A Unified Framework for State Estimation of Nonlinear Stochastic Systems with Unknown Inputs

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***Abstract*—This paper considers the unknown input filtering problem of nonlinear stochastic systems with arbitrary unknown inputs. It is known that the celebrated extended Kalman filter (EKF) may have poor performance in solving this problem due to the lack of the true dynamics of the unknown input. A possible remedy to improve the performance is to apply an EKF-like nonlinear version of the recently developed ERTSF (NERTSF), which however may only yield a specific linear combination of the unknown input vector. In this paper, an unknown-input decoupled nonlinear estimation framework is proposed, through which specific derivative-based and -free estimators are derived to provide both the estimable and unestimable unknown input estimates. Applications to rederive the existing literature results are provided to illustrate the usefulness of the proposed results.**

I. Introduction

The problem of estimating the state vector of a discrete-time stochastic linear system with arbitrary unknown inputs, known as the unknown input filtering (UIF) problem, has received much research attention ever since the original work of Kitanidis [1] first appeared. This is due to the fact that many practical systems are subject to disturbances, modeling errors, constraints, and system uncertainties. The main aim of the UIF problem is to optimally estimate the system state without knowing any unknown input model. Recently [2], a globally optimal filtering framework is developed for unbiased minimum-variance (UMV) state estimation for systems with unknown inputs that simultaneously affect the system state and the output. Through this result, the recent proposed UMV filters in [3]-[6] all can be verified as globally optimal solutions of the UIF problem concerning different filter structures.

Nonlinear filtering problems arise in many practical applications, e.g., financial estimation, biological and industrial processes, target localization and tracking, robots and robotic manipulators, and traffic state estimation. As is well known, a general approach to solve these problems is generalizing the Kalman filter paradigm for nonlinear systems, e.g., the extended Kalman filter (EKF). It is noted that the EKF is a first-order filter that propagates only the mean and covariance of the filtering densities, which however may diverge or provide poor state estimates due to its inherent first-order Taylor approximation of the nonlinear model. Other efforts to improve on the EKF have been developed, e.g., the DD2 filter [7], the unscented Kalman filter (UKF) [8], the derivative-free version of the EKF [9], the Gaussian particle filter (GPF) [10], the cubature Kalman filters (CKF) [11], and the derivative-free estimation method [12]. As addressed in [9], all the above filters attempt to improve the EKF by representing state uncertainty with a different ensemble set of state vectors. To the best of the author's knowledge, all the above-mentioned nonlinear estimator design methods are not yet applied to UIF problem for nonlinear stochastic systems.

A heuristic approach of applying the aforementioned filters to solve the UIF problem of nonlinear stochastic systems is to augment the system state with the unknown inputs, and then apply the dedicated filtering method to the obtained augmented system. Notice that in this approach the unknown input model is always needed and assumed beforehand; as shown in the previous works [13]-[15], this approach may not perform well for arbitrary unknown inputs. A possible solution to remedy this problem is to propose ERTSF-like recursive algorithms that can optimally estimate the system state in light of arbitrary unknown input values [13]-[15]. However, it should be stressed that all these unknown-input decoupled nonlinear estimators (UIDNEs) are derived based on a direct application and extension of the ERTSF [5], which in general may only yield a specific linear combination of the unknown input vector. In other words, only the estimable unknown input estimates from the measured outputs are provided and the remaining unestimable unknown input estimates are ignored. On the other hand, few research results concern simultaneous state and input estimation for nonlinear systems [16], which only considered linear measurement and limited system nonlinearity. Thus, the state estimation problem of applying the UIF method for general nonlinear stochastic systems still remains open.

In this paper, we extend the previous works [13]-[15] and continue the research line in investigating the applications of the UIF method to solve the addressed state estimation problem of nonlinear stochastic systems with unknown inputs. Specifically, the main aim of this paper is to present a unified framework of UIDNE designs for nonlinear stochastic systems with unknown inputs, through which the existing nonlinear filtering methods can be easily applied.

The paper is organized as follows. In Section II, the statement of the problem is addressed and the problems encountered in the existing methods are briefly introduced. In Section III, the proposed unified framework of UIDNEs is

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presented. Two applications of applying the proposed unified framework to design a specific UIDNE estimator for general nonlinear systems with unknown inputs are also given: one is the derivative-based estimator design and the other is the derivative-free estimator design. Illustrations of applying the proposed framework to rederive the existing literature results are given in Section IV to show the usefulness of the proposed results. Finally, conclusions are highlighted in the last section.

II. Statement of the Problem

Consider the general nonlinear system with unknown inputs as follows:

$$\begin{aligned} x_{k+1} &= f(x_k, d_k, u_k) + w_k, &(1)\\ y_k &= g(x_k, d_k, u_k) + v_k, &(2)\end{aligned}$$

where $x_k \in R^n$, $d_k \in R^p$, $u_k \in R^q$, and $y_k \in R^m$ are, respectively, the state, unknown input, control input, and measured output. The process noise w_k and the measurement noise v_k are uncorrelated zero mean white sequences with covariance matrices $Q_k > 0$ and $R_k > 0$, respectively. The initial state x_0 is with unbiased mean $\bar{x}_0$ and covariance $\bar{P}_0^x$ and is independent of w_k and v_k. The estimation problem of the paper focuses on optimal estimating the system state from the measurements y_k, denoted as $\hat{x}_{k|k}$, while without knowing any dynamics of the unknown input d_k.

A common approach used for the state estimation of the nonlinear system (1), (2) is to apply the augmented state approach, where the following augmented state vector:

$$X_k = \left[\begin{array}{cc} x_k^T & d_k^T \end{array} \right]^T,$$

is formed and the assumption that the unknown input d_k can be effectively modeled as a random-walk stochastic process is usually employed. Thus, system (1)-(2) can be rewritten as the following:

$$\begin{aligned} X_{k+1} &= f^a(X_k, u_k) + w_k^a, &(3)\\ y_k &= g(X_k, u_k) + v_k, &(4)\end{aligned}$$

where

$$w_k^a = \left[\begin{array}{c} w_k \\ w_k^d \end{array} \right], \quad f^a(X_k, u_k) = \left[\begin{array}{c} f(X_k, u_k) \\ d_k \end{array} \right],$$

with which existing nonlinear filtering methods such as those addressed in the Introduction can be directly applied. Unfortunately, this direct application of existing methods to solve UIDNE design problem may not perform well as illustrated in intending to apply the well-known EKF in previous works [13]-[15]. This is mainly due to the lack of the true dynamics of the unknown input d_k.

A possible remedy to the aforementioned performance degradation problem is to apply ERTSF-like nonlinear filtering approaches [14]-[15], where the original system (1)-(2) is first transformed into the following approximated system with explicit unknown inputs:

$$\begin{aligned} x_{k+1} &\approx f(x_k, d_k^e, u_k) + G_k(I - \Phi_k)d_k + w_k, &(5)\\ y_k &\approx \tilde{g}(x_k, \hat{d}_{k-1|k-1}^e, u_k) + H_k d_k + v_k, &(6)\end{aligned}$$

where $d_k^e \in R^p$ represents the linear combination of d_k that can be estimated from the set $y^k = \{y_0, y_1, \ldots, y_k\}$, matrix Φ_k satisfies the relationship $H_k \Phi_k = H_k$,

$$\begin{aligned} G_k &= \frac{\partial f(x_k, d_k, u_k)}{\partial d_k}\Big|_{d_k = \hat{d}_{k|k}^e}, &(7)\\ H_k &= \frac{\partial g(x_k, d_k, u_k)}{\partial d_k}\Big|_{d_k = \hat{d}_{k-1|k-1}^e}, &(8)\\ \tilde{g}(\bullet) &= g(\bullet) - H_k \hat{d}_{k-1|k-1}^e. &(9)\end{aligned}$$

Then, applying the ERTSF to (5)-(6), one obtains the corresponding UIDNE filter. One drawback of the above approach is that only the estimates of the estimable unknown input vector, i.e., $\Phi_k d_k$, can be obtained. Furthermore, both of (5)-(6) are approximations of the original terms (1)-(2).

The main aims of this paper are: 1) to further develop a unified estimation framework for nonlinear stochastic systems with unknown inputs that can simultaneously yield the estimable and unestimable unknown input estimates, 2) to design two specific derivative-based and -free UIDNEs for general nonlinear systems, and 3) to rederive existing literature results via the proposed unified framework.

III. Unknown-Input Decoupled Nonlinear Estimators Design

A. A Unified Framework

In this subsection, a unified framework of deriving UIDNEs is developed. The main idea of achieving the aim is based on descriptor system formulation and UIDNE design for standard systems, which is highlighted as below.

First, we assume that the unknown input vector d_k can be transformed into two parts: $\bar{d}_k$ and $\tilde{d}_k$, which represent the estimable and unestimable parts of d_k from y^k, respectively, as follows:

$$\left[\begin{array}{c} \bar{d}_k \\ \tilde{d}_k \end{array} \right] = \left[\begin{array}{c} \bar{\Phi}_k \\ \tilde{\Phi}_k \end{array} \right] d_k, \tag{10}$$

where matrix $\bar{\Phi}_k$ is assumed to be known (see Remark 1 for details) and $\tilde{\Phi}_k$ is a design parameter such that matrix $\left[\begin{array}{cc} \bar{\Phi}_k^T & \tilde{\Phi}_k^T \end{array} \right]^T$ is nonsingular. Using (10), we have

$$d_k = \left[\begin{array}{c} \bar{\Phi}_k \\ \tilde{\Phi}_k \end{array} \right]^{-1} \left[\begin{array}{c} \bar{d}_k \\ \tilde{d}_k \end{array} \right] = \bar{\Psi}_k \bar{d}_k + \tilde{\Psi}_k \tilde{d}_k. \tag{11}$$

Second, using (11) system (1)-(2) can be rewritten as follows:

$$\begin{aligned} x_{k+1} &= f^*(x_k, \bar{d}_k, u_k) + \tilde{f}(x_k, d_k, u_k) + w_k, &(12)\\ y_k &= g^*(x_k, \bar{d}_k, u_k) + v_k, &(13)\end{aligned}$$

where

$$\begin{aligned} f^*(x_k, \bar{d}_k, u_k) &= f(x_k, \bar{\Psi}_k \bar{d}_k, u_k), &(14)\\ \tilde{f}(x_k, d_k, u_k) &= f(x_k, d_k, u_k) - f^*(x_k, \bar{d}_k, u_k), &(15)\\ g^*(x_k, \bar{d}_k, u_k) &= g(x_k, \bar{\Psi}_k \bar{d}_k, u_k). &(16)\end{aligned}$$

Third, we intend to reform $\tilde{f}(x_k, d_k, u_k)$. Using the first order Taylor expansion, (15) can be approximated as follows:

$$\tilde{f}(x_k, d_k, u_k) = \begin{bmatrix} \tilde{f}_1(x_k, d_k, u_k) & \cdots & \tilde{f}_n(x_k, d_k, u_k) \end{bmatrix}^T,$$

where

$$\begin{aligned}\tilde{f}_i(x_k, d_k, u_k) &\approx \frac{\partial f_i(x_k, d_k, u_k)}{\partial d_k}|_{d_k \leftarrow \bar{\Psi}_k \bar{d}_k} \cdot \tilde{\Psi}_k \tilde{d}_k \\ &= G_k^i (I - \bar{\Psi}_k \bar{\Phi}_k) d_k. \end{aligned} \quad (17)$$

Then, $\tilde{f}(x_k, d_k, u_k)$ can be approximated as follows:

$$\tilde{f}(x_k, d_k, u_k) \approx \Pi_k d_k^a, \quad (18)$$

where $d_k^a = d_k$ and

$$\Pi_k = \begin{bmatrix} (G_k^1)^T & \cdots & (G_k^n)^T \end{bmatrix}^T (I - \bar{\Psi}_k \bar{\Phi}_k). \quad (19)$$

However, for some applications (17) may not be feasible or accurate to implement $\tilde{f}_i(\bullet)$. One possible remedy to this problem is to employ the following second order Stirling's interpolation formula [12]:

$$\begin{aligned}\tilde{f}_i &\approx \sum_{j=1}^{p} \tilde{d}_k^j \frac{\bar{f}_i(\bar{d}_k + \Delta e_p^j) - \bar{f}_i(\bar{d}_k - \Delta e_p^j)}{2\Delta} \\ &+ \sum_{j=1}^{p} (\tilde{d}_k^j)^2 \frac{\bar{f}_i(\bar{d}_k + \Delta e_p^j) + \bar{f}_i(\bar{d}_k - \Delta e_p^j) - 2\bar{f}_i(\bar{d}_k)}{2\Delta^2},\end{aligned}$$

where $\tilde{d}_k^j$ is the j-th element of $\tilde{d}_k$, Δ is a suitable chosen small value, e_p^j is the jth column of the identity matrix with dimension p,

$$\bar{f}_i(\bar{d}_k) = f_i(x_k, \bar{\Psi}_k \bar{d}_k, u_k).$$

Using the following notations:

$$\begin{aligned}G_{k,1}^{i,j}(\bar{d}_k) &= \frac{\bar{f}_i(\bar{d}_k + \Delta e_p^j) - \bar{f}_i(\bar{d}_k - \Delta e_p^j)}{2\Delta}, \\ G_{k,2}^{i,j}(\bar{d}_k) &= \frac{\bar{f}_i(\bar{d}_k + \Delta e_p^j) + \bar{f}_i(\bar{d}_k - \Delta e_p^j) - 2\bar{f}_i(\bar{d}_k)}{2\Delta^2}, \\ G_{k,l}^i &= \begin{bmatrix} G_{k,l}^{i,1} & \cdots & G_{k,l}^{i,p} \end{bmatrix}, \quad l = 1, 2, \\ d_k^\dagger &= \begin{bmatrix} (\tilde{d}_k^1)^2 & \cdots & (\tilde{d}_k^p)^2 \end{bmatrix}^T,\end{aligned}$$

(18) can be expressed as follows:

$$\begin{aligned}\tilde{f}(x_k, d_k, u_k) &\approx \begin{bmatrix} G_{k,1}^1 (I - \bar{\Psi}_k \bar{\Phi}_k) & G_{k,2}^1 \\ \vdots & \vdots \\ G_{k,1}^n (I - \bar{\Psi}_k \bar{\Phi}_k) & G_{k,2}^n \end{bmatrix} \begin{bmatrix} d_k \\ d_k^\dagger \end{bmatrix} \\ &= \Pi_k d_k^a,\end{aligned}$$

where $d_k^a = \begin{bmatrix} d_k^T & (d_k^\dagger)^T \end{bmatrix}^T$. Conceptually, the above approach to implement $\tilde{f}(\bullet)$ can be extended to high order approximation in order to obtain more accurate result. It is also noted that the above matrix Π_k is a function of the system state x_k and the estimable unknown input $\bar{d}_k$, which are in general unknown. To solve this problem, we use the following estimated value:

$$\hat{\Pi}_k = \Pi_k(x_k \leftarrow \hat{x}_{k|k}, d_k \leftarrow \bar{\Psi}_k \hat{\bar{d}}_{k|k}).$$

In summary, we can effectively express $\tilde{f}(\bullet)$ as follows:

$$\tilde{f}(x_k, d_k, u_k) = \hat{\Pi}_k d_k^a. \quad (20)$$

Fourth, using (20) and defining an augmented system state $X_k \in R^{n+dim(\bar{d})}$ as $X_k = \begin{bmatrix} x_k^T & \bar{d}_k^T \end{bmatrix}^T$, (12) and (13) can be rewritten, respectively, as the following descriptor system:

$$\begin{aligned}E_{k+1} X_{k+1} &= f^*(X_k, u_k) + \hat{\Pi}_k d_k^a + w_k, \quad (21) \\ y_k &= g^*(X_k, u_k) + v_k, \quad (22)\end{aligned}$$

where $E_{k+1} = \begin{bmatrix} I & 0 \end{bmatrix}$.

Fifth, we intend to transform the above descriptor system into a standard system with unknown inputs. To solve this objective, we first propose the following theorem without proof (see [17] for details).

Theorem 1: Consider the following nonlinear descriptor system with unknown inputs:

$$E_{k+1} X_{k+1} = f^*(X_k, u_k) + \hat{\Pi}_k d_k^a + w_k.$$

If matrix E_{k+1} is of full-row rank, then the above descriptor system can be equivalently transformed into the following standard system with unknown inputs:

$$X_{k+1} = E_{k+1}^+ (f^*(X_k, u_k) + \hat{\Pi}_k d_k^a + w_k) + U_k^* d_k^*,$$

where M^+ is the Moore-Penrose pseudo-inverse of M,

$$U_k^* = (I - E_{k+1}^+ E_{k+1}) \gamma_k, \quad d_k^* = (U_k^*)^+ X_{k+1},$$

with matrix parameter γ_k being chosen so that U_k^* is of full-column rank.

Then, applying Theorem 1 to (21) we obtain the following standard augmented state system with unknown inputs:

$$X_{k+1} = E_{k+1}^+ f^*(X_k, u_k) + \bar{U}_k \check{d}_k + E_{k+1}^+ w_k, \quad (23)$$

where

$$\begin{aligned}\bar{U}_k &= \begin{bmatrix} U_k^* & E_{k+1}^+ \hat{\Pi}_k \end{bmatrix} = \begin{bmatrix} 0 & \hat{\Pi}_k \\ I & 0 \end{bmatrix}, \quad (24) \\ \check{d}_k &= \begin{bmatrix} (d_k^*)^T & (d_k^a)^T \end{bmatrix}^T. \quad (25)\end{aligned}$$

From [17] and using (11) and (24), we have the following relationship:

$$d_k^* = (U_k^*)^+ X_{k+1} = \bar{d}_{k+1} = \bar{\Phi}_{k+1} d_{k+1}. \quad (26)$$

Using (24)-(26) in (23) yields

$$X_{k+1} = E_{k+1}^+ f^*(X_k, u_k) + U_k \check{d}_k + E_{k+1}^+ w_k, \quad (27)$$

where

$$U_k = \begin{bmatrix} 0 & \hat{\Pi}_k \\ \bar{\Phi}_{k+1} & 0 \end{bmatrix}, \quad \check{d}_k = \begin{bmatrix} d_{k+1} \\ d_k^a \end{bmatrix}. \quad (28)$$

Next, direct applying the RTSKF [18] to (22) and (27) and considering the estimation framework in [12], we can readily obtain the following unified filtering algorithm.

Step 1: Unknown input update

$$\begin{aligned} S_k &= \nabla_X g^*(\hat{X}_{k|k-1}, u_k) \cdot U_{k-1}, & (29)\\ P^{\breve{d}}_{k|k} &= \left(S_k^T (P^y_{k|k-1})^{-1} S_k\right)^+, & (30)\\ K^{\breve{d}}_k &= P^{\breve{d}}_{k|k} S_k^T (P^y_{k|k-1})^{-1}, & (31)\\ \breve{d}_{k|k} &= K^{\breve{d}}_k (y_k - \hat{y}_{k|k-1}), & (32) \end{aligned}$$

where ∇_X is the gradient with respect to X,

$$\begin{aligned} \hat{X}_{k|k-1} &= \bar{X}_{k|k-1} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \hat{X}_{k-1|k-1}, & (33)\\ P^y_{k|k-1} &= R_k + E[(g^*(X_k, u_k) - \hat{y}_{k|k-1}) \\ &\quad \times (g^*(X_k, u_k) - \hat{y}_{k|k-1})^T | y^{k-1}], & (34)\\ \hat{y}_{k|k-1} &= E[g^*(X_k, u_k) | y^{k-1}]. & (35) \end{aligned}$$

Step 2: Measurement update

$$\begin{aligned} K_k &= P^{Xy}_{k|k-1} (P^y_{k|k-1})^{-1}, & (36)\\ V_k &= U_{k-1} - K_k S_k, & (37)\\ \hat{X}_{k|k} &= \bar{X}_{k|k-1} + K_k (y_k - \hat{y}_{k|k-1}) + V_k \breve{d}_{k|k}, & (38)\\ P^X_{k|k} &= P^{\bar{X}}_{k|k-1} - K_k P^y_{k|k-1} K_k^T + V_k P^{\breve{d}}_{k|k} V_k^T, & (39) \end{aligned}$$

where

$$\begin{aligned} P^{Xy}_{k|k-1} &= E[(X_k - \bar{X}_{k|k-1}) \\ &\quad \times (g^*(X_k, u_k) - \hat{y}_{k|k-1})^T | y^{k-1}]. & (40) \end{aligned}$$

Step 3: Time update

$$\begin{aligned} \bar{X}_{k+1|k} &= E^+_{k+1} E[f^*(X_k, u_k) | y^k], & (41)\\ P^{\bar{X}}_{k+1|k} &= E[(E^+_{k+1} f^*(X_k, u_k) - \bar{X}_{k+1|k}) \\ &\quad \times (E^+_{k+1} f^*(X_k, u_k) - \bar{X}_{k+1|k})^T | y^k] \\ &\quad + E^+_{k+1} Q_k (E^+_{k+1})^T. & (42) \end{aligned}$$

Finally, under the following rank condition:

$$rank[S_k] = rank[U_{k-1}] = rank[\bar{\Phi}_k] + rank[\hat{\Pi}_{k-1}], \quad (43)$$

the system state estimate $\hat{x}_{k|k}$ and the estimable unknown input estimate $\hat{\bar{d}}_{k|k}$ can be obtained, respectively, as follows:

$$\begin{aligned} \hat{x}_{k|k} &= \begin{bmatrix} I & 0 \end{bmatrix} \hat{X}_{k|k}, & (44)\\ \hat{\bar{d}}_{k|k} &= \begin{bmatrix} 0 & I \end{bmatrix} \hat{X}_{k|k}. & (45) \end{aligned}$$

Moreover, the one time-step delayed estimate of the unknown input can be obtained as follows [18], [19]:

$$\hat{d}_{k-1|k} = \begin{bmatrix} 0 & I \end{bmatrix} \breve{d}_{k|k}. \quad (46)$$

Remark 1. In the special case that the gradient of $g(\bullet)$ with respect to d is of full-column rank [16], one has $\bar{\Phi}_k = I$. For other cases [6], one has $\bar{\Phi}_k = \tilde{H}_k$ where $\tilde{H}_k$ is of full-row rank and is the full-rank factorization of matrix H_k obtained by (8) or its corresponding Stirling's interpolation.

Remark 2. In view of (21)-(22) and the result proposed in [17], the above unified framework can be easily adapted to solve state estimation problem for nonlinear descriptor systems with unknown inputs.

B. A Derivative-Based Estimator Design

A specific estimator design using model derivatives for nonlinear systems with unknown inputs can be facilitated via the proposed unified framework. First, we represent the nonlinear functions $f^*(\bullet)$ and $g^*(\bullet)$, respectively, as follows:

$$\begin{aligned} f^*(X_k, u_k) &= B_k X_k + f_L(X_k, u_k), & (47)\\ g^*(X_k, u_k) &= N_k X_k + g_L(X_k, u_k), & (48) \end{aligned}$$

where

$$\begin{aligned} B_k &= \frac{\partial f^*(X_k, u_k)}{\partial X_k}\Big|_{X_k = \hat{X}_{k|k}}, & (49)\\ N_k &= \frac{\partial g^*(X_k, u_k)}{\partial X_k}\Big|_{X_k = \hat{X}_{k|k-1}}, & (50)\\ f_L(X_k, u_k) &= f^*(X_k, u_k) - B_k X_k, & (51)\\ g_L(X_k, u_k) &= g^*(X_k, u_k) - N_k X_k. & (52) \end{aligned}$$

Second, based on (48) we can implement the matrix S_k given by (29) as follows:

$$S_k = N_k U_{k-1}. \quad (53)$$

Next, we assume that the following nonlinear functions:

$$\begin{aligned} \tilde{f}_L(X_k, u_k) &= f_L(X_k, u_k) - f_L(\hat{X}_{k|k}, u_k), & (54)\\ \tilde{g}_L(X_k, u_k) &= g_L(X_k, u_k) - g_L(\bar{X}_{k|k-1}, u_k), & (55) \end{aligned}$$

are zero mean and uncorrelated with each other, and such that

$$\begin{aligned} E[\tilde{f}_L(X_k, u_k) \tilde{f}_L^T(X_k, u_k)] &\le \rho_f I, & (56)\\ E[\tilde{g}_L(X_k, u_k) \tilde{g}_L^T(X_k, u_k)] &\le \rho_g I, & (57) \end{aligned}$$

for some known positive scalars ρ_f and ρ_g.

Finally, using the well-known matrix inequality [16] $zy^T + yz^T \le \delta yy^T + (1/\delta)zz^T$, where δ is a positive scalar and z and y are matrices of appropriate dimensions, the obtained estimator is given by (29)-(46) with that (34)-(35), (40), and (41)-(42) are replaced, respectively, by

$$\begin{aligned} P^y_{k|k-1} &= R_k + (1+\delta_1) N_k P^{\bar{X}}_{k|k-1} N_k^T \\ &\quad + \rho_g (1 + (1/\delta_1)) I, & (58)\\ \hat{y}_{k|k-1} &= g^*(\bar{X}_{k|k-1}, u_k), & (59)\\ P^{Xy}_{k|k-1} &= P^{\bar{X}}_{k|k-1} N_k^T, & (60)\\ \bar{X}_{k+1|k} &= E^+_{k+1} f^*(\hat{X}_{k|k}, u_k), & (61)\\ P^{\bar{X}}_{k+1|k} &= E^+_{k+1}((1+\delta_2) B_k P^X_{k|k} B_k^T \\ &\quad + \rho_f (1 + (1/\delta_2)) I + Q_k)(E^+_{k+1})^T. & (62) \end{aligned}$$

Remark 3. Using the following special values:

$$\delta_1 = 0, \quad \rho_g = 0, \quad \delta_2 = 0, \quad \rho_f = 0,$$

in (58) and (62) reduces to the corresponding terms in [15].

C. A Derivative-Free Ensemble Implementation

One possible derivative-free ensemble implementation of the aforementioned derivative based estimator is based on the proposed unified framework and the weighted samples determination in [15]. The obtained derivative-free estimator is given by (29)-(46) with that (34)-(35), (40), and (41)-(42) are replaced, respectively, by

$$P^{y}_{k|k-1} = R_k + \alpha^2 \sum_{i=0}^{n_{\mathcal{X}}} W_i (y_{i,k|k-1} - \hat{y}_{k|k-1}) \times (y_{i,k|k-1} - \hat{y}_{k|k-1})^T, \tag{63}$$

$$\hat{y}_{k|k-1} = \sum_{i=0}^{n_{\mathcal{X}}} W_i y_{i,k|k-1}, \tag{64}$$

$$P^{Xy}_{k|k-1} = \alpha^2 \sum_{i=0}^{n_{\mathcal{X}}} W_i (\mathcal{X}_{i,k|k-1} - \bar{X}_{k|k-1}) \times (y_{i,k|k-1} - \hat{y}_{k|k-1})^T, \tag{65}$$

$$\bar{X}_{k+1|k} = \sum_{i=0}^{n_{\mathcal{X}}} W_i \mathcal{X}_{i,k+1|k}, \tag{66}$$

$$P^{\bar{X}}_{k+1|k} = E^{+}_{k+1} Q_k (E^{+}_{k+1})^T + \alpha^2 \sum_{i=0}^{n_{\mathcal{X}}} W_i \times (\mathcal{X}_{i,k+1|k} - \bar{X}_{k+1|k})(\mathcal{X}_{i,k+1|k} - \bar{X}_{k+1|k})^T, \tag{67}$$

with

$$\begin{aligned} y_{i,k|k-1} &= g^*(\mathcal{X}_{i,k|k-1}, u_k), && (68)\\ \mathcal{X}_{i,k|k} &= \hat{X}_{k|k} + \frac{1}{\alpha}\sqrt{\frac{1}{W_i} \cdot P^{X}_{k|k}} \times \pi_i, && (69)\\ \mathcal{X}_{i,k+1|k} &= E^{+}_{k+1} f^*(\mathcal{X}_{i,k|k}, u_k). && (70) \end{aligned}$$

Here, $n_{\mathcal{X}}$ is the number of samples, α is a scaling constant, W_i is the weight associated with the ith component which is constrained as $\sum_{i=0}^{n_{\mathcal{X}}} W_i = 1$, and π_i $(0 \le i \le n_{\mathcal{X}})$ are specific n_X dimensional sampling vectors satisfying the following relationships:

$$\pi_0 = 0, \quad \sum_{i=1}^{n_{\mathcal{X}}} \pi_i \pi_i^T = I_{n_X}, \tag{71}$$

where I_{n_X} is the identity matrix with dimension n_X.

In view of (53) and (60), the matrix S_k given by (29) is obtained as follows:

$$S_k = \left((P^{\bar{X}}_{k|k-1})^{+} P^{Xy}_{k|k-1} \right)^T U_{k-1}. \tag{72}$$

IV. Applications to Rederive the Existing Literature Results

To illustrate the proposed results, in this section we show the applications of using the proposed unified framework to rederive the ERTSF [5], the RTSF [3], and the RTSKF [18].

A. Hsieh (2009)

Consider the following linear system with unknown inputs:

$$\begin{aligned} x_{k+1} &= A_k x_k + G_k d_k + w_k,\\ y_k &= C_k x_k + H_k d_k + v_k, \end{aligned}$$

which can be transformed into (12)-(13) with:

$$\begin{aligned} f^*(x_k, \bar{d}_k, u_k) &= A_k x_k + G_k \tilde{H}^{+}_k \bar{d}_k, && (73)\\ \tilde{f}(x_k, d_k, u_k) &= G_k (I - \tilde{H}^{+}_k \tilde{H}_k) d_k, && (74)\\ g^*(x_k, \bar{d}_k, u_k) &= C_k x_k + \bar{H}_k \bar{d}_k, && (75) \end{aligned}$$

where the full-rank factorization of matrix H_k is employed, i.e., $H_k = \bar{H}_k \tilde{H}_k$, and the following matrix parameters are used: $\bar{\Phi}_k = \tilde{H}_k$ and $\bar{\Psi}_k = \tilde{H}^{+}_k$.

Then, using (73)-(75), (27) and (22), respectively, become

$$\begin{aligned} X_{k+1} &= \begin{bmatrix} [A_k & G_k \tilde{H}^{+}_k] \\ & 0 \end{bmatrix} X_k + U_k \breve{d}_k + \begin{bmatrix} I \\ 0 \end{bmatrix} w_k, && (76)\\ y_k &= \begin{bmatrix} C_k & \bar{H}_k \end{bmatrix} X_k + v_k, && (77) \end{aligned}$$

where

$$U_k = \begin{bmatrix} 0 & \hat{\Pi}_k \\ \tilde{H}_{k+1} & 0 \end{bmatrix}, \quad \hat{\Pi}_k = G_k (I - \tilde{H}^{+}_k \tilde{H}_k). \tag{78}$$

Applying (29)-(42) to (76)-(78), one obtains

Step 1: Unknown input update

$$\begin{aligned} S_k &= \begin{bmatrix} H_k & C_k \hat{\Pi}_{k-1} \end{bmatrix}, && (79)\\ P^{\breve{d}}_{k|k} &= \left(S_k^T (P^{y}_{k|k-1})^{-1} S_k \right)^{+}, && (80)\\ K^{\breve{d}}_k &= P^{\breve{d}}_{k|k} S_k^T (P^{y}_{k|k-1})^{-1}, && (81)\\ \breve{d}_{k|k} &= K^{\breve{d}}_k (y_k - C_k \hat{x}_{k|k-1}), && (82) \end{aligned}$$

where

$$P^{y}_{k|k-1} = C_k P^{x}_{k|k-1} C_k^T + R_k. \tag{83}$$

Step 2: Measurement update

$$\begin{aligned} K_k &= \begin{bmatrix} K^{x}_k \\ 0 \end{bmatrix}, \; K^{x}_k = P^{x}_{k|k-1} C_k^T (P^{y}_{k|k-1})^{-1}, && (84)\\ V_k &= \begin{bmatrix} -K^{x}_k H_k & (I - K^{x}_k C_k)\hat{\Pi}_{k-1} \\ \tilde{H}_k & 0 \end{bmatrix}, && (85)\\ \hat{X}_{k|k} &= \bar{X}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1}) + V_k \breve{d}_{k|k}, && (86)\\ P^{X}_{k|k} &= P^{\bar{X}}_{k|k-1} - K_k P^{y}_{k|k-1} K_k^T + V_k P^{\breve{d}}_{k|k} V_k^T. && (87) \end{aligned}$$

Step 3: Time update

$$\bar{X}_{k+1|k} = \begin{bmatrix} \hat{x}_{k+1|k} \\ 0 \end{bmatrix}, \quad P^{\bar{X}}_{k+1|k} = \begin{bmatrix} P^{x}_{k+1|k} & 0 \\ 0 & 0 \end{bmatrix}, \tag{88}$$

where

$$\hat{x}_{k+1|k} = \begin{bmatrix} A_k & G_k \tilde{H}^{+}_k \end{bmatrix} \hat{X}_{k|k}, \tag{89}$$

$$P^{x}_{k+1|k} = \begin{bmatrix} A_k & G_k \tilde{H}^{+}_k \end{bmatrix} P^{X}_{k|k} \begin{bmatrix} A_k^T \\ (G_k \tilde{H}^{+}_k)^T \end{bmatrix} + Q_k. \tag{90}$$

Finally, using (80)-(82), (84)-(85), and (88)-(90) in (86)-(87) yields the ERTSF, an alternative to the original result in [5], as follows:

$$\hat{X}_{k|k} = \begin{bmatrix} \hat{x}_{k|k-1} + L_k(y_k - C_k\hat{x}_{k|k-1}) \\ M_k(y_k - C_k\hat{x}_{k|k-1}) \end{bmatrix}, \quad (91)$$

$$P^X_{k|k} = \begin{bmatrix} P^x_{k|k} & P^{x\bar{d}}_{k|k} \\ (P^{x\bar{d}}_{k|k})^T & P^{\bar{d}}_{k|k} \end{bmatrix}, \quad (92)$$

where

$$\begin{aligned} L_k &= K^x_k + (\Gamma_k - K^x_k S_k)K^{\breve{d}}_k, & (93)\\ M_k &= \begin{bmatrix} \tilde{H}_k & 0 \end{bmatrix} K^{\breve{d}}_k, & (94)\\ P^x_{k|k} &= P^x_{k|k-1} - L_k P^y_{k|k-1} L^T_k + L_k\Psi_k + (L_k\Psi_k)^T, & (95)\\ P^{x\bar{d}}_{k|k} &= (M_k\Psi_k)^T, & (96)\\ P^{\bar{d}}_{k|k} &= M_k P^y_{k|k-1} M^T_k, & (97) \end{aligned}$$

with

$$\begin{aligned} \Gamma_k &= \begin{bmatrix} 0 & \hat{\Pi}_{k-1} \end{bmatrix}, & (98)\\ \Psi_k &= P^y_{k|k-1} L^T_k - C_k P^x_{k|k-1}. \end{aligned}$$

Based on (43), (78), and (79), the existence condition of the above ERTSF is given as follows:

$$\begin{aligned} rank[S_k] &= rank[\tilde{H}_k] + rank[G_{k-1}(I - \tilde{H}^+_{k-1}\tilde{H}_{k-1})], \\ &= rank[H_k] + rank[G_{k-1}(I - H^+_k H_k)], \end{aligned}$$

which is exactly the unbiasedness condition derived in the original result [5].

B. Gillijns et al. (2007)

In the special case that matrix H_k has full-column rank, one has

$$\bar{\Phi}_k = I, \quad S_k = \begin{bmatrix} H_k & 0 \end{bmatrix}, \quad \Gamma_k = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad (99)$$

$$K^{\breve{d}}_k = \begin{bmatrix} (H^T_k(P^y_{k|k-1})^{-1}H_k)^{-1}H^T_k(P^y_{k|k-1})^{-1} \\ 0 \end{bmatrix}, \quad (100)$$

due to (78)-(81) and (98). It can be easily checked that using (99)-(100) in (91)-(97) yields the recursive three-step filter (RTSF) [3].

C. Hsieh (2000)

For the special case $H_k = 0$, one has

$$\begin{aligned} \bar{\Phi}_k &= 0, \quad S_k = \begin{bmatrix} 0 & C_k G_{k-1} \end{bmatrix}, & (101)\\ M_k &= 0, \quad \Gamma_k = \begin{bmatrix} 0 & G_{k-1} \end{bmatrix}, & (102)\\ K^{\breve{d}}_k &= \begin{bmatrix} 0 \\ (\Lambda^T_k(P^y_{k|k-1})^{-1}\Lambda_k)^{-1}\Lambda^T_k(P^y_{k|k-1})^{-1} \end{bmatrix}, & (103) \end{aligned}$$

where $\Lambda_k = C_k G_{k-1}$. It can be easily checked that using (101)-(103) in (91)-(97) yields the unbiased minimum-variance filter (UMVF) [1]. Moreover, using (82) and (103) we can obtain the unknown input estimate of the RTSKF [18] as given by (46).

V. Conclusion

In this paper, a unified UIF estimator framework (algorithm) is proposed to solve the dual state and unknown inputs estimation problem. Specifically, for the unknown inputs estimates, both the estimable and unestimable parts are considered. Two specific derivative-based and -free UIDNEs are derived as a direct application of applying the proposed unified framework. Through the proposed results, the existing results in Hsieh [5], Gillijns *et al.* [3], and Hsieh [18] are rederived in a unified way.

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