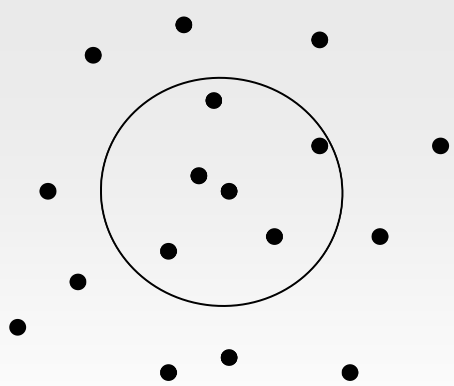
Smoothed Particle Hydrodynamics (SPH)

Some particle properties are determined by taking an average over neighboring particles

The fluid is represented by a particle system

Fluid dynamics



1. Only particles inside circle contribute to the average
2. Close particles should contribute more than distant particles

In the average: Use a weight function

Before we consider the details…

How do we describe our particle system?

Each particle is specified by a state list:

mass, velocity, position, force,

density, pressure, color

Particle i ➡ $(m_i, \mathbf{v}_i, r_i, \mathbf{F}_i, \rho_i, p_i, C_i)$

The Goal

The acceleration of a particle is

$$\boxed{\frac{dv_i}{dt} = a_i^{pressure} + a_i^{viscosity} + a_i^{interactive} + a_i^{gravity}}$$

Remember that $a_i = \frac{F_i}{m_i}$

Let us now learn how to set up the particle list…

Particle mass

In our simulation we choose to have the same mass for all particles, $m_i = m$

The mass m is calculated by

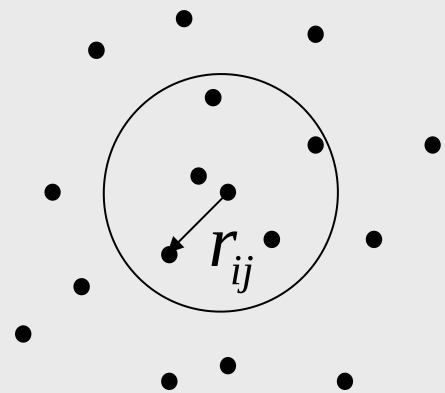
$$m = \frac{(Density\ of\ \ fluid)\cdot(Total\ volume)}{Total\ number\ of\ \ particles}$$

Note! Do not change the mass during the simulation

Let us now go back to the weighted averages…

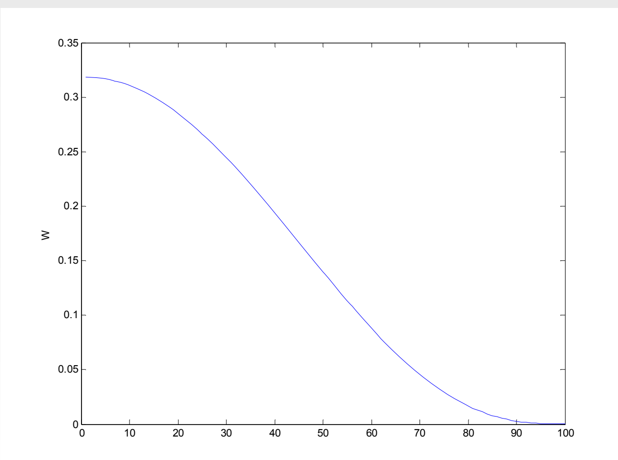
How do we determine the density of a particle?

$$\rho_i = \sum_j m_j W(r_{ij})$$



The sum include particle i

Weight function or Kernel function

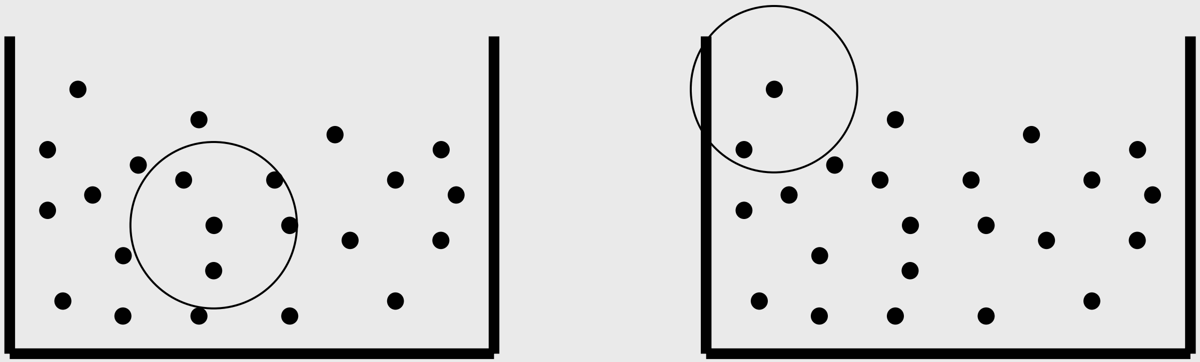


What happens if particle *i* has no neighbours?

- The density calculation is done every time step
- The neighbor list must be updated every time step

Surface tracking

It is not trivial to know where the surface is…



- We can find the surface by monitoring the density
- If the density at a particle deviates too much compared to expected density we tag it as a surface particle

Pressure

We get the pressure from the relation:

$$p_i = c_s^{\ 2}\left(\rho_i - \rho_0\right)$$

where c_s is the speed of sound and ρ_0 is the fluid reference density

Let us take a look on the particle property list again

$$(\boxed{m_i}, \mathbf{v}_i, r_i, \mathbf{F}_i, \boxed{\rho_i}, \boxed{p_i}, C_i)$$

- Note that velocities and positions are calculated from the forces in a way similar to an ordinary particle system
- The next property we focus on is thus the force

- But before we go into that we need to learn more about taking averages…

In SPH we formally define averages in the following way:

$$\langle A(r) \rangle = \int_V A(r')W(r-r')dr'$$

In practice we use a discrete version of this:

$$\langle A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j W(r_{ij})$$

$$\langle \nabla A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla W(r_{ij})$$

$$\langle \nabla^2 A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla^2 W(r_{ij})$$

Example

$$\langle \rho \rangle_i \approx \sum_j \frac{m_j}{\rho_j} \rho_j W(r_{ij}) \approx \sum_j m_j W(r_{ij})$$

Meshless method!!

Velocities and Forces

Motion equation in elasticity: $$\frac{dv}{dt} = \frac{1}{\rho}\nabla \cdot \sigma + \frac{1}{\rho}F_{ext}$$

We also had: $$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Now we instead use: $$\sigma_{ij} = -p\mathrm{I}_{ij} + \mu\dot{\varepsilon}_{ij}$$

All this together produces the following fluid equation called Navier-Stokes equation

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla \cdot \nabla \mathbf{v} + \frac{f_{ext}}{m} + \mathbf{g}$$

Note: f_{ext} could for example be an interactive force

- Our task is now to convert each term on the RHS in Navier-Stokes to SPH-averages
  - First term (pressure) becomes:

$$\left\langle -\frac{1}{\rho}\nabla p \right\rangle_i \approx \sum_j P_{ij} \nabla W(r_{ij})$$

where

$$P_{ij} = -m_j \left(\frac{p_i}{\rho_i^{\,2}} + \frac{p_j}{\rho_j^{\,2}} \right)$$

- The second term (viscosity):

$$\left\langle \frac{\mu}{\rho} \nabla \cdot \nabla \mathbf{v} \right\rangle_i \approx \sum_j \mathbf{V}_{ij} \nabla^2 W(r_{ij})$$

where $$\mathbf{V}_{ij} = \mu m_j \frac{\mathbf{v}_j - \mathbf{v}_i}{\rho_i \rho_j}$$

Note: $\nabla \cdot \nabla \mathbf{v} = \nabla^2 \mathbf{v}$

Summary

The <u>acceleration</u> of a particle can now be written:

$$\boxed{\frac{dv_i}{dt} = a_i^{pressure} + a_i^{viscosity} + a_i^{interactive} + a_i^{gravity}}$$

$$a_i^{pressure} \approx \sum_j P_{ij} \nabla W(r_{ij})$$

$$a_i^{viscosity} \approx \sum_j V_{ij} \nabla^2 W(r_{ij})$$

$$a_i^{interactive} \approx \frac{1}{m_i} f_i^{interactive}$$

$$a_i^{gravity} \approx (0,0,-g)$$

Remember that

$$F_i = m_i a_i$$

NOTE!

Sometimes one uses different kernels for each term in RHS

Simulation loop

For each time step:

1. Find neighbors to each particle and store in a list
2. Calculate density for each particle
3. Calculate pressure for each particle
4. Calculate all type of accelerations for each particle, and sum it up
5. Find new velocities and positions by using the same integration method as before…

Parameter values

Consider a container filled with 10 litre of water.
We will use SI-units for all parameters.

Number of SPH-particles: $N = 1000$

Mass: $m = 0.01 kg$

Density: $\rho_0 = 1000 \frac{kg}{m^3}$ (water)

Interaction radius: h is chosen such that 15-20 particles are interacting on average

Dynamical viscosity: $\mu = 0.001 \frac{\text{kg}}{\text{ms}}$

Speed of sound: $c_s = 1500 \frac{m}{s}$ (water) => **Time step:** $dt = 0.0001 s$

Speed of sound: $c_s = 1 - 10 \frac{m}{s}$ => **Time step:** $dt = 0.01 - 0.03 s$

Comments on Bonus assignments

Full list of Bonus assignments can be found in the SPH-lab specifications.

- Implementation of color field method for finding surface particles
- Use *Powray* to raytrace images or animations
- Implementation of flame propagation
- Implementation of surface tension

The color property?

What is the use of this property?

- We can use it to detect the **position of the surface** of our fluid
- We can also use it to find the **normal vectors** at the surface (important for rendering!)
- The normal vectors allow us to implement surface tension
- By adding several color fields we can for example implement a simple model of **flame propagation**

The color field

- The color parameter is a quantity that is zero everywhere except at the particle where it has value one

- Similar to how we calculated density we now calculate the average color at particle i as

$$\langle C \rangle_i \approx \sum_j \frac{m_j}{\rho_j} C_j W(r_{ij})$$

- Deviations of the color field show us where the surface is, and in this case we choose to study the derivative of color field

- The gradient of a color field is

$$\langle \nabla C \rangle_i \approx \sum_j \frac{m_j}{\rho_j} C_j \nabla W(r_{ij})$$

- When the magnitude of the gradient is larger than a certain value, we tag the particle as a surface particle

Flame propagation

It is easy to introduce a simplified fire model by using the color field technique…

1. Add three new particle properties
   i. Burnable
   ii. Burn (on/off)
   iii. Burning time
2. For all surface particles that are burning
   i. Calculate an average color field gradient using the variable Burnable as particle color
   ii. Find all surface particles with average color field gradient higher than a certain value and flag them as burning (Burn on)
   iii. Decrease Burning time and check if Burn should be off
3. Visualize burning particles

Surface tension

The force that tends to make surfaces smooth (like a drop of liquid) can be modeled in the following way:

$$a_i^{tension} = -\frac{\sigma_s}{\rho_i} \left\langle \nabla^2 C \right\rangle_i \frac{\mathbf{n}_i}{\left|\mathbf{n_i}\right|}$$

where $\mathbf{n}_i = \left\langle \nabla C \right\rangle_i$ and $\left\langle \nabla^2 C \right\rangle_i \approx \sum_j \frac{m_j}{\rho_j} C_j \nabla^2 W(r_{ij})$

Note: If the magnitude of n_i is small we can get numerical problem in the division above. To avoid this we only calculate $\mathbf{n}_i / \left|\mathbf{n_i}\right|$ if the magnitude of n_i exceeds a certain threshold.