State Estimation in Discrete-Time Nonlinear Stochastic Systems Subject to Random Data Loss

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Abstract— **This paper focuses on observer design problem in discrete-time nonlinear stochastic systems subject to random data loss. A sufficient condition is derived which guarantees exponential mean-square stability of the estimation error. An efficient algorithm is also proposed to obtain the observer gain. The effectiveness of the proposed observer design technique is evaluated by applying it to a multi-input multi-output aerodyne flight system.**

I. Introduction

A significant issue in state estimation for stochastic systems arises from the fact that real-time calculation of states' conditional Probability Density Function (PDF) is impossible in most cases [1]. Several methodologies have been proposed to address this issue. In some early attempts, various approximation techniques were suggested. In [5] and [18], the Gram-Charlier expansion was employed. The Edgeworth expansion was utilized in [19] to obtain an approximation of the system states' conditional PDF. Extended Kalman filters and statistically linearized filters are also of the approximation based observation techniques which were proposed in [7] and [11], respectively. Other techniques include bound-optimal filters [17], minimum variance filters [24], error covariance assignment theory [26], and exponentially bounded filters [20], [25] and [21].

In all of the aforementioned design methodologies, it is assumed that the measured data from the system is always available to the observer. In some applications however, this information may not be available at certain time instances. For example, in over network control and fault diagnosis systems [3], [6], which have received much attention in recent years, the measured data from the system may randomly be lost while being transmitted through the communication link.

In recent years, tremendous efforts have been made towards the design of observers that can perform satisfactorily despite random data loss. However, these efforts have been mostly directed towards deterministic systems, e.g. proposed techniques in [2], [4], [8], [9], [10], [12], [16], and [23]. Only [22] considers state estimation in linear stochastic systems subject to the packet loss.

Observer design problem for nonlinear stochastic systems subject to data loss remains to be an important area for further research. This paper is an extension of [25] to networked systems, focusing on observer design problem in discrete-time nonlinear stochastic systems subject to random data loss.

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A. Nomenclature

In the remainder of this paper $\mathbb{E}\{x\}$ stands for the expectation of the random variable x. $\mathbb{I}_0^+$ is the set of positive integers including zero, i.e., $\mathbb{I}_0^+ = \{0, 1, 2, 3, \cdots\}$. $\mathbb{R}$ and $\mathbb{R}^n$ denote the real number and n-dimensional Euclidean space, respectively. $\mathbb{R}^{n\times m}$ is a set of all $n \times m$ real matrices. The notation $\| x \|$ refers to the Euclidean vector norm of x which is $\| x \| = \left(x^T x\right)^{1/2}$. If A is a matrix, $\overline{\lambda}(A)$ and $\underline{\lambda}(A)$ respectively denote the largest and smallest eigenvalues of A. $A > 0 (< 0)$ means A is a positive (negative) definite matrix. $A \geq 0 (\leq 0)$ means A is a positive (negative) semi-definite matrix. The end of definitions, assumptions, lemma, theorems, proofs, and remarks are highlighted by the '□' sign.

II. System Model and Problem Formulation

Consider the following nonlinear discrete-time stochastic system:

$$x(k+1) = f(x(k), u(k)) + g_1(x)v(k) \tag{1}$$

$$y(k) = h(x(k)) + g_2(x)w(k) \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ respectively denote state, control and output vectors. $k \in \mathbb{I}_0^+$ is the time index and $v \in \mathbb{R}$ and $w \in \mathbb{R}$ are zero mean uncorrelated normalized Gaussian random variables with:

$$\mathbb{E}\{\|v(k)\|^2\} < \alpha_v, \text{ and } \mathbb{E}\{\|w(k)\|^2\} < \alpha_w \tag{3}$$

where, α_v and α_w are positive real numbers. It is also assumed that there exist real scalars $b_1 \geq 0$ and $b_2 \geq 0$ such that the nonlinear vector functions g_1 and g_2 satisfy

$$\|g_1(x)\| \leq b_1, \ \ \|g_2(x)\| \leq b_2, \text{for all } x \in \mathbb{R}^n. \tag{4}$$

The system output $y(k)$ is digitized and transmitted through the network to the observer. The received data to the observer module is given by:

$$z(k) = \beta(k)y(k) + g_3(x)s(k) \tag{5}$$

where, $z \in \mathbb{R}^l$ and $\beta(k)$ is a random variable. Its value is 1 when $y(k)$ is successfully received, otherwise it is 0. It is assumed that the random variable $\beta(k)$ is a Bernoulli distributed sequence with

$$\text{Prob}\{\beta(k) = 1\} = \mathbb{E}\{\beta(k)\} = \overline{\beta} \tag{6}$$

where $\overline{\beta}$ is a positive real number within the interval $(0, 1]$. In (5), $s \in \mathbb{R}$ represents the noise at the receiver which is a

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zero mean normalized Gaussian random number uncorrelated with v and w satisfying

$$\mathbb{E}\{\|s(k)\|^2\} < \alpha_s \tag{7}$$

where, α_s is a positive real number. Quantization error arising from converting analog to digital values and vice versa is considered in $s(k)$. Similar to g_1 and g_2, it is assumed that there exists a real scalar $b_3 \geq 0$ such that the nonlinear vector function g_3 satisfies the following inequality,

$$\|g_3(x)\| \leq b_3, \text{for all } x \in \mathbb{R}^n. \tag{8}$$

By defining $\hat{x}(k)$ as an estimate of $x(k)$ at time index k, the main objective of this work is to design an observer in the form of

$$\hat{x}(k+1) = f(\hat{x}(k), u(k)) + L\left[z(k) - \bar{\beta}h(\hat{x}(k), 0)\right] \tag{9}$$

by using $z(k)$, such that the error between actual and estimated state vectors, i.e.,

$$e(k) = x(k) - \hat{x}(k) \tag{10}$$

is as small as possible. The design parameter in (9) is $L \in \mathbb{R}^{n\times m}$ which is so-called as the observer gain in the literature.

III. The Preliminaries

Due to existence of random variables $v(k)$, $w(k)$, $s(k)$ and $\beta(k)$ in (1), (2) and (5), it is almost impossible to guarantee convergence of the estimation error to zero. Instead, we shall focus on approaches which ensure exponential bounded response of $e(k)$. Motivated by the minimum mean-square concept which is frequently used in estimation theory, sufficient conditions are obtained which guarantee exponential mean-square stability of the observer. In the following, a definition of the exponential mean-square stability and some lemmas are given and are utilized in the rest of the paper.

Definition 1: (Definition 2 in [20]) The stochastic system (1)-(2) is said to be asymptotically mean-square stable if there exist real scalars $0 < \varphi \leq 1$, $\mu_1 \geq 0$ and $\mu_2 > 0$, such that:

$$\mathbb{E}\left\{\|x(k)\|^2\right\} \leq \mu_1 + \mu_2(1-\varphi)^k, \text{for } k \in \mathbb{I}_0^+, \tag{11}$$

and then $x(k)$ is said to be exponentially bounded in mean-square with exponent φ. □

Typically Lyapunov's methodology is employed for the study of system stability. The following Lemma presents sufficient conditions for the mean-square asymptotic stability of a stochastic system in terms of the Lyapunov functional.

Lemma 1: (Theorem 2 in [20]) Let $V(x(k))$ be a Lyapunov function. If there exist real scalars $\theta_1 > 0$, $\theta_2 > 0$, $\eta \geq 0$ and $0 < \varphi \leq 1$ such that

$$\theta_1 \|x(k)\|^2 \leq V(x(k)) \leq \theta_2 \|x(k)\|^2 \tag{12}$$

and

$$\mathbb{E}\{V(x(k+1)|x_k, \ldots, x_0)\} - V(x(k)) \leq \eta - \varphi V(x(k)), \tag{13}$$

then the sequence $x(k)$ satisfies

$$\mathbb{E}\left\{\|x(k)\|^2\right\} \leq \frac{\theta_2}{\theta_1}\|x(0)\|^2(1-\varphi)^k + \frac{\eta}{\theta_1\varphi} \tag{14}$$

Proof: See Theorem 2 in [20]. □

Lemma 1 implies that if there exists a Lyapunov function satisfying (12) and (13), then the stochastic system is mean-square asymptotically stable simply by defining $\mu_2 = \theta_2\|x(0)\|^2/\theta_1$ and $\mu_1 = \eta/(\theta_1\varphi)$ and using Definition 1.

Lemma 2: (Schur's Complement Lemma, [27]) Suppose A, B and C are matrices with appropriate dimensions, and C is invertible. Then,

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \geq 0 \text{ if and only if } C \geq 0 \text{ and } A - BC^{-1}B^T \geq 0$$

Proof: See [27]. □

Lemma 3: Suppose A, B, C and D are matrices with appropriate dimensions and $D \geq 0$. Then

$$\begin{aligned}&\begin{bmatrix} x^T & y^T \end{bmatrix}\begin{bmatrix} A-B & C \\ C^T & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \leq \\ &\begin{bmatrix} x^T & y^T \end{bmatrix}\begin{bmatrix} (1+\alpha)A-B & 0 \\ 0 & (1+\alpha^{-1})D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\end{aligned} \tag{15}$$

if $A - CD^{-1}C^T \geq 0$

Proof: The inequality (15) holds if

$$\begin{aligned}\begin{bmatrix} (1+\alpha)A-B & 0 \\ 0 & (1+\alpha^{-1})D \end{bmatrix} - \begin{bmatrix} A-B & C \\ C^T & D \end{bmatrix} = \\ \begin{bmatrix} \alpha A & -C \\ -C^T & \alpha^{-1}D \end{bmatrix} \geq 0.\end{aligned} \tag{16}$$

By using the Schur Lemma it is simple to show that (16) with $D \geq 0$ holds if $A - CD^{-1}C^T \geq 0$. □

IV. Observer Design

A sufficient condition is derived guaranteeing the exponential mean-square stability of $e(k)$. Expansion of (10) by using (1), (2), (5) and (9) results in:

$$\begin{aligned} e(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= f(x(k), u(k)) - f(\hat{x}(k), u(k)) + g_1(x(k))v(k) \\ &\quad - Lg_3(x(k))s(k) + \bar{\beta}Lh(\hat{x}(k)) \\ &\quad - \beta(k)L\left[h(x(k)) + g_2(x(k))w(k)\right] \end{aligned} \tag{17}$$

After some manipulations, $e(k+1)$ can be written as follows:

$$\begin{aligned} e(k+1) &= (A - \bar{\beta}LC)e(k) + M(k) - \bar{\beta}LN(k) - \\ &\left(\beta(k) - \bar{\beta}\right)Lh(x(k)) + g_1(x(k))v(k) - Lg_3(x(k))s(k) \\ &- \left(\beta(k) - \bar{\beta}\right)Lg_2(x(k))w(k) - \bar{\beta}Lg_2(x(k))w(k) \end{aligned} \tag{18}$$

where, $M(k)$ and $N(k)$ are given by:

$$M(k) = f(x(k), u(k)) - f(\hat{x}(k), u(k)) - Ae(k) \tag{19}$$

$$N(k) = h(x(k)) - h(\hat{x}(k)) - Ce(k) \tag{20}$$

In this paper, it is assumed that there exist $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and real scalars $a_{ij} \geq 0$, for $i, j = 1, 2$ such that the following incrementally inequalities are satisfied for any ϵ and δ deviations from the operating point (x, u).

$$\left\| f(x+\epsilon, u+\delta) - f(x,u) - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} \right\| \leq a_{11} \left\| \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} \right\| + a_{12} \quad (21)$$

$$\|h(x+\epsilon) - h(x) - C\epsilon\| \leq a_{21} \|\epsilon\| + a_{22} \quad (22)$$

This is applicable to so many applications.

A. Asymptotic Mean-Square Stability of the Observer

Theorem 1: Consider the nonlinear networked system given by (1)-(8). Assume that there exists an observable pair (A, C) satisfying (21) and (22). For a given observer gain L such that

$$(1+\alpha)^{1/2}\rho(A - \bar{\beta}LC) < 1 \quad (23)$$

for a real scalar $\alpha > 0$, then the estimation error (18) is exponentially mean-square stable if

$$\bar{\lambda}(P)a_{11}^2 + \gamma^{-1}\bar{\beta}^2\bar{\lambda}(L^TPL)a_{21}^2 < 0.5\frac{\alpha}{(1+\alpha)(1+\gamma)} \quad (24)$$

for a real scalar $\gamma > 0$, which P is a positive-definite matrix and a solution of the following algebraic Riccati equation

$$(1+\alpha)(A - \bar{\beta}LC)^T P(A - \bar{\beta}LC) - P = -I \quad (25)$$

where I is an $n \times n$ identity matrix. □

Proof: A similar approach as in [25] is employed. Consider the following Lyapunov function

$$V(e(k)) = e(k)^T Pe(k). \quad (26)$$

with $P > 0$. It follows from (18) and (26) that:

$$\begin{aligned}\Delta V(k) = {} & \mathbb{E}\{V(e(k+1)|e(k), \cdots, e(0))\} - V(e(k)) = \\ & \mathbb{E}\{[(A - \bar{\beta}LC)e(k) + M(k) - \bar{\beta}LN(k) - \\ & (\beta(k) - \bar{\beta})Lh(x(k)) + g_1v(k) - Lg_3s(k) - \\ & (\beta(k) - \bar{\beta})Lg_2w(k) - \bar{\beta}Lg_2w(k)]^T P[(A - \bar{\beta}LC)e(k) + \\ & M(k) - \bar{\beta}LN(k) - (\beta(k) - \bar{\beta})Lh(x(k)) + g_1v(k) - Lg_3s(k) \\ & - (\beta(k) - \bar{\beta})Lg_2w(k) - \bar{\beta}Lg_2w(k)]\} - e(k)^T Pe(k) \quad (27)\end{aligned}$$

For the sake of simplicity, the argument of g_1, g_2, and g_3 are omitted in the following. Since $\beta(k)$ is a Bernoulli probability distribution function with expected value $\bar{\beta}$, then $\mathbb{E}\{\beta(k) - \bar{\beta}\} = 0$ and $\mathbb{E}\{(\beta(k) - \bar{\beta})^2\} = \bar{\beta}(1 - \bar{\beta})$. On the other hand, $\mathbb{E}\{v(k)\} = \mathbb{E}\{w(k)\} = \mathbb{E}\{s(k)\} = 0$. Thus, expansion of (27) results in:

$$\begin{aligned}\Delta V(k) = {} & e(k)^T\left((A - \bar{\beta}LC)^T P(A - \bar{\beta}LC) - P\right)e(k) + \\ & e(k)^T(A - \bar{\beta}LC)^T P(M(k) - \bar{\beta}LN(k)) + \\ & (M(k) - \bar{\beta}LN(k))^T P(A - \bar{\beta}LC)e(k) + \\ & (M(k) - \bar{\beta}LN(k))^T P(M(k) - \bar{\beta}LN(k)) + \\ & \bar{\beta}(1 - \bar{\beta})h^T(x(k))L^TPLh(x(k)) + \mathbb{E}\{v(k)^T g_1^T Pg_1v(k)\} + \\ & \mathbb{E}\{s(k)^T g_3^T L^TPLg_3s(k)\} + \bar{\beta}\mathbb{E}\{w(k)^T g_2^T L^TPLg_2w(k)\} \quad (28)\end{aligned}$$

Applying Lemma 3 and (25) yields:

$$\begin{aligned}\Delta V(k) \leq {} & - e(k)^T e(k) + (1+\alpha^{-1})(1+\gamma)M(k)^T PM(k) + \\ & (1+\alpha^{-1})(1+\gamma^{-1})\bar{\beta}^2 N(k)^T L^T PLN(k) + \\ & \bar{\beta}(1 - \bar{\beta})h^T(x(k))L^TPLh(x(k)) + \\ & \mathbb{E}\{v(k)^T g_1^T Pg_1v(k)\} + \mathbb{E}\{s(k)^T g_3^T L^TPLg_3s(k)\} + \\ & \bar{\beta}\mathbb{E}\{w(k)^T g_2^T L^TPLg_2w(k)\} \quad (29)\end{aligned}$$

where, α and γ are real positive numbers, i.e., $\alpha > 0$ and $\gamma > 0$. It is concluded from (3)-(4), (7)-(8), (21) and (22) that:

$$\begin{aligned}M(k)^T PM(k) \leq {} & \bar{\lambda}(P)\left(a_{11}\|e(k)\| + a_{12}\right)^2 \leq \\ & \bar{\lambda}(P)\left(2a_{11}^2\|e(k)\|^2 + 2a_{12}^2\right) \quad (30)\end{aligned}$$

$$\begin{aligned}N(k)^T L^T PLN(k) \leq {} & \bar{\lambda}(L^TPL)\left(a_{21}\|e(k)\| + a_{22}\right)^2 \leq \\ & \bar{\lambda}(L^TPL)\left(2a_{21}^2\|e(k)\|^2 + 2a_{22}^2\right) \quad (31)\end{aligned}$$

$$\mathbb{E}\left\{v(k)^T g_1^T Pg_1v(k)\right\} \leq \bar{\lambda}(P)b_1\alpha_v \quad (32)$$

$$\mathbb{E}\left\{w(k)^T g_2^T L^TPLg_2w(k)\right\} \leq \bar{\lambda}(L^TPL)b_2\alpha_w \quad (33)$$

$$\mathbb{E}\left\{s(k)^T g_3^T L^TPLg_3s(k)\right\} \leq \bar{\lambda}(L^TPL)b_3\alpha_s \quad (34)$$

The fact that $h^T(x(k))L^TPLh(x(k)) \leq \bar{\lambda}(L^TPL)d^2$ which d is a real scalar yields:

$$\begin{aligned}\Delta V(k) \leq {} & - e(k)^T e(k) + \\ & (1+\alpha^{-1})(1+\gamma)\bar{\lambda}(P)\left(2a_{11}^2\|e(k)\|^2 + 2a_{12}^2\right) + \\ & \bar{\beta}^2(1+\alpha^{-1})(1+\gamma^{-1})\bar{\lambda}(L^TPL)\left(2a_{21}^2\|e(k)\|^2 + 2a_{22}^2\right) + \\ & \bar{\beta}(1-\bar{\beta})\bar{\lambda}(L^TPL)d^2 + \bar{\lambda}(P)b_1\alpha_v + \bar{\beta}\bar{\lambda}(L^TPL)b_2\alpha_w + \\ & \bar{\lambda}(L^TPL)b_3\alpha_s = -(1-\varphi_1)\|e(k)\|^2 + \eta \quad (35)\end{aligned}$$

where,

$$\begin{aligned}\varphi_1 = {} & 2(1+\alpha^{-1})(1+\gamma)\bar{\lambda}(P)a_{11}^2 + \\ & 2\bar{\beta}^2(1+\alpha^{-1})(1+\gamma^{-1})\bar{\lambda}(L^TPL)a_{21}^2 \quad (36)\end{aligned}$$

and,

$$\begin{aligned}\eta = {} & 2(1+\alpha^{-1})(1+\gamma)\bar{\lambda}(P)a_{12}^2 + \\ & 2\bar{\beta}^2(1+\alpha^{-1})(1+\gamma^{-1})\bar{\lambda}(L^TPL)a_{22}^2 + \bar{\beta}(1-\bar{\beta})\bar{\lambda}(L^TPL)d^2 + \\ & \bar{\lambda}(P)b_1\alpha_v + \bar{\beta}\bar{\lambda}(L^TPL)b_2\alpha_w + \bar{\lambda}(L^TPL)b_3\alpha_s \quad (37)\end{aligned}$$

If $\varphi_1 < 1$, then by choosing $\varphi = \min\{\varphi_1, \bar{\lambda}(P)\}$,

$$\Delta V(k) \leq -\varphi V(k) + \eta \quad (38)$$

where $0 < \varphi \leq 1$. Since $\eta \geq 0$ and, $\underline{\lambda}(P)\|e(k)\|^2 \leq V(k) \leq \bar{\lambda}(P)\|e(k)\|^2$ then, the sufficient conditions of Lemma 1 are satisfied and $e(k)$ is therefore exponentially mean-square stable. The inequality $\varphi_1 < 1$ implies the sufficient condition (24) and the proof is complete. □

B. Design Procedure

Assuming that *a priori* discrete-time model of the system is available, the following algorithm is proposed for the design of the observer's gain.

Design Algorithm 1:

Step 1: Find A, B, C, a_{11} and a_{21} such that (21) and (22) are satisfied.

Step 2: Find an observer gain such that $\rho(A-\bar{\beta}LC)<1$, i.e., all poles are located inside the unit circle in z-plane. This is required to guarantee stability of the linear part of the estimation error dynamic.

Step 3: Find the maximum value of α, denoted by $\bar{\alpha}$, as follows:

$$\bar{\alpha}=\left(\frac{1}{\rho(A-\bar{\beta}LC)}\right)^2-1 \tag{39}$$

Step 4: If $\bar{\alpha}\leq 0$ then, there is no feasible solution for the sufficient condition (24). Go to Step 2 and find another L until a positive $\bar{\alpha}$ is obtained. If $\bar{\alpha}>0$ then, increase α from 0 to $\bar{\alpha}$ with a step size $\Delta\alpha$ in a "for" loop.

Step 5: For every α, look for a $\gamma>0$ such that (24) is satisfied. It is simply executable by defining another "for" loop for γ and increasing it by a step size $\Delta\gamma$ from 0 to a sufficiently large positive number. If a γ could be found then, exit from the algorithm and the estimation error is exponentially mean-square stable based on Theorem 1. If no γ could be found for all $0<\alpha<\bar{\alpha}$ such that (24) is satisfied then, the given L does not guarantee the exponential mean-square stability of the estimation error. In such a case, go to Step 2 and repeat the algorithm for another L. If no L could be found then, the problem does not have a solution.

End

V. Illustrative Example

The effectiveness of the proposed observer design is assessed on an aerodyne with the following dynamics [15]:

$$\begin{aligned}\dot{\upsilon}(t)&=u_1(t)-\sin\phi(t)+v(t)\\ \dot{\phi}(t)&=\frac{u_2(t)\upsilon^2(t)-\cos\phi(t)}{\upsilon(t)}+10v(t)\end{aligned} \tag{40}$$

where $\upsilon\in\mathbb{R}\geq 0$ is the speed and $\phi\in\mathbb{R}$ is the flight-path angle. The control signals are the propulsive balance u_1 and the lift u_2. v represents the state noise which is assumed to be a normal distribution with mean 0 and standard deviation 0.02. It was shown in [14] that the following controller makes the equilibrium point $(\upsilon,\phi)^*=(\upsilon_0,0)$ exponentially stable.

$$u_1(t)=\cos\phi(t)-\left(\frac{\upsilon(t)}{\upsilon_0}\right)^2,\ u_2(t)=\frac{1}{\upsilon_0^2}-\frac{\upsilon^2(t)}{\upsilon_0^4}\sin\phi(t) \tag{41}$$

By defining $[x_1\ x_2]=[\upsilon\ \phi]$, and applying (41), a discrete-time state-space approximation of the system is given by:

$$\begin{aligned}x_1(k+1)&=x_1(k)+\\ &T_s\left[-\left(\frac{x_1(k)}{\upsilon_0}\right)^2+\cos x_2(k)-\sin x_2(k)+v(k)\right]\\ x_2(k+1)&=x_2(k)+\\ &T_s\left[\frac{x_1(k)}{\upsilon_0^2}-\frac{x_1^3(k)}{\upsilon_0^4}\sin x_2(k)-\frac{1}{x_1(k)}\cos x_2(k)+10v(k)\right]\end{aligned} \tag{42}$$

In (42), T_s has been omitted from the argument of x_1 and x_2. The system outputs are given by

$$y_i(k)=0.1\times i\times x_i(k)+w(k),\ \ i=1,2 \tag{43}$$

where, w is normal distribution with mean 0 and standard deviation 0.03. The available system information at the observer module is given by:

$$z_i(k)=\beta(k)y_i(k)+s(k),\ \ i=1,2 \tag{44}$$

where, s is a normal distribution with mean 0 and standard deviation 0.05. $\beta(k)$ is a Bernoulli PDF with expectation value of 0.8. The system sample time is set to $T_s=0.02s$. υ_0 is set to 0.8. The objective is to estimate x_1 and x_2 from z_1 and z_2.

At the first step, A, B, C and a_{ij}'s, $i,j=1,2$ are calculated. A is given by

$$A=\begin{bmatrix}1-\frac{2T_s}{\upsilon_0} & T_s\\ \frac{2T_s}{\upsilon_0^2} & 1-T_s\upsilon_0^2\end{bmatrix}. \tag{45}$$

Since u_1 and u_2 do not appear in (42), then $B=0$.

a_{11} was chosen as:

$$a_{11}=\left\|\begin{bmatrix}0 & -T_s\\ -\frac{T_s}{\upsilon_0^2} & T_s\upsilon_0^2\end{bmatrix}\right\|. \tag{46}$$

The selection of C as the Jacobian matrix of $h(x_k)$ at the equilibrium point $(x,u)^*=(\upsilon_0,0)$, i.e., $C=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, leads to the conclusion that the inequality (22) is always satisfied for every $a_{21}\geq 0$ and $a_{22}\geq 0$. In the following a_{21} is arbitrary chosen to be a small value 0.01.

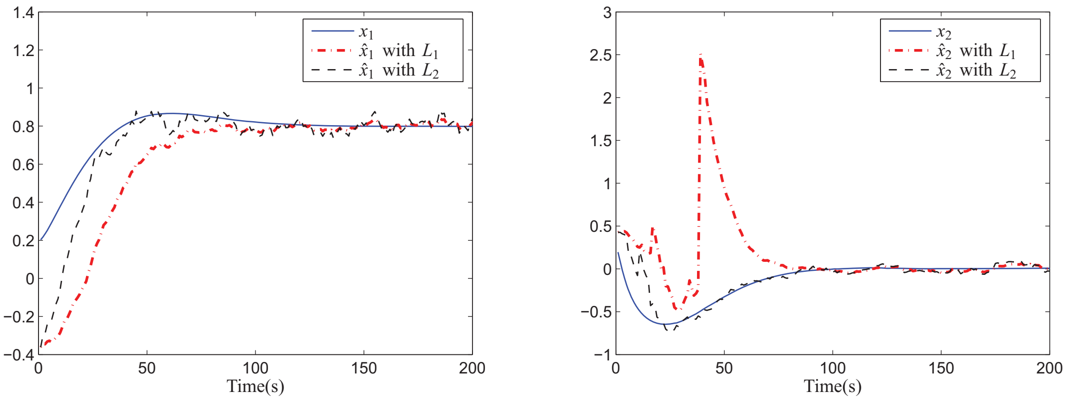
First, it is assumed that there is no packet drop in the system i.e., $\bar{\beta}=1$. Following Algorithm 1, a possible L satisfying all conditions of Theorem 1 is given by:

$$\text{Observer Gain (No Packet Drop): } L_1=\begin{bmatrix}0.45 & 0\\ 0 & 0.5\end{bmatrix} \tag{47}$$

Next, the expectation of packet drop is chosen to be 0.8, i.e., $\bar{\beta}=0.8$. There is no new point of principle for different value of $\bar{\beta}$. By using Algorithm 1, steps 2 to 4, it is simple to show that the sufficient condition (24) has no feasible solution with (47). It is then required to re-tune L. Through a few trail and error, a possible observer gain satisfying all sufficient conditions with $\bar{\beta}=0.8$ is obtained as follows:

$$\text{Observer Gain (Under Packet Drop): } L_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{48}$$

A Monte Carlo based simulation is run for performance evaluation of the obtained observer gains. The system is run 20 times with different initial conditions. In each experiment, i) a distributed random number with the expectation value of 0.8 is generated which simulates the packet drop, and ii) the system responses are recorded for both L_1 and L_2. In the



(a) Average of the original and estimated signals x_1 and $\hat{x}_1$.

(b) Average of the original and estimated signals x_2 and $\hat{x}_2$.

Fig. 1. Observer response under pack drop condition for $\bar{\beta} = 0.8$.

following, the average system response and absolute value of the estimation errors is calculated and plotted.

Figure 1 shows the system responses. It confirms that the observer gain L_2, which has been designed by taking packet drop into account, results in significant improvement in the performance of the state estimator.

VI. Conclusions and Future Works

State estimation of nonlinear stochastic discrete-time systems over a network that can experience packet drops was considered. A sufficient condition was derived which guarantees mean-square exponential stability of the estimation error despite the packet drops. An efficient algorithm was also proposed for the design of the observer gain. An example demonstrates significant improvement in the observer performance.

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