Numerical Solution for a Class of Pursuit-Evasion Problem in Low Earth Orbit

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***Abstract*—A two spacecraft pursuit-evasion problem in a low earth orbit with two payoffs, is investigated by an integrated approach using the semi-direct nonlinear programming and the multiple shooting method. The problem is formulated by a zero-sum differential game. The miss distance at a fixed terminal time and the capture time are defined as the payoffs. The pursuer strives to minimize the payoff while the evader attempts to maximize it. Semi-direct nonlinear programming serves as a preprocessor in which control is parameterized in piecewise form. Its solution is then used as the initial values for the multiple shooting method and thus a refined solution is obtained for a two-point boundary-value problem arising from the necessary conditions. The optimal trajectory and optimal control using the semi-direct nonlinear programming and the multiple shooting method are computed and compared. Numerical equivalence of the semi-direct method and the hybrid method with respect to the differential game is evidenced by a realisti- cally modeled pursuit-evasion test case. This proposed integrated approach is shown to be robust, accurate and more efficient than using only a single method.**

I. Introduction

Generally, comprehensive analysis of the pursuit-evasion problem of spacecraft is a challenging work, since a number of difficulties are induced using a two-sided problem in which two players possess independent control within a complex dynamical system. Conflict between an optimal evading spacecraft and an optimal pursuing spacecraft in which both parties have independent control is best modeled by using two competing players. This problem can be described by a zero-sum two-player differential game. This type of game was first introduced by Isaacs in 1965 [1] and pioneer researchers Friedman [2] and Berkovitz [3] rigorously derived necessary conditions and the existence of an open-loop representation of a saddle-point trajectory.

A two-point boundary-value problem (TPBVP) including costates and boundaries of two adversaries can be raised by the necessary condition of differential game on a low earth orbit. Since the number of differential equations of TPBVP is two times that of the state equations and the equations are nonlinear with complex boundary conditions, the numerical solution is the only choice. For this type of boundary problem, collocation (e.g. Dickmanns and Well [4]) or multiple shooting method (e.g. Stoer and Bulirsch [5]) may be used to derive solution to the boundary problem. Contrary to collocation method, multiple shooting method which is also called indirect method, has the advantage that many constraints are allowed and very accurate results can be obtained. However, the multiple shooting method is more sensitive to initial values of adjoint variables and state variables. Good initial values for the saddle trajectories of two spacecraft are required, and considerable work with respect to boundary structure has to be done by the user. In order to derive the adjoint differential equations, the user must have deep insight into the physical and mathematical nature of the optimization problem (e.g. Breitner, Pesch and Grimm [6], [7]).

For a general optimal control problem, "direct" approaches can be presented without solving the TPBVP such as a collocation nonlinear programming method (e.g. Herman and Conway [8]) and a piecewise nonlinear programming method (e.g. Goh and Teo [9]). However, since these methods are characteristic of one player, direct methods cannot be directly applied to solve the differential game problem. In this regard, the concept of "semi-direct" collocation nonlinear programming method was first presented by Horie and Conway [10] to solve the differential game. The primary advantage of the semi-direct approach is that it has more convergent property than the multiple shooting method. The disadvantages are: (1) several minima occur for the discretized problems; (2) the solution is dependent on the genetic algorithm.

Taking into account pros and cons from previous research, the goal of this paper is to overcome the disadvantages by combining the indirect with the semi-direct method.

This paper is organized as follows. Section II describes the pursuit-evasion problem on a low earth orbit. Section III explains the semi-direct method and approach to initial value determination. Section IV presents the hybrid method and the philosophy behind it. Section V provides the numerical results for two payoffs. Finally, the conclusions emerge in Section VI.

II. Problem definition

This study explores an optimal interception of an optimally evading spacecraft by a pursuing spacecraft with two kinds of payoff. Both spacecraft have low-thrust capability to perform their respective maneuvers. In describing the pursuit-evasion of spacecraft, the following are assumed:

1) Pursuit-evasion occurs in a low earth orbit and a low, constant, thrust-to-mass ratio is assumed for both spacecraft.
2) The two spacecraft start maneuvering simultaneously at initial time $t_0 = 0$, and each of them is assumed to

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possess complete and instantaneous information on the state of the opponent player.

Realistically, the evader is unlikely to possess such complete information, however, optimal evasion is needed under this worst case scenario.

Fig. 1 shows the relationship of the two players in a rotating coordinate system. The reference orbit coordinate is described

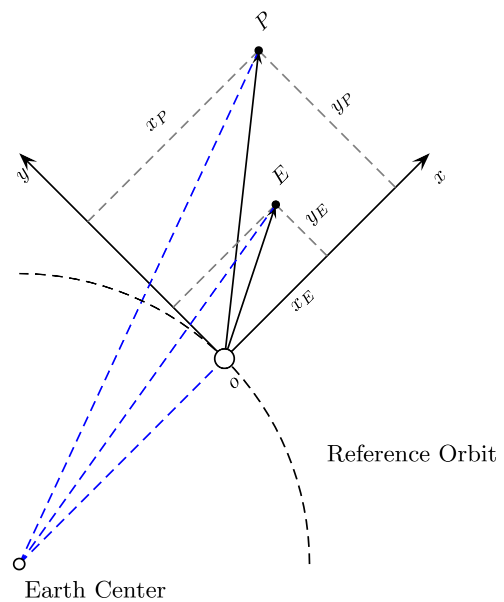


Fig. 1. Relation of pursuing spacecraft, evading spacecraft and origin.

by x, y, z, where x is the direction from earth center to a rotating reference point defined as the origin of coordinate on reference orbit, and y is perpendicular to x along the direction of the origin's velocity on the orbital plane and z corresponds to the right-hand rule with x and y. The subscript i represents the pursuing spacecraft P and the evading spacecraft E, respectively. The second derivative of state for each spacecraft is denoted by $\ddot{x}_i$, $\ddot{y}_i$ and $\ddot{z}_i$. Under the problem definition, the motion of two spacecraft is described by the equations as follows.

$$\begin{cases} \ddot{x}_i = 2\dfrac{\mu}{r^3(t)}x_i + 2\omega(t)\dot{y}_i + \dot{\omega}(t)y_i + \omega^2 x_i + \dfrac{T_i(t)}{m_i}u_{ix} \\ \ddot{y}_i = -\dfrac{\mu}{r^3(t)}y_i - 2\omega(t)\dot{x}_i - \dot{\omega}(t)x_i + \omega^2 y_i + \dfrac{T_i(t)}{m_i}u_{iy} \\ \ddot{z}_i = -\omega^2 z_i + \dfrac{T_i(t)}{m_i}u_{iz} \qquad\qquad i = P, E \end{cases} \tag{1}$$

These equations of motion are known as C-W equations, where $\frac{T_i}{m_i}$ represents the thrust-to-mass ratio (constant) for each spacecraft. Eq. (1) includes the angular velocity ω with regard to the origin, earth planetary constant μ and the distance r from earth center to the origin. The control is performed with the thrust direction on coordinate axis, defined by u_{ix}, u_{iy} and u_{iz} and can be designed to solve the problem by imposing the bounds with $i = P$ or E:

$$U_i = (u_{ix}, u_{iy}, u_{iz}), \quad \|U_i\| \le 1 \tag{2}$$

In Eq. (2), $\|\cdot\|$ is the Euclidean norm. Let X_i defines the state variable with respect to P and E, respectively,

$$X_i(t) = (x_i(t), y_i(t), z_i(t), \dot{x}_i(t), \dot{y}_i(t), \dot{z}_i(t))^T \quad (i = P, E) \tag{3}$$

where $\dot{x}_i$, $\dot{y}_i$ and $\dot{z}_i$ denote the first derivative of x_i, y_i, and z_i with respect to time t. Then the kinetic equations are written as:

$$\dot{X}_i = f_i(X_i, u_i) \tag{4}$$

Here, considering the following pursuit-evasion problem with two payoffs.

Problem 1 *Under the dynamical system* (1)*, pursuer P and evader E choose the control variables to maximize and minimize the miss distance at a fixed time, then the problem is*

$$\begin{aligned} &u_P^* \in U_P, \qquad u_E^* \in U_E \\ &J(u_P^*, u_E^*) = \min_{u_P}\max_{u_E} \frac{1}{2}\|D\cdot[X_P(t_f) - X_E(t_f)]\|^2 \end{aligned} \tag{5}$$

where

$$D = \begin{bmatrix} E_3 & 0 \end{bmatrix}_{6\times 3} \tag{6}$$

Here, E_3 is the identity matrix of the third order. The next problem is closely related to the Problem 1.

Problem 2 *The condition of this problem is analogous to the Problem 1, but the payoff is capture time, then the problem can be described by*

$$\begin{aligned} &u_P^* \in U_P, \qquad u_E^* \in U_E \\ &(X_P(t_0), X_E(t_0), t_0) \to \theta \\ &J(u_P^*, u_E^*) = \min_{u_P}\max_{u_E} t_f \end{aligned} \tag{7}$$

where

$$\theta = \{(X_P, X_E, t) : \|D\cdot[X_p(t_f) - X_E(t_f)]\| \le l,\, l \ge 0\} \tag{8}$$

In Problem 1 the game ends when the pursuer reaches the fixed terminal $t = t_f$, and the terminal distance is payoff. The pursuit-evasion problem stated as Problem 1 is called game of fixed duration [2]. In Problem 2 the set θ represents a terminal set with constant $l \ge 0$. Namely when a fixed initial value $(X_P(t_0), X_E(t0), t_0)$ enters θ along the optimal trajectory, then the game described by Problem 2 ends. Problem 2 is also called game of pursuit-evasion [2]. According to Berkovitz [3], the necessary conditions of an open-loop representation of a optimal solution can be provided; to state these conditions, a Hamiltonian H is introduced as:

$$H = \lambda_P^T f_P + \lambda_E^T f_E \tag{9}$$

where λ_P and λ_E are the adjoint variables of the pursuer and the evader, respectively. Therefore, the necessary conditions for an open-loop representation of an optimal saddle-point solution include the adjoint equations for the Lagrange multipliers λ_P and λ_E :

$$\dot{\lambda}_P = -\left(\frac{\partial H}{\partial X_P}\right)^T - \left(\frac{\partial H}{\partial u_P}\right)\left(\frac{\partial u_P}{\partial X_P}\right)^T \tag{10a}$$

$$\dot{\lambda}_E = -\left(\frac{\partial H}{\partial X_E}\right)^T - \left(\frac{\partial H}{\partial u_E}\right)\left(\frac{\partial u_E}{\partial X_E}\right)^T \tag{10b}$$

with the respective boundary conditions at t_f:

$$\lambda_P(t_f) = \frac{\partial \Psi}{\partial X_P(t_f)} \tag{11a}$$

$$\lambda_E(t_f) = \frac{\partial \Psi}{\partial X_E(t_f)} \tag{11b}$$

Here, Ψ represents the distance payoff in Problem 1 and terminal set in Problem 2, respectively. In addition, a saddle-point solution must satisfy the following pair of equations:

$$u_P^* = \arg\min_{u_P \in U_P} H = \arg\min_{u_P \in U_P} \lambda^T \cdot f_P \tag{12a}$$

$$u_E^* = \arg\max_{u_E \in U_E} H = \arg\max_{u_E \in U_E} \lambda^T \cdot f_E \tag{12b}$$

Eqs (1), (10), (11), (12) consist of a TPVBP. A hybrid method is presented for this kind of pursuit-evasion problem.

III. Semi-Direct Method

In the following, the piecewise nonlinear programming method from Goh and Teo [9] is adopted, since the method can obtain one-one costates to the indirect method, the piecewise method has less burden work of computation, and it properly represents the piecewise thrust in real situation. In order to use the semi-direct algorithm, the transformation from a two-side problem to an optimal control problem (one-sided) is introduced by Horie [11]. The control for the evader requires a transformation to a function of X_E, λ_E, t namely $u_E = u_E(X_E, \lambda_E, t)$. Then, an extended state vector introduced by Conway [10], $\tilde{X}$, is defined after inserting the adjoint variable λ_E:

$$\tilde{X} = \left[X_P^T, X_E^T, \lambda_E^T\right]^T \tag{13}$$

The corresponding dynamic equation including Eq. (10b) is given by

$$\dot{\tilde{X}}(t) = \tilde{f} = \left[f_P^T, f_E^T\left(X_E, u_E(X_E, \lambda_E, t)\right), \dot{\lambda}_E^T\right]^T \tag{14}$$

where the extended control variable $\tilde{u}$ is u_P only. Since Eq. (11b) provides the boundary condition for the adjoint variable λ_E, each of these equations can be collected in the extended boundaries Ψ:

$$\tilde{\Psi} = \begin{bmatrix} \Psi & \lambda_E(t_f) - \dfrac{\partial \Psi}{\partial X_E(t_f)} \end{bmatrix} \tag{15}$$

So far, the zero-sum game has been converted into a non-classical optimal control problem. The numerical optimizer can be used to solve the problem.

$$\min_{\tilde{u}(t)} \dot{\tilde{X}} = \tilde{f} \qquad \text{subject to} : \tilde{\Psi} = 0 \tag{16}$$

Nevertheless, in Eq. (16) initial value for some components' for the extended variable $\tilde{x}$ corresponding with the inserting adjoint λ_E is yet unknown.

When the nonlinear programming method is applied to solve optimal control problems, the method of solution requires discretization of the continuous variables. So the time interval is partitioned into N subarcs : $[t_{i-1}, t_i]_{i=1,\cdots,N}$ with $t_N = t_f$. Essentially, each control component $\tilde{u}_k(t)$ is approximated by a piecewise constant function defined upon a set of knots $\left\{t_j^k = 1, 2, \cdots k_i, t_1^k = t_0, t_i^k = t_f\right\}$, which could be unequally spaced and different for each control. Here, t_j^k which represents time on the knot t_j with $j = 1, \cdots, k_i$ can be independent of t_i with $i = 0, \cdots, N$. The $k - th$ control as a sum of basic functions with coefficients or parameters $\left\{\sigma_j^k, j = 1, \cdots, k_i\right\}$ can formally be written as:

$$\tilde{u}_k(t) = \sum_{j=0}^{k_i} \sigma_{kj} \chi_{kj}(t) \tag{17}$$

where $\chi_{kj}(t)$ is, for piecewise constant control, the characteristic function for the $j-th$ interval of the $k-th$ set of knots, defined by

$$\chi_{kj}(t) = \begin{cases} 1, & t \in \left[t_{j-1}^k, t_j^k\right) \\ 0, & t \in X - \left[t_{j-1}^k, t_j^k\right) \end{cases} \tag{18}$$

the problem discretized with respect to states and control variables consists of a mathematical programming problem which is solved by sequential quadratic programming algorithm, and through the use of an NLP solver, for example, NLPQL by Schittkowski [12].

From Eq. (14), the initial value of λ_E and $\tilde{u}$ are needed for the solution of (16). An initial guess for this optimal control problem, which is from the solution of the TPBVP in Section II, is supplied by NSGA-II (e.g. Deb.K [13]), since the genetic algorithm (GA) code is due to its robustness in searching for a feasible set with a relevant number of constraints [14].

IV. The Hybrid Method

It would be desirable to combine the good convergence properties of the semi-direct method with the reliability and accuracy of the multiple shooting method. However, a proper choice of multiple shooting nodes, initial values of the adjoint variables λ_P and λ_E are needed in advance. As for the combination of the methods, the grid points of the semi-direct method, yield a good choice for the multiple shooting nodes, and there is an additional advantage, reliable estimates for the adjoint variables can be obtained from costates in output of the NLP solver, as the costate solution of Eq. (16) are explicitly one-one to the adjoint variables according to the literature [9]. This methodological approach produces an accurate and robust solution.

The multiple shooting method is characterized by stability, efficiency, and accuracy. It is modified for solving the boundary value problem, based on single shooting and is a reliable method for the numerical solution of boundary-value problems. In this problem, solutions $x(t)$ and $\lambda(t)$ of a system of 2n ordinary differential equations can be achieved through,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{pmatrix} = G(x(t), \lambda(t)) = \begin{cases} G_1(t, x(t), \lambda) \\ G_2(t, x(t), \lambda) \\ \vdots \\ G_{2n}(t, x(t), \lambda) \end{cases} \tag{19a}$$

$$r_i(x(t_0), x(t_f), \lambda(t_0), \lambda(t_f)) = 0, \quad 1 \le i \le n_b \tag{19b}$$

The idea of multiple shooting is to choose a grid of so-called multiple shooting nodes on fixed subdivision [5] of the interval $[t_0, t_f]$ with $\hat{t}_j$, $j = 1 \cdots n_m$.

$$t_0 = \hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_{n_m} = t_f$$

The multiple shooting method is based on reducing the solution of the boundary-value problem to the solution of a series of initial-value problems. However, the initial values for the variables $x(t)$ and $\lambda(t)$ at the nodes $\hat{t}_j$ are determined by a guess. Let Z_j denote an initial guess for the vectors $\left(x^T(\hat{t}_j), \lambda^T(\hat{t}_j)\right)^T$, for $j = 1, \cdots, n_m$. Thus, for $j = 1, \cdots, n_m - 1$, the numerical solutions of the initial-value problems are:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{pmatrix} = G(x(t), \lambda(t)), \quad \hat{t}_j \le t \le \hat{t}_{j+1} \tag{20a}$$

$$\left(x^T(\hat{t}_j), \lambda^T(\hat{t}_j)\right)^T = Z_j \tag{20b}$$

$x(t)$ and $\lambda(t)$ are solutions to the above TPBVP, if and only if the vector $\bar{Z} = (\bar{Z}_1, \cdots, \bar{Z}_{n_m-1})^T$ is zero as function $\mathscr{G}$

$$\mathscr{G}(\bar{Z}) = \begin{pmatrix} \mathscr{G}_1 & \cdots \mathscr{G}_{n_m-1} \end{pmatrix}^T = 0 \tag{21}$$

Here, the components of $\mathscr{G}$ include the matching conditions

$$\begin{aligned} \mathscr{G}_j(\bar{Z}_j, \bar{Z}_{j+1}) = \left(x^T(\hat{t}_{j+1}; \hat{t}_j, \bar{Z}_j), \lambda^T(\hat{t}_{j+1}; \hat{t}_j, \bar{Z}_j)\right)^T \\ - \bar{Z}_{j+1} \qquad\qquad (1 \le j \le n_m - 2) \end{aligned} \tag{22}$$

and the boundary conditions

$$\mathscr{G}_{n_m-1}(\bar{Z}_1, \bar{Z}_{n_m-1}) = R(\bar{Z}_1, \bar{Z}_{n_m-1}) \tag{23}$$

with

$$R(\bar{Z}_1, \bar{Z}_{n_m-1}) = [r_i(\bar{Z}_1, z(\hat{t}_{n_m}; \hat{t}_{n_m-1}, \bar{Z}_{n_m-1}))]_{i=1,\cdots,n_b}$$

A zero of the above system of nonlinear equations is determined by the modified Newton method. The multiple shooting algorithm described above is implemented in the FORTRAN code BNDSCO [15]. The integration method for the numerical solution of the initial-value problem from Eq. (20) is so-called Gragg-Bulirsch-Stoer extrapolation method [5].

V. Numerical Results

According to Eq. (1), the in-plane motion (i.e., the $x - y$ subsystem) and the out-of-plane motion (i.e., the z subsystem) can be considered separately since they are independent. In this regard, the pursuit and evasion occur on the co-planar low earth orbit, and the derivative of angular velocity at the origin vanishes (i.e., $\dot{\omega} = 0$). In this research, normalization is used in the numerical solution because it reduces the influence of minor parameters in the dynamic systems. The essential parameters are set for all the test cases as follows: (1) The r_0 is assumed as the distance unit, where the r_0 is the summation of the Earth radius $R_E = 6371.004$ km and initial altitude of the reference origin. (2) The terminal time t_f is such that the normalized time is in the interval $[0, 1]$. An initial altitude of 500 km for the reference origin is used in test case. The following values for the thrust-to-mass ratios are assumed for the test case with $g = 9.8 \times 10^3\,\text{km/s}^2$:

$$\frac{T_P}{m_P} = 0.033g \quad \frac{T_E}{m_E} = 0.01g \tag{24}$$

The initial values of the state components normalized at initial time are listed in Table I.

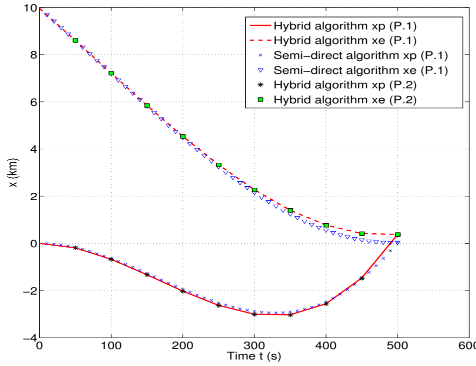
TABLE I
INITIAL STATE NORMALIZED ON THE ALTITUDE OF 500KM

$i = P$ or E	Pursuring spacecraft	Evading spacecraft
$\tilde{x}_i$	0	1.449×10^{-3}
$\tilde{y}_i$	0	3.501×10^{-3}
$\dot{\tilde{x}}_i$	0	-1.940×10^{-3}
$\dot{\tilde{y}}_i$	0	-4.060×10^{-4}
$\frac{\tilde{T}_i}{m_i}$	0.0118	0.00036

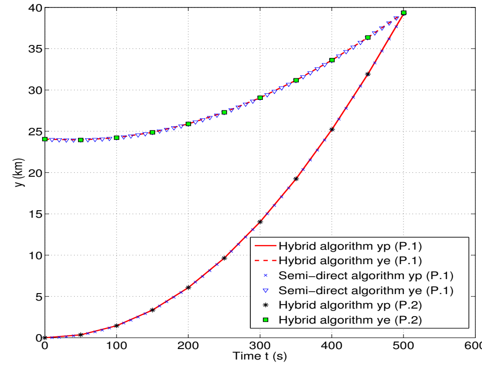
As mentioned in Section III and IV, the result provided by the GA was then used by the semi-direct algorithm, whose results were further used as the initial values for multiple shooting method. In other words, the output of the solver for the semi-direct regarding the states and costate served as the initial values of multiple shooting method for the TPBVP. The simulation results for Problem 1 and Problem 2 can be obtain by the hybrid method, respectively, as described in Section IV. By considering the close relationship of Problem 1 and Problem 2, when the result of Problem 1 is solved by the hybrid, the result of Problem 2 is obtained by homotopy method, and vice versa.

Figure 2 shows the simulation results with respect to the hybrid algorithm and homotopy. In Fig. 2, the pursuer can approach the evader in 500 second, and the trajectory of the semi-direct algorithm is close to that of the hybrid algorithm via multiple shooting, and some state components corresponding to the saddle-point solution are illustrated in Fig. 2: x_i and y_i , $(i = P, E)$. The solution of Problem 2 is qualitatively similar to the Problem 1, here $l = 0.1/r_0$ distance unit, the payoff J is 500.3947 second.

Figure 3 illustrates control laws of semi-direct and the hybrid for both spacecraft. As far as the control is concerned in

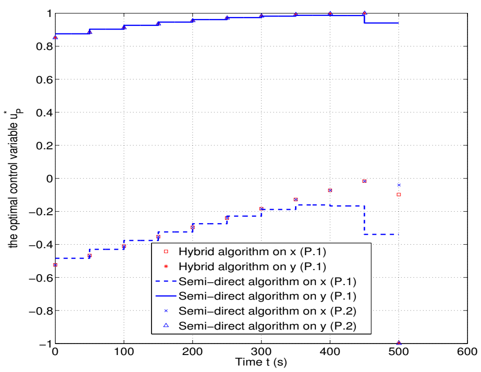


(a) x_i time history of the semi-direct and the hybrid method

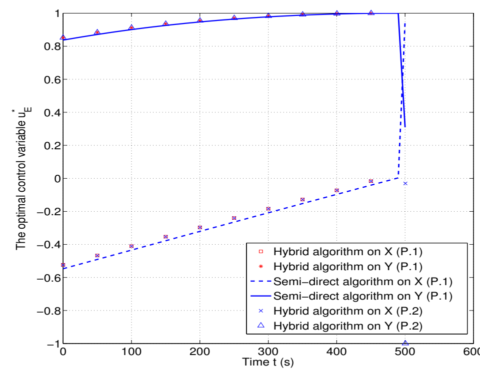


(b) y_i time history of the semi-direct and the hybrid method

Fig. 2. The time history of trajectory of P and E on Problem 1 and Problem 2

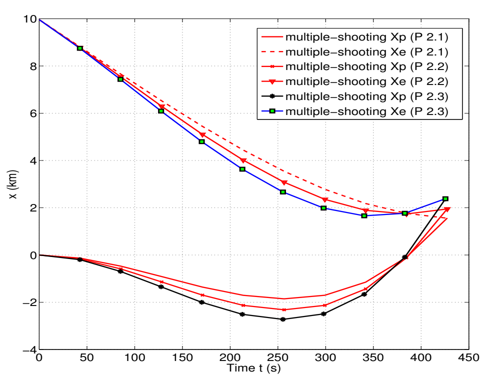


(a) Time history of pursuer's control on x regarding P and E

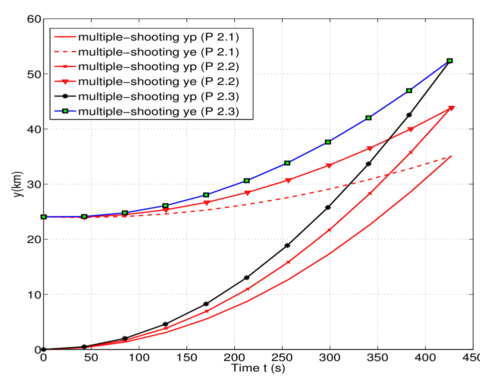


(b) Time history of evader's control on y regarding P and E

Fig. 3. The time history of the optimal control



(a) x_i time history of the hybrid method



(b) y_i time history of the hybrid method

Fig. 4. The trajectories with respect to different thrust-to-mass ratio

Fig. 3, the optimal solution of semi-direct algorithm displays exactly the same behavior as the optimal control of the hybrid method in Problem 1. It is noteworthy that there is a jump point at the endpoint of the final interval. As for the jump point, it is not the essential of the pursuit-evasion problem, but that costates tend to zero at the terminal time which makes the optimal control singular. By comparison with Fig. 3a and Fig. 3b, the pursuer and evader have identically similar strategies in the process of pursuit-evasion.

For the convergence of the solution with respect to the semi-direct and the hybrid method, the constraint errors for boundary conditions are shown in Table II. It is shown that the errors of semi-direct is on the order of 10^{-7} and 10^{-9} , and those of the hybrid method is on the order of 10^{-13}.

TABLE II
CONSTRAINT ERROR OF THE SEMI-DIRCT AND THE HYBRID METHOD

	Constraint Errors	
	Semi-direct method	Hybrid method
Problem 1	4.7750×10^{-9}	4.5826×10^{-13}
Problem 2	2.6738×10^{-7}	2.0976×10^{-13}

In addition, the game of pursuit-evasion of different thrust-to-mass ratios with the same initial value can be obtained by the homotopy method as shown in Table III. For all the cases in Table III, the sum of constraint error of terminal conditions and grid nodes are on the order of 10^{-14} , and the corresponding trajectories are illustrated in Fig. 4.

TABLE III
CONSTRAINT EERRORS OF THE HYBRID METHOD

Problem 2	Thrust-to-mass ratio	Constraint errors
(P. 2.1)	$\frac{T_P}{m_P} = 0.04g$ $\frac{T_E}{m_E} = 0.01g$	8.3666×10^{-14}
(P. 2.2)	$\frac{T_P}{m_P} = 0.05g$ $\frac{T_E}{m_E} = 0.02g$	1.7029×10^{-14}
(P. 2.3)	$\frac{T_P}{m_P} = 0.06g$ $\frac{T_E}{m_E} = 0.03g$	5.8310×10^{-14}

In Table III, The differences of the thrust-to mass ratios of two spacecraft are $0.03g$, from the Fig. 4 corresponding to the Table, It is shown that the capture time is almost identical.

VI. CONCLUSION

In this paper, a numerical solution combining the semi-direct and indirect method is developed for the two spacecraft pursuit-evasion problem with two payoffs in a low earth orbit. The hybrid numerical approach is adopted for a two-point boundary-value problem arising from the necessary conditions of the differential game for which a high accurate solution is obtained. Under the assumed conditions, if the difference of thrust-to-mass ratio of two spacecraft is constant with the same initial values of state variables, then the capture time of the pursuit-evasion is almost invariant. With the semi-direct nonlinear programming algorithm, robustness of the solution from multiple shooting method can be improved such that a final refined solution is achieved by the method. By comparing the results of the hybrid approach with that of the semi-direct nonlinear programming algorithm, it is shown that the two numerical algorithms are approximately equivalent. According to the test case, the proposed hybrid approach exhibits both robustness and high accuracy. A feasible scheme for solving the two-point boundary-value problem using the hybrid approach is evidenced by the test cases with two different payoffs. This further suggests that the combined methodology can be applied to a more realistic dynamic model.

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