e. the relative order is one. In this case, define a strictly decreasing nonnegative real function ω_0: $\mathbf{R}_{0+}\to\mathbf{R}_{0+}$, that is, $\omega_0=\omega_0(\varepsilon)\to+\infty$ as $\varepsilon\to0^+$. Thus, (19) leads to:

$$\begin{aligned}
0&\le\min_{t_k\in STI,\,\omega\in\mathbf{R}_{0+}}\left\{\mathrm{Re}\left\{G_{\sigma(t_k)}(j\omega)\right\}\right\}\sum_{t_k\in STI(t_i)}\left(\int_{t_{k-1}}^{t_k}u^2(\tau)\,d\tau\right)\\
&=\varepsilon\sum_{t_k\in STI(t_i)}\left(\int_0^{T_{k-1}}u^2(t_{k-1}+\tau)\,d\tau\right)\\
&\le\sum_{t_k\in STI(t_i)}\min_{\omega(\le\omega_0(\varepsilon))\in\mathbf{R}_{0+}}\left\{\mathrm{Re}\left\{G_{\sigma(t_k)}(j\omega)\right\}\right\}\left(\int_0^{T_{k-1}}u^2(t_{k-1}+\tau)\,d\tau\right)\\
&\le\gamma_0-2\sum_{t_k\in STI(t_i)}\int_{\omega_0(\varepsilon)}^{\infty}\mathrm{Re}\left\{G_{\sigma(t_k)}(j\omega)\right\}\left|U_{\sigma_0(t_{i-1})}(j\omega)\right|^2d\omega<\infty\ ,
\end{aligned}\tag{22}$$

$\forall t_i\in STI$ and for any positive finite real constants ε , ς and γ_0 so that

$$\begin{aligned}
\infty>\gamma_0&>\liminf_{\varepsilon\to0^+}\left\{\gamma+\varsigma+2\sum_{t_k\in STI(t_i)}\int_{\omega_0(\varepsilon)}^{\infty}\mathrm{Re}\left\{G_{\sigma(t_k)}(j\omega)\right\}\left|U_{\sigma_0(t_{i-1})}(j\omega)\right|^2d\omega\right\}\\
&=\gamma+\varsigma>0\ ,
\end{aligned}\tag{23}$$

since Popov's inequality (2) implies that $\int_0^t\varphi_{\sigma_0(\tau)}(\tau)\,y(\tau)\,d\tau\ge-\gamma\ge-\gamma_0>-\infty$ for any finite real constant $\gamma_0\ge\gamma$ and $\lim_{\varepsilon\to0}\{\omega_0(\varepsilon)\}=+\infty$. Thus, $\exists\lim_{t\to\infty}\{u(t)\}=0$ remains valid if the strictly positive real transfer functions have relative order equal to one. Property (i) has been proven.

Property (ii) follows from similar considerations as those in the proof of Property (i) from (12) and (15) under the maximum allowable residence time constraint (13) for any active non positive real active parameterizations provided that the first active parameterization after an arbitrary finite time is strictly positive real.

Property (iii) is proven by noting that:

1) $t_i\in STI_z$ does not need to be accounted for in Popov's inequality or equivalent feed-forward loop since the minimum real value of its associate feed-forward transfer function is zero, and

2) one gets the following relations for parameterizations which are not positive real under no critical pole at $s=0$, what implies $\left|\mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\}\right|<\infty$, together with the input saturation-vanishing constraint $|u(t)|\le Ke^{-\lambda t_i}\ \ \forall t\in[t_i,\,t_{i+1})$ for $t_i\in STI_n$:

$$\begin{aligned}
&-\sum_{t_i\in STI_n}\int_{t_i}^{t_{i+1}}y(\tau)\,\varphi_{\sigma_0(t_i)}(y(\tau),\tau)\,d\tau=\sum_{t_i\in STI_n}\int_{t_i}^{t_{i+1}}u(\tau)\,y(\tau)\,d\tau\\
&\le\frac{1}{2\pi}\sum_{t_i\in STI_n}\max_{\omega\in\mathbf{R}_{0+}}\left\{\left|\mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\}\right|\right\}\int_{-\infty}^{\infty}\left|U_{\sigma_0(t_i)}(j\omega)\right|^2\,d\omega\\
&\le\max_{\omega\in\mathbf{R}_{0+},\,t_i\in STI_n}\left\{\left|\mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\}\right|\right\}\sum_{t_i\in STI_n}\int_{t_i}^{t_{i+1}}u^2(\tau)\,d\tau\\
&\le K^2\max_{\omega\in\mathbf{R}_{0+},\,t_i\in STI_n}\left\{\left|\mathrm{Re}\left\{G_{\sigma(t_i)}(j\omega)\right\}\right|\right\}\sum_{t_i\in STI_n}T_ie^{-2\lambda t_i}\le\frac{K'^2}{1-e^{-2\lambda_0t^*}}<\infty\ ,
\end{aligned}\tag{24}$$

since a λ constant fulfilling $\infty>\lambda>\max\left\{\lambda_0,\max_{t_i\in STI_n}\left\{\frac{\ln(T_i)}{2T_i}\right\}\right\}$ exists since $\max_{t_i\in STI_n}\left\{\frac{\ln(T_i)}{2T_i}\right\}<\infty$, equivalently $\max_{t_i\in STI_n}\left\{T_ie^{-2\lambda T_i}\right\}<1$, for $T_i\in[0,\,\infty)$ and $\limsup_{T_i\to\infty}\left\{\max\left\{\lambda_0,\,\max_{t_i\in STI_n}\left\{\frac{\ln(T_i)}{2T_i}\right\}\right\}\right\}<\infty$ where $t^*=\min\{t\colon t\in STI_n\}$. The combination of (19)-(22) with Popov's inequality equivalent versions (2) and (3), guaranteed under (17)-(18), Lemma 2 and (9) yields:

$$0<\min_{t_i\in STI_p}\left\{G_{\sigma(t_i)}\right\}\int_0^{\infty}u^2(\tau)\,\mu(\tau)\,d\tau<E(t)\le\gamma+\frac{K'^2}{1-e^{-2\lambda_0t^*}}<\infty\ ,\tag{25}$$

after separating the bounded contribution of non strictly positive real parameterizations from the strictly positive real ones of zero relative order. Since the switching law possesses infinitely many strictly positive real active parameterizations by hypothesis, it follows that $\int_0^t u^2(\tau)\mu(\tau)\,d\tau$ cannot be a strictly increasing function of time, since, otherwise, a contradiction would follow from (25), thus $\exists \lim_{t\to\infty}\{u(t)\}=0$. For the case of relative order equal to one, the proof follows closely to such a case in the proof of Property (i). ***

Remark 3: Lemma 3 (iii) admits infinitely many non strictly positive real parameterizations in the switching law but subject to a time-dependent saturation-vanishing input constraint in the feed-forward loop while excluding transfer functions with poles at the origin. Then, positive real transfer functions which are not strictly positive real with simple poles at the origin are excluded of the switching law. ***

IV. Asymptotic Hyperstability of the Nonlinear Switched System

This section gives some results on asymptotic hyperstability for switching laws among different linear parameterizations in the feed-forward loop for nonlinear feedback devices subject also to switching laws and satisfying Popov's inequality (2). The proofs of the following theorems are omitted for space reasons.

Theorem 1: The following properties hold:

(i) Let $\boldsymbol{A}=\{A_i :\, i\in\overline{p}\}$ be a set of p Hurwitz matrices. Then all matrices in $\boldsymbol{A}$ have a common Lyapunov function if and only if $\sum_{i=1}^{p}\left(A_i X_i + X_i A_i^T\right)<0$ for any given n-matrices $X_i \ge 0$ for $i\in\overline{p}$ ("$\ge$" denoting matrix positive semidefiniteness). The open-loop switched system (1), i.e. $u\equiv 0$, is globally asymptotically stable for any given arbitrary switching law $\sigma\!:\ \mathbf{R}_{0+}\to\overline{p}$.

(ii) The matrices in $\boldsymbol{A}$ do not have a common Lyapunov function if and only if there is a set of n-matrices $X_i \ge 0$ for $i\in\overline{p}$ such that $\sum_{i=1}^{p}\left(A_i X_i + X_i A_i^T\right)=0$.

(iii) The matrices in $\boldsymbol{A}$ do not have a common Lyapunov function if at least one non-Hurwitz matrix exists in the set $\overline{\boldsymbol{A}}=\boldsymbol{A}\cup\{A_i^{-1} :\, i\in\overline{p}\}$.

(iv) The set $\boldsymbol{A}=\{A_i :\, i\in\overline{p}\}$ of Hurwitz matrices has a common Lyapunov function only if $\sum_{i=1}^{p}\left(\alpha_i A_i + \beta_i A_i^{-1}\right)$ is a Hurwitz matrix $\forall\,\alpha_i,\,\beta_i\in\mathbf{R}_{0+}$ such that $\sum_{i=1}^{p}\left(\alpha_i A_i + \beta_i A_i^{-1}\right)\in\mathbf{R}_+^{n\times n}$.

(v) The set $\boldsymbol{A}=\{A_i :\, i\in\overline{p}\}$ of Hurwitz matrices has a common Lyapunov function $V(x(t))=x^T(t)Px(t)$, where $P=P^T>0$ is a positive definite real n-matrix, if:

$$\|A_i - A_k\|_2 < \frac{\lambda_{\min}(Q_k)\left(|\lambda_{\max}(A_k)|-\varepsilon\right)}{K_k^2\lambda_{\max}(Q_k)} = \frac{\lambda_{\max}(A_k^T P + P A_k)\left(|\lambda_{\max}(A_k)|-\varepsilon\right)}{K_k^2\lambda_{\min}(A_k^T P + P A_k)}, \tag{26}$$

$\forall i\in\overline{p}$ for any given $k\in\overline{p}$, for any given arbitrary real constant $\varepsilon\in\left(0,\,|\lambda_{\max}(A_k)|\right)$ and some testable real constant $K_k\ge 1$, where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ stand for the $(\cdot)$-matrix and $Q_k=-(A_k^T P+PA_k)$. The open-loop system (1) is globally exponentially stable for any arbitrary switching law $\sigma\!:\ \mathbf{R}_{0+}\to\overline{p}\ \ \forall p\in\mathbf{N}$. ***

Theorem 2: Let $\{\lambda_{ij} :\, j\in\overline{n}\}$ be the spectrum of $A_i\ \ \forall i\in\overline{p}$ and let $K_i(\in\mathbf{R}_+)\ge 1$ and $\rho_i\in\mathbf{R}$ with $i\in\overline{p}$ be constants such that $\max_{j\in\overline{n}}\{\mathrm{Re}\{\lambda_{ij}\}\}\le-\rho_i$ ($\max_{j\in\overline{n}}\{\mathrm{Re}\{\lambda_{ij}\}\}<-\rho_i$ if there is some eigenvalue of A_i of multiplicity larger than one for any $i\in\overline{p}$), and $\|e^{A_i t}\|\le K_i e^{-\rho_i t}\ \ \forall i\in\overline{p}$. Consider the open-loop system with a switching law $\sigma\!:\ \mathbf{R}_{0+}\to\overline{p}$ generating the following sequence of switching time instants:

$$STI = STI(\sigma) = \Big\{t_{i_0},\ t_{i_0+1},\ \ldots,\ t_{i_0+i_1} = t_1^*,\ t_{i_0+i_1+1},\ \ldots,\ t_{i_0+i_1+i_2} = t_2^*,\ \ldots,\ t_{\sum_{j=0}^{k} i_j} = t_k^*,\ \ldots\Big\}, \tag{27}$$

with $\{i_j\}$ being a finite or infinite strictly increasing sequence of nonnegative integers subject to $i_0=0$ and $|i_{j+1}-i_j|\le\xi<\infty$ under the following assumptions:

A.1. The set $\boldsymbol{A}$ contains at least one Hurwitz matrix.

A.2. $STI\supset STI^*=\{t_i^* :\, i\in\overline{N}^*\cup\{0\}\}$, $\overline{N}^*=\{1,\ 2,\ \ldots,\ N^*\}\subset\mathbf{N}$, for some N^*, is a set of marked switching time instants chosen provided that either a minimum residence time constraint is guaranteed if $N^*=\infty$ for each current marked active parameterization $\sigma(t_i^*)\ \ \forall i\in\overline{N}^*\cup\{0\}$, being necessarily stable, according to:

$$T^*_{\sigma\left(t_{\sum_{j=0}^{k} i_j+\ell}\right)} \ge \frac{1}{\rho}\left(\sum_{\ell=0}^{i_{k+1}-1}\left|\ln\left(K_{\sigma\left(t_{\sum_{j=0}^{k} i_j+\ell}\right)}\right)\right|\right) - \sum_{\ell=1}^{i_{k+1}-1}\rho_{\sigma\left(t_{\sum_{j=0}^{k} i_j+\ell}\right)} T_{\sigma\left(t_{\sum_{j=0}^{k} i_j+\ell}\right)} - |\ln(\delta)|, \tag{28}$$

for some real constant $\delta\in(0,1)$, or $N^*<\infty$ with the last active parameterization $\left[t_{N^*},\,\infty\right)$ being Hurwitz. Thus, the open-loop switched system is globally exponentially stable. ***

Remark 4: Note that Theorem 2 does not require explicitly that the residence time constraint has to be kept for each stable active parameterization or for that being stable if there is just one stable. The test can be applied at finite time intervals of lengths being subject to prescribed upper-bounds. This can translate in the marked residence time resulting increased as such testing time intervals become larger. ***

Theorems 1-2 on global exponential stability of the open-loop system are combined with Lemma 3 to generate some asymptotic hyperstability-type results for the closed-loop system (1) subject to a Popov's-type integral inequality (2).

Theorem 3: Consider a switched feedback system (1) with the nonlinear feedback device satisfying a Popov's inequality (17), eventually both constraints (17)-(18) in the case of a finite number of switches. The following properties hold:

(i) Assume that all the transfer functions of the feed-forward loop being built with the various matrices of the set $\boldsymbol{A}$ are strictly positive real (then also Hurwitz) and, furthermore, (26) holds, under the necessary and sufficient condition $\sum_{i=1}^{p}\left(A_i X_i + X_i A_i^T\right) < 0$ for any given n-matrices $X_i \geq 0$ with $i \in \overline{p}$. Then, the closed-loop system is asymptotically hyperstable for any switching law $\sigma\colon \mathbf{R}_{0+} \to \overline{p}$, for any $p \in \mathbf{N}$, such that the nonlinear feedback device satisfies Lemma 2. If, in particular, $\varphi_{\sigma(t)}(t) = \varphi(t) \quad \forall t \in \mathbf{R}_{0+}$, i.e. the nonlinear device does not depend on switching and satisfies Popov's inequality (2), then the closed-loop systems is unconditionally asymptotically hyperstable for any arbitrary switching law $\sigma\colon \mathbf{R}_{0+} \to \overline{p}$.

(ii) Assume that all the matrices of $\boldsymbol{A}$ are Hurwitz with their associate transfer functions being strictly positive real. If $\sum_{i=1}^{p}\left(A_i X_i + X_i A_i^T\right) = 0$ for some n-matrices $X_i \geq 0$ with $i \in \overline{p}$ or if $\sum_{i=1}^{p}\left(\alpha_i A_i + \beta_i A_i^{-1}\right)$ is not Hurwitz for some $\alpha_i, \beta_i \in \mathbf{R}_{0+}$ subject to $\sum_{i=1}^{p}\left(\alpha_i A_i + \beta_i A_i^{-1}\right) \in \mathbf{R}_{+}^{n \times n}$, then the closed-loop switched system (1) is not unconditionally asymptotically hyperstable for a switching-independent feedback nonlinear device $\varphi_{\sigma(t)}(t) = \varphi(t) \quad \forall t \in \mathbf{R}_{0+}$ under any given switching law $\sigma\colon \mathbf{R}_{0+} \to \overline{p}$ even if the transfer functions of all the feed-forward parameterizations are strictly positive real.

(iii) Consider a switched system (1) possessing (at least) one strictly positive real parameterization of the feed-forward loop under some switching law $\sigma\colon \mathbf{R}_{0+} \to \overline{p}$ whose associate set of switching time instants $STI = STI(\sigma)$ is subject to the subsequent constraints:

1) The first active parameterization through time after an arbitrary finite time is strictly positive real.

2) All active non positive real parameterization, if any, satisfies the constraint of *maximum allowable residence time constraint* (13), via (12), and it is preceded by a strictly positive real one.

3) There is a set $STI^* \subset STI$ of marked switching time instants for some of the active strictly positive real parameterizations satisfying the *minimum residence time constraint* (28) for the time interval $\left[t_k^*, t_{k+1}^*\right)$ defined for each two consecutive marked switching instants.

4) The nonlinear feedback device satisfies Popov's inequality (17) for the set STI, eventually (17)-(18) if it generates only a finite number of switches.

Then, the closed-loop system is asymptotically hyperstable for such a switching law $\sigma\colon \mathbf{R}_{0+} \to \overline{p}$. ***

Remark 5: Theorem 3 [(i)-(ii)] refers to unconditional and conditional switching when all the active parameterizations are strictly positive real for the cases of existence or non existence of a common Lyapunov function for the active parameterization. The sufficiency-type asymptotic hyperstability conditions of Theorem 3 (iii) involve constraints on the switching time instants to two levels, namely, maximum allowable residence time for active non positive real parameterizations and minimum residence time for certain test active strictly positive real parameterizations for any feedback nonlinear devices satisfying Popov's inequalities. ***

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