ESE 605 Homework 1

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Problems from Boyd & Vandenberghe:
2.10.

(a) Show that C is convex if $A \succeq 0$

For $x, y \in C$,

$$x^T Ax + b^T x + c \leq 0$$
$$y^T Ay + b^T y + c \leq 0$$

For $0 < \theta < 1, A \succeq 0$

$$\begin{aligned}
&(\theta x + (1-\theta)y)^T A(\theta x + (1-\theta)y) + b^T(\theta x + (1-\theta)y) + C \\
=&\theta^2 x^T Ax + \theta(1-\theta)(x^T Ay + y^T Ax) + (1-\theta)^2 y^T Ay + \theta b^T x + (1-\theta)b^T y + c \\
\leq&\theta^2 x^T Ax + \theta(1-\theta)(x^T Ay + y^T Ax) + (1-\theta)^2 y^T Ay + \theta(-x^T Ax - c) + (1-\theta)(-y^T Ay - c) + c \\
=&-\theta(1-\theta)(x-y)^T A(x-y) \\
\leq&0
\end{aligned}$$

$\theta x + (1-\theta)y \in C$, so C is convex.

The converse is not true. A counter-example is for $x \in \mathbf{R}, A = -1, b = c = 0$. In this case, C is convex, but $A \not\succeq 0$.

(b)

$$C \cap H = \left\{x \in \mathbf{R}^n \mid x^T Ax + b^T x + c \leq 0, g^T x + h = 0\right\}$$

Its intersection with an arbitrary line $\{\hat{x} + tv \mid t \in \mathbb{R}\}$ is:

$$\left\{\hat{x} + tv \mid (\hat{x} + tv^T)A(\hat{x} + tv) + b^T(\hat{x} + tv) + c \leq 0, g^T(\hat{x} + tv) + h = 0\right\}$$

$$\begin{aligned}
&(\hat{x} + tv)^T A(\hat{x} + tv) + b^T(\hat{x} + tv) + c \\
=&(v^T Av)t^2 + (\hat{x}^T Av + v^T A\hat{x} + b^T v)t + (\hat{x}^T A\hat{x} + b^T\hat{x} + c)
\end{aligned}$$

Since we can assume $\hat{x} \in H$

$$g^T(\hat{x} + tv) + h = (g^T v)t + (g^T\hat{x} + h) = (g^T v)t$$

If $g^T v \neq 0$, then $t = 0$, and the intersection with line becomes $\{x\}$ if $\hat{x}^T A \hat{x} + b^T \hat{x} + c \leq 0$, or empty otherwise.

If $g^T v = 0$, then the intersection with line becomes

$$\left\{ \hat{x} + tv \mid (v^T A v) t^2 + (\hat{x}^T A v + v^T A \hat{x} + b^T v) t + (\hat{x}^T A \hat{x} + b^T \hat{x} + c) \leq 0 \right\}$$

If $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbf{R}$, then $v^T A v \geq 0$. C's intersection with an arbitrary line is convex, and C is convex.

Thus, $C \cap H$ is convex.

The converse is not true. A counter-example is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, c = -1, g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, h = 0$$

where $A + \lambda g g^T \not\succeq 0$ and $C \cap H$ is convex.

2.13.

XX^T is a semi-definite matrix $\in \mathbf{R}^{n \times n}$, and

$$rank(XX^T) = rank(X) = k$$

The conic hull is

$$\left\{ \sum_i^m \theta_i A_i \mid \theta_i \geq 0, A_i \in S_{n+}, rank(A_i) = k, i = 1, \cdots, m \right\}$$

We now want to prove the rank of the conic hull is greater than K.
Suppose $v \in \mathcal{N}(A + B)$, then

$$(A + B)v = 0 \Leftrightarrow v^T (A + B) v = 0 \Leftrightarrow v^T A v + v^T B v = 0$$

this implies,

$$v^T A v = 0 \Leftrightarrow Av = 0, \quad v^T B v = 0 \Leftrightarrow Bv = 0$$

So any vector $\in \mathcal{N}(A + B)$ must also $\in \mathcal{N}(A)$ and $\in \mathcal{N}(B)$, which means a positive combination of positive semi-definite matrices can only gain rank.

Thus, the conic hull can be simplified as

$$\text{semi-definite matrix } \in \mathbf{R}^{n \times n} \text{ that has rank between } k \text{ and } n$$

2.15.

(a) is convex in p.

$$P = \left\{ p \mid \mathbf{1}^T p = 1, p \succeq 0 \right\} \text{ is a polyhedron and is a convex set}$$

Since

$$\mathbf{E}f(x)=\sum_{i=1}^{n} p_i f\left(a_i\right) \geq \alpha \text{ is linear and convex}$$
$$\mathbf{E}f(x)=\sum_{i=1}^{n} p_i f\left(a_i\right) \leq \beta \text{ is linear and convex}$$

Then $\alpha \leq \mathbf{E}f(x) \leq \beta$ is convex.

(b) is convex in p.

$$\operatorname{Prob}(x>\alpha)=\sum_{i;\alpha_i>\alpha} p_i \leq \beta \text{ is linear and convex}$$

(c) is convex in p.

$$\mathbf{E}\left|x^3\right|=\sum_{i=1}^{n} p_i\left|a_i^3\right| \leq \alpha \mathbf{E}|x|=\alpha \sum_{i=1}^{n} p_i\left|a_i\right|$$
$$\sum_{i=1}^{n}(\left|a_i^3\right|-\alpha\left|a_i\right|) p_i \leq 0 \text{ is linear and convex}$$

(d) is convex in p.

$$\mathbf{E}x^2=\sum_{i=1}^{n} p_i a_i^2 \leq \alpha \text{ is linear and convex}$$

(e) is convex in p.

$$\mathbf{E}x^2=\sum_{i=1}^{n} p_i a_i^2 \geq \alpha \text{ is linear and convex}$$

(f) is generally not convex in p.

$$\mathbf{var}(x)=\mathbf{E}(x-\mathbf{E}x)^2=\mathbf{E}x^2-(\mathbf{E}x)^2=\sum_{i=1}^{n} p_i a_i^2-(\sum_{i=1}^{n} p_i a_i)^2 \leq \alpha$$

A counter-example is when $n=2, a_1=0, a_2=1, \alpha=0.1$. $p=(0,1)$ and $p=(1,0)$ lies in the set $\{p \mid \mathbf{var}(x) \leq \alpha\}$, but the mid-point $p=(\frac{1}{2}, \frac{1}{2})$ is not in the set.

2.19.

(a)

$$\begin{aligned} f^{-1}(C) &=\{x \in \operatorname{dom} f \mid f(x) \in C\} \\ &=\{x \in \operatorname{dom} f \mid g^T f(x) \leq h\} \\ &=\{x \mid g^T(Ax+b) / \left(c^T x+d\right) \leq h, c^T x+d>0\} \\ &=\{x \mid (A^T g-h^T c)^T x+(g^T b-dh) \leq 0, c^T x+d>0\} \end{aligned}$$

$f^{-1}(C)$ is a halfspace intersected with $\operatorname{dom} f$.

(b)

$$\begin{aligned} f^{-1}(C) &= \{x \mid G(Ax+b)/\left(c^T x + d\right) \preceq h, c^T x + d > 0\} \\ &= \{x \mid (A^T G^T - ch^T)^T x + (Gb - hd) \preceq 0, c^T x + d > 0\} \end{aligned}$$

$f^{-1}(C)$ is a polyhedron intersected with $\mathbf{dom}\, f$.

(c)

$$\begin{aligned} f^{-1}(C) &= \{x \mid \frac{(Ax+b)^T P^{-1}(Ax+b)}{(c^T x+d)^T(c^T x+d)} \le 1, c^T x + d > 0\} \\ &= \{x \mid x^T(A^T P^{-1} A - cc^T)x + 2(b^T P^{-1} A - d^T c^T)x + b^T P^{-1} b - d^T d \le 0, c^T x + d > 0\} \end{aligned}$$

If $A^T P^{-1} A - cc^T \in \mathbf{S}^n_{++}$, $f^{-1}(C)$ is an ellipsoid intersected with $\mathbf{dom}\, f$. $f^{-1}(C)$ is a polyhedron intersected with $\mathbf{dom}\, f$.

(d)

$$\begin{aligned} f^{-1}(C) &= \{x \mid (a_1^T x + b_1)A_1 + \cdots + (a_n^T x + b_n)A_n \preceq B(c^T x + d), c^T x + d > 0\} \\ &= \{x \mid (a_{11}A_1 + \cdots + a_{n1}A_n - c_1 B)x_1 + \cdots + (a_{1n}A_1 + \cdots + a_{nn}A_n - c_n B)x_n \\ &\quad + (b_1 A_1 + \cdots + b_n A_n - dB) \preceq 0, c^T x + d > 0\} \end{aligned}$$

$f^{-1}(C)$ is the solution set of a linear matrix inequality intersected with $\mathbf{dom}\, f$.

2.24.

(a)

$$\left\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \ge 1\right\}$$

A boundary point x_0 is in the form of $(t, 1/t)$, where $t > 0$. The tangent to the boundry is $(1, -1/t^2)$. The outer normal vector a is normal to the tangent vector, so $a^T(1, -1/t^2) = 0$, and then we have $a = (1/t^2, 1)$.

The supporting hyperplane is:

$$\begin{aligned} &\left\{x \mid a^T x = a^T x_0\right\} \\ \Rightarrow &\left\{x \mid \frac{x_1}{t^2} + x_2 = \frac{2}{t}\right\} \end{aligned}$$

Therefore the set could be expressed as an intersection of halfspaces:

$$\bigcap_{t>0} \left\{x \in \mathbf{R}^2 \mid \frac{x_1}{t^2} + x_2 \ge \frac{2}{t}\right\}$$

(b) The supporting hyperplane is:

$$\begin{gathered} \left\{x \mid a^T x = a^T \hat{x}\right\} \\ \hat{x}_i = 1,\ a_i > 0 \\ \hat{x}_i = -1,\ a_i > 0 \\ -1 < \hat{x}_i < 1,\ a_i = 0 \end{gathered}$$

2.28.

- Positive semidefinite cone for $n = 1$:

$$x_1 \geq 0$$

- Positive semidefinite cone for $n = 2$:

$$z^T X z = z_1^2 x_1 + 2z_1 z_2 x_2 + z_2^2 x_3 \geq 0$$

  Apparently, $x_1 \geq 0, x_3 \geq 0$, also

$$\partial_{z_1}(\cdot) = 2x_1 z_1 + 2x_2 z_2 = 0 \Rightarrow z_1 = -\frac{x_2 z_2}{x_1}$$

  Put back into the above inequation,

$$\begin{gathered} x_3 - \frac{x_2^2}{x_1} \geq 0 \\ x_1 x_3 - x_2^2 \geq 0 \end{gathered}$$

  Together we have: $x_1 \geq 0, x_3 \geq 0, x_1 x_3 - x_2^2 \geq 0$.

- Positive semidefinite cone for $n = 3$:

$$z^T X z = z_1^2 x_1 + z_2^2 x_4 + Z_3^2 X_6 + 2z_1 z_2 x_2 + 2z_1 z_3 x_3 + 2z_2 z_3 x_5 \geq 0$$

  From the theorem of semidefinite cone, all principle minors are non-negative, which means

$$\begin{aligned} & x_1 \geq 0, x_4 \geq 0, x_6 \geq 0, x_4 x_6 - x_5^2 \geq 0, x_1 x_6 - x_3^2 \geq 0, x_1 x_4 - x_2^2 \geq 0 \\ & x_1(x_4 x_6 - x_5^2) - x_2(x_2 x_6 - x_3 x_5) + x_3(x_2 x_5 - x_3 x_4) \\ & = x_1 x_4 x_6 + 2x_2 x_3 x_5 - x_1 x_5^2 - x_2^2 x_6 - x_3^2 x_4 \geq 0 \end{aligned}$$

Problems from additional exercises:
A2.7.

dual cone of a cone K :

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

(a) $k = \{0\}$

  Since $\forall y, y^T 0 = 0$, the dual cone $K^* = \mathbf{R}^2$.

(b) $k = \mathbf{R}^2$

  only $y = 0$ satisfies $\forall x \in \mathbf{R}^2, y^T x = 0$, so $K^* = \{0\}$.

(c) $K = \{(x_1, x_2) \,||x_1 \,|\leq x_2\}$

We can draw this cone in $\mathbf{R}^2$, and $y^T x \geq 0$ means rotate the board line segments less than 90 degree.

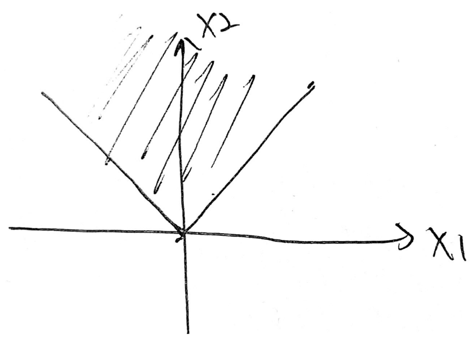
We can see that the dual cone is exactly the primary cone. So

$$K^* = \{(x_1, x_2) \,||x_1 \,|\leq x_2\}$$

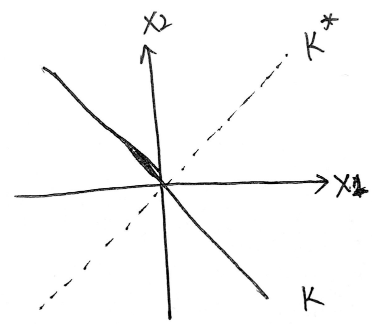
(d) $K = \{(x_1, x_2) \,|x_1 + x_2 = 0\}$

We can draw this cone in $\mathbf{R}^2$, which is a line, and the dual cone is its orthogonal line.

$$K^* = \{(x_1, x_2) \,|x_1 - x_2 = 0\}$$



(a) A2.7(c)



(b) A2.7(d)

Problems from Boyd & Vandenberghe:
3.8.

First consider the case $f : \mathbf{R} \to \mathbf{R}$
From the first-order condition for convexity, f is convex iff $dom f$ is convex and

$$f(y) \geq f(x) + f'(x)(y - x)$$
$$\text{likewise, } f(x) \geq f(y) + f'(y)(x - y)$$

So we have

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$$
$$f'(x)(y - x) \leq f'(y)(y - x)$$
$$\frac{f'(y) - f'(x)}{y - x} \geq 0$$
$$\lim_{y - x \to 0} \frac{f'(y) - f'(x)}{y - x} \geq 0$$
$$f''(x) \geq 0$$

To prove sufficiency, assume the function satisfies $dom f$ is convex and $f''(x) \geq 0$.

$$
\begin{aligned}
&\int_x^y f''(t)(y-t)dt \\
&= f'(t)(y-t)|_x^y + f'(t)|_x^y \\
&= -f'(x)(y-x) + f(y) - f(x) \\
&\geq 0
\end{aligned}
$$

which is equal to the first-order condition:

$$f(y) \geq f(x) + f'(x)(y-x)$$

and therefore we can conclude convexity.

To prove the general case, with $f : \mathbf{R}^n \to \mathbf{R}$, Let $x, y \in \mathbf{R}^n$ and consider f restricted to the line passing through them, i.e., the function defined by $g(t) = f(ty + (1-t)x)$

So f is convex iff $ty + (1-t)x \in dom f$ and

$$g''(t) = (y-x)^T f''(ty + (1-t)x)(y-x) \geq 0$$

which is equivalent to

$$f''(ty + (1-t)x) \succeq 0$$

3.13.

$$D_{\mathrm{kl}}(u, v) = f(u) - f(v) - \nabla f(v)^T (u - v),$$

where $f(v) = \sum_{i=1}^n v_i \log v_i$.

$f(v)$ is strictly convex. Taking its second-order derivative:

$$\nabla^2_{v_i v_j} f(v) = \begin{cases} \dfrac{1}{v_i} \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

$$\Rightarrow \nabla^2 f(x) \succ 0$$

Since negative entropy is strictly convex, we have:

$$
\begin{aligned}
f(u) &\geq f(v) + \nabla f(v)^T (u - v) \\
D_{\mathrm{kl}}(u, v) = f(u) - f(v) - \nabla f(v)^T (u - v) &\geq 0
\end{aligned}
$$

$f(v)$ is strictly convex, which means for all $u, v \in \mathbf{R}^n_{++}, x \neq y$

$$
\begin{aligned}
f(u) &> f(v) + \nabla f(v)^T (u - v) \\
D_{\mathrm{kl}}(u, v) = f(u) - f(v) - \nabla f(v)^T (u - v) &> 0
\end{aligned}
$$

Therefore, $D_{\mathrm{kl}}(u, v) = 0$ iff $u = v$.

3.14.

(a) Since $f(x,z)$ is a concave function of z, $\nabla^2_{zz} f(x,z) \le 0$.

$f(x,z)$ is a convex function of x, $\nabla^2_{xx} f(x,z) \ge 0$.

Also $\nabla^2_{xz} f(x,z) = \nabla^2_{zx} f(x,z)$.

The Hessian matrix is symmetric and positive semi-definite in the top left block and negative semi-definite in the bottom right block.

(b) For each fixed x, $f(x,z)$ is a concave function of z, which means

$$f(\tilde{x}, z) \le f(\tilde{x}, \tilde{z})$$

For each fixed z, $f(x,z)$ is a convex function of x, which means

$$f(\tilde{x}, \tilde{z}) \le f(x, \tilde{z})$$

Together we have:

$$f(\tilde{x}, z) \le f(\tilde{x}, \tilde{z}) \le f(x, \tilde{z})$$

The above inequality implies:

$$\begin{gathered}
\inf_x f(x,z) = f(\tilde{x}, z) \\
\sup_z \inf_x f(x,z) = f(\tilde{x}, \tilde{z}) \\
\sup_z f(x,z) = f(x, \tilde{z}) \\
\inf_x \sup_z f(x,z) = (\tilde{x}, \tilde{z}) \\
\Rightarrow \sup_z \inf_x f(x,z) = \inf_x \sup_z f(x,z)
\end{gathered}$$

(c) If for all z, $f(\tilde{x}, z) \le f(\tilde{x}, \tilde{z})$, then $\nabla_z f(\tilde{x}, \tilde{z}) = 0$.

Similarly, if for all x, $f(\tilde{x}, \tilde{z}) \le f(x, \tilde{z})$, then $\nabla_x f(\tilde{x}, \tilde{z}) = 0$.

In all, $\nabla f(\tilde{x}, \tilde{z}) = 0$

Bonus questions:
Problem A2.5 from additional exercises.

(a) Since C and D are convex, and the intersection of 2 convex sets is convex, then $C \cap D$ is convex. Also, for $x \in C \cap D$, $x \in C \Rightarrow \theta x \in C$ and $x \in D \Rightarrow \theta x \in D$, which implies $\theta x \in C \cap D$, therefore $C \cap D$ is a cone. To combine up, $C \cap D$ is a convex cone.

Since dual cones for convex cones are convex cones, C^* and D^* are convex cones. For $x = u_1 + v_1 \in C^* + D^*$, $y = u_2 + v_2 \in C^* + D^*$,

$$\begin{aligned}
\theta_1 x + \theta_2 y &= \theta_1(u_1 + v_1) + \theta_2(u_2 + v_2) \\
&= \theta_1 u_1 + \theta_2 u_2 + \theta_1 v_1 + \theta_2 v_2
\end{aligned}$$

For $u_1, u_2 \in C^*$, $\theta_1 u_1 + \theta_2 u_2 \in C^*$. For $v_1, v_2 \in D^*$, $\theta_1 v_1 + \theta_2 v_2 \in D^*$. Therefore $\theta_1 x + \theta_2 y \in C^* + D^*$, and $C^* + D^*$ is a convex cone.

(b) For $x = u + v \in C^* + D^*$, we want to show $x \in (C \cap D)^*$. For $y \in C \cap D$, we want to show $x^T y \geq 0$:

$$\begin{gathered} x^T y = u^T y + v^T y \\ u \in C^*, y \in C \Rightarrow u^T y \geq 0 \\ v \in D^*, y \in D \Rightarrow v^T y \geq 0 \\ \Rightarrow x^T y = u^T y + v^T y \geq 0 \end{gathered}$$

We have shown $x \in (C \cap D)^*$, therefore,

$$(C \cap D)^* \supseteq C^* + D^*$$

(c) $C \cap D$ and $C^* + D^*$ are closed convex cones, so

$$(C \cap D)^* \subseteq C^* + D^* \Longleftrightarrow C \cap D \supseteq (C^* + D^*)^*$$

For $x \in (C^* + D^*)^*$, $y = u + v \in C^* + D^*$,

$$x^T y = x^T u + x^T v \geq 0$$

We know that C^*, D^* are pointed. Set $v = 0$, and then $u = 0$, we have:

$$\begin{gathered} x^T u \geq 0 \\ x^T v \geq 0 \end{gathered}$$

Which implies $x \in C^{**}$ or $x \in C$. And $x \in D^{**}$ or $x \in D$. Together, $x \in C \cap D$.

$$C \cap D \supseteq (C^* + D^*)^*$$

We have shown $(C \cap D)^* \subseteq C^* + D^*$ and $(C \cap D)^* \supseteq C^* + D^*$. So

$$(C \cap D)^* = C^* + D^*$$

(d)

$$\begin{aligned} V &= \{x \mid Ax \succeq 0\} \\ &= \{x \mid a_1 x \geq 0\} \cap \cdots \cap \{x \mid a_n x \geq 0\} \end{aligned}$$

$$\begin{aligned} V^* &= \{x \mid a_1 x \geq 0\}^* + \cdots + \{x \mid a_n x \geq 0\}^* \\ &= \{v_1 a_1 \mid v_1 \geq 0\} + \cdots + \{v_n a_n \mid v_n \geq 0\} \\ &= \{v_1 a_1 + \cdots + v_n a_n \mid v_1, \cdots, v_n \geq 0\} \\ &= \{A^T v \mid v \succeq 0\} \end{aligned}$$