Approximate Solutions to the Hamilton-Jacobi Equations for Generating Functions: the General Cost Function Case

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***Abstract*—Recently, the method based on generating functions is proposed for nonlinear optimal control problems. For a finite time optimal control problem with given boundary condition, once a generating function for a fixed boundary condition is obtained, any optimal trajectory of the same system for different boundary conditions can be generated easily. An algorithm to compute an approximate solution to the Hamilton-Jacobi equation with respect to the generating function for a nonlinear optimal control problem is developed in this paper. Numerical examples illustrate the effectiveness of the proposed method.**

I. Introduction

The Hamilton-Jacobi equation (HJE) plays a fundamental role in the analysis and control of nonlinear systems. For an optimal control problem, the conventional method is solving HJE for the value function. Since it is difficult to find an exact value function for a HJE, several numerical methods have been proposed to find it, e.g., Galerkin's spectral method [1], Chebyshev polynomials approximation method [2], the shooting methods [3] and so on. However, since the basic principle of the shooting method is computing the trajectory repeatedly so that the exact one satisfying the given boundary values is obtained, we need to solve the HJE for the value function again for different boundary conditions. Similarly to the shooting method, we must solve the nonlinear optimal control problem separately for each set of initial and/or terminal state values if we use the conventional methods. In addition, all of them are iterative methods. Therefore, online computation of the HJE for a value function for different boundary condition imposes a heavy computational burden for online optimal trajectory generation problems.

A lot of research have been done to obtain a series solution to the HJE by solving a sequence of ODEs recursively [4]–[7] and so on. An approximate stabilizing solution of the HJE is obtained by using symplectic geometry and the Hamiltonian perturbation technique for feedback optimal control problems [8]. However, all of these methods are for the value function. In particular, papers [4], [5], [7], [8] are for the infinite time optimal control problems. For a finite time nonlinear optimal control problem with fixed initial and terminal state values, it is reduced to a Two-Point Boundary-Value Problem (TPBVP) for an ordinary differential equation (ODE) with respect to a Hamiltonian system [9]. Recently, Guibout and Scheeres solved a TPBVPs for conservative systems by solving the HJE for generating functions [10], whose solution is applied to spacecraft formation flight transfers. Park et al. clarified the special nature of the relationship between the optimal cost function of the optimal control problem and the generating function [11], suggesting that a family of value functions can be derived from a generating function. Therefore, if we solve the HJE for a generating function, it is easy to achieve a family of optimal trajectories for the optimal control problems for several different boundary conditions. The generating function for the HJE can be found by taking the infimum of the optimal cost function, and it can be effectively solved by the Galerkin spectral method with Chebyshev polynomials [12]. However, this method requires that the Hamiltonian function of the nonlinear optimal control problem has a special form.

Inspired by [10] and [5], the authors have proposed an algorithm to calculate an approximate solution to the HJE for the generating function for the nonlinear optimal control problems with a quadratic cost function with respect to the input [13]. In this paper, this method is extended for the nonlinear optimal control problems with a general cost function. For a nonlinear finite time optimal control problem with a set of boundary conditions, a series solution to the HJE for the generating function is calculated approximately with arbitrary accuracy by solving a sequence of first order ODEs recursively. Particularly, for a certain class of cost functions, the exact coefficients of the Taylor series of the generating function can be obtained. Furthermore, numerical examples illustrate the effectiveness of the proposed method.

We introduce the problem to be solved in Section 2. Section 3 reviews how to derive the the solution of the HJEs from an optimal control problem. Section 4 proposes the algorithm to solve the HJEs in detail. Numerical examples illustrate the effectiveness of the proposed method in Section 5.

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II. Problem statement

A nonlinear optimal control problem to be solved and how it is rendered to a TPBVP for ODEs as in [14] are elaborated in this section. Consider a system

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x(t) \in \Omega \subset \mathbb{R}^n$ is the state, Ω is a neighborhood of the origin in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n, g : \Omega \to \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^m$ is the input. Define a cost function

$$J(x_0, u) = \int_{t_0}^{t_f} (q(x,t) + \frac{1}{2}u^{\mathrm{T}}R(x,t)u)\mathrm{d}t, \tag{2}$$

where $q(x,t)$ is a non-negative function and $R(x,t) > 0$ is a continuous real matrix function of dimension $m \times m$. The objective is to find an optimal control input u^* minimizing the cost function J as

$$u^* = \arg\min_u J(x_0, u) \tag{3}$$

subject to the boundary condition

$$x(t_0) = x_0, \qquad x(t_f) = x_{t_f}. \tag{4}$$

Here, the symbols $x_0 \in \Omega$ and $x_{t_f} \in \Omega$ are the fixed initial value and the terminal value of the state.

Let us introduce a column vector $\lambda \in \mathbb{R}^n$ to play a role of costate, according to Pontryagin's minimum principle, the necessary conditions for minimization of the performance index (2) are

$$\dot{x} = H_\lambda(x,\lambda,t), \qquad \dot{\lambda} = -H_x(x,\lambda,t), \tag{5}$$

$$u^* = -R^{-1}(x,t)g(x)^{\mathrm{T}}\lambda, \tag{6}$$

where $H_{(\cdot)}$ denotes the partial derivative $\partial H/\partial(\cdot)$, and $H(\cdot)$ is the Hamiltonian function which is defined as follows:

$$H(x,\lambda,t) = q(x,t) + \lambda^{\mathrm{T}} f(x) - \frac{1}{2}\lambda^{\mathrm{T}} g(x)R(x,t)^{-1}g(x)^{\mathrm{T}}\lambda. \tag{7}$$

The boundary value is given as in (4).

Therefore, the optimal control problem (1)-(4) is equivalent to the TPBVP for ODEs (5). Since it is difficult to solve the TPBVP, we will use the generating function technique to obtain an approximation of its solution.

III. Canonical transformations and Hamilton-Jacobi equations

This section explains that a TPBVP for a optimal control problem is characterized by the HJEs for the generating functions. Given the Hamiltonian system (5), we can treat state-costate $(x(t),\lambda(t))$ as a canonical transformation from the initial state-costate (x_0,λ_0) to the current state-costate (x,λ). Thus, there exist a generating functions $S(x,\lambda_0,t)$ for the following canonical transformations [15], [10]:

$$\lambda = \frac{\partial S(x,\lambda_0,t)}{\partial x}, \qquad x_0 = \frac{\partial S(x,\lambda_0,t)}{\partial \lambda_0}, \tag{8}$$

and $S(x,\lambda_0,t)$ satisfies its own HJE

$$\frac{\partial S(x,\lambda_0,t)}{\partial t} + H(x, \frac{\partial S}{\partial x}) = 0. \tag{9}$$

The generating functions $S(x,\lambda_0,t)$ is well-defined at initial time $t = t_0$ for (9) as follows due to (8):

$$S(x,\lambda_0,t_0) = {\lambda_0}^{\mathrm{T}} x_0. \tag{10}$$

If we can find a function $S(x,\lambda_0,t)$ satisfying the HJE (9), since x_0 and x_{t_f} are given in advance, we can obtain λ_0 by solving the second equation in the canonical transformations (8) at the terminal time $t = t_f$, which is

$$x_0 = \left.\frac{\partial S(x,\lambda_0,t)}{\partial \lambda_0}\right|_{t=t_f}. \tag{11}$$

Then it is easy to obtain the costate $\lambda(x)$ by the first equation in the canonical transformations (8). Finally, we can obtain the optimal control $u^*(x)$ by (6). Therefore, we will use the generating function $S(x,\lambda_0,t)$ to solve the TPBVP for the ODEs (5). There are the generating function in the other form, for example, $S_1(x,\hat{x},t)$, $S_3(\lambda,\hat{x},t)$, and $S_4(\lambda,\hat{\lambda},t)$ (the above function $S(x_{t_f},\lambda_0,t_f)$ is called $S_2(x_{t_f},\lambda_0,t_f)$). However, we can not derive the costate λ directly from S_3 or S_4 using their own canonical transformations like in (8). Although there are canonical transformations like in (8) for S_1, it is not defined well at the initial time [10].

IV. Solutions to Hamilton-Jacobi equations

The method solving the HJE (9) for the generating function $S(x,\lambda_0,t)$ is proposed in this section. First, like Einstein notation, when an index variable appears twice in a single term, it implies that we are summing over all of its possible values. For example, $a_{i_1i_2i_3}x_{i_1}y_{i_2}z_{i_3}$ exactly means

$$a_{i_1i_2i_3}x_{i_1}y_{i_2}z_{i_3} = \sum_{i_1}\sum_{i_2}\sum_{i_3} a_{i_1i_2i_3}x_{i_1}y_{i_2}z_{i_3}. \tag{12}$$

The state $x \in \mathbb{R}^n$ and the initial value of the the costate $\lambda_0 \in \mathbb{R}^n$ are written as $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $\lambda_0 = (\lambda_{01}, \lambda_{02}, \cdots, \lambda_{0n})^{\mathrm{T}}$ respectively. To simplify the notation, we define

$$X \overset{\triangle}{=} (x^{\mathrm{T}}, {\lambda_0}^{\mathrm{T}})^{\mathrm{T}}. \tag{13}$$

We need the following assumption to guarantee that the HJE (9) has a solution $S(x,\lambda_0,t)$ and it can be expanded in the Taylor series up to any order about $X = 0$.

Assumption 1: A function $S(x,\lambda_0,t)$ satisfying the HJE (9) exists and is unique. It is well defined for all $t \geq t_0$, and analytic w.r.t. x and λ_0 at the origin in $\mathbb{R}^{2n}$.

A generating function is possibly singular at a certain time indeed. When it is singular at a certain time, it can be transformed to another generating function employing the idea in [10]. For example, it can be transformed to $S_1(x,x_0,t_0,t)$. According to Assumption 1, expanding the generating function $S(x,\lambda_0,t)$ in the Taylor series expansion w.r.t. X about $X = 0$ as

$$S(x,\lambda_0,t) = \sum_{v=0}^{\infty}\sum_{w=0}^{v} s^{v,w}_{i_1i_2\cdots i_{v-w}j_1j_2\cdots j_w}(t)\cdot x_{i_1}x_{i_2}\cdots x_{i_{v-w}}\lambda_{0j_1}\lambda_{0j_2}\cdots\lambda_{0j_w}, \tag{14}$$

where, $s^{v,w}_{i_1i_2\cdots i_{v-w}j_1j_2\cdots j_w}(t)$'s are the coefficients of the terms of the Taylor series of the generating function S, whose order w.r.t. x is $v-w$ and that w.r.t. λ_0 is w, $1 \le i_1 \le i_2 \le \cdots \le i_{v-w} \le n$, and $1 \le j_1 \le j_2 \le \cdots \le j_w \le n$. Due to the Taylor series expansion of the generating function (14), the initial value of the generating function $S(x,\lambda_0,t)$ in (10) reduces to

$$\begin{cases} s^{2,1}_{i_1 j_1}(t_0) & = 1, \quad (i_1 = j_1), \\ s^{v,w}_{i_1 i_2 \cdots i_{v-w} j_1 j_2 \cdots j_w}(t_0) & = 0, \quad (\text{other cases}). \end{cases} \tag{15}$$

To simplify notations, in what follows, $S^{(v,w)}(x,\lambda_0,t)$ denotes the terms of the Taylor series of the generating function S, whose order w.r.t. x is $v-w$ and that w.r.t. λ_0 is w, then

$$S(x,\lambda_0,t) = \sum_{v=0}^{\infty}\sum_{w=0}^{v} S^{(v,w)}(x,\lambda_0,t), \tag{16}$$

the function $\Gamma(S;x,\lambda_0,t)$ denotes the HJE (9) as follows.

$$\Gamma(S;x,\lambda_0,t) \triangleq \frac{\partial S}{\partial t} + H(x,\frac{\partial S}{\partial x}) = 0 \tag{17}$$

We focus on the local solution to (17) in a neighborhood of the origin. Expanding every element of $f(x)$ and $-g(x)R(x,t)^{-1}g(x)^{\mathrm{T}}/2$ in their Taylor series w.r.t. x about $x=0$, in the remainder of this paper, we use the column vectors $f^{(v)}(x)$'s $(v=1,2,\cdots)$ to denote the terms of the Taylor series of vector function $f(x)$, whose elements' order w.r.t. x is v, the symmetric matrix $r^{(v)}(x,t)$'s $(v=0,1,\cdots)$ to denote the terms of the Taylor series of the matrix value function $-g(x)R(x,t)^{-1}g(x)^{\mathrm{T}}/2$, whose elements' order w.r.t. x is v, $q^{(v)}(x,t)$'s $(v=2,3,\cdots)$ to denote the terms of the Taylor series of the matrix function $q(x,t)$, whose elements' order w.r.t. x is v. Then, $f(x)$, $-g(x)R(x,t)^{-1}g(x)^{\mathrm{T}}/2$ and $q(x,t)$ can be written as

$$f(x) = \sum_{v=1}^{\infty} f^{(v)}(x), \tag{18}$$

$$-\frac{1}{2}g(x)R(x,t)^{-1}g(x)^{\mathrm{T}} = \sum_{v=0}^{\infty} r^{(v)}(x,t), \tag{19}$$

$$q(x,t) = \sum_{v=2}^{\infty} q^{(v)}(x,t). \tag{20}$$

Therefore, the Hamiltonian function $H(x,\lambda)$ (7) can be written as

$$H(x,\lambda) = \sum_{v=2}^{\infty} q^{(v)}(x,t) + \lambda^{\mathrm{T}}\sum_{v=1}^{\infty} f^{(v)}(x) + \lambda^{\mathrm{T}}\sum_{v=0}^{\infty} r^{(v)}(x,t)\lambda. \tag{21}$$

Due to the simplified notation of the Taylor series of the generating function S in (16), the canonical transformations in (8) reduce to

$$\lambda = \frac{\partial S(x,\lambda_0,t)}{\partial x} = \sum_{v=1}^{\infty}\sum_{w=0}^{v-1} \frac{\partial S^{(v,w)}(x,\lambda_0,t)}{\partial x}, \tag{22}$$

$$x_0 = \frac{\partial S(x,\lambda_0,t)}{\partial \lambda} = \sum_{v=1}^{\infty}\sum_{w=1}^{v} \frac{\partial S^{(v,w)}(x,\lambda_0,t)}{\partial \lambda}. \tag{23}$$

Substituting the costate $\partial S/\partial \lambda$ in (22) and the Hamiltonian function (21) for the function Γ in (17), $\Gamma(S;x,\lambda_0,t)$ can be written as a series w.r.t. X as follows.

$$\Gamma(S;x,\lambda_0,t) = \sum_{v=0}^{\infty}\sum_{w=0}^{v} \Gamma^{(v,w)}(x,\lambda_0,t) = 0 \tag{24}$$

where $\Gamma^{(v,w)}(x,\lambda_0,t)$'s denote the terms of the Taylor series of the function $\Gamma(S;x,\lambda_0,t)$, whose order w.r.t. x is $v-w$ and that w.r.t. λ_0 is w, the expressions of $\Gamma^{(v,w)}(x,\lambda_0,t)$'s are as follows.

$$\begin{aligned} \Gamma^{(v,w)}(x,\lambda_0,t) &= \frac{\partial S^{(v,w)}}{\partial t} + \sum_{v_1=w+1}^{v} \frac{\partial S^{(v_1,w)}}{\partial x}^{\mathrm{T}} f^{(v-v_1+1)}(x) \\ &+ \sum_{v_1=1}^{v+1}\sum_{v_2=1}^{v-v_1+2}\sum_{w_1=\underline{w}_1}^{\overline{w}_1} \frac{\partial S^{(v_1,w_1)}}{\partial x}^{\mathrm{T}} r^{(v-v_1-v_2+2)}(x,t)\frac{\partial S^{(v_2,w-w_1)}}{\partial x} \\ &+ (1-sgn^2(w))q^{(v)}(x,t) \end{aligned} \tag{25}$$

Here $\underline{w}_1 = \max\{0, w-v_2+1\}$, $\overline{w}_1 = \min\{w, v_1-1\}$ and

$$sgn(w) = \begin{cases} 0 & , \quad w = 0; \\ 1 & , \quad w > 0. \end{cases} \tag{26}$$

Since X is independent, $\Gamma^{(v,w)}(x,\lambda_0,t)$'s are the terms of the Taylor series of the function $\Gamma(S;x,\lambda_0,t)$, solving the HJE in (17) for the generating function S is equal to solving

$$\Gamma^{(v,w)}(x,\lambda_0,t) = 0, \quad 0 \le w \le v \tag{27}$$

for all $S^{(v,w)}(x,\lambda_0,t)$. Unfortunately, $\partial S^{(v,w)}(x,\lambda_0,t)/\partial t$ couples with the higher order term $S^{(v+1,w)}(x,\lambda_0,t)$ due to (25). Therefore, it is impossible to solve the equations in (27) recursively with respect to the order $v = 0,1,2,\cdots$. Of course, if we truncate the Taylor series of the generating function $S(x,\lambda_0,t)$ up to order N, we can solve all the equations in (27) simultaneously. However, the total number of the unknown terms $S^{(v,w)}(x,\lambda_0,t)$ $(0 \le v \le N, 0 \le w \le v)$ becomes larger and larger as N increases, therefore solving all the equations in (27) for $S^{(v,w)}(x,\lambda_0,t)$ simultaneously becomes more and more difficult as N becomes larger. If the coupling relationship between $\partial S^{(v,w)}(x,\lambda_0,t)/\partial t$ and the higher order term $S^{(v+1,w)}(x,\lambda_0,t)$ does not exist, then we can solve equations in (27) recursively w.r.t. w and v.

Let us define a function $S^{[N]}$ as the truncated Taylor series of S up to the order N as follows.

$$S^{[N]}(x,\lambda_0,t) \triangleq \sum_{v=0}^{N}\sum_{w=0}^{v} S^{(v,w)}(x,\lambda_0,t). \tag{28}$$

Because of the HJE in (24), the following equation holds.

$$\Gamma(S;x,\lambda_0,t) = \Gamma(S^{[N]};x,\lambda_0,t) + o\left(\|X\|^N\right) = 0 \tag{29}$$

This means that the function $S^{[N]}$ is an approximate solution of the HJE $\Gamma(S;x,\lambda_0,t) = 0$. The following lemma tells us that $S^{(v,w)}(x,\lambda_0,t) \equiv 0$ $(0 \le w \le v < 2)$ $\forall t \ge t_0$ in (28). This property eliminates the coupling relationship between $\partial S^{(v,w)}(x,\lambda_0,t)/\partial t$ and the higher order term $S^{(v+1,w)}(x,\lambda_0,t)$.

Lemma 1: The terms $S^{(v,w)}(x,\lambda_0,t)$ in (28) satisfy $S^{(v,w)}(x,\lambda_0,t) \equiv 0$, $\forall t \ge t_0$, $\forall 0 \le w \le v < 2$. In particular, if $q(x,t)$ is equal to zero, $S^{(v,0)}(x,\lambda_0,t) \equiv 0$ holds $\forall t \ge t_0$ and $\forall v$.

The proof of Lemma 1 is omitted here. Due to (25), Lemma 1 implies that $\partial S^{(v,w)}(x,\lambda_0,t)/\partial t$ does not depend on $S^{(v+1,w)}(x,\lambda_0,t)$. Therefore, we can prove the following theorem. It shows that solving (29) for an approximate generating function $S^{[N]}$ is equivalent to solving a sequence of first-order differential equations.

Theorem 1: The function $S^{[N]}(x,\lambda_0,t)$ satisfying (29) can be obtained by solving the following equations.

$$\frac{\partial S^{(v,w)}(x,\lambda_0,t)}{\partial t} = F^{v,w}(S^{(2,0)},S^{(2,1)},\cdots,S^{(v,w-1)},S^{(v,w)}) \quad (30)$$

where

$$F^{v,w} = (1-sgn^2(w))q^{(v)}(x,t) - \sum_{v_1=w+1}^{v} \frac{\partial S^{(v_1,w)}}{\partial x}^{\mathrm{T}} f^{(v-v_1+1)}(x) + \sum_{v_1=2}^{v}\sum_{v_2=2}^{v-v_1+2}\sum_{w_1=\max\{0,w-v_2+1\}}^{\min\{w,v_1-1\}} \frac{\partial S^{(v_1,w_1)}}{\partial x}^{\mathrm{T}} r^{(v-v_1-v_2+2)}(x,t)\frac{\partial S^{(v_2,w-w_1)}}{\partial x}, \quad (31)$$

$0 \le w \le v$, $2 \le v \le N$. In particular, if $q(x,t)$ is equal to zero, and $R(x,t)^{-1}$ is a matrix function of polynomials of t, we can obtain the exact solutions to all $S^{(v,w)}(x,\lambda_0,t)$.

The proof of this theorem is easy using Lemma 1, so it is omitted here. This theorem tells us that we can solve (29) for all $S^{(v,w)}(x,\lambda_0,t)$ recursively as $v=2,3,\cdots,N, w=0,1,\cdots,v$. That means, when expanding the Taylor series of the generating function $S(x,\lambda_0,t)$ up to the order N ($N \ge 2$) as in (28), we can obtain an approximate generating function $S^{[N]}$ satisfying the HJE (29) by solving equations in (30). By this theorem, we can calculate an approximate solution $S^{[N]}$ to HJE (17) up to any order N recursively w.r.t. w and v. The following elaborates how to solve equations in (30) for coefficients of $S^{(v,w)}(x,\lambda_0,t)$ in detail. We introduce some notations for the subscripts of $s^{v,w}_{i_1 i_2\cdots i_{v-w} j_1 j_2\cdots j_w}(t)$ to make the following content easy to understand. For a fixed $s^{v,w}_{i_1 i_2\cdots i_{v-w} j_1 j_2\cdots j_w}(t)$, the set $\Phi_{v,w}$ and Ψ_w are defined respectively as follows:

$$\Phi_{v,w} = \{i_1,i_2,\cdots,i_{v-w}\}, \qquad \Psi_w = \{j_1,j_2,\cdots,j_w\}. \quad (32)$$

The symbol $|\,(\cdot)\,|$ denotes the number of the elements of a set $(\cdot)$, and the symbol $\tau(\cdot)$ denotes any permutation of the set $(\cdot)$ satisfying

$$\tau(\{a_1,a_2,\cdots,a_n\}) = a_i a_j \cdots a_k, \quad (a_i \le a_j \le \cdots \le a_k;\ i,j,\cdots,k \in \{1,2,\cdots,n\}). \quad (33)$$

Since x and λ_0 are independent, because of Lemma 1 (i), (27) implies

$$\dot{s}^{v,w}_{i_1 i_2\cdots i_{v-w} j_1 j_2\cdots j_w}(t) = -\sum_{v_1=\max\{2,w+1\}}^{v} \Big(\sum_{\substack{\phi\subset\Phi_{v,w}\\ |\phi|=v_1-w-1}}\Big) \left(f_{l_1,\tau(\Phi_{v,w}\setminus\phi)} \cdot s^{v_1,w}_{\tau(\phi\cup\{l_1\})\tau(\Psi_w)}(t)\right) - \sum_{v_1=2}^{v}\sum_{v_2=2}^{v-v_1+2}\sum_{w_1=\max\{0,w-v_2+1\}}^{\min\{w,v_1-1\}} \Big(\sum_{\substack{\phi\subset\Phi_{v,w}\\ |\phi|=v_1-w_1-1}}\Big)\Big(\sum_{\substack{\omega\subset\Phi_{v,w}\setminus\phi\\ |\omega|=v_2-w_2-1}}\Big)\Big(\sum_{\substack{\psi\subset\Psi_w\\ |\psi|=w_1}}\Big) \left(r_{l_1,l_2,\tau(\Phi_{v,w}\setminus\phi\setminus\omega)}(t)\cdot s^{v_1,w_1}_{\tau(\phi\cup\{l_1\})\tau(\psi)}(t) s^{v_2,w-w_1}_{\tau(\omega\cup\{l_2\})\tau(\Psi_w\setminus\psi)}(t)\right) - (1-\mathrm{sgn}^2(w))q_{i_1 i_2\cdots i_v}(t), \quad (34)$$

where $0 \le w \le v$, $2 \le v \le N$, $1 \le i_1 \le i_2 \le \cdots \le i_{v-w} \le n$, $1 \le j_1 \le j_2 \le \cdots \le j_w \le n$, $1 \le l_1 \le n, 1 \le l_2 \le n$, $f_{l_1,i_1 i_2\cdots i_k}$'s are coefficients of the l_1-th element of the vector function $f^{(k)}(x)$ (the term of the Taylor series of $f(x)$ whose order w.r.t. x is k as in (18)), $r_{l_1,l_2,i_1 i_2\cdots i_k}(t)$'s are coefficients of the l_1-th row l_2-th column element of the matrix function $r^{(k)}(x,t)$ (the term of the Taylor series of $-g(x)R(x,t)^{-1}g(x)^{\mathrm{T}}/2$ whose order w.r.t. x is k as in (19)), and $q_{i_1 i_2\cdots i_k}(t)$'s are coefficients of the function $q^{(k)}(x,t)$ (the term of the Taylor series of $q(x,t)$ whose order w.r.t. x is k as in (20)). It is easy to know the equations in (34) are first order ODEs.

Since the optimal control input $u^* = -R^{-1}(x,t)g(x)^{\mathrm{T}}\lambda$, the costate λ is necessary for generating the optimal control input to the system (1)–(4). According to the canonical transformations, we define $\lambda_N(x,\lambda_0,t)$ as

$$\lambda_N \stackrel{\triangle}{=} \partial S^{[N]}(x,\lambda_0,t)/\partial x, \quad (35)$$

therefore,

$$\begin{aligned}\lambda = \partial S/\partial x &= \partial\left(S^{[N]} + o\left(\|X\|^N\right)\right)/\partial x \\ &= \lambda_N + o\left(\|X\|^{N-1}\right).\end{aligned} \quad (36)$$

That is, the vector λ_N is an approximation of λ up to the order $N-1$.

According to (35), the vector λ_N is the function of the initial value of the costate λ_0, but it is unknown. Of course, if we can obtain the initial value of the costate λ_0, it is easy to obtain the state trajectory of the system (1)–(4) by integrating equation in (1) using the optimal control input

$$u^*(x,\lambda_0,t) = -R(x,t)^{-1}g(x)^{\mathrm{T}}\lambda_N(x,\lambda_0,t). \quad (37)$$

We adopt the following idea to obtain λ_0.

As in (8), another transformation associated with the canonical transformation (35) at the terminal time $t=t_f$ is

$$x_0 = \partial S^{[N]}(x_{t_f},\lambda_0,t_f)/\partial\lambda_0. \quad (38)$$

As in (23), the equations in (38) are polynomial vector value functions of λ_0. We define λ_{0N} as the solution to (38). When N is small, equations in (38) are low-order polynomial equations. Hence it is easy to solve it directly. But if we want to obtain a more accurate control input and state trajectory, we need to select a larger N. Then it becomes more difficult to solve (38), since its order becomes higher.

Since (38) has $N-1$ solutions in general, supposing that there always exist real solutions among them. We select one among real solutions which makes $\|\lambda_{0N}\|$ be the smallest. Because our method is based on the Taylor series expansion, the state should be in a neighborhood of the origin. From now on, the symbol λ_{0N} denotes the specific solution to (38) thus defined. Since Theorem 1, we can solve HJE for an approximate generating function by solving a sequence of ODEs recursively. The following Algorithm 1 summarizes the proposed method solving a nonlinear optimal trajectory generation problem on a finite time interval.

Algorithm 1:

1. Set $v = 2$;

2. For $w = 0 : v$, solve ODE (34) for $s^{v,w}_{i_1 i_2 \cdots i_{v-w} j_1 j_2 \cdots j_w}(t)$;

3. Do $v = v + 1$;

4. If $v \le N_{\max}$, then go to step 2; else, stop;

5. Solve (38) for λ_{0v} in the case of given x_0 and x_{t_f};

6. Integrate $\dot{x} = f(x) + g(x)u$ using $u = -R(x,t)^{-1} \cdot g(x)^{\mathrm{T}} \lambda_v(x, \lambda_{0v}, t)$;

7. If $\left\| x(t_f) - x_{t_f} \right\| \le \varepsilon$, then stop; else, go to step 3.

where $\varepsilon > 0$ is the tolerable error and $N_{\max} \ge 2$ is the setting highest order of the Taylor series of the generating function to be calculated. If the accuracy is not enough, we can select a larger $N_{\max}$ to calculate the higher order terms of the generating function. Algorithm 1 gives us an arbitrarily accurate optimal control input for a given nonlinear optimal control problem. Note that, if we change the initial value x_0 or the terminal value x_{t_f} or both of them, we only need to solve (38) again for λ_0.

V. Numerical examples

Using the method proposed in Section 4, we will solve the following optimal control problems.

Example 1 (One dimensional problem): Consider the following nonlinear optimal control problem, the dynamic equation is

$$\dot{x} = 2x + x^2 + u, \tag{39}$$

the cost function is

$$J = \frac{1}{2} \int_{t_0}^{t_f} u^2 \mathrm{d}t. \tag{40}$$

The purpose is to find an optimal control input u^* minimizing the cost function in (40) subject to the boundary condition

$$x(t_0) = x_0, \quad x(t_f) = x_{t_f}. \tag{41}$$

We can obtain the analytic expressions of $S^{v,w}(x, \lambda_0, t)$. The coefficients of $S^{v,w}(x, \lambda_0, t)$ of the Taylor series of the generating function up to the third order are as follows. All of the following coefficients' subscripts are always equal to one, we omit them.

$$
\begin{aligned}
&s^{2,1}(t) = e^{-2(t-t_0)}, \quad s^{2,2}(t) = -\frac{1}{8} e^{-4(t-t_0)} + \frac{1}{8} \\
&s^{3,1}(t) = -\frac{1}{2} e^{-2(t-t_0)} + \frac{1}{2} e^{-4(t-t_0)} \\
&s^{3,2}(t) = -\frac{1}{4} e^{-2(t-t_0)} + \frac{1}{2} e^{-4(t-t_0)} - \frac{1}{4} e^{-6(t-t_0)} \\
&s^{3,3}(t) = \frac{1}{32} e^{-8(t-t_0)} - \frac{1}{12} e^{-6(t-t_0)} + \frac{1}{16} e^{-4(t-t_0)} - \frac{1}{96} \\
&s^{0,0}(t) = s^{1,0}(t) = s^{1,1}(t) = s^{2,0}(t) = s^{3,0}(t) = 0
\end{aligned}
$$

This example verifies the later claim of Theorem 1 under the corresponding assumption.

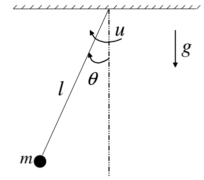


Fig. 1. A Pendulum

Example 2 (A Pendulum): Fig.1 depicts a pendulum. The dynamic equation of this nonlinear optimal control problem is

$$\dot{x} = \begin{pmatrix} x_2 \\ -(mgl \sin x_1 + \mu x_2^2)/(ml^2) \end{pmatrix} + \begin{pmatrix} 0 \\ 1/(ml^2) \end{pmatrix} u, \tag{42}$$

where $x = [x_1, x_2]^{\mathrm{T}}$, $x_1 = \theta, x_2 = \dot{\theta}$, m is the mass of the pendulum, l is the length of the pendulum, μ is the friction factor, and g is the gravity acceleration. Consider the cost function as

$$J = \frac{1}{2} \int_{t_0}^{t_f} (x^{\mathrm{T}} x + u^2) \mathrm{d}t. \tag{43}$$

The purpose is to find an optimal control input u^* minimizing the cost function in (43) subject to the boundary condition. Consider the case of $m = l = \mu = 1, t_0 = 0$ and $t_f = 1$. The designed initial state value is

$$x(t_0) = (-0.1, 0)^{\mathrm{T}}. \tag{44}$$

Consider the two different terminal state values, the one is

$$x(t_f) = (0.1, 0.1)^{\mathrm{T}}, \tag{45}$$

another is

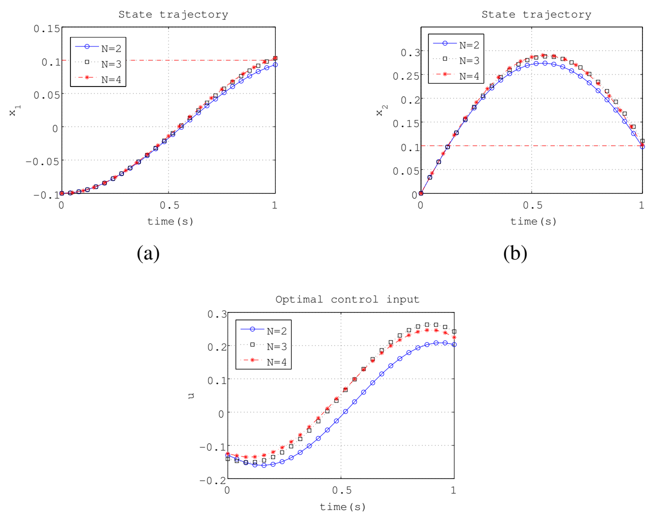
$$x(t_f) = (0.15, 0.15)^{\mathrm{T}}. \tag{46}$$

For the different terminal state values, the state trajectory and the optimal input can be seen in Fig.2 and Fig.3 when $N = 2, 3, 4$. Here we expand the generating function in its Taylor series up to order N. Fig.2(a), Fig.2(b), Fig.3(a) and Fig.3(b) show that, the states achieve to the designed value at the terminal time. Table I and Table II show the error of the terminal state for $N = 2, 3, 4$. The biggest $N = 4$ achieves the best accuracy for $x(t_f) = (0.1, 0.1)^{\mathrm{T}}$, and the case of $N = 3$ achieves the best accuracy for $x(t_f) = (0.15, 0.15)^{\mathrm{T}}$. From Table I and Table II, we can see the accuracy is not monotonic with respect to N, this is because of the property of the Taylor series.

TABLE I
THE ERROR OF THE TERMINAL STATE VALUE IN THE CASE OF $x(t_0) = (-0.1, 0)^{\mathrm{T}}$, $x(t_f) = (0.1, 0.1)^{\mathrm{T}}$

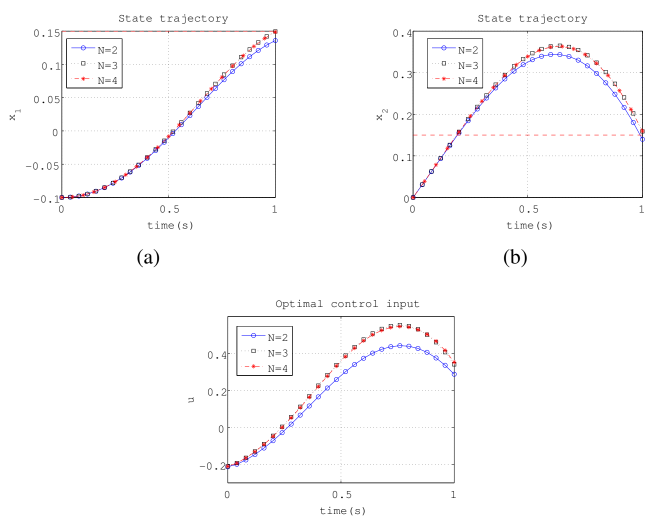
	$N = 2$	$N = 3$	$N = 4$
$\|\|x_N(t_f) - x(t_f)\|\|$	4.93×10^{-5}	1.14×10^{-4}	1.60×10^{-5}

To illustrate the advantage of the proposed method, we will use a single generating function (up to the order 4) to generate a family of optimal trajectories for different terminal state values. The different terminal state values are showed in Fig.4. All of the generated trajectories achieve to designed value at



(a) (b) (c)

Fig. 2. The state and the optimal input trajectory for the boundary condition $x(t_0) = (-0.1, 0)^{\mathrm{T}}$, $x(t_f) = (0.1, 0.1)^{\mathrm{T}}$



(a) (b) (c)

Fig. 3. The state and the optimal input trajectory for the boundary condition $x(t_0) = (-0.1, 0)^{\mathrm{T}}$, $x(t_f) = (0.15, 0.15)^{\mathrm{T}}$

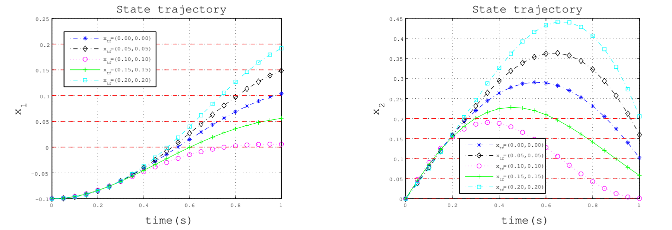
TABLE II
THE ERROR OF THE TERMINAL STATE VALUE IN THE CASE OF $x(t_0) = (-0.1, 0)^{\mathrm{T}}$, $x(t_f) = (0.15, 0.15)^{\mathrm{T}}$

	$N=2$	$N=3$	$N=4$
$\lVert x_N(t_f) - x(t_f) \rVert$	3.03×10^{-4}	7.81×10^{-5}	9.47×10^{-5}

the terminal time from Fig.4. This example of the pendulum illustrates the effectiveness and the advantage of the proposed method.

VI. Conclusion

In this paper, we have developed an algorithm to obtain an approximate solution to the HJE for generating functions by solving a sequence of first order ordinary differential equations



(a) (b)

Fig. 4. A family of optimal trajectories generated by the generating function for a set of different terminal state values

recursively. Although this method double the dimension of the problem, a generating function can generate a family of optimal trajectories for different boundary conditions of a given system. In addition, the generating function can be calculated off-line. Therefore, the proposed method reduces the online computational cost for a given optimal control system, it would be very useful to online control problems such as online trajectory generation. Of course, we can use other bases functions to obtain generating functions. This problem will be studied in the future.

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