Controllability of Higher Order Switched Boolean Control Networks

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***Abstract*—This paper investigates controllability of higher order switched Boolean control networks (SBCNs). First, by using semi-tensor product (STP) method, two algebraic forms of higher order SBCNs are derived. And a necessary and sufficient condition of controllability for higher order SBCNs is obtained. Then based on the second algebraic form, the corresponding control and switching-law which drive a point to a given reachable point are designed. Finally, an illustrative example is given to show the validity of the main result.**

I. Introduction

The Boolean network which was firstly introduced by Kauffman in literature [1] becomes a hot topic in system biology, physics and system science. Control problem is an important topic for any control systems. Boolean control networks (BCNs) were proposed in paper [2] to describe gene-regulatory networks. References [3], [4], [5] have studied controllability and obserability of BCNs. In fact, a set of Boolean networks can be switched according to a defined probability, which are called probabilistic Boolean networks. Papers [6], [7] have already given a strict proof for the controllability of probabilistic Boolean control networks.

In practice, time-delay is a very common case in the real world. Higher order networks are used to describe these kinds of phenomena. References [8], [9] have investigated higher order Boolean networks about their structure and controllability, respectively. In the multi-model cases such as switched Boolean networks and probabilistic Boolean networks, there have rather few results because of complexity. In this note, we shall present some results on controllability of higher order SBCNs through model-input-state (MIS) matrices approach. The MIS matrix, which is a useful tool to study various control problems for SBCNs, is introduced in reference [10].

This paper is organized as follows. Some necessary preliminaries including notations and the MIS matrices are presented in Section 2. Section 3 gives two kinds of algebraic forms of higher order SBCNs. Controllability and trajectory tracking problems of higher order SBCNs are investigated in Section 4. An illustrative example is provided in Section 5. Finally, Section 6 concludes the paper briefly.

II. Preliminaries

A. Notations

Some necessary notations, which will be used in the sequel:

- $\mathbb{Z}_+$ is the set of nonnegtive integers.
- $\mathcal{M}_{m\times n}$ is the set of $m\times n$ real matrices.
- $\mathcal{D}=\{0,1\}$.
- δ_n^i is the i-th column of the identity matrix I_n.
- $\Delta_n:=\{\delta_n^1,\delta_n^2,\cdots,\delta_n^n\}$.
- A matrix $M\in\mathcal{M}_{n\times s}$ is called a logical matrix if all the columns of M, denoted by $Col(M)$ are of the form of δ_n^k. That is

$$Col(M)\subset\Delta_n.$$

Denoted by $\mathcal{L}_{n\times s}$ the set of $n\times s$ logical matrices. If $M\in\mathcal{L}_{n\times s}$, then it can be expressed as $M=[\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_s}]$. For the sake of compactness, it is briefly denoted by

$$M=\delta_n[i_1\ i_2\ \cdots\ i_s].$$

- $A=[a_{ij}]\in\mathcal{M}_{m\times n}, B\in\mathcal{M}_{p\times q}$. The Kronecker product of matrices A and B is

$$A\otimes B:=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B\\ a_{21}B & a_{22}B & \cdots & a_{2n}B\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B\end{bmatrix}. \quad (1)$$

- $A\in\mathcal{M}_{m\times n}, B\in\mathcal{M}_{p\times q}$. Then the STP of A and B is

$$A\ltimes B:=(A\otimes I_{s/n})(B\otimes I_{s/p}), \quad (2)$$

where $s=lcm(n,p)$ is the least common multiple of n and p. The STP of matrices is a generalization of the conventional matrix product. Thus, we can simply call it "product" and omit the symbol "$\ltimes$" if there is no confusion.

- A swap matrix $W_{[m,n]}$ is an $mn\times mn$ matrix constructed in the following way: label its columns by $(11,12,\cdots,1n,\cdots,m1,m2,\cdots,mn)$ and its rows by $(11,21,\cdots,m1,\cdots,1n,2n,\cdots,mn)$. Then its element in the position $((I,J),(i,j))$ is assigned as

$$w_{(I,J),(i,j)}=\delta_{i,j}^{I,J}=\begin{cases}1, & I=i \text{ and } J=j;\\ 0, & \text{otherwise.}\end{cases}$$

When $m=n$, we briefly denote $W_{[n]}=W_{[n,n]}$.

- Define Boolean addition $+_\mathcal{B}$ and Boolean product $\times_\mathcal{B}$ as

$$a+_\mathcal{B} b:=a\vee b,\ a\times_\mathcal{B} b:=a\wedge b,\quad a,b\in\mathbb{R}. \quad (3)$$

Let $A,B\in\mathcal{M}_{p\times q}$. Denote

$$A+_\mathcal{B} B=[a_{ij}+_\mathcal{B} b_{ij}]_{p\times q}.$$

Let $A_i \in \mathcal{M}_{p\times q}, i = 1, 2, \cdots, n$. Denote

$$\sum_{i=1}^{n}{}_{\mathcal{B}}\, A_i = A_1 +_{\mathcal{B}} A_2 +_{\mathcal{B}} \cdots +_{\mathcal{B}} A_n.$$

If $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, then

$$A \times_{\mathcal{B}} B = [c_{ij}] \in \mathcal{M}_{m\times p},$$

where

$$c_{ij} = a_{i1} \times_{\mathcal{B}} b_{1j} +_{\mathcal{B}} \cdots +_{\mathcal{B}} a_{in} \times_{\mathcal{B}} b_{nj} := \sum_{k=1}^{n}{}_{\mathcal{B}}\, a_{ik} \times_{\mathcal{B}} b_{kj}.$$

If $A \in \mathcal{M}_{n\times n}$, we denote

$$A^{(k)} = \underbrace{A \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} A}_{k} \in \mathcal{M}_{n\times n}.$$

• $A \in \mathcal{M}_{m\times sn}$. Equally split the columns of A into s blocks, then $Blk_i(A)$ is the i-th $m \times n$ block of $A, i = 1, 2, \cdots, s$, i.e.

$$A = [Blk_1(A)\ Blk_2(A) \cdots Blk_s(A)].$$

B. MIS matrices of SBCNs

In this part, we introduce MIS matrices and refer to paper [10] for more details. This matrix plays an important role in describing dynamic process of SBCNs. Firstly, we start from a none switched BCN. A BCN with n network nodes and m inputs can be described as

$$\begin{cases} x_1(t+1) &= f_1(u_1(t), u_2(t), \cdots, u_m(t), x_1(t), x_2(t), \\ & \quad \cdots, x_n(t)), \\ x_2(t+1) &= f_2(u_1(t), u_2(t), \cdots, u_m(t), x_1(t), x_2(t), \\ & \quad \cdots, x_n(t)), \\ & \cdots \\ x_n(t+1) &= f_n(u_1(t), u_2(t), \cdots, u_m(t), x_1(t), x_2(t), \\ & \quad \cdots, x_n(t)), \end{cases} \tag{4}$$

where $x_i, u_j \in \mathcal{D}, i = 1, 2, \cdots, n, j = 1, \cdots, m$, $f_i : \mathcal{D}^{m+n} := \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{m+n} \to \mathcal{D}, i = 1, 2, \cdots, n$ are logical functions.

Let

$$X(t) = (x_1(t)\ \cdots\ x_n(t))^T,\ U(t) = (u_1(t)\ \cdots\ u_m(t))^T,$$

then (4) can be briefly expressed as

$$X(t+1) = F(U(t), X(t)), \tag{5}$$

where

$$F(U(t), X(t)) = \begin{bmatrix} f_1(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \\ f_2(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \\ \vdots \\ f_n(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \end{bmatrix}. \tag{6}$$

Then we can consider an SBCN like this

$$X(t+1) = F_{\sigma(t)}(U(t), X(t)), \tag{7}$$

where

$$\sigma(t) : \mathbb{Z}_+ \to \Lambda = \{1, 2, \cdots, N\} \tag{8}$$

is the switching law. The switching functions $F_i, i = 1, 2, \cdots, N$ are denoted by

$$F_i(U(t), X(t)) = \begin{bmatrix} f_1^i(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \\ f_2^i(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \\ \vdots \\ f_n^i(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \end{bmatrix}. \tag{9}$$

Assume the switching law (8) is designable. Let Σ_i denote the model corresponding to F_i.

In order to use matrix expression, we identify $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, equivalently, $D \sim \Delta_2$. Thus we can equivalently consider the mapping $F : \mathcal{D}^{m+n} \to \mathcal{D}^n$ as a mapping $F : \Delta_{2^{m+n}} \to \Delta_{2^n}$. From reference [11], the BCN (4) can be expressed into its algebraic form as

$$x(t+1) = Lu(t)x(t), \tag{10}$$

where $x = \ltimes_{i=1}^n x_i \in \Delta_{2^n}$, $u = \ltimes_{j=1}^m u_j \in \Delta_{2^m}$ and $L \in \mathcal{L}_{2^n\times 2^{m+n}}$ is the network transition matrix of system (4). Let L_i be the network transition matrix corresponding to the model Σ_i. Then the algebraic form of SBCN (7) can be expressed as

$$x(t+1) = L_{\sigma(t)}u(t)x(t), \tag{11}$$

where $\{L_{\sigma(t)}, t = 1, 2, \cdots\}$ is a sequence of logical matrices. $L_{\sigma(t)}$ is the network transition matrix corresponding to the model $\Sigma_{\sigma(t)}$ at time t.

Denote $\mathcal{U}$ as the corresponding input set and χ as the state set of (11). First, we define the set of points in the product space as $\{Q_i \mid i = 1, \cdots, N_Q\} = \{\Sigma_\lambda \mid \lambda = 1, 2 \cdots, N\} \times \mathcal{U} \times \chi$, where $N_Q = N \cdot 2^{m+n}$. Now we construct MIS matrix $\mathcal{J} \in \mathcal{B}_{N_Q\times N_Q}$ of (11) in the following way:

$$\mathcal{J}_{ij} = \begin{cases} 1, & \text{if there exsits an edge from } Q_j \text{ to } Q_i, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

From the constructing procedure of $\mathcal{J}$, it is easy to obtain the following result.

Lemma 2.1: [10] $\mathcal{J}$ is the MIS matrix of SBCN (11), then

$$\mathcal{J} = \left.\begin{bmatrix} \mathcal{J}_0 \\ \vdots \\ \mathcal{J}_0 \end{bmatrix}\right\} 2^m N, \tag{13}$$

where $\mathcal{J}_0 = [\ L_1\ L_2\ \cdots\ L_N\ ] \in \mathcal{L}_{2^n\times N_Q}$. $\mathcal{J}_0$ is called the basic block of $\mathcal{J}$.

Example 2.2: Let $X(t) = (x_1(t)\ x_2(t))^T, U(t) = u(t)$. Assume the dynamic of an SBCN is

$$X(t+1) = F_{\sigma(t)}(U(t), X(t)), \tag{14}$$

where $\sigma(t) : \mathbb{Z}_+ \to \{1, 2, 3, 4\}$. The functions F_i, $i = 1, 2, 3, 4$ are

$$F_1 = \begin{bmatrix} u \wedge x_1 \\ u \to x_1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} u \wedge x_1 \\ u \to x_2 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} u \wedge x_2 \\ u \to x_1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} u \wedge x_2 \\ u \to x_2 \end{bmatrix}.$$

The corresponding structure matrices are

$$\begin{array}{l} L_1 = \delta_4[1\ 1\ 4\ 4\ 3\ 3\ 3\ 3],\ L_2 = \delta_4[1\ 2\ 3\ 4\ 3\ 3\ 3\ 3], \\ L_3 = \delta_4[1\ 3\ 2\ 4\ 3\ 3\ 3\ 3],\ L_4 = \delta_4[1\ 4\ 1\ 4\ 3\ 3\ 3\ 3]. \end{array} \tag{15}$$

The MIS matrix of system (14) is

$$\begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 \end{bmatrix}.$$

III. Two algebraic forms of higher order SBCNs

In this section, dynamic of higher order SBCNs is presented first. Then, two of its algebraic forms are given via STP approach.

A τ-th order Boolean control network can be described as

$$\begin{cases} x_1(t+1) &= f_1(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t), \\ & \quad \cdots, x_1(t-\tau+1), \cdots, x_n(t-\tau+1)), \\ x_2(t+1) &= f_2(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t), \\ & \quad \cdots, x_1(t-\tau+1), \cdots, x_n(t-\tau+1)), \\ \cdots & \\ x_n(t+1) &= f_n(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t), \\ & \quad \cdots, x_1(t-\tau+1), \cdots, x_n(t-\tau+1)), \\ & \quad t \geqslant \tau - 1, \end{cases} \tag{16}$$

where $x_i, u_j \in \mathcal{D}$, $i = 1, \cdots, n, j = 1, \cdots, m$ are states and inputs, respectively. $f_l, l = 1, \cdots, n$ are logical functions. Let

$$\begin{array}{rcl} X(t) &=& (x_1(t)\ \cdots\ x_n(t))^T, \\ U(t) &=& (u_1(t)\ \cdots\ u_m(t))^T, \end{array} \tag{17}$$

then (16) can be briefly expressed as

$$X(t+1) = F(U(t), X(t), X(t-1), \cdots, X(t-\tau+1)), \tag{18}$$

where

$$\begin{array}{l} F(U(t), X(t), X(t-1), \cdots, X(t-\tau+1)) \\ = \begin{bmatrix} f_1(u_j(t), x_k(t), x_k(t-\tau+1)) \\ f_2(u_j(t), x_k(t), x_k(t-\tau+1)) \\ \cdots \\ f_n(u_j(t), x_k(t), x_k(t-\tau+1)) \end{bmatrix}. \end{array} \tag{19}$$

$j = 1, \cdots, m;\ k = 1, \cdots, n$. Then, we can consider a τ-th order SBCN like this

$$X(t+1) = F_{\sigma(t)}(U(t), X(t), X(t-1), \cdots, X(t-\tau+1)), \tag{20}$$

where

$$\sigma(t) : \mathbb{Z}_+ \to \Lambda = \{1, 2, \cdots, N\} \tag{21}$$

is the switching-law. Functions $F_i, i = 1, 2, \cdots, N$ of switching models are determined by

$$F_i(U(t), X(t)) = \begin{bmatrix} f_1^i(u_j(t), x_k(t), x_k(t-\tau+1)) \\ f_2^i(u_j(t), x_k(t), x_k(t-\tau+1)) \\ \cdots \\ f_n^i(u_j(t), x_k(t), x_k(t-\tau+1)) \end{bmatrix}. \tag{22}$$

Assume the switching-law (21) is designable. Let Σ_i denotes the model corresponding to F_i. Using the same way as the algebraic form in **Section 2.2** from reference [11], the τ-th order BCN (18) can be expressed into its algebraic form as

$$x(t+1) = Lu(t)x(t-\tau+1)x(t-\tau+2)\cdots x(t), \tag{23}$$

where $t \geqslant \tau - 1, x = \ltimes_{i=1}^n x_i \in \Delta_{2^n}$, $u = \ltimes_{j=1}^m u_j \in \Delta_{2^m}$ and $L \in \mathcal{L}_{2^n \times 2^{m+\tau n}}$ is the transition matrix of system (18). Let L_λ be the transition matrix corresponding to the model $\Sigma_\lambda, \lambda = 1, \cdots, N$. Then the first algebraic form of τ-th order SBCN (20) can be expressed as

$$x(t+1) = L_{\sigma(t)}u(t)x(t-\tau+1)x(t-\tau+2)\cdots x(t), \tag{24}$$

where $\{L_{\sigma(t)}, t \geqslant \tau - 1\}$ is a sequence of logical matrices. $L_{\sigma(t)}$ is the transition matrix corresponding to the model $\Sigma_{\sigma(t)}$ at time t.

Next, we will present the second algebraic form of τ-th order SBCN (20), which is similar to that of (11). Before this, some lemmas are listed which will be needed in the sequel.

Lemma 3.1: [12] Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then,

$$W_{[m,n]}XY = YX, \quad W_{[n,m]}YX = XY. \tag{25}$$

Lemma 3.2: [13] Let

$$E_d = \delta_2[1\ 2\ 1\ 2]. \tag{26}$$

Then, for any two logical variables $X, Y \in \Delta_2$,

$$E_d XY = Y, \quad E_d W_{[2]} XY = X. \tag{27}$$

Lemma 3.3: [12] Let

$$\Phi_k = \begin{bmatrix} \delta_k^1 & \mathbf{0}_k & \cdots & \mathbf{0}_k \\ \mathbf{0}_k & \delta_k^2 & \cdots & \mathbf{0}_k \\ \vdots & & \ddots & \\ \mathbf{0}_k & \mathbf{0}_k & \cdots & \delta_k^k \end{bmatrix}. \tag{28}$$

Then, for any $X \in \Delta_k$,

$$X^2 = \Phi_k X. \tag{29}$$

Lemma 3.4: [12] Let $X \in \mathbb{R}^t$ be a column vector and $A \in \mathcal{M}_{m \times n}$ then,

$$XA = (I_t \otimes A)X, \tag{30}$$

where $I_t \in \mathcal{M}_{t \times t}$ is an identity matrix.
Set

$$z(t) = x(t-\tau+1)x(t-\tau+2)\cdots x(t), \tag{31}$$

then

$$z(t+1) = x(t-\tau+2)x(t-\tau+3)\cdots x(t)x(t+1). \tag{32}$$

From (24) and the lemmas above we have

$$\begin{array}{rl} & z(t+1) \\ = & x(t-\tau+2)\cdots x(t)L_{\sigma(t)}u(t)x(t-\tau+1)\cdots x(t) \\ = & (E_d)^n x(t-\tau+1)\cdots x(t)L_{\sigma(t)}u(t)z(t) \\ = & (E_d)^n z(t)L_{\sigma(t)}u(t)z(t) \\ = & (E_d)^n (I_{2^{\tau n}}\otimes L_{\sigma(t)})z(t)u(t)z(t) \\ = & (E_d)^n (I_{2^{\tau n}}\otimes L_{\sigma(t)})W_{[2^m,2^{\tau n}]}u(t)(z(t))^2 \\ = & (E_d)^n (I_{2^{\tau n}}\otimes L_{\sigma(t)})W_{[2^m,2^{\tau n}]}u(t)\Phi_{2^{\tau n}}z(t) \\ = & (E_d)^n (I_{2^{\tau n}}\otimes L_{\sigma(t)})W_{[2^m,2^{\tau n}]}(I_{2^m}\otimes\Phi_{2^{\tau n}})u(t)z(t). \end{array} \tag{33}$$

Thus, we can obtain the second algebraic form of τ-th order SBCN (20)

$$z(t+1) = M_{\sigma(t)}u(t)z(t) \tag{34}$$

where $M_{\sigma(t)} = (E_d)^n(I_{2^{\tau n}}\otimes L_{\sigma(t)})W_{[2^m,2^{\tau n}]}(I_{2^m}\otimes\Phi_{2^{\tau n}}) \in \mathcal{L}_{2^{\tau n}\times 2^{\tau n+m}}$.

IV. Controllability of higher order SBCNs

This part investigates controllability of higher order SBCNs. We study system (34) instead of system (24) in order to use MIS matrix approach. Thus, we have to prove that the controllability of system (34) is equivalent to that of system (24). To this end, we should provide definition of controllability of higher order SBCNs first.

Definition 4.1: Consider the higher order SBCN in the form of (20).

1) Given initial state $A_0 = \{X(i) \mid i = 0,1,\cdots,\tau-1, X(i)\in\mathcal{D}^n\}$. The higher order SBCN is said to be controllable from A_0 to $X_d \in \mathcal{D}^n$, if there exist a switching-law $\sigma(t): \mathbb{Z}_+ \to \{1,2,\cdots,N\}$ and a control sequence $\{U(t) \mid t = \tau-1,\tau,\cdots,\tau-1+s\}$, such that the corresponding model-input pairs $\{\Sigma(t) \mid U(t), t = \tau-1,\tau,\cdots,\tau-1+s\}$, where $0 < s < \infty$ and $s\in\mathbb{Z}_+$, can drive the trajectory from A_0 to $X(\tau-1+s) = X_d$;
2) The higher order SBCN is said to be controllable at initial state A_0, if it is controllable from A_0 to any $X_d \in \mathcal{D}^n$;
3) The higher order SBCN is said to be controllable, if it is controllable at any initial state A_0.

Next, we prove that the controllability of system (34) is equivalent to that of system (24). The following proposition is necessary.

Proposition 4.2: Set $X, Y \in \Delta_m, Z\in\Delta_n$. If

$$Z \ltimes X = Z \ltimes Y \tag{35}$$

we have

$$X = Y.$$

Its proof is obvious from the definition of STP. This proposition does not hold for STP of general real matrices. Here we give a counter-example to show the necessity of **Proposition 4.2**.

Example 4.3: Set

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here $A \in \Delta_2, C\bar{\in}\Delta_2$ and obviously $A \neq C$, but for any $B\in\mathcal{M}_{1\times 16}$, we have

$$B\ltimes A = B\ltimes C \in \mathcal{M}_{1\times 8}.$$

Theorem 4.4: The controllability of system (34) is equivalent to that of system (24).

Proof (Sufficiency) For any initial state $z(\tau-1)$ and given objective $z(T)$, we know that the corresponding states are A_0 and $x(T)$ from equation (31). By the controllability of system (24), there exists a model-input sequence

$$\{(\Sigma(\tau-1), u(\tau-1)), \cdots, (\Sigma(T-1), u(T-1))\} \tag{36}$$

such that

$$\begin{array}{rcl} x(\tau) & = & L_{\sigma(\tau-1)}u(\tau-1)x(0)\cdots x(\tau-1), \\ x(\tau+1) & = & L_{\sigma(\tau)}u(\tau)x(1)\cdots x(\tau), \\ & \vdots & \\ x(T) & = & L_{\sigma(T-1)}u(T-1)x(T-\tau)\cdots x(T-1). \end{array} \tag{37}$$

By equations (33) and (34), it follows that

$$\begin{array}{rcl} z(\tau) & = & M_{\sigma(\tau-1)}u(\tau-1)z(\tau-1), \\ z(\tau+1) & = & M_{\sigma(\tau)}u(\tau)z(\tau), \\ & \vdots & \\ z(T) & = & M_{\sigma(T-1)}u(T-1)z(T-1). \end{array} \tag{38}$$

Thus, we get a proper model-input sequence for any initial state $z(\tau-1)$ and $z(T)$.

(Necessity) For any initial state A_0 and given objective $x(T)$, if $x(T-1), x(T-2), \cdots, x(T-\tau+1)$ are arbitrarily selected, we can derive that that the corresponding states are $z(\tau-1)$ and $z(T)$ from equations (31). By the controllability of system (34), there exists a sequence (36) such that equation (38) holds. Compare the first row of (33) and (32), by **Proposition 4.2**, we obtain equations (37). Thus, the model-input sequence (36) can drive any initial value A_0 to any given $x(T)$. □

For higher order SBCNs we have the following result on controllability.

Lemma 4.5: [10] Assume the MIS matrix of system (11) is $\mathcal{J}$ and $\mathcal{J}_0$ is the basic block of $\mathcal{J}$. System (11) is controllable, iff

$$\mathcal{C} := \sum_{s=1}^{2^{(m+n)}N-1}{}_{\mathcal{B}}\ M_{\mathcal{B}}^{(s)} > 0, \quad M_{\mathcal{B}} = \sum_{i=1}^{2^m N}{}_{\mathcal{B}}\ Blk_i(\mathcal{J}_0). \tag{39}$$

Via **Theorem 4.4** and **Lemma 4.5** we derive the main result.

Theorem 4.6: Assume the MIS matrix of system (34) is $\mathcal{K}$ and $\mathcal{K}_0$ is the basic block of $\mathcal{K}$. System (24) is controllable, iff

$$\mathcal{F} := \sum_{s=1}^{2^{(m+\tau n)}N-1}{}_{\mathcal{B}}\ G_{\mathcal{B}}^{(s)} > 0, \quad G_{\mathcal{B}} = \sum_{i=1}^{2^m N}{}_{\mathcal{B}}\ Blk_i(\mathcal{K}_0). \tag{40}$$

$\mathcal{F}\in\mathcal{L}_{2^{\tau n}\times 2^{\tau n}}$ is called controllability matrix of system (34).

The purpose of the following part is to find a control and a proper switching-law, which drive A_0 to X_d. First, we introduce a lemma.

Lemma 4.7: [8] There is a one-to-one correspondence between the trajectories of (24) and the trajectories of (34).

Denote the sets of trajectories of (24) and (34) by T_x and T_z, respectively. The bijective between T_x and T_z can be defined as $\varphi : T_x \to T_z$, which maps each trajectory in T_x with initial value $\{x(0), \cdots, x(\tau-1)\}$ to a trajectory in T_z with initial value $z(\tau-1)$. Thus, we can find a model-input pair sequence for system (34) by the following algorithm.

Algorithm 4.8: Let the (i,j)-th element of the controllability matrix $[\mathcal{F}]_{ij} > 0$.

- Step 1: Find the smallest s, such that in the block decomposed form
$$\mathcal{K}_0^{(s)} = [\ Blk_1(\mathcal{K}_0^{(s)}), \cdots, Blk_{2^m N}(\mathcal{K}_0^{(s)})\ ], \tag{41}$$
there exists a block, $Blk_\alpha(\mathcal{K}_0^{(s)})$, whose (i,j)-th element satisfies
$$[Blk_\alpha(\mathcal{K}_0^{(s)})]_{ij} > 0. \tag{42}$$
Change single-index $(\alpha(\alpha_1, \alpha_2); 2^m N)$ into double-index $(\alpha_1, \alpha_2; N, 2^m)$. Then, set $\Sigma_{\sigma(0)} = \Sigma_{\alpha_1},\ u(0) = \delta_{2^m}^{\alpha_2}$ and $x(s) = \delta_{2^n}^{i}$. If $s = 1$, stop. Else, go to next step.
- Step 2: Find k, β, such that
$$[Blk_\beta(\mathcal{K}_0)]_{ik} > 0; \quad [Blk_\alpha(\mathcal{K}_0^{(s-1)})]_{kj} > 0. \tag{43}$$
Change single-index $(\beta(\beta_1, \beta_2); 2^m N)$ into 2-index $(\beta_1, \beta_2; N, 2^m)$. Then, set $\Sigma_{\sigma(s-1)} = \Sigma_{\beta_1},\ u(s-1) = \delta_{2^m}^{\beta_2}$ and $x(s-1) = \delta_{2^n}^{k}$.
- Step 3: If $s - 1 = 1$, stop. Else, replace s by $s-1$ and replace i by k, and go back to Step 2.

If we need to find the trajectory between A_0 and X_d in system (20). By equation (31), the corresponding states in system (34) are z_0 and z_d. Using **Algorithm 4.8**, we obtain a trajectory T_z and model-input sequence $\mathcal{MI}$. Then the trajectory and model-input sequence we want to find are $\varphi^{-1}(T_z)$ and $\mathcal{MI}$, respectively.

V. An illustrative example

Let $X(t) = (x_1(t)\ x_2(t))^T, U(t) = u(t)$. Assume the dynamics of a 2nd SBCN is
$$X(t+1) = F_{\sigma(t)}(U(t), X(t), X(t-1)), \tag{44}$$
where $\sigma(t) : \mathbb{Z}_+ \to \{1, 2\}$. The switching functions $F_i,\ i = 1, 2$ are
$$F_1 = \left[\begin{array}{c} u(t) \wedge x_1(t-1) \wedge x_2(t) \\ u(t) \to x_1(t-1) \wedge x_2(t) \end{array} \right],$$
$$F_2 = \left[\begin{array}{c} u(t) \wedge x_2(t-1) \vee x_1(t) \\ u(t) \to x_2(t-1) \vee x_1(t) \end{array} \right].$$
Algebraic form of system (44) is
$$x(t+1) = L_{\sigma(t)} u(t) x(t-1) x(t), \tag{45}$$
where $x(t) = x_1(t) \ltimes x_2(t)$ and
$$L_1 = \delta_4[1\ 4\ 1\ 4\ 1\ 4\ 1\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4],$$
$$L_2 = \delta_4[1\ 1\ 1\ 1\ 1\ 4\ 1\ 4\ 1\ 1\ 1\ 1\ 1\ 4\ 1\ 4\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3].$$
Set
$$z(t) = x(t-1)x(t). \tag{46}$$
By equation (33), we obtain another equivalent algebraic form of system (44):
$$z(t+1) = M_{\sigma(t)} u(t) z(t), \tag{47}$$
where
$$M_1 = \delta_{16}[1\ 8\ 9\ 16\ 1\ 8\ 9\ 16\ 4\ 8\ 12\ 16\ 4\ 8\ 12\ 16\ 3\ 8\ 11\ 16\ 3\ 8\ 11\ 16\ 3\ 8\ 11\ 16\ 3\ 8\ 11\ 16],$$
$$M_2 = \delta_{16}[1\ 5\ 9\ 13\ 1\ 8\ 9\ 16\ 1\ 5\ 9\ 13\ 1\ 8\ 9\ 16\ 1\ 7\ 9\ 15\ 1\ 7\ 9\ 15\ 1\ 7\ 9\ 15\ 1\ 7\ 9\ 15].$$
By **Lemma 2.1**, MIS matrix of system (44) is
$$\mathcal{K} = \left[\begin{array}{c} \mathcal{K}_0 \\ \mathcal{K}_0 \\ \mathcal{K}_0 \\ \mathcal{K}_0 \end{array} \right],$$
where $\mathcal{K}_0 = [M_1\ \ M_2]$. Then we have
$$\mathcal{F} = \left[\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0 \\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{array} \right].$$
By **Theorem 4.6**, system (44) is not controllable.

VI. Conclusion

Controllability of higher order SBCNs has been investigated in this paper. First, two kinds of algebraic forms have been presented via STP approach. Then, we have proved the equivalence of the controllability between the two algebraic forms. Using MIS matrix method, the necessary and sufficient condition of controllability of higher order SBCNs has been obtained. Trajectory tracking problem has been studied for higher order SBCNs. In the end, an illustrative example has been given to show the validity of the main results. The weakness of the approach proposed in this paper is the computational complexity. We can use STP toolbox based on MATLAB to run calculation in personal computer. However, even in STP toolbox we can not deal with the matrix with dimension bigger than 2^{25}.

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