A Pseudospectral Approach to Ascent Trajectory Optimization for Hypersonic Air-Breathing Vehicles

Wei Shi, Zhongliang Jing, Yongsheng Yang, Hui Ge
School of Aeronautics and Astronautics
Shanghai Jiao Tong University
Shanghai, China 200240
Email: wei@sjtu.edu.cn

***Abstract*—The objective of this paper is to investigate a reliable method to generate optimal ascent trajectory for hypersonic air-breathing vehicles. When solving optimal trajectories in endo-atmospheric flight, most of traditional indirect methods suffer from difficulties in finding an appropriate initial guess and getting a convergent solution for the two point boundary value problem (TPBVP). With improvements on on-board computer performance, direct methods such as pseudospectral method show promising potential for real-time optimal guidance. It removes the need for computing analytical gradients of aerodynamic coefficients, and remains high accurate solution similar to indirect methods. In this work, optimal ascent trajectory generation problem was formulated as a fuel-optimal control problem. Gauss pseudospectral method (GPM) was presented to generate the optimal ascent trajectories. Optimal solutions from GPM were compared with an indirect method based on finite difference method (FDM). Numerical simulations were studied with various initial conditions to investigate the optimal trajectory characteristics for hypersonic air-breathing vehicles. The results verified the validity and accuracy of GPM for ascent trajectory optimization.**

Keywords*—Trajectory optimization; optimal control; flight guidance ; pseudospectral method; hypersonic vehicles.

I. INTRODUCTION

The hypersonic flight regime presents a very challenging problem in ascent trajectory optimization. First, aerodynamics, propulsion, and vehicle attitude are strongly coupled during the ascent phase of hypersonic air-breathing vehicles. Second, online guidance lacks a kind of fast and effective numerical method to obtain an optimal solution. Finally, different constraints make the trajectory optimization more difficult.

Calise and Corban conducted some of the earliest work in the vertical plane [1-3]. Singular perturbation theory was applied to reduce the equations. In order to reduce the real-time computation, the method separated the energy and mass dynamics from the altitude and flight path angle dynamics. Gath proposed optimal 3D ascent guidance algorithm based on vector calculus for rocket-powered launch vehicles [4]. Dukeman developed a closed-form ascent guidance algorithm, cyclically to solve TPBVP during endo-atmospheric flight [5, 6]. Sun and Lu brought optimal endo-atmospheric ascent guidance for launch vehicles closer to reality [7]. Pan improved the closed-loop guidance algorithms for multi-burns [8, 9].

The motivation of this paper is from Lu and Oscar's work for hypersonic air-breathing vehicles. They first derived the necessary conditions of the optimal 3D ascent trajectory [10]. Closed-loop ascent guidance had been realized by solving the TPBVP based on indirect scheme. However, their indirect method for ascent trajectory optimization still have some disadvantages. First, to derive analytic expressions with vector calculus can become quite complicated. Second, the region of convergence for the TPBVP may be quite small. Furthermore, it's not easy to get good initial guesses especially for the co-state. Pseudospectral method attracts our attention. It was first used as a numerical algorithm to solve partial differential equations. Fahroo and Ross extended the Legendre pseudospectral method to estimate co-states [11, 12]. Benson and Huntington investigated and compared different methods for pseudospectral transcription [13]. More recently, Rao A.V. et al. proposed h-p adaptive scheme to automatically select the transcription mesh [14, 15]. Pseudospectral method removes the need for computing analytical gradients of aerodynamic coefficients, and remains high accurate solution similar to indirect methods.

The remainder of this paper is organized as follows. In Section II, we build up ascent dynamics for hypersonic air-breathing vehicles. Ascent trajectory problem is formulated by applying optimal control theory. Section III elaborates Gauss Pseudospectral Method for numerically solving optimal control problem. Section IV describes the Generic Hypersonic Aerodynamic Model Example (GHAME) used in this research. Section V presents the results to the ascent trajectory optimization. Section VI summarizes the work done in this paper.

II. ASCENT TRAJECTORY PROBLEM

All the equations of ascent motion are normalized for better numerical conditioning. The radius is normalized by r_0, the radius of the Earth at equator. Time is normalized by $\sqrt{r_0/g_0}$. The velocity is normalized by $\sqrt{r_0 g_0}$, the circular velocity around the Earth at r_0. The mass is normalized by m_0, the initial mass of the vehicle. The magnitudes of the aerodynamic forces are in **g**'s, and the expressions are as follow:

$$L = \frac{r_0}{2m_0 g_0}\rho(r)V^2 S_{ref} Cl(Ma,\alpha) \tag{1}$$

$$D = \frac{r_0}{2m_0 g_0}\rho(r)V^2 S_{ref} Cd(Ma,\alpha) \tag{2}$$

Where *Ma* is Mach number, *Cl* and *Cd* are the lift and the drag coefficient respectively. Notice V is a non-dimensional variable here, while atmospheric density ρ and the vehicle's reference area S_{ref} remain dimensional. We assume that the atmospheric density is an exponential function of altitude. Because the thrust is a function of both throttle and the vehicle state, we can express thrust as follows,

$$T = f\left(I_{sp}, Ma, \bar{x}, \bar{u}\right) \tag{3}$$

Where $\bar{x}$ and $\bar{u}$ are respectively the state and the control of the vehicle, I_{sp} is specific impulse, which is normalized by time scale.

A. Ascent dynamics in vertical plane

Assuming a spherical, nonrotating earth and Newtonian gravitational field, note $g_0 = \mu / r_0^2$, we can write the dimensionless dynamic equations in vertical plane

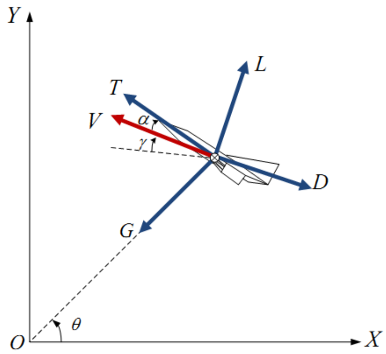


Fig. 1. Forces of hypersonic vehicles in vertical plane

$$\begin{aligned}
\dot{r} &= V\sin\gamma \\
\dot{\theta} &= \frac{V\cos\gamma}{r} \\
\dot{V} &= \frac{T\cos\alpha - D}{m} - \frac{\sin\gamma}{r^2} \\
\dot{\gamma} &= \frac{1}{V}\left(\frac{T\sin\alpha + L}{m} - \frac{1}{r^2}\cos\gamma\right) + \frac{V}{r}\cos\gamma \\
\dot{m} &= -\frac{T}{I_{SP}}
\end{aligned} \tag{4}$$

Where r is the radius from the centre of the Earth to the vehicle; θ the polar angle; V the velocity; γ the flight path angle and m the total mass.

B. Optimal Ascent Trajectory Problem

The objective of trajectory optimization problem is to find an optimal solution to minimize a selected performance index. For ascent trajectory optimization, the performance index is typically chosen to fuel consumption and thereby maximizing the payload. Denote the performance index by

$$J = \Phi\left(\bar{x}_f, t_f\right) = -m_f$$

The initial conditions and terminal conditions are assumed to be known in this paper. We write them as k (k is the number of dynamic equations) algebraic conditions

$$\bar{\Psi}\left[\bar{x}\left(t_f\right)\right] = 0, \bar{\Psi} \in R^K \tag{6}$$

The Hamiltonian function is

$$\begin{aligned}
H_{2D} = {} & P_r\left(V\sin\gamma\right) + P_\theta\left(\frac{V\cos\gamma}{r}\right) + P_V\left(\frac{T\cos\alpha - D}{m} - \frac{\sin\gamma}{r^2}\right) \\
& + P_\gamma\left[\frac{1}{V}\left(\frac{T\sin\alpha + L}{m} - \frac{1}{r^2}\cos\gamma\right) + \frac{V}{r}\cos\gamma\right] - P_m \cdot \frac{T}{I_{SP}}
\end{aligned} \tag{7}$$

Therefore, the necessary condition for the optimal solution can be represented as

$$H\left(\bar{x}, \bar{p}, \bar{u}^*, t\right) = \max_{\bar{u}} H\left(\bar{x}, \bar{p}, \bar{u}, t\right) \tag{8}$$

Where $\dot{\bar{p}} = -\frac{\partial H}{\partial \bar{x}}$.

In addition to the terminal conditions given above, the optimal solution must meet the following transversality conditions

$$\begin{aligned}
\bar{p}\left(t_f\right) &= -\frac{\partial\Phi\left(\bar{x}_f, t_f\right)}{\partial \bar{x}_f} + \left(\frac{\partial\Psi}{\partial \bar{x}_f}\right)^T \bar{v} \\
H\left(\bar{x}, \bar{p}, \bar{u}^*, t\right)\Big|_{t_f} &= \frac{\partial\Phi\left(\bar{x}_f, t_f\right)}{\partial t_f}
\end{aligned} \tag{9}$$

Where $\bar{v} \in R^k$ is a constant multiplier vector. The final condition is used for free final time problem.

III. Gauss Pseudospectral Method

Gauss Pseudospectral Method approximates the state using a basis of global interpolating polynomials at a series of Legendre-Gauss (LG) points. The global interpolating polynomials can also be differentiated to approximate the left-hand side of the dynamic equations. Therefore constraints of differential equations are converted to a group of constraints of algebraic equations. The terminal states and the integral part of the cost are determined using Gauss quadrature. Eventually, the optimal control problem is converted to a nonlinear programming problem (NLP) with a series of algebraic equations constraints.

The First step in pseudospectral transcription is to change the time interval of the optimal control problem from $t \in [t_0, t_f]$ to $\tau \in [-1,1]$. This is done using the mapping

$$\tau = \frac{t_f - t_0}{2} + \frac{t_f + t_0}{2} \tag{10}$$

The mapping is used to replace the optimal control problem, minimizing the cost

$$J = \Phi\left(X(1), t_f\right) + \frac{\left(t_f - t_0\right)}{2} \int_{-1}^{1} L\left(x(\tau), u(\tau), \tau\right) d\tau \tag{11}$$

subject to the following dynamic constraints, boundary conditions and path constraints

$$\frac{dX}{d\tau} = \frac{\left(t_f - t_0\right)}{2} f\left(X(\tau), u(\tau), \tau\right) \tag{12}$$

$$\phi\left(X(-1), t_0, x(1), t_f\right) = 0 \tag{13}$$

$$C\left(X(\tau), u(\tau), \tau; t_0, t_f\right) \leq 0 \tag{14}$$

The state and control are approximated using a basis of N+1 Lagrange interpolating polynomials at the LG points

$$x(\tau) \approx X(\tau) = \sum_{i=0}^{N} X(\tau_i) L_i(\tau) \tag{15}$$

$$u(\tau) \approx U(\tau) = \sum_{i=0}^{N} X(\tau_i) \tilde{L}_i(\tau) \tag{16}$$

Where $L_i(\tau)$ are Lagrange polynomials

$$L_i(\tau) = \prod_{j=0, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j} \tag{17}$$

The left-hand side of the dynamic equations is approximated by differentiating the state at the Legendre-Gauss points as follows

$$\dot{x}(\tau) \approx \dot{X}(\tau) = \sum_{i=0}^{N} \dot{L}_i(\tau) X(\tau_i) = \sum_{i=0}^{N} D_{ki}(\tau) X(\tau_{ki}), \quad (k, i = 1, \ldots. N) \tag{18}$$

The costates play an important role in determine the optimality condition, so it's necessary to estimate the costates for optimality verification. The costates approximation, $\Lambda(\tau)$, utilizes a basis of N+1 Lagrange interpolating polynomials to estimate the costates and the derivative as

$$\lambda(\tau) \approx \Lambda(\tau) = \sum_{i=1}^{N+1} \lambda(\tau_i) L_i'(\tau) \tag{19}$$

$$\dot{\lambda}(\tau) \approx \dot{\Lambda}(\tau) = \sum_{i=1}^{N+1} \lambda(\tau_i) L_i'(\tau) = \sum_{i=1}^{N+1} \lambda(\tau_i) D_{ki}' \tag{20}$$

The derivative of Lagrange polynomial at LG points can be determined offline. GPM discretizes and transcribes the optimal control problem to a nonlinear programming problem.

IV. Hypersonic Vehicle Characteristics

The Generic Hypersonic Aerodynamics Model Example (GHAME) for computer simulation is used in this study. The GHAME model from Dryden Flight Research Centre provides general users with aerodynamic and engine data representative of a single-stage-to-orbit (SSTO) hypersonic aircraft that can take off horizontally from conventional runways, accelerate to orbital velocities and insert into a low Earth orbit (LEO).

The vehicle geometry is built up of simple geometric shapes. The primary structure is based off a cylinder 6.1 m in diameter and 36.6 m long. This comprises the volume required for tankage of the liquid hydrogen. A pair of 10° half-angle cones is attached onto the cylinder to form the nose and tail of the vehicle. The wing and vertical tail are modelled as triangular plates. The aerodynamic references are as follows: the reference area, S_{ref}, is 557.4 m2, the reference cord, c, is 22.9 m, and the reference span, b, is 24.4 m. The overall length of the vehicle is 71.4 m. The vehicle is estimated to have a gross take-off weight of 136080 kg. The configuration of the vehicle is shown in Fig. 2.

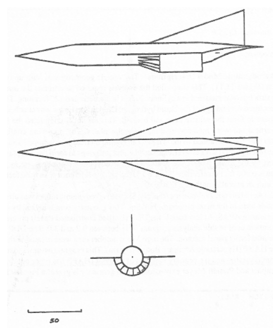


Fig. 2. GHAME configuration

A. GHAME Aerodynamic Data

The aerodynamic parameters are linearized about specific α for a given Mach number. The lift and drag coefficients, Cl and Cd respectively, are given by

$$Cl = Cl_0(Ma, \alpha) + Cl_\alpha(Ma, \alpha)\alpha + Cl_{\delta e}(Ma, \alpha)\delta e \tag{21}$$

$$Cd = Cd_0(Ma,\alpha) + Cd_\alpha(Ma,\alpha)\alpha \tag{22}$$

Where α is angle of attack of the vehicle, Ma is Mach number, δe is the elevon deflection. Cl0 is determined by fitting from look up table, where the range for Mach is $0<Ma<24$, and for angle of attack is $-3° < \alpha < 21°$. Details on the GHAME modelling can refer to [16, 17].

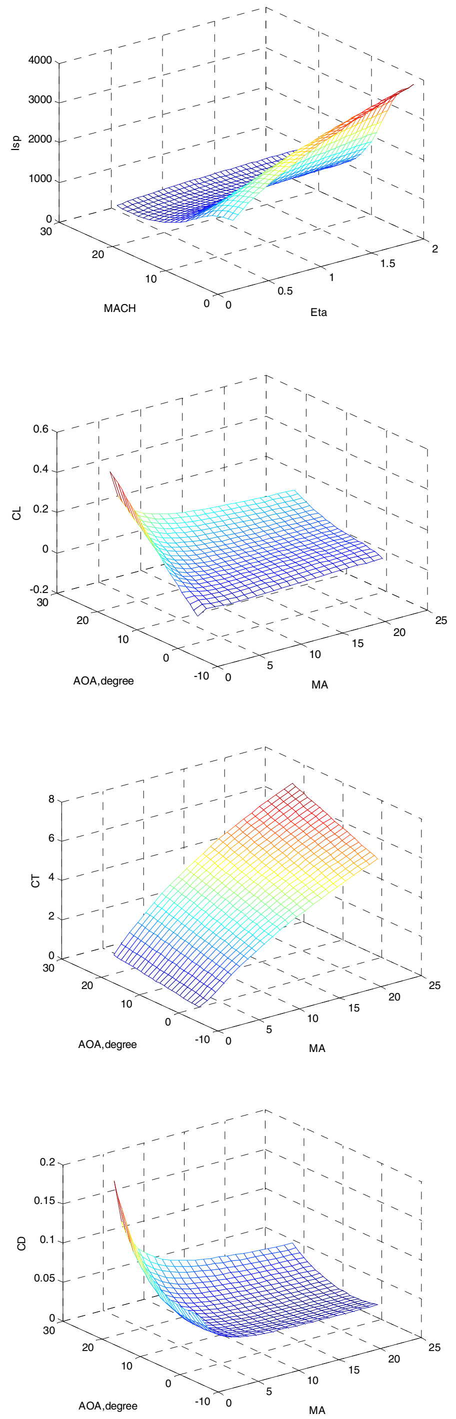


Fig. 3. Aerodynamic and propulsion coefficients of the GHAME

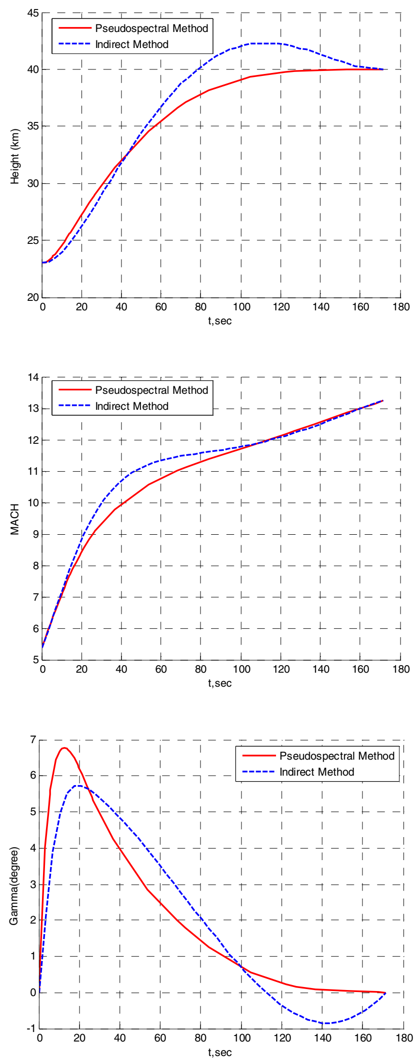
B. GHAME Engine Model

The engine is a multi-cycle, variable inlet geometry type. The break points for the cycles are: turbojet from Mach 0 to Mach 2; ramjet from Mach 2 to Mach 6; and a supersonic combustion ramjet (SCRAMJET) from Mach 6 and above. The thrust of the engine can be written as

$$T = \frac{r_0}{m_0} 0.029\eta \cdot Isp(Ma,\eta) \cdot \rho(r) \cdot V \cdot Ct(Ma,\alpha) \cdot Ac \tag{23}$$

V. Simulation Results

For the ascent trajectory problem, we compare optimal trajectories in vertical plane from GPM with an indirect method based on Finite Difference Method (FDM). GPOPT toolkit [15] is used to implement GPM to generate optimal trajectories. All the simulations are based on the aerodynamic and propulsion data of the GHAME. The following initial conditions are used: initial altitude of 23 km; initial velocity of 1600 m/s; initial mass of 136080 kg. The control variables are throttle command and angle of attack. The terminal conditions are: final attitude of 40 km; final velocity of 42000 m/s; flight path angle of 0°.



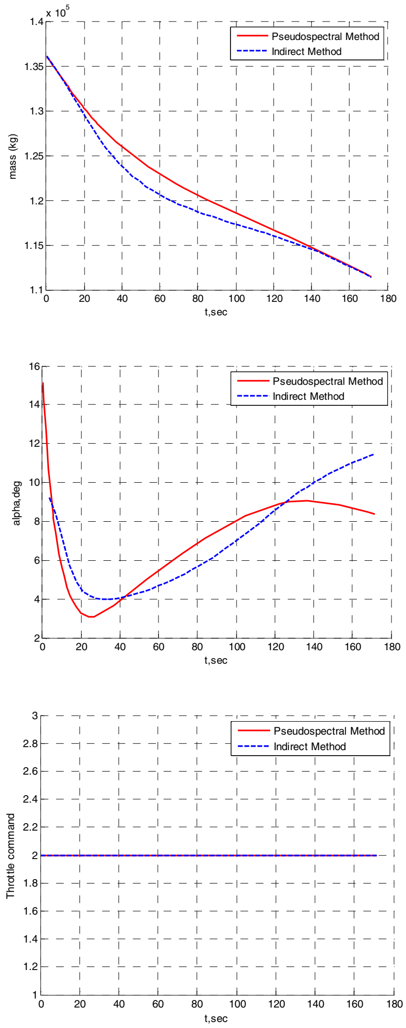


Fig. 4. Comparison of optimal ascent profiles

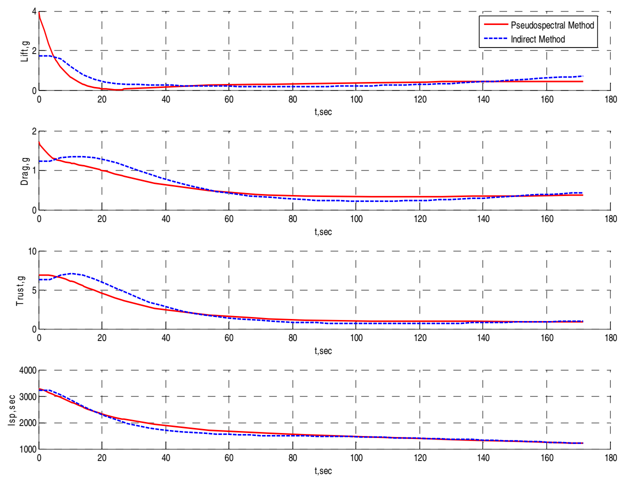


Fig. 5. Comparison of forces and specific impulse

Although both the methods can complete the same mission, the trajectory profiles show some difference. GPM overshoots the final altitude less than the FDM. The rate of angle of attack in GPM is a little larger than the FDM. We can also see that throttle should always be the maximum. The Mach number increases rapidly for the first 60s after launch and more smoothly as closer to the final condition. The comparison with FDM supports the validity of GPM, and their performances are quite close.

We show in Fig. 5 the force values, including lift, drag and thrust, and specific impulse of the vehicle as well. The peaks of the forces occur during the first 20s after launch. Hamiltonian values are plotted in Fig. 6 at each mesh point in the range of angle of attack. The results show that the Hamiltonian values equal zero when obtaining the optimal trajectory, which coincides with the inference from the optimal necessary conditions.

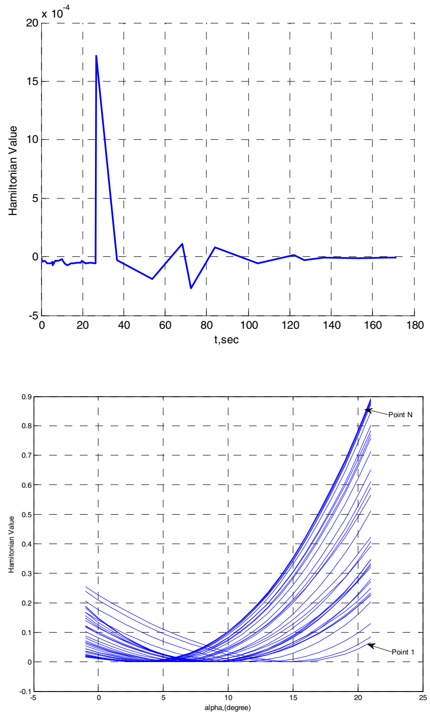


Fig. 6. Hamiltonian value trends

The trajectory trends for hypersonic vehicles are investigated from different initial conditions. The set of solutions have fixed terminal conditions, while the initial altitude is varied from 17 km to 27 km. As the vehicle initial altitude increases, the total time increases yet final weight increases. The reason for angle-of-attack decreasing is that the density decreases rapidly as altitude increases, so the vehicle at higher altitudes has a smaller acceleration than at lower altitudes, and generally a small accelerate needs a small angle-of-attack.

TABLE I. OPTIMAL TRAJECTORY WITH DIFFERENT INITIAL ALTITUDES

Initial Altitude *km*	***Final Time*** *s*	***Final Weight*** *kg*

Initial Altitude *km*	*Final Tine* *s*	*Final Weight* *kg*
17	169.1	110307.8
19	161.2	110831.8
21	166.1	111212.4
23	171.5	111471.0
25	177.5	111624.4
27	183.7	111702.6

The optimization performance of GPM and FDM is compared for the 6 ascent missions. Average time consumption and convergence probability are selected as two major performance indexes. Both optimization algorithms are performed under the same hardware condition. Table II shows the comparison results for the two methods. GPM costs much more time for ascent trajectory optimization than FDM on average, while GPM has a better convergence performance than FDM.

TABLE II. PERFORMANCE COMPARISON BETWEEN GPM AND FDM

	Average Tine consumption *s*	*convergence probability* *%*
GPM	6.17	100 (6/6)
FDM	0.32	83.3 (5/6)

VI. CONCLUSION

Gauss Pseudospectral Method was presented to generate optimal ascent trajectories for hypersonic air-breathing vehicles. The GHAME model was selected as the vehicle aerodynamic and propulsion model. Numerical simulations indicated GPM has greater convergent performance and remain high accuracy for ascent trajectory optimization. The comparison with Finite Difference Method demonstrated the validity of pseudospectral method. The optimality condition of general optimal control problem was also well verified by GPM. Optimal solutions with different initial conditions were shown to describe various characteristics of the ascent trajectories for hypersonic air-breathing vehicles. Pseudospectral method still requires further improvement on computation efficiency to apply to on-board trajectory optimization.

ACKNOWLEDGMENT

The authors would like to thank the supports of the National Natural Science Foundation of China (Grant No. 61175028) and the Ph.D. Programs Foundation of China Ministry of Education (Grant No. 20090073110045).

REFERENCES

[1] A. Calise, J. Corban, and G. Flandro, "Trajectory optimization and guidance law development for national aerospace plane applications," *Acta Cosmologica,* vol. 2, pp. 1406-1411, 1988.

[2] J. Corban, A. Calise, and G. Flandro, "Optimal guidance and propulsion control for transatmospheric vehicles," *NASA STI/Recon Technical Report A,* vol. 89, p. 52719, 1989.

[3] J. Corban, A. Calise, and G. Flandro, "Rapid near-optimal aerospace plane trajectory generation and guidance," *Journal of guidance, control, and dynamics,* vol. 14, pp. 1181-1190, 1991.

[4] P. F. Gath and A. J. Calise, "Optimization of Launch Vehicle Ascent Trajectories with Path Constraints and Coast Arcs," *Journal of guidance, control, and dynamics,* vol. 24, pp. 296-304, 2001.

[5] G. A. Dukeman and A. J. Calise, "Enhancements to an atmospheric ascent guidance algorithm," in *AIAA Guidance, Navigation, and Control Conference and Exhibit*, Austin, TX, 2003.

[6] G. A. Dukeman, "Atmospheric ascent guidance for rocket-powered launch vehicles," in *AIAA Guidance, Navigation, and Control Conference and Exhibit*, Monterey, CA, 2002.

[7] P. Lu, H. Sun, and B. Tsai, "Closed-loop endoatmospheric ascent guidance," *Journal of guidance, control, and dynamics,* vol. 26, pp. 283-294, 2003.

[8] P. Lu and B. Pan, "Highly Constrained Optimal Launch Ascent Guidance," *Journal of guidance, control, and dynamics,* vol. 33, pp. 404-414, 2010.

[9] B. Pan, P. Lu, and Z. Chen, "Three-dimensional closed-form costate solutions in optimal coast," *Acta Astronautica,* vol. 77, pp. 156-166, 2012.

[10] O. J. Murillo and P. Lu, "Fast Ascent Trajectory Optimization for Hypersonic Air-Breathing Vehicles," presented at the AIAA Guidance, Navigation, and Control Conference and Exhibit, Toronto, Ontario, 2010.

[11] I. Ross and F. Fahroo, "Legendre pseudospectral approximations of optimal control problems," *New Trends in Nonlinear Dynamics and Control and their Applications,* pp. 327-342, 2003.

[12] I. M. Ross and F. Fahroo, "Issues in the real-time computation of optimal control," *Mathematical and computer modelling,* vol. 43, pp. 1172-1188, 2006.

[13] D. Benson, "A Gauss pseudospectral transcription for optimal control," PhD, Aerospace Engineering, MIT. , 2005.

[14] A. V. Rao, D. A. Benson, C. Darby, M. A. Patterson, C. Francolin, I. Sanders, and G. T. Huntington, "Algorithm 902: GPOPS, a MATLAB software for solving multiple-phase optimal control problems using the Gauss pseudospectral method," *ACM Transactions on Mathematical Software,* vol. 37, pp. 1-39, 2010.

[15] C. L. Darby, W. W. Hager, and A. V. Rao, "An hp-adaptive pseudospectral method for solving optimal control problems," *Optimal Control Applications and Methods,* vol. 32, pp. 476-502, 2011.

[16] D. A. White, A. Bowers, K. Ili, and J. Menousek, "Flight, Propulsion, and Thermal Control of Advanced Aircraft and Hypersonic Vehicles," in *Handbook of Intelligent Control: Neural, Fuzzy, and Adaptive Approaches.* , ed New York: Multiscience Press, Inc., 1992, pp. 357-465.

[17] P. H. Zipfel, *Modeling and simulation of aerospace vehicle dynamics*. Reston, VA: American Institute of Aeronautics and Astronautics, 2007.