Event-trigged Control for Discrete-time Multi-agent Networks

Lulu Li, Daniel W. C. Ho, Yuanyuan Zou, Chi Huang, and Jianquan Lu

Abstract—**In this paper, a new control strategy was proposed to deal with the discrete-time multi-agent consensus problem. Two types of protocols are discussed in this paper: i) networks of single-integrators without delay under centralized event-triggered control and ii) networks of single-integrators with delay under distributed event-triggered control. For each consensus protocol, we prove that the multi-agent network will achieve consensus asymptotically. Numerical examples are provided to demonstrate the effectiveness of the obtained theoretical results.**

Index Terms—**Multi-agent networks, consensus, event-triggered control, time delay.**

I. Introduction

Recent years have witnessed a thriving research activity on how to assemble and coordinate individual agent into a coherent whole to perform a common task in multi-agent network. The central point of multi-agent network is the coordination and cooperative control, at the level of the interconnected system, of evolution patterns absent in the simple local systems. Several significant results on multi-agent coordination and cooperative control have been reported in [1]–[6], which include consensus, formation control, distributed filtering and estimation, etc.

Consensus in multi-agent network means all agents reach an agreement on certain quantities of interest which has become one of the most focused problems in distributed coordination control of multi-agent networks and has attracted great attention [1]–[5]. In 1995, Vicsek et al: [7] proposed a discrete-time model of n agents which move in a plane with same speed but different headings. It was proved by simulation that all agents would reach an agreement under the assumption of large population density and small noise. A theoretical explanation for simulation result of Vicsek model was given in [8]. Olfati-Saber and Murray introduced two consensus protocol for continuous-time multi-agent networks with or without time-delays and three different kinds of problems were respectively discussed [1]. Moreau [9] provided a new consensus protocol and introduced a novel method based on the concept of convexity. In the past few years, there have been plentiful results including average consensus [1], [2], cluster consensus [10], asynchronous information consensus [11], leader-follower consensus [12], consensus with switching topology and time-delays [4], [6], [13], and references therein.

Since the broad bandwidth of networks is unavoidable in some cases, sampled control for multi-agent system is more coincident with the applications in our real life. Under fixed undirected/ directed interaction, two consensus algorithms for double-integrator dynamics within a uniform sampled-data setting were proposed in [14]. The results of reference [14] are extends to the dynamic network topology cases in [15]. Unlike time-driven control approach (i.e., periodic sampling), event-triggered control approach means the control signals are kept constant until the certain condition is violated and then the control signal will be updated (or re-computed). Some related results about event-triggered control have been reported in [16]–[22].

In [21], event-triggered control is applied to the continuous-time multi-agent consensus problem. Centralized and distributed triggering conditions which ensure the consensus of multi-agent network are presented respectively. In [22], Seyboth et al. also studied the multi-agent consensus under event-triggered control and three kinds problems of networks of single integrator agents with and without communication delays, and networks of double-integrator agents were discussed. It is worth noting that the framework in [21] and [22] about event-triggered multi-agent consensus only considers the continuous-time cases and the network topology is assumed to be undirected or balanced.

In this paper, we will propose the event-triggered cooperative control strategy for discrete-time directed multi-agent network. Both the centralized event-triggered control without delay and distributed event-triggered control with delay are analyzed. Under the assumption that the multi-agent network is strongly connected, consensus is achieved for the discrete-time network.

The organization of the remaining part is given as follows. In Section II some preliminaries about algebra graph theory and the system models are presented. In Section III and IV, consensus analysis of the proposed protocol are presented in detail. In Section V, two numerical simulation examples are given to show the effectiveness of the theoretical results. In Section VI, concluding remarks are drawn.

L. L. Li is with the Department of Mathematics, City University of Hong Kong, Hong Kong. (e-mail:lilulu01@gmail.com). Daniel W. C. Ho is with the Department of Mathematics, City University of Hong Kong, Hong Kong. (e-mail: madaniel@cityu.edu.hk). Y. Y. Zou is with the Key Laboratory of Advanced Control and Optimization for Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai, 200237, China (e-mail:yyzou@ecust.edu.cn). Chi Huang is with the Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China. (e-mail: huangchima@gmail.com). J. Q. Lu is with the Department of Mathematics, Southeast University, Nanjing 210096, China. (e-mail:jqluma@seu.edu.cn).

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II. Preliminaries and problem formulation

A. Basic graph theory

Let $\mathcal{G}(\mathcal{V},\,\mathcal{E},\,\mathcal{A})$ be a *weighted directed graph* with the set of nodes $\mathcal{V}=\{v_1,\,v_2,\,\ldots,v_N\}$, the set of edges $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ and a weighted adjacency matrix $\mathcal{A}=[\bar{a}_{ij}]$ with nonnegative adjacency elements $\bar{a}_{ij}$. An edge of $\mathcal{G}$ is denoted by $e_{ij}=(v_i,\,v_j)$, where v_i and v_j are called the parent and child vertices, respectively. For adjacency matrix $\mathcal{A}$, $(v_i,\,v_j)\in\mathcal{E}\Longleftrightarrow\bar{a}_{ji}>0$. For the undirected graph, one has $\bar{a}_{ij}=\bar{a}_{ji}$. The set of *neighbors* of v_i in $\mathcal{G}$ is denoted by $\mathcal{N}_i(v_i)=\{v_j:(v_j,\,v_i)\in\mathcal{E}\}$. A *path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices are an directed edge of the digraph. A directed graph $\mathcal{G}$ is *strongly connected* if there is a path for any pair of distinct vertices in $\mathcal{G}$.

B. System Model

Let $\mathcal{N}=\{1,\ldots,\,N\}$. In this section, we present two consensus protocols that solve consensus problems in a multi-agent network of discrete-time integrator agents with dynamics

$$x_i(k+1)=x_i(k)+\epsilon u_i(k),\qquad i\in\mathcal{N},\tag{1}$$

where $x_i(k)\in R$ is the state of the agent i and $\epsilon>0$ is the step-size.

In the event-triggered cooperative control strategy, suppose $k_0^i,\,k_1^i,\,\cdots,\,k_l^i,\,\cdots$ is the sequence of the event times of the agent i which is defined based on the event-triggering condition. The state measurement error of agent i is defined as

$$e_i(k)=x_i(k)-\widehat{x}_i(k),\tag{2}$$

where $\widehat{x}_i(k)$ is the latest broadcast state of agent i which is given by $\widehat{x}_i(k)=x_i(k_l^i)$, $k\in[k_l^i,\,k_{l+1}^i)$. The agent i will broadcast its latest state to its neighbors when the state measurement error of agent i exceeds the prescribed level (i.e., the event-triggering condition).

The following two scenarios are considered in this paper.

i) Fixed directed network topology and zero communication delay under centralized event-triggered control (i.e., all the agent broadcast its latest state to its neighbors simultaneously):

$$u_i(k)=\sum_{j\in\mathcal{N}_i}\overline{a}_{ij}(\widehat{x}_j(k)-\widehat{x}_i(k)).\tag{3}$$

ii) Fixed directed network topology with communication delay under distributed event-triggered control (i.e., the agent broadcast its latest state to its neighbors asynchronous):

$$u_i(k)=\sum_{j\in\mathcal{N}_i}\overline{a}_{ij}(\widehat{x}_j(k-\tau)-\widehat{x}_i(k)),\tag{4}$$

where $\tau>0$ is the communication delay from agent j to agent i, $x_j(k-\tau)=x_j(k_{l'}^j)$, $k-\tau\in[k_{l'}^j,\,k_{l'+1}^j)$, and $\widehat{x}_i(k)=x_i(k_l^i)$, $k\in[k_l^i,\,k_{l+1}^i)$.

Let $A=[a_{ij}]$ with $a_{ij}=\epsilon\bar{a}_{ij}\geq 0$ for $i\neq j$ and $a_{ii}=1-\epsilon\sum_{j=1,\,j\neq i}^{N}a_{ij}$.

Then, the dynamics of multi-agent network (1) with a control law (3) and (4) can be written respectively into

$$x_i(k+1)=e_i(k)+\sum_{j=1}^{N}a_{ij}\widehat{x}_j(k),\quad i\in\mathcal{N},\tag{5}$$

and

$$x_i(k+1)=e_i(k)+\sum_{j\in\mathcal{N}_i}a_{ij}\widehat{x}_j(k-\tau)+a_{ii}\widehat{x}_i(k),\quad i\in\mathcal{N}.\tag{6}$$

We make the following assumption in this paper.

Assumption 1: $a_{ii}>0$ for any $i\in\mathcal{N}$.

III. Centralized Event-triggered Approach

In this section, we mainly consider the centralized event-triggered control for the multi-agent network without communication delay. Let $\xi=\{\xi_1,\,\xi_2,\,\ldots,\,\xi_N\}$ be the normalized left eigenvector of matrix A with respect to the eigenvalue 1 satisfying $\sum_{i=1}^{N}\xi_i=1$. It can be obtained that $\xi_i>0$ from Perron-Frobenius theorem (see [23]).

Theorem 1: Consider the multi-agent network (1) with the control law (3) and assume the communication graph G is strongly connected. Suppose that $0<\sigma<1$ and the initial conditions associated with (1) are given as $x_i(0)$, $(i\in\mathcal{N})$. Then, the network will achieve consensus asymptotically under the triggering condition given by

$$\begin{aligned}\sum_{i=1}^{N}\xi_i\frac{4(1-a_{ii})}{a_{ii}}e_i^2(k)\quad>\quad&\sigma\sum_{i=1}^{N}\xi_i\sum_{j=1,\,j\neq i}^{N}a_{ij}a_{ii}(\widehat{x}_j(k)\\&-\widehat{x}_i(k))^2,\end{aligned}\tag{7}$$

Moreover, the final consensus value is $\sum_{i=1}^{N}\xi_i x_i(0)$.

Proof: Consider the Lyapunov functional as

$$V(k)=\sum_{i=1}^{N}\xi_i x_i^2(k).\tag{8}$$

$$\begin{aligned}\Delta V(k)\quad=\quad&\sum_{i=1}^{N}\xi_i x_i^2(k+1)-\sum_{i=1}^{N}\xi_i x_i^2(k)\\=\quad&\sum_{i=1}^{N}\xi_i[e_i(k)+\sum_{j=1}^{N}a_{ij}\widehat{x}_j(k)]^2-\sum_{i=1}^{N}\xi_i x_i^2(k)\\=\quad&\sum_{i=1}^{N}\xi_i[\sum_{j=1}^{N}a_{ij}^2\widehat{x}_j^2(k)+\sum_{j=1}^{N}\sum_{l>j}a_{ij}a_{il}(\widehat{x}_j^2(k)\end{aligned}$$

$$+\widehat{x}_l^2(k)) - \widehat{x}_i^2(k) + \sum_{j=1}^{N}\sum_{l>j} a_{ij}a_{il}(-\widehat{x}_j^2(k)$$

$$-\widehat{x}_l^2(k) + 2\widehat{x}_j(k)\widehat{x}_l(k)) + 2\sum_{j=1,\, j\neq i}^{N} a_{ij}e_i(k)\cdot$$

$$(\widehat{x}_j(k) - \widehat{x}_i(k)), \tag{9}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{N}\xi_i[\sum_{j=1}^{N} a_{ij}^2\widehat{x}_j^2(k) + \sum_{j=1}^{N}\sum_{l>j} a_{ij}a_{il}(\widehat{x}_j^2(k) + \widehat{x}_l^2(k)) \\
&-\widehat{x}_i^2(k)] \\
=\ &\sum_{i=1}^{N}\xi_i[\sum_{j=1}^{N} a_{ij}^2\widehat{x}_j^2(k) + \sum_{j=1}^{N}\sum_{l=1,\, l\neq j}^{N} a_{ij}a_{il}\widehat{x}_j^2(k) \\
&-\widehat{x}_i^2(k)] \\
=\ &\sum_{i=1}^{N}\xi_i[\sum_{j=1}^{N}\sum_{l=1}^{N} a_{ij}a_{il}\widehat{x}_j^2(k) - \widehat{x}_i^2(k)] \\
=\ &\sum_{j=1}^{N}\xi_j\widehat{x}_j^2(k) - \sum_{i=1}^{N}\xi_i\widehat{x}_i^2(k) \\
=\ &0,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&2\sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} a_{ij}e_i(k)(\widehat{x}_j(k) - \widehat{x}_i(k))] \\
\leq\ &\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} 2\xi_i a_{ij}[\frac{1}{2\alpha_i}e_i^2(k) + \frac{\alpha_i}{2}(\widehat{x}_j(k) - \widehat{x}_i(k))^2] \\
=\ &\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N} a_{ij}[\frac{1}{\alpha_i}e_i^2(k) + \alpha_i(\widehat{x}_j(k) - \widehat{x}_i(k))^2].
\end{aligned} \tag{11}$$

Substituting (10) and (11) into (8), we have

$$\begin{aligned}
\Delta V(k) =\ & -\sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N} a_{ij}a_{il}(\widehat{x}_j(k) \\
&-\widehat{x}_l(k))^2 + \sum_{j=1,\, j\neq i}^{N} a_{ij}a_{ii}(\widehat{x}_j(k) \\
&-\widehat{x}_i(k))^2] + 2\sum_{i=1}^{N}\xi_i\cdot[\sum_{j=1,\, j\neq i}^{N} a_{ij}e_i(k)\cdot \\
&(\widehat{x}_j(k) - \widehat{x}_i(k))] \\
\leq\ & -\sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N} a_{ij}a_{il}(\widehat{x}_j(k) \\
&-\widehat{x}_l(k))^2 + \sum_{j=1,\, j\neq i}^{N} a_{ij}(a_{ii} - \alpha_i)(\widehat{x}_j^2(k) \\
&-\widehat{x}_i(k))^2] + \sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}\frac{a_{ij}}{\alpha_i}e_i^2(k)
\end{aligned} \tag{12}$$

Thus, by choosing $\alpha_i = \frac{a_{ii}}{2}$, a sufficient condition for $\Delta V(k) \leq 0$ is given by

$$\sum_{i=1}^{N}\xi_i\frac{4(1-a_{ii})}{a_{ii}}e_i^2(k) \leq \sigma\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N} a_{ij}a_{ii}(\widehat{x}_j(k) -\widehat{x}_i(k))^2. \tag{13}$$

We choose the events trigger condition as

$$\sum_{i=1}^{N}\xi_i\frac{4(1-a_{ii})}{a_{ii}}e_i^2(k) > \sigma\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N} a_{ij}a_{ii}(\widehat{x}_j(k) -\widehat{x}_i(k))^2. \tag{14}$$

Furthermore, the sequence of event-times $0 = k_0 < k_1 < \cdots < k_l < \cdots$ for the network is defined iteratively as

$$k_{l+1} = inf\{k:\, k > k_l, f(k,\, x_k,\, x_{k_l}) > 0\}, \tag{15}$$

where

$$f(k,\, x_k,\, x_{k_l}) = \sum_{i=1}^{N}\xi_i[\frac{4(1-a_{ii})}{a_{ii}}e_i^2(k) - \sigma\sum_{j=1,\, j\neq i}^{N} a_{ij}a_{ii} (x_j(x_{k_l}) - x_i(x_{k_l}))^2]\} \tag{16}$$

Hence, under the trigger condition (14), we have $\Delta V(k) \leq 0,\ \forall k \geq 0$. According to the LaSalle's invariance principle, we can get that all the agents in the network will converge to the set $\{x(k) \in \mathbb{R}^n | \Delta V(k) = 0\}$. It follows from G is strongly connected that $\Delta V(k) = 0$ if and only if $\widehat{x}_j(k) = \widehat{x}_i(k)$ and $e_i(k) = 0, \forall i,\, j \in \mathcal{N}$.

Therefore, we have

$$\lim_{k\to\infty}(x_j(k) - x_i(k)) = 0. \tag{17}$$

Since Let $\eta(k) = \sum_{i=1}^{N}\xi_i x_i(k)$. We can calculate the difference of $\eta(k)$ as follows:

$$\begin{aligned}
\Delta\eta(k) &= \sum_{i=1}^{N}\xi_i(x_i(k+1) - x_i(k)) \\
&= -\sum_{i=1}^{N}\xi_i\widehat{x}_i(k) + \sum_{i=1}^{N}\xi_i a_{ij}\sum_{j=1}^{N}\widehat{x}_j(k) \\
&= 0,
\end{aligned}$$

which implies that $\eta(k)$ is a constant. That is, $\eta(k) = \eta(0) = \sum_{i=1}^{N}\xi_i x_i(0)$.

Therefore,

$$\lim_{k\to\infty} x_i(k) = \sum_{i=1}^{N}\xi_i x_i(0),\ \forall i \in \mathcal{N}. \tag{18}$$

This completes the proof of this theorem. ■

From Theorem 1, It can be noted that all the agent will broadcast its latest state to its neighbors simultaneously in the centralized event-triggered control. Moreover, the event-triggering condition in centralized control uses the global information of the network. In the section IV, the distributed event-triggered control in the multi-agent consensus problem with communication delays will be discussed.

Remark 1: It can be seen from the event-triggered condition (14) that the size of inter-event times $\{k_{l+1}-k_l\}$ depends on the the value of the step size ϵ. Moreover, the inter-event times of the protocol (3) will be larger than 2 if the step size ϵ is small enough.

IV. Distributed Event-triggered Approach

Time delay is a very important communication constraint in the process of information exchange which should be considered in the consensus protocol. In the following part, we shall apply the event-triggered control to the problem of multi-agent network (6) with communication delays. Part of the proof will be omitted here due to the length limit. Denote $X = \{\psi : \{-\tau, -\tau+1, \cdots, -1, 0\} \to \mathbb{R}^N\}$ and suppose the initial condition of the network is $\phi_i \in X, i \in \mathcal{N}$.

Theorem 2: Consider the multi-agent network (1) with a control law (4) and assume the communication graph G is strongly connected. Then, for any finite communication delay τ, the network will achieve consensus asymptotically under the triggering condition given by

$$e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j \in \mathcal{N}_i} a_{ij}(\widehat{x}_j(k-\tau) - \widehat{x}_i(k))^2, \quad i \in \mathcal{N}, \tag{19}$$

where $0 < \sigma < 1$ is a constant. Moreover, the final consensus value is

$$\frac{\sum_{i=1}^N \xi_i x_i(0) + \sum_{i=1}^N \xi_i \sum_{j=1,\, j\neq i}^N a_{ij} \sum_{s=-\tau}^{-1} x_j(s)}{1 + \sum_{i=1}^N \xi_i (1-a_{ii})\tau}.$$

Proof: Consider the Lyapunov functional as

$$V(k) = V_1(k) + V_2(k), \tag{20}$$

where

$$V_1(k) = \sum_{i=1}^N \xi_i x_i^2(k), \tag{21}$$

and

$$V_2(k) = \sum_{i=1}^N \sum_{j=1}^N a_{ij} \sum_{s=k-\tau}^{k-1} \widehat{x}_j^2(s). \tag{22}$$

Difference $V(k)$ along the solution of (10) gives that

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k). \tag{23}$$

By some simple computation, we can obtain that

$$\begin{aligned}
\Delta V(k) = & \sum_{i=1}^N \xi_i [\sum_{j=1}^N a_{ij}^2 \widehat{x}_j^2(k) + \sum_{j=1,\, j\neq i}^N \sum_{l>j,\, l\neq i}^N a_{ij} a_{il} \cdot \\
& (\widehat{x}_j^2(k) + \widehat{x}_l^2(k)) - \widehat{x}_i^2(k) + \sum_{j=1,\, j\neq i}^N a_{ij} a_{ii} \cdot \\
& (\widehat{x}_j^2(k) + \widehat{x}_i^2(k)) - \sum_{j=1,\, j\neq i}^N a_{ij} a_{ii} (\widehat{x}_j(k-\tau) \\
& - \widehat{x}_i(k))^2 - \sum_{j=1,\, j\neq i}^N \sum_{l>j,\, l\neq i}^N a_{ij} a_{il} (\widehat{x}_j(k-\tau) \\
& - \widehat{x}_l(k-\tau))^2] + 2\sum_{i=1}^N \xi_i [\sum_{j=1,\, j\neq i}^N a_{ij} e_i(k) \\
& (\widehat{x}_j(k-\tau) - \widehat{x}_i(k))]
\end{aligned} \tag{24}$$

Using a similar argument as the proof of Theorem 1, we can obtain that

$$\begin{aligned}
\Delta V(k) = & -\sum_{i=1}^N \xi_i [\sum_{j=1,\, j\neq i}^N \sum_{l>j,\, l\neq i}^N a_{ij} a_{il} (\widehat{x}_j(k-\tau) \\
& - \widehat{x}_l(k-\tau))^2 + \sum_{j=1,\, j\neq i}^N a_{ij} a_{ii} (\widehat{x}_j(k-\tau) \\
& - \widehat{x}_i(k))^2] + 2\sum_{i=1}^N \xi_i \cdot [\sum_{j=1,\, j\neq i}^N a_{ij} e_i(k) \cdot \\
& (\widehat{x}_j(k-\tau) - \widehat{x}_i(k))] \\
\leq & -\sum_{i=1}^N \xi_i [\sum_{j=1,\, j\neq i}^N \sum_{l>j,\, l\neq i}^N a_{ij} a_{il} (\widehat{x}_j(k-\tau) \\
& - \widehat{x}_l(k-\tau))^2 + \sum_{j=1,\, j\neq i}^N a_{ij} (a_{ii} - \alpha_i) \cdot \\
& (\widehat{x}_j(k-\tau) - \widehat{x}_i(k))^2] + \sum_{i=1}^N \xi_i \cdot \\
& \sum_{j=1,\, j\neq i}^N \frac{a_{ij}}{\alpha_i} e_i^2(k)
\end{aligned} \tag{25}$$

Thus, a sufficient condition for $\Delta V(k) \leq 0$ is given by

$$e_i^2(k) \leq \frac{\sigma \alpha_i (a_{ii} - \alpha_i)}{1 - a_{ii}} \sum_{j=1,\, j\neq i}^N a_{ij} (\widehat{x}_j(k-\tau) - \widehat{x}_i(k))^2, \tag{26}$$

Let $f(\alpha_i) = \frac{\alpha_i (a_{ii} - \alpha_i)}{1 - a_{ii}}$. Then, we can easily obtain the maximum of $f(\alpha_i)$ by taking $\alpha = \frac{a_{ii}}{2}$, which makes (26) become

$$e_i^2(k) \leq \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^N a_{ij} (\widehat{x}_j(k-\tau) - \widehat{x}_i(k))^2. \tag{27}$$

Hence, we can choose the trigger condition

$$e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(k-\tau)-\widehat{x}_i(k))^2, \quad i \in \mathcal{N}. \tag{28}$$

Hence, under the trigger condition (28), we have $\Delta V(k) \leq 0,\ \forall k \geq 0$. According to the LaSalle's invariance principle, we can get that all the agents in the network will converge to the maximal positively invariant set of the set $\Phi = \{x(k+\theta) \in X : \Delta V(k) = 0\}$ asymptotically. Note that $\Delta V(k) = 0$ if and only if $e_i(k) = 0$ and

$$\widehat{x}_j(k-\tau) = \widehat{x}_i(k),\ \forall i,\, j \in \mathcal{N}_i. \tag{29}$$

Substituting (29) into (6) yields that

$$x_i(k+1) = x_i(k),\ \forall i \in \mathcal{N}. \tag{30}$$

Hence, we have

$$x_i(k) = \widehat{x}_i(k) = \widehat{x}_j(k-\tau) = x_j(k-\tau) = x_j(k),\ \forall j \in \mathcal{N}_i. \tag{31}$$

It follows from G is strongly connected that

$$x_i(k) = x_j(k),\ k \geq -\tau, \forall i,\, j \in \mathcal{N}, \tag{32}$$

Therefore, by the LaSalle's invariance principle, we have

$$\lim_{k\to\infty}(x_j(k) - x_i(k)) = 0. \tag{33}$$

Next, we shall give the consensus value c of the multi-agent network which is shown to be depending on the initial values of the multi-agent network. Let $\eta(k) = \sum_{i=1}^{N} \xi_i x_i(k) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \sum_{k-\tau}^{k-1} \widehat{x}_j(s)$. By simple calculations, we can obtain that

$$\begin{aligned} \Delta\eta(k) &= \eta(k+1) - \eta(k) \\ &= 0. \end{aligned}$$

Due to $\Delta\eta(k) = 0$ for $k \geq 0$, it can be easily obtained that $\eta(k)$ is a constant. That is, $\eta(k) = \eta(0) = \sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \sum_{-\tau}^{-1} \widehat{x}_j(s) = \sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \sum_{-\tau}^{-1} x_j(s)$.

Hence,

$$\eta(0) = \lim_{t\longrightarrow\infty} \eta(k) \quad = \quad c + \sum_{i=1}^{N} \xi_i (1-a_{ii})\tau c.$$

Therefore, we can conclude that

$$c = \frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \sum_{s=-\tau}^{-1} x_j(s)}{1 + \sum_{i=1}^{N} \xi_i (1-a_{ii})\tau}.$$

This completes the proof of this theorem. ■

Remark 2: It should be emphasized that the event-triggered condition (28) is verified by each agent only based on each own and its received neighboring agents' information, i.e., only local information is used to verify the event-trigger condition.

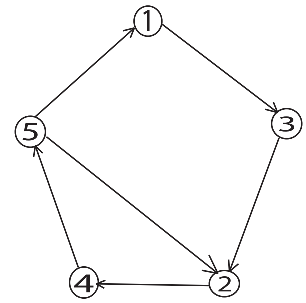

Fig. 1. Network topology in Example 1.

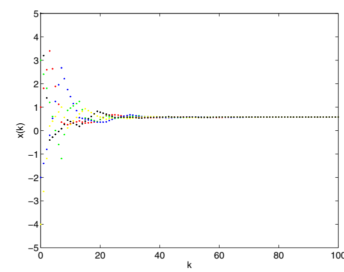

Fig. 2. The states of the system in Example 1.

V. Numerical examples

In this section, two examples are given to illustrate the correctness of the theoretical results.

Example 1: Consider the multi-agent system (1) with five agents. The directed network topology is displayed in Figure 1, and the weight of each edge is set as 1, i.e. $\overline{a}_{ij} = 1$. The step size ϵ and constant σ are set to be $\frac{1}{5}$ and 0.9. The initial conditions are selected as $[1,\ 3,\ -2,\ -4,\ 5]^T$.

Figure 2 shows the simulation result for the centralized event-triggered control for multi-agent network (1). In this simulation, the number of control events is 29. From Figure 2, we can also find that the multi-agent network will achieve consensus under the control protocol (2), which illustrates Theorem 1 very well.

Example 2: In order to illustrate the results of Theorem 2, the same multi-agent network as Example 1 is considered. Assume $\tau = 1$.

The state trajectory of the network is given in Fig. 3 and we can see from Fig. 3 that the multi-agent system reaches consensus. It can be seen from Fig.4 that the event are triggered totally 36, 39, 37, 36, 35 times respectively during the evolve of the agents. This example demonstrates that the event-triggered control approach is an effective method for the multi-agent network consensus problem.

VI. Conclusion

In this paper, we have investigated the discrete-time consensus problem of multi-agent network where each agent transmit its current state to its neighbors only when certain

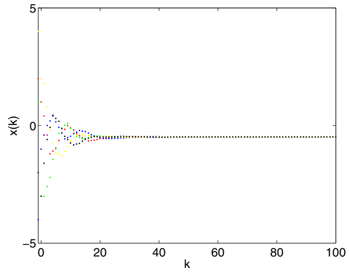

Fig. 3. The states of the system in Example 2.

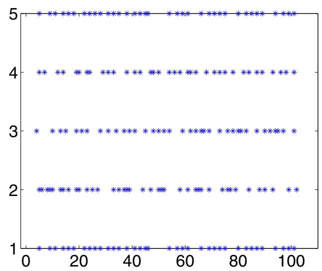

Fig. 4. Event trigger times in Example 2.

"event" occurs. The network under study is directed and contains communication delays. A centralized formulation of the problem without considering communication delay is studied first and the results are then extended to the distributed cases with communication delays. The theoretical results are well illustrated by two numerical examples.

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