Robust State Estimation via the Descriptor Kalman Filtering Method

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***Abstract*—This paper considers robust state estimation problem for uncertain descriptor systems subject to bounded uncertainties on the basis of the descriptor Kalman filtering (DKF) method. A new robust filtering framework (RFF), which divides the uncertain augmented output equation (AOE) into two parts: one is the nominal part and the other is the uncertain part, is proposed to facilitate the robust filter design. In the sequel, a robust descriptor Kalman filter (RDKF) is derived based on the proposed RFF and the DKF method. Some simplified versions of the RDKF are also proposed for special cases. The motivation of this research is to show that the AOE reformulation imbedded in the recursive ML estimation method serves as a useful mean to yield the dedicated robust filters. An extension of the proposed result to solve state estimation for uncertain descriptor systems with unknown inputs is also provided.**

I. Introduction

Robust state estimation is one of the most important research topics in literature because system parameter uncertainties and external disturbances are generally unavoidable in practical system design. For both standard state-space and descriptor systems, the regularized least-squares (RLS) based framework developed in [1] is widely accepted as a useful mean to design robust state estimator to encompass systems with norm bounded uncertainties [2]-[5]. A more general paradigm in robust filter design, which is based on penalizing the sensitivity of estimation errors to parameter variations, is also proposed [6]. It is noted that all the above results are not directly applied to uncertain descriptor systems with unknown inputs (external disturbances).

Unknown input filtering (UIF) has also played a significant role in many robust applications, e.g., geophysical and environmental applications [7], fault detection and isolation problems [8], and functional filtering [9]. It is noticed that the descriptor Kalman filtering (DKF) [10], which was originally proposed to optimally estimate the nominal system state for descriptor systems, serves as a general estimation framework to yield H_∞/H_2 Kalman filter and predictor for descriptor and standard systems with or without unknown inputs [11]-[13]. However, to the best of the author's knowledge, the DKF is not directly applicable for the addressed robust state estimation problem. Recently [14], a potential robust filtering method, which intends to divide an uncertain measurement equation into two parts: one is the nominal output and the other is associated with the uncertain part, is proposed to facilitate the H_∞/H_2 filter design.

In this paper, we extend the previous works [11]-[14] and continue the research line in investigating the applications of the DKF in [10] to solve the addressed robust state estimation problem. Specifically, the previous proposed robust filtering method in [14] is extended and fully explored. The main aim of this paper is to develop a new robust filtering framework (RFF), which first divides the uncertain augmented output equation (AOE) [11] imbedded in the DKF design into two parts: one is the nominal part and the other is the uncertain part, and then yields an equivalent AOE which does not contain the uncertainties, to facilitate the robust filter design. In the sequel, a robust descriptor Kalman filter (RDKF) is derived based on the proposed RFF and the DKF. The motivation of this research is to show that the AOE reformulation imbedded in the recursive maximum likelihood (ML) estimation method serves as a useful mean to yield the dedicated robust filters.

The paper is organized as follows. In Section II, the statement of the problem is addressed and the existing RLS based robust filter design is briefly introduced. In Section III, the proposed RFF is presented and a robust DKF (RDKF) for uncertain descriptor system is designed. It is shown in Theorem 1 that the RDKF can be equivalent to the DR filter in [2]. In Section IV, some simplified versions of the RDKF for a special case are presented, through which the results in [1] and [2] are rederived and compared. An extension to design a robust filter for uncertain descriptor systems with unknown inputs is addressed in Section V. Finally, conclusions are highlighted in the last section.

II. Statement of the Problem

Consider the uncertain discrete-time linear stochastic descriptor system as follows:

$$(E_{k+1} + \delta E_{k+1})x_{k+1} = (A_k + \delta A_k)x_k + w_k, \quad (1)$$

$$y_k = (C_k + \delta C_k)x_k + v_k, \quad (2)$$

where $x_k \in R^n$ is the descriptor vector and $y_k \in R^m$ is the measured output. The process noise $w_k \in R^p$ and the measurement noise v_k are uncorrelated zero-mean white sequences with covariance matrices $Q_k > 0$ and $R_k > 0$, respectively. The initial state x_0 is with unbiased mean $\bar{x}_0$ and covariance $\bar{P}_0^x$ and is independent of w_k and v_k. E_{k+1}, A_k, and C_k are the known nominal system matrices and δE_{k+1},

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δA_k, and δC_k are time-varying uncertainties corresponding to the nominal system matrices. In this paper, we assume that the uncertainties are given as the following structures:

$$\begin{aligned} \delta A_k = M_{a,k}\Delta_k^x N_{a,k}, \quad \delta E_{k+1} = M_{a,k}\Delta_k^x N_{e,k+1}, \\ \delta C_k = M_{c,k}\Delta_k^y N_{c,k}, \quad ||\Delta_k^x|| \leq 1, \quad ||\Delta_k^y|| \leq 1, \end{aligned} \tag{3}$$

where $M_{a,k}$, $M_{c,k}$, $N_{a,k}$, $N_{e,k+1}$, and $N_{c,k}$ are known matrices, and Δ_k^x and Δ_k^y are both arbitrary bounded matrices with norms less or equal to 1. The robust estimation problem of the paper is focused on finding an optimal recursive robust state estimation for the addressed descriptor system with modeling uncertainties.

A well known approach to solve the aforementioned robust estimation problem is to consider the following RLS problem [2], [3], [4]:

$$\min_x \max_{\delta A, \delta b} \left[||x||_Q^2 + ||(A+\delta A)x - (b+\delta b)||_W^2 \right], \tag{4}$$

with the following maps

$$\begin{bmatrix} -A_k & E_{k+1} \\ 0 & C_{k+1} \end{bmatrix} \rightarrow A, \begin{bmatrix} -\delta A_k & \delta E_{k+1} \\ 0 & \delta C_{k+1} \end{bmatrix} \rightarrow \delta A,$$

$$\begin{bmatrix} x_k - \hat{x}_{k|k} \\ x_{k+1} \end{bmatrix} \rightarrow x, \begin{bmatrix} A_k \hat{x}_{k|k} \\ y_{k+1} \end{bmatrix} \rightarrow b, \begin{bmatrix} \delta A_k \hat{x}_{k|k} \\ 0 \end{bmatrix} \rightarrow \delta b,$$

$$\begin{bmatrix} P_{k|k}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \rightarrow Q, \begin{bmatrix} Q_k^{-1} & 0 \\ 0 & R_{k+1}^{-1} \end{bmatrix} \rightarrow W.$$

Then, the solution of (4) is given by

$$\hat{x} = x(\hat{\lambda}), \quad \hat{\lambda} = arg \min_{\lambda \geq ||H^T W H||} G(\lambda), \tag{5}$$

where

$$\begin{aligned} x(\lambda) &= \left(Q(\lambda) + A^T W(\lambda) A\right)^{-1} \\ &\quad \times \left(A^T W(\lambda) b + \lambda N_A^T N_b\right), &(6)\\ G(\lambda) &= ||x(\lambda)||_Q^2 + \lambda ||N_A x(\lambda) - N_b||^2 \\ &\quad + ||Ax(\lambda) - b||_{W(\lambda)}^2, &(7)\\ Q(\lambda) &= Q + \lambda N_A^T N_A, &(8)\\ W(\lambda) &= W + WH(\lambda I - H^T W H)^+ H^T W, &(9)\\ N_A &= \begin{bmatrix} -N_{a,k} & N_{e,k+1} \\ 0 & N_{c,k+1} \end{bmatrix}, N_b = \begin{bmatrix} N_{a,k}\hat{x}_{k|k} \\ 0 \end{bmatrix}, &(10)\\ H &= \begin{bmatrix} M_{a,k} & 0 \\ 0 & M_{c,k+1} \end{bmatrix}, &(11) \end{aligned}$$

in which M^+ is the Moore-Penrose pseudo-inverse of M. As shown in [2], the optimum robust filtered estimates $\hat{x}_{k|k}$ resulting from (4) can be derived by solving an appropriate cost function (also known as the optimum data fitting problem), which yields the following DR filter:

$$\begin{aligned} \hat{x}_{k+1|k+1} &= P_{k+1|k+1}\left(\hat{E}_{k+1}^T(\hat{Q}_k + \hat{A}_k P_{k|k}\hat{A}_k^T)^{-1} \right. \\ &\quad \left. \times \hat{A}_k \hat{x}_{k|k} + \hat{C}_{k+1}^T \hat{R}_{k+1|k+1}^{-1} y_{k+1}^*\right), &(12)\\ P_{k+1|k+1} &= \left(\hat{E}_{k+1}^T(\hat{Q}_k + \hat{A}_k P_{k|k}\hat{A}_k^T)^{-1}\hat{E}_{k+1} \right. \\ &\quad \left. + \hat{C}_{k+1}^T \hat{R}_{k+1|k+1}^{-1}\hat{C}_{k+1}\right)^{-1}, &(13) \end{aligned}$$

where $y_{k+1}^* = \begin{bmatrix} y_{k+1}^T & 0 \end{bmatrix}^T$,

$$\begin{aligned} \hat{E}_{k+1} &= \begin{bmatrix} E_{k+1} \\ \sqrt{\hat{\lambda}_k} N_{e,k+1} \end{bmatrix}, &(14)\\ \hat{C}_{k+1} &= \begin{bmatrix} C_{k+1} \\ \sqrt{\hat{\lambda}_k} N_{c,k+1} \end{bmatrix}, &(15)\\ \hat{R}_{k+1} &= \begin{bmatrix} R_{k+1} - \hat{\lambda}_k^{-1} M_{c,k+1} M_{c,k+1}^T & 0 \\ 0 & I \end{bmatrix}, &(16)\\ \hat{Q}_k &= \begin{bmatrix} Q_k - \hat{\lambda}_k^{-1} M_{a,k} M_{a,k}^T & 0 \\ 0 & I \end{bmatrix}, &(17)\\ \hat{A}_k &= \begin{bmatrix} A_k \\ \sqrt{\hat{\lambda}_k} N_{a,k} \end{bmatrix}, &(18) \end{aligned}$$

in which the optimum scalar parameter $\hat{\lambda}_k$ is obtained by minimizing the function $G(\lambda)$ of (7) over the following searching interval:

$$\hat{\lambda}_k > ||diag\{M_{a,k}^T Q_k^{-1} M_{a,k}, M_{c,k+1}^T R_{k+1}^{-1} M_{c,k+1}\}||. \tag{19}$$

Note that the robust estimation performance of (12)-(13) can be improved through re-scaling the inputs and outputs of the structured modeling errors by applying off-line convex optimization of the semi-axes of an ellipsoid that contains all the possible outputs of the uncertainty matrices [4].

As previously stated in Introduction, the DKF serves as a general estimation framework to yield H_∞/H_2 Kalman filter and predictor for descriptor and standard systems with or without unknown inputs. However, it is also noted that the DKF is not directly applicable for the aforementioned robust state estimation problem. Thus, the main aim of this paper is to rederive (12)-(13) via the proposed RFF and the DKF. The obtained robust filter is named as the robust DKF (RDKF).

III. ROBUST STATE ESTIMATORS DESIGN

A. ML Estimation for Uncertain Measurement

In this subsection, we show the main idea how to derive a robust version of maximum likelihood (ML) linear estimation beginning with the simple problem of estimating an unknown vector x based on the uncertain measurement vector y as follows:

$$y = (C + M\Delta N)x + v, \quad ||\Delta|| \leq 1, \tag{20}$$

where v is a zero-mean random vector with covariance R.

First, we divide (20) into two parts, one is the nominal part, i.e. $\Delta = 0$, and the other is the remaining part which highlights the uncertain term, as follows:

$$\begin{aligned} y &= Cx + \hat{v}, &(21)\\ 0 &= M\Delta N x + \tilde{v}, &(22) \end{aligned}$$

where $\hat{v}$ and $\tilde{v}$ are uncorrelated zero-mean random vectors with undetermined covariances $\hat{R}$ and $\tilde{R}$, respectively.

Second, we left multiply (22) by $M^T R^{-1}$ as follows:

$$0 = (M^T R^{-1} M)\Delta N x + M^T R^{-1}\tilde{v}. \tag{23}$$

Choose $\tilde{v}$ as follows:

$$\tilde{v} = \hat{\lambda}^{-1/2} M n, \tag{24}$$

where $\hat{\lambda} > ||M^T R^{-1} M||$ and n is a zero-mean random vector with unity covariance. Using (24) in (23) yields

$$0 = (M^T R^{-1} M)\Delta N x + \hat{\lambda}^{-1/2}(M^T R^{-1} M) n, \tag{25}$$

which holds if the following constraint is satisfied:

$$0 = \sqrt{\hat{\lambda}}\Delta N x + n. \tag{26}$$

From the last term of (20), it is clear that

$$||\Delta N x||^2 \leq (Nx)^T (Nx),$$

which renders (26) to the following more useful expression:

$$0 = \sqrt{\hat{\lambda}} N x + n, \tag{27}$$

if one intends to maximize the uncertainty.

Third, comparing (20) with (21)-(22) and using (24), we have

$$\hat{v} = v - \tilde{v} = v - \hat{\lambda}^{-1/2} M n, \tag{28}$$

which has the following covariance:

$$\begin{aligned} cov(\hat{v}) &= cov(v - \tilde{v}) \\ &= cov(v) + cov(\tilde{v}) - E[v\tilde{v}^T] - E[\tilde{v}v^T] \\ &= R - \lambda^{-1} M M^T, \end{aligned} \tag{29}$$

where the following relationships are used:

$$E[\hat{v}\tilde{v}^T] = 0, \quad E[\tilde{v}\hat{v}^T] = 0.$$

Next, summarizing the above arguments we can transform the original uncertain system (20) into the following equivalent system:

$$y^* = \hat{C} x + \hat{v}^*, \tag{30}$$

where

$$y^* = \begin{bmatrix} y \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C \\ \sqrt{\hat{\lambda}} N \end{bmatrix} x, \quad \hat{v}^* = \begin{bmatrix} \hat{v} \\ n \end{bmatrix}. \tag{31}$$

The covariance of $\hat{v}^*$ is given by

$$cov(\hat{v}^*) = \hat{R} = \begin{bmatrix} R - \hat{\lambda}^{-1} M M^T & 0 \\ 0 & I \end{bmatrix}. \tag{32}$$

Finally, the ML estimate of x based on (30) is given as follows:

$$\hat{x}_{ML} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{C} \\ \hat{C}^T & 0 \end{bmatrix}^+ \begin{bmatrix} y^* \\ 0 \end{bmatrix}, \tag{33}$$

$$P_{ML} = -\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{C} \\ \hat{C}^T & 0 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{34}$$

In the following lemma, we show that the above ML estimator can be equivalent to the DR filter corresponding to (20).

Lemma 1: Consider the uncertain observations as given by (30). Then the ML estimate of x based on (30), that is referenced by (33)-(34), is the same as the DR filter corresponding to (20) if the following rank condition is achieved:

$$rank[\hat{C}] = dim(x). \tag{35}$$

Proof: The proof is a special case of that of Theorem 1.

B. ML Estimation for Uncertain Descriptor State

In this subsection, we further show how to derive a robust version of maximum likelihood (ML) linear estimation for an uncertain system state transition. Consider the following uncertain descriptor system state update at time k:

$$\begin{aligned} &(E_{k+1} + M_{a,k}\Delta_k N_{e,k+1}) x_{k+1} \\ &= (A_k + M_{a,k}\Delta_k N_{a,k}) x_k + w_k, \quad ||\Delta_k|| \leq 1, \end{aligned} \tag{36}$$

where w_k is a zero-mean white sequence with covariance Q_k. We assume that at step k we are given an *a priori* estimate for the state x_k denoted by $\hat{x}_k$ and with the covariance P_k^x.

First, with the appropriate definition of the AOE [11] corresponding to (36), we obtain

$$\begin{bmatrix} \bar{x}_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} E_{k+1} \\ \sqrt{\hat{\lambda}_k} N_{e,k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} \bar{\eta}_{k+1} \\ \tilde{\eta}_{k+1} \end{bmatrix}, \tag{37}$$

where

$$\bar{x}_{k+1} = A_k \hat{x}_k, \tag{38}$$

$$\tilde{x}_{k+1} = \sqrt{\hat{\lambda}_k} N_{a,k} \hat{x}_k, \tag{39}$$

$$\bar{\eta}_{k+1} = -A_k(x_k - \hat{x}_k) - (w_k - \hat{\lambda}_k^{-1/2} M_{a,k} n_k^x), \tag{40}$$

$$\tilde{\eta}_{k+1} = -\sqrt{\hat{\lambda}_k} N_{a,k}(x_k - \hat{x}_k) - n_k^x, \tag{41}$$

in which n_k^x is a zero-mean white sequence with unity covariance. The covariances of $\bar{\eta}_{k+1}$ and $\tilde{\eta}_{k+1}$ are given, respectively, as follows:

$$\bar{P}_{k+1} = A_k P_k^x A_k^T + Q_k - \hat{\lambda}_k^{-1} M_{a,k} M_{a,k}^T, \tag{42}$$

$$\tilde{P}_{k+1} = \hat{\lambda}_k N_{a,k} P_k^x N_{a,k}^T + I. \tag{43}$$

Second, using (14), (17)-(18), and (42)-(43), (37) can be expressed more compactly as follows:

$$x_{k+1}^* = \hat{E}_{k+1} x_{k+1} + \eta_{k+1}^*, \tag{44}$$

where

$$x_{k+1}^* = \begin{bmatrix} \bar{x}_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \hat{A}_k \hat{x}_k, \quad \eta_{k+1}^* = \begin{bmatrix} \bar{\eta}_{k+1} \\ \tilde{\eta}_{k+1} \end{bmatrix}. \tag{45}$$

Furthermore, the covariance of η_{k+1}^* is given as follows:

$$cov(\eta_{k+1}^*) = \hat{P}_{k+1} = \hat{Q}_k + \hat{A}_k P_k^x \hat{A}_k^T. \tag{46}$$

Finally, the ML estimate of x based on (44) is given as follows:

$$\hat{x}_{k+1,ML} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_{k+1} & \hat{E}_{k+1} \\ \hat{E}_{k+1}^T & 0 \end{bmatrix}^+ \begin{bmatrix} x_{k+1}^* \\ 0 \end{bmatrix}, \tag{47}$$

$$P_{k+1,ML} = -\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_{k+1} & \hat{E}_{k+1} \\ \hat{E}_{k+1}^T & 0 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{48}$$

In the following lemma, we show that the above ML estimator can be equivalent to the DR filter corresponding to (33).

Lemma 2: Consider the uncertain descriptor state transition as given by (33). Then the ML estimate of x based on (44), that is referenced by (47)-(48), is the same as the DR filter corresponding to (33) if the following rank condition is achieved:

$$rank[\hat{E}_{k+1}] = dim(x_{k+1}). \tag{49}$$

Proof: The proof is a special case of that of Theorem 1.

C. Robust DKF for Uncertain Descriptor Systems

The recursive equation for the robust filtered estimates corresponding to (1)-(3) can be directly obtained by combining the results of the previous subsections. First, we can define the AOE corresponding to (1)-(2) as follows:

$$\begin{bmatrix} x^*_{k+1} \\ y^*_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{E}_{k+1} \\ \hat{C}_{k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} \eta^*_{k+1} \\ \hat{v}^*_{k+1} \end{bmatrix}, \tag{50}$$

where

$$\hat{v}^*_{k+1} = \begin{bmatrix} \hat{v}_{k+1} \\ n^y_{k+1} \end{bmatrix}, \quad cov(\hat{v}^*_{k+1}) = \hat{R}_{k+1}, \tag{51}$$

in which n^y_{k+1} is a zero-mean white sequence with unity covariance.

Next, applying the DKF to (50), we obtain the following robust DKF (RDKF):

$$\begin{aligned} \hat{x}_{k+1} &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_{k+1} & 0 & \hat{E}_{k+1} \\ 0 & \hat{R}_{k+1} & \hat{C}_{k+1} \\ \hat{E}^T_{k+1} & \hat{C}^T_{k+1} & 0 \end{bmatrix}^+ \\ &\quad \times \begin{bmatrix} (x^*_{k+1})^T & (y^*_{k+1})^T & 0 \end{bmatrix}^T, \qquad (52) \\ P^x_{k+1} &= -\begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_{k+1} & 0 & \hat{E}_{k+1} \\ 0 & \hat{R}_{k+1} & \hat{C}_{k+1} \\ \hat{E}^T_{k+1} & \hat{C}^T_{k+1} & 0 \end{bmatrix}^+ \\ &\quad \times \begin{bmatrix} 0 & 0 & I \end{bmatrix}^T. \qquad (53) \end{aligned}$$

Finally, the equivalence of the DR filter and the RDKF is highlighted in the following theorem.

Theorem 1: If the following rank condition holds:

$$rank \begin{bmatrix} \hat{E}_{k+1} \\ \hat{C}_{k+1} \end{bmatrix} = dim(x_{k+1}), \tag{54}$$

the DR filter, which is given by (12)-(13), is equivalent to the RDKF, that is referenced by (52)-(53).

Proof. If rank condition (54) holds, the pseudo inverse in (52)-(53) can be implemented alternatively by using matrix inverse. Thus, we have [11]:

$$\begin{bmatrix} \hat{P}_k & 0 & \hat{E}_k \\ 0 & \hat{R}_k & \hat{C}_k \\ \hat{E}^T_k & \hat{C}^T_k & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \Gamma_k & \Psi_k & \Upsilon_k \end{bmatrix}, \tag{55}$$

where the entries marked as "×" are irrelevant to the discussion,

$$\begin{aligned} \Gamma_k &= -\Upsilon_k \hat{E}^T_k (\hat{P}_k)^{-1}, \quad \Psi_k = -\Upsilon_k \hat{C}^T_k (\hat{R}_k)^{-1}, \qquad (56) \\ \Upsilon_k &= -(\hat{E}^T_k \hat{P}^{-1}_k \hat{E}_k + \hat{C}^T_k \hat{R}^{-1}_k \hat{C}_k)^{-1}. \qquad (57) \end{aligned}$$

Using (55)-(57) in (52)-(53) and comparing the results with (12)-(13), it is clear that the RDKF is equivalent to the DR filter. This has completed the proof.

Remark 1. From (52)-(53), it is clear that for nominal descriptor systems, i.e., $M_{a,k} = 0$, $M_{c,k} = 0$, $N_{e,k+1} = 0$, $N_{a,k} = 0$, and $N_{c,k} = 0$, the above RDKF is reduced to the conventional "3-block" DKF in [10].

IV. Simplified Versions of the RDKF

In this section, we will derive some simplified versions of the RDKF subject to a special case $\delta E_{k+1} = 0$. In achieving this, we consider the following uncertain system model:

$$\begin{aligned} E_{k+1} x_{k+1} &= (A_k + M_{a,k} \Delta^x_k N_{a,k}) x_k + w_k, \qquad (58) \\ y_k &= (C_k + M_{c,k} \Delta^y_k N_{c,k}) x_k + v_k. \qquad (59) \end{aligned}$$

We assume that at time k the *a posteriori* estimate for the state x_k can be obtained as $\hat{x}_k$, which has covariance P^x_k, i.e.,

$$x_k = \hat{x}_k + \eta^x_k, \quad cov(\eta^x_k) = P^x_k. \tag{60}$$

The robust estimation problem at hand is to estimate x_{k+1} based on y_{k+1} with arbitrary contractions Δ^x_k and Δ^y_k, where $||\Delta^x_k|| \leq 1$ and $||\Delta^x_k|| \leq 1$.

A. Equivalently Reduced-order AOE

First, we present the AOE corresponding to (58)-(59) based on Section III as follows:

$$\begin{bmatrix} x^*_{k+1} \\ y^*_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{E}_{k+1} \\ \hat{C}_{k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} \eta^*_{k+1} \\ \hat{v}^*_{k+1} \end{bmatrix}, \tag{61}$$

where $\hat{E}_{k+1} = \begin{bmatrix} E^T_{k+1} & 0 \end{bmatrix}^T$. The covariances of η^*_{k+1} and $\hat{v}^*_{k+1}$ denoted by $\hat{P}_{k+1}$ and $\hat{R}_{k+1}$, respectively, are given by (46) and (16), respectively. In the following discussions, we assume that rank condition (54) holds.

Second, using the following identities:

$$\begin{aligned} \hat{E}^T_{k+1} \hat{P}^{-1}_{k+1} \hat{E}_{k+1} &= E^T_{k+1} (A_k \hat{P}^x_k A^T_k + Q^*_k)^{-1} E_{k+1}, \qquad (62) \\ \hat{E}^T_{k+1} \hat{P}^{-1}_{k+1} \hat{A}_k \hat{x}_k &= E^T_{k+1} (A_k \hat{P}^x_k A^T_k + Q^*_k)^{-1} A^*_k \hat{x}_k, \qquad (63) \end{aligned}$$

where

$$\begin{aligned} \hat{P}^x_k &= \left((P^x_k)^{-1} + \hat{\lambda}_k N^T_{a,k} N_{a,k} \right)^{-1}, \qquad (64) \\ Q^*_k &= Q_k - \hat{\lambda}^{-1}_k M_{a,k} M^T_{a,k}, \qquad (65) \\ A^*_k &= A_k \hat{P}^x_k (P^x_k)^{-1}, \qquad (66) \end{aligned}$$

in which $\hat{\lambda}_k$ is given by (19), the AOE (61) can be simplified as follows:

$$\begin{bmatrix} A^*_k \hat{x}_k \\ y^*_{k+1} \end{bmatrix} = \begin{bmatrix} E_{k+1} \\ \hat{C}_{k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} -w^*_k \\ \hat{v}^*_{k+1} \end{bmatrix}, \tag{67}$$

where

$$E[w^*_k] = 0, \quad cov(w^*_k) = A_k \hat{P}^x_k A^T_k + Q^*_k. \tag{68}$$

Finally, the simplified version of the RDKF can be obtained by using the following maps in (52)-(53).

$$A_k^*\hat{x}_k \to x_{k+1}^*, \quad E_{k+1} \to \hat{E}_{k+1}, \quad cov(w_k^*) \to \hat{P}_{k+1}. \quad (69)$$

Remark 2. An equivalent system which has the same AOE as given by (61) can be obtained as follows:

$$\begin{aligned} \hat{E}_{k+1}x_{k+1} &= \hat{A}_k x_k + \hat{w}_k, & (70)\\ y_k^* &= \hat{C}_k x_k + \hat{v}_k^*, & (71) \end{aligned}$$

where $\hat{w}_k$ is a zero-mean white sequence with covariance $\hat{Q}_k$. However, unlike that for (61) there is no corresponding equivalent system for the reduced-order AOE (67). This illustrates one advantage of transforming the original system model into an equivalent AOE, which is imbedded in the ML estimation method.

B. An Alternative Solution for $E_{k+1} = I$

In this subsection, we give an alternative simplified solution to (58)-(59) subject to $E_{k+1} = I$ based on the approach in [1]. To facilitate the following derivation, the measurement (59) is rewritten by using (58) as follows:

$$\begin{aligned} y_{k+1} &= C_{k+1}A_k x_k + C_{k+1}w_k + v_{k+1} \\ &\quad + (C_{k+1}M_{a,k}\Delta_k^x N_{a,k} + M_{c,k+1}\Delta_{k+1}^y N_{c,k+1}A_k)x_k. \quad (72) \end{aligned}$$

Define the augmented state X_k as $X_k = \begin{bmatrix} x_k^T & w_k^T \end{bmatrix}^T$. Then, the measurement (72) can be rewritten as follows:

$$y_{k+1}^* = \hat{C}_{k+1}^* X_k + v_{k+1}^*, \quad (73)$$

where

$$\begin{aligned} y_{k+1}^* &= \begin{bmatrix} y_{k+1} \\ 0 \end{bmatrix}, & (74)\\ \hat{C}_{k+1}^* &= \begin{bmatrix} C_{k+1}A_k & C_{k+1} \\ \sqrt{\hat{\lambda}_k}N_{k+1} & 0 \end{bmatrix}, & (75)\\ v_{k+1}^* &= \begin{bmatrix} v_{k+1} - \hat{\lambda}_k^{-1/2}M_{k+1}n_{k+1}^y \\ n_{k+1}^y \end{bmatrix}, & (76)\\ N_{k+1} &= \begin{bmatrix} N_{a,k} \\ N_{c,k+1}A_k \end{bmatrix}, & (77)\\ M_{k+1} &= \begin{bmatrix} C_{k+1}M_{a,k} & M_{c,k+1} \end{bmatrix}, & (78) \end{aligned}$$

in which $\hat{\lambda}_k > ||M_{k+1}^T R_{k+1}^{-1} M_{k+1}||$.

Thus, the AOE corresponding to (60) and (73) can be expressed as follows:

$$\begin{bmatrix} X_k^* \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} E_{k+1}^* \\ C_{k+1}^* \end{bmatrix} X_k + \begin{bmatrix} \eta_{k+1}^\dagger \\ \hat{v}_{k+1} \end{bmatrix}, \quad (79)$$

where

$$\begin{aligned} X_k^* &= \begin{bmatrix} \hat{x}_k \\ 0 \\ 0 \end{bmatrix}, \quad E_{k+1}^* = \begin{bmatrix} I & 0 \\ \sqrt{\hat{\lambda}_k}N_{k+1} & 0 \\ 0 & I \end{bmatrix}, & (80)\\ \eta_{k+1}^\dagger &= \begin{bmatrix} -\eta_k^x \\ n_{k+1}^y \\ -w_k \end{bmatrix}, \quad C_{k+1}^* = C_{k+1}\begin{bmatrix} A_k & I \end{bmatrix}, & (81)\\ \hat{v}_{k+1} &= v_{k+1} - \hat{\lambda}_k^{-1/2}M_{k+1}n_{k+1}^y. & (82) \end{aligned}$$

The covariances of $\eta_{k+1}^\dagger$ and $\hat{v}_{k+1}$ are given, respectively, by

$$\begin{aligned} cov(\eta_{k+1}^\dagger) &= P_{k+1}^\dagger = \begin{bmatrix} P_k^x & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_k \end{bmatrix}, & (83)\\ cov(\hat{v}_{k+1}) &= R_{k+1}^* = R_{k+1} - \hat{\lambda}^{-1}M_{k+1}M_{k+1}^T. & (84) \end{aligned}$$

Using the following relationships:

$$\begin{aligned} (E_{k+1}^*)^T(P_{k+1}^\dagger)^{-1}E_{k+1}^* &= \begin{bmatrix} (\bar{P}_k^x)^{-1} & 0 \\ 0 & Q_k^{-1} \end{bmatrix}, & (85)\\ (E_{k+1}^*)^T(P_{k+1}^\dagger)^{-1}X_k^* &= \begin{bmatrix} (P_k^x)^{-1} \\ 0 \end{bmatrix}\hat{x}_k, & (86) \end{aligned}$$

where

$$\bar{P}_k^x = \left((P_k^x)^{-1} + \hat{\lambda}_k N_{k+1}^T N_{k+1}\right)^{-1}, \quad (87)$$

the AOE (79) can be replaced by

$$\begin{bmatrix} x_k^* \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} I \\ C_{k+1}^* \end{bmatrix} X_k + \begin{bmatrix} \bar{\eta}_{k+1}^* \\ \hat{v}_{k+1} \end{bmatrix}, \quad (88)$$

where

$$\begin{aligned} x_k^* &= \begin{bmatrix} (P_k^x)^{-1} \\ 0 \end{bmatrix}\hat{x}_k, & (89)\\ E[\bar{\eta}_{k+1}^*] &= 0, \quad cov(\bar{\eta}_{k+1}^*) = \begin{bmatrix} \bar{P}_k^x & 0 \\ 0 & Q_k \end{bmatrix}. & (90) \end{aligned}$$

Finally, using the following state transformation:

$$x_{k+1} = \begin{bmatrix} A_k & I \end{bmatrix} X_k, \quad (91)$$

the AOE (88) can be rewritten as follows:

$$\begin{bmatrix} \bar{A}_k\hat{x}_k \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} I \\ C_{k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} -\bar{w}_k^* \\ \hat{v}_{k+1} \end{bmatrix}, \quad (92)$$

where

$$\begin{aligned} \bar{A}_k &= A_k\bar{P}_k^x(P_k^x)^{-1}, & (93)\\ E[\bar{w}_k^*] &= 0, \quad cov(\bar{w}_k^*) = A_k\bar{P}_k^x A_k^T + Q_k. & (94) \end{aligned}$$

The simplified version of the RDKF can then be obtained by using the following maps in (52)-(53).

$$\begin{aligned} &\bar{A}_k\hat{x}_k \to x_{k+1}^*, \quad I \to \hat{E}_{k+1}, \quad cov(\bar{w}_k^*) \to \hat{P}_{k+1}, \\ &y_{k+1} \to y_{k+1}^*, \quad C_{k+1} \to \hat{C}_{k+1}, \quad R_{k+1}^* \to \hat{R}_{k+1}. \quad (95) \end{aligned}$$

Remark 3. For the special case that $E_{k+1} = I$ and $\delta C_k = 0$, the AOE (67) is reduced to

$$\begin{bmatrix} A_k^*\hat{x}_k \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} I \\ C_{k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} -w_k^* \\ v_{k+1} \end{bmatrix}. \quad (96)$$

Comparing (96) with (92), it is clear that the RDKFs derived in Subsections IV-A and IV-B have almost the same filter structures, which have differences only in the modified covariances affected by the uncertainties and the intervals of optimization for the parameter $\hat{\lambda}_k$ [2]. Note that, as shown in [4], [5], the performances of both filters are almost equivalent.

V. EXTENSION TO UNCERTAIN DESCRIPTOR SYSTEMS WITH UNKNOWN INPUTS

In this section, we show how to apply the proposed robust state estimation method to solve more general uncertain system which involves both norm bounded system uncertainties and arbitrary unknown inputs. To this aim, consider the extension of uncertain descriptor system (1)-(3) with unknown inputs as follows:

$$(E_{k+1}+\delta E_{k+1})x_{k+1} = (A_k+\delta A_k)x_k + G_k d_k + w_k, \quad (97)$$

$$y_k = (C_k+\delta C_k)x_k + H_k d_k + v_k. \quad (98)$$

The problem at hand is to optimally estimate the system state x_{k+1} regardless the values of d_{k+1}. As illustrated in the previous section, the main idea to solve the aforementioned state estimation problem is to properly formulate the AOE corresponding to (97)-(98), which is given as below.

First, using the result in Section III.C, we can define the AOE corresponding to (97)-(98) as follows:

$$\begin{aligned}\begin{bmatrix} x^*_{k+1} \\ y^*_{k+1} \end{bmatrix} &= \begin{bmatrix} \hat{E}_{k+1} \\ \hat{C}_{k+1} \end{bmatrix} x_{k+1} + \begin{bmatrix} 0 \\ \hat{H}_{k+1} \end{bmatrix} d_{k+1} \\ &\quad + \begin{bmatrix} -\hat{G}_k \\ 0 \end{bmatrix} d_k + \begin{bmatrix} \eta^*_{k+1} \\ \hat{v}^*_{k+1} \end{bmatrix}, \end{aligned} \quad (99)$$

where

$$\hat{H}_{k+1} = \begin{bmatrix} H_{k+1} \\ 0 \end{bmatrix}, \quad \hat{G}_k = \begin{bmatrix} G_k \\ 0 \end{bmatrix}. \quad (100)$$

Second, using the same approach as given in [11], [12], we define the augmented state X_k as $X_k = \begin{bmatrix} x_k^T & \bar{d}_k \end{bmatrix}^T$, where $\bar{d}_k = H_k^+ H_k d_k$ represents the estimable unknown input vector based on the measurement y_k. Then, the AOE (99) can be rewritten as follows:

$$\begin{bmatrix} X^*_{k+1} \\ y^*_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{E}^a_{k+1} & -\breve{G}_k \\ \hat{C}^a_{k+1} & 0 \end{bmatrix} \begin{bmatrix} X_{k+1} \\ d_k \end{bmatrix} + \begin{bmatrix} \eta^{*,a}_{k+1} \\ \hat{v}^*_{k+1} \end{bmatrix}, \quad (101)$$

where

$$\begin{aligned} X^*_{k+1} &= x^*_{k+1} + \hat{G}_k \bar{\bar{d}}_k, && (102)\\ \hat{E}^a_{k+1} &= \begin{bmatrix} \hat{E}_{k+1} & 0 \end{bmatrix}, && (103)\\ \hat{C}^a_{k+1} &= \begin{bmatrix} \hat{C}_{k+1} & \hat{H}_{k+1} \end{bmatrix}, && (104)\\ \breve{G}_k &= \hat{G}_k(I - H_k^+ H_k), && (105)\\ \eta^{*,a}_{k+1} &= \eta^*_{k+1} - \hat{G}_k(d_k - \bar{\bar{d}}_k). && (106) \end{aligned}$$

Third, based on (101) the extended RDKF is obtained by using the following maps in (52)-(53).

$$\begin{gathered} X_{k+1} \to x_{k+1}, \quad X^*_{k+1} \to x^*_{k+1}, \quad \hat{P}^X_{k+1} \to \hat{P}_{k+1}, \\ \begin{bmatrix} \hat{E}^a_{k+1} & -\breve{G}_k \end{bmatrix} \to \hat{E}_{k+1}, \quad \begin{bmatrix} \hat{C}^a_{k+1} & 0 \end{bmatrix} \to \hat{C}_{k+1}, \end{gathered}$$

where $\hat{P}^X_{k+1}$ is the covariance of $\eta^{*,a}_{k+1}$ and is given by

$$\hat{P}^X_{k+1} = \hat{Q}_k + \begin{bmatrix} \hat{A}_k & \hat{G}_k \end{bmatrix} P^X_k \begin{bmatrix} \hat{A}_k & \hat{G}_k \end{bmatrix}^T. \quad (107)$$

Finally, the dedicated robust state estimator is then given as follows:

$$\hat{x}_{k+1} = \begin{bmatrix} I \\ 0 \end{bmatrix}^T \hat{X}_{k+1}, \; P^x_{k+1} = \begin{bmatrix} I \\ 0 \end{bmatrix}^T P^X_{k+1} \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (108)$$

VI. CONCLUSION

In this paper, a new robust filtering framework (RFF) is developed and fully explored to solve the robust state estimation for descriptor systems with bounded norm uncertainties. A robust descriptor Kalman filter (RDKF) has been derived on the basis of the RFF and the DKF in [10]. It is shown in a theorem that the proposed RDKF is equivalent to the DR filter developed in [2] subject to that a rank condition holds. The issue of simplifying the implementation of the RDKF is addressed. Specifically, to illustrate the usefulness of the proposed results the robust filter in [1] is rederived. Through this result and considering the special case that uncertainties only in the system dynamics, it is shown in this paper that the robust filtering methods in [1] and [2] obtain almost the same robust filter structures, which yield almost equivalent performances. This research shows that both the robust state estimation and unknown input filtering problems can be solved in a unified way.

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