ESE 605 Homework 1

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Problems from Boyd & Vandenberghe:
3.15.

(a) According to L'Hospital's Rule,

$$\lim_{\alpha\to 0} u_\alpha = \lim_{\alpha\to 0} \frac{\frac{\partial}{\partial\alpha}(x^\alpha - 1)}{\frac{\partial}{\partial\alpha}\alpha} = \lim_{\alpha\to 0} \frac{x^\alpha logx}{1} = logx$$

(b) For $0 < \alpha < 1$

$$\frac{\partial^2}{\partial x^2} u_\alpha(x) = (\alpha - 1)x^{\alpha-2} < 0$$

So $u_\alpha(x)$ is concave.

$$\frac{\partial}{\partial x} u_\alpha(x) = x^{\alpha-1} > 0$$

So $u_\alpha(x)$ is monotone increasing.

$$0 < \alpha \le 1,\ u_\alpha(1) = \frac{1-1}{\alpha} = 0$$
$$\alpha = 0,\ u_\alpha(1) = log1 = 0$$

So $u_\alpha(1) = 0$.

3.18.

(a) Considering an arbitrary line, given by $X = Z + tV$, where $Z, V \in S_n$. We define $g(t) = f(Z + tV)$, and restrict g to the interval of values of t for which $Z + tV \succ 0$. Without loss of generality, we can assume that $t = 0$ is inside this interval. define $g(t) = f(Z + tV)$

$$\begin{aligned} g(t) &= tr((Z + tV)^{-1}) \\ &= tr(Z^{-1/2}(I + t(Z^{-1/2}V(Z^{-1/2})^{-1}(Z^{-1/2}) \end{aligned}$$

Let $Z^{-1/2}VZ^{-1/2} = Q\Lambda Q^T$ be the Schur decomposition, $Q^TQ = QQ^T = I$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$,

$$\begin{aligned}
g(t) &= tr(Z^{-1}Q(I + t\Lambda)^{-1}Q^T) \\
&= tr(QZ^{-1}Q^T(I + t\Lambda)^{-1}) \\
&= \sum_{i=1}^{n} \left(Q^T Z^{-1} Q\right)_{ii} (1 + t\lambda_i)^{-1}
\end{aligned}$$

Since $(1 + t\lambda_i)^{-1}$ is convex over t and $\left(Q^T Z^{-1} Q\right)_{ii}$ is positive, $g(t)$ is convex. Thus $f(X)$ is convex on its dom.

(b) Similarly, define $g(t) = f(Z + tV)$. Let $Z^{-1/2}VZ^{-1/2} = Q\Lambda Q^T$. We have

$$\begin{aligned}
g(t) &= (\det(Z + tV))^{1/n} \\
&= (\det(Z^{1/2}(I + tQ\Lambda Q^T)Z^{1/2})^{1/n} \\
&= (\det Z)^{1/n} \left(\prod_{i=1}^{n} (1 + t\lambda_i)\right)^{1/n}
\end{aligned}$$

Since $(\det Z)^{1/n}$ is positive and geometric mean $(\prod_{i=1}^{n}(t))^{1/n}$ is concave, $g(t)$ is concave. Thus $f(X)$ is concave on its dom.

3.20.

(a) Since norm function is convex and composition with affine function preserves convexity, $f(x)$ is convex.

(b) First we want to show that $h(X) = -(\det X)^{1/m}, X \succ 0$ is convex. Considering an arbitrary line, given by $X = Z + tV$, we define $g(t) = h(Z + tV)$. Let $Z^{-1/2}VZ^{-1/2} = Q\Lambda Q^T$ be the Schur decomposition.

$$\begin{aligned}
g(t) &= -(\det(Z + tV))^{1/m} \\
&= -\left(\det(Z^{1/2}(I + tQ\Lambda Q^T)Z^{1/2})\right)^{1/m} \\
&= -(\det Z)^{1/m} \left(\prod_{i=1}^{n} (1 + t\lambda_i)\right)^{1/m}
\end{aligned}$$

Since $-(\det Z)^{1/m}$ is negative and geometric mean $(\prod_{i=1}^{n}(t))^{1/n}$ is concave, $g(t)$ is convex. Thus $h(X)$ is convex.

$X = A_0 + x_1A_1 + \cdots + x_nA_n$ is affine, therefore $f(x)$ is convex.

(c) We have shown in 3.18 that $tr(X^{-1})$ is convex. Since $X = A_0 + x_1A_1 + \cdots + x_nA_n$ is affine, $f(x)$ is convex.

3.22.

(c)

$$h(x) = -\log x \text{ is convex on } \mathbf{R}_{++} \text{ and nonincreasing}$$
$$g(x,u,v) = uv - x^T x \text{ is positive and concave}$$

So the composition $f(x,u,v) = h(g(x,v,v))$ is convex.

(d)

$$f(x) = -t^{p-1/p}(t - \|x\|_p^p / t^{p-1})^{1/p}$$
$$h(x,y) = -x^{1/p} y^{1-1/p} \text{ is convex and nonincreasing}$$
$$g_1(t) = t \text{ is concave}$$
$$g_2(x,t) = t - \|x\|_p^p / t^{p-1} \text{ is concave for } t^{p-1} > 0$$

So the composition $f(x,t) = h(g(x,t))$ is convex.

(e)

$$f(x) = -\log\left(t^{p-1}(t - \|x\|_p^p/t^{p-1})\right) = -(p-1)\log t - \log\left(t - \|x\|_p^p/t^{p-1}\right)$$
$$-(p-1)\log t \text{ is concave}$$
$$h(x) = -\log x \text{ is convex and nonincreasing}$$
$$g(x) = t - \|x\|_p^p/t^{p-1} \text{ is concave for } t^{p-1} > 0$$
$$\text{so, } h\left(g(x)\right) = -\log\left(t - \|x\|_p^p/t^{p-1}\right) \text{ is convex}$$

So the combination $f(x)$ is convex.

3.27.

Since $X \in \mathbf{S}_{++}^n$, Y is SPD and Y^{-1} is SPD. So $z^T Y^{-1} z$ is convex in z and Y. Then, $w - z^T Y^{-1} z$ is concave in (w, z, Y), in other word, concave in X.

$$f(X) = (w - z^T Y^{-1} z)^{1/2}$$
$$h(X) = X^{1/2} \text{ is concave and nondecreasing}$$
$$g(X) = w - z^T Y^{-1} z \text{ is concave}$$

From the composition rule, $(w - z^T Y^{-1} z)^{1/2}$ is concave of X.

3.36.

(a)

$$\begin{aligned} f^*(y) &= \sup_{x_i \in \mathbf{R}^n} (y^T x - \max x_i) \\ &\leq \sup_{x_i \in \mathbf{R}^n} (\max x_i \|y\| - \max x_i) \\ &= \sup_{x_i \in \mathbf{R}^n} (\max x_i (\|y\| - 1)) \end{aligned}$$

By setting $x_i = \max x_i$ goes to positive or negative infinity, we can show that if $f^*(y)$ has a upper bound, then $\|y\| = 1$.

Also, if $y \leq 0$, then by setting $\max x_i = -\infty$ and other entries all zero, $f^*(y) \to \infty$.

Therefore, we have

$$f^*(y) = \begin{cases} 0 & \|y\| = 1, y \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

(d) When $p > 1$,

$$f^*(y) = \sup_{x \in \mathbf{R}_{++}} (xy - x^p)$$
$$\frac{\partial}{\partial x}(xy - x^p) = y - px^{p-1} = 0 \Rightarrow x = (\frac{y}{p})^{\frac{1}{p-1}}$$
$$f^*(y) = \begin{cases} (p-1)(\frac{y}{p})^{\frac{p}{p-1}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

When $p < 0$,

$$f^*(y) = \sup_{x \in \mathbf{R}_{++}} (xy - x^p)$$
$$\frac{\partial}{\partial x}(xy - x^p) = y - px^{p-1} = 0 \Rightarrow x = (\frac{y}{p})^{\frac{1}{p-1}}$$
$$f^*(y) = \begin{cases} (p-1)(\frac{y}{p})^{\frac{p}{p-1}} & y < 0 \\ 0 & \text{otherwise} \end{cases}$$

(f)

$$f^*(y, u) = \sup_{(x,t) \in \mathbf{dom} f} (y^T x + ut + \log(t^2 - x^T x))$$
$$\frac{\partial}{\partial x} f^*(y) = y - \frac{2x}{t^2 - x^T x} = 0$$
$$\frac{\partial}{\partial t} f^*(y) = u + \frac{2t}{t^2 - x^T x} = 0$$
$$f^*(y, u) = \begin{cases} -2 + \log 4 + \log(y^T y - u^2) & \|y\|_2 < -u \\ \infty & \text{otherwise} \end{cases}$$

3.39.

(a)

$$\begin{aligned} g^*(y) &= \sup_x (y^T x - f(x) - c^T x - d) \\ &= \sup_x \left((y - c)^T x - f(x)\right) - d \\ &= f^*(y - c) - d \end{aligned}$$

(c)

$$\begin{aligned}
g^*(y) &= \sup_x (y^T x - \inf_z f(x,z)) \\
&= \sup_x (y^T x + \sup_z (-f(x,z))) \\
&= \sup_{x,z} (y^T x - f(x,z)) \\
&= f^*(y, 0)
\end{aligned}$$

If we express $g(x) = \inf_z \{h(z) \mid Az + b = x\}$, then

$$f(x,z) = h(z), Az + b = x$$

$$\begin{aligned}
f^*(y,v) &= \sup_{x,z} (y^T x + v^T z - f(x,z)) \\
&= \sup_{Az+b=x} \left(y^T x + v^T z - h(z)\right) \\
&= \sup_z \left(y^T (Az + b) + v^T z - h(z)\right) \\
&= h^*(A^T y + v) + y^T b
\end{aligned}$$

We know that

$$g^*(y) = f^*(y, 0) = h^*(A^T y) + y^T b$$

(d)

$$f^*(y) = \sup_x (y^T x - f(x))$$

We want to show

$$f^{**}(y) = \sup_x (y^T x - f^*(y)) = f(x)$$

by showing that

$$g(x) = y^T x - f^*(y)$$

generates all affine functions that are global underestimators of f.

Immediately we know that $g(x) \le f(x)$ and $g(x)$ is affine. Further, if an affine function $h(x) = a^T x + b \le f(x)$, then $a \in \mathbf{dom} f^*$ and $f^*(a) = \sup_x (a^T x - f(x)) \le -b$. So $h(x) = g(x)$. We have shown $g(x)$ generates all affine functions that are global underestimators of f.

From the result exercise 3.28, we proved the conjugate of the conjugate of a closed convex function is itself.

3.49.

(a)

$$\log f(x) = \log(e^x/(1+e^x)) = x - \log(1+e^x)$$

We can show $\log(1+e^x)$ is convex by showing it has positive second derivative

$$\frac{\partial^2}{\partial x^2}\log(1+e^x) = \frac{e^x}{(1+e^x)^2} > 0$$

Since x is concave, $\log(1+e^x)$ is convex, $\log f(x)$ is concave.

(b)

$$\log f(x) = -\log(1/x_1 + \cdots + 1/x_n)$$

$$\frac{\partial^2}{\partial x_i x_j}(-\log(1/x_1+\cdots+1/x_n)) = \frac{1/(x_i^2 x_j^2)}{(1/x_1+\cdots+1/x_n)^2}$$
$$\frac{\partial^2}{\partial x_i x_j}(-\log(1/x_1+\cdots+1/x_n)) = \frac{1/x_i^4}{(1/x_1+\cdots+1/x_n)^2} - \frac{2/x_i^3}{1/x_1+\cdots+1/x_n}$$
$$\Rightarrow \nabla^2 \log f(x) = \frac{1}{(1/x_1+\cdots+1/x_n)^2}\left(qq^T - (1/x_1+\cdots+1/x_n)\mathbf{diag}(2/x_1^3,\cdots,2/x_n^3)\right)$$

where $q_i = 1/x_i^2$.

We want to show $H = \nabla^2(-\log(1/x_1+\cdots+1/x_n)) \preceq 0$, we must verify that $v^T H v \leq 0$ for all v,

$$v^T H v = \frac{1}{(1/x_1+\cdots+1/x_n)^2}\left(\left(\sum_{i=1}^n v_i/x_i^2\right)^2 - 2\sum_{i=1}^n 1/x_i \sum_{i=1}^n v_i^2/x_i^3\right)$$

From Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^n v_i/x_i^2\right)^2 \leq 2\sum_{i=1}^n 1/x_i \sum_{i=1}^n v_i^2/x_i^3$$

We have shown that $H \preceq 0$ and therefore $\log f(x)$ is concave.

(c)

$$\log f(x) = \sum_{i=1}^n \log x_i - \log(\sum_{i=1}^n x_i)$$

Consider its intersection with an arbitrary line:

$$\begin{aligned} g(t) &= \log f(x+tv) \\ &= \sum_{i=1}^n \log(x_i+tv_i) - \log(\sum_{i=1}^n (x_i+tv_i)) \end{aligned}$$

$\mathbf{dom} g = \{t \mid x + tv > 0\}$

we want to check the convexity of $g(t)$,

$$g''(t) = -\sum_{i=1}^{n} \frac{v_i^2}{(x_i + tv_i)^2} + \frac{(\sum_{i=1}^{n} v_i)^2}{(\sum_{i=1}^{n} (x_i + tv_i))^2}$$

We can just consider $t = 0$, since for any fixed t_0, we can always redifine $x_0 = x + t_0 v$.

$$g''(0) = -\sum_{i=1}^{n} \frac{v_i^2}{x_i^2} + \frac{(\sum_{i=1}^{n} v_i)^2}{(\sum_{i=1}^{n} x_i)^2}$$

We notice that increase v by a factor c will lead to $g''(0)$ increase by a factor c^2, which means scale of v will not change the sign of $g''(0)$. Without loss of generality, we can assume $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} x_i$.

Now we want to show $\sum_{i=1}^{n} \frac{v_i^2}{x_i^2} \geq 1$ by solving the following problem:

$$\begin{aligned} \min . & \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} \\ \text{s.t.} & \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} x_i \end{aligned}$$

The Lagrangian is:

$$L(v, \lambda) = \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} - \lambda(\sum_{i=1}^{n} v_i - \sum_{i=1}^{n} x_i)$$

$$\frac{\partial}{\partial v_i} L(v, \lambda) = \frac{2v_i}{x_i^2} - \lambda = 0 \Rightarrow v_i = \frac{\lambda x_i^2}{2}$$

Since $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} x_i$, $\lambda = 2 \sum x_i / \sum x_i^2$.

Therefore the minimal value is:

$$\sum_{i=1}^{n} \frac{v_i^2}{x_i^2} = \left(\frac{\sum x_i}{\sum x_i^2} \right)^2 \sum x_i^2 = \frac{(\sum x_i)^2}{\sum x_i^2} \geq 1$$

We have proved $g''(0) \leq 0$ and thus $g(t)$ is concave. So $\log f(x)$ is concave.

4.4.

(a)

$$Q_j \bar{x} = (1/k) \sum_{i=1}^{k} Q_i Q_j \bar{x}$$

Since $\mathcal{G}$ is closed under products, $Q_i Q_j \in \mathcal{G}$. Therefore $Q_j \bar{x} = \bar{x}$, and $\bar{x} \in \mathcal{F}$.

(b) According to the definition of convexity,

$$f(\bar{x}) = f\left((1/k)\sum_{i=1}^{k} Q_i x\right) \leq (1/k)\sum_{i=1}^{k} f(x) = f(x)$$

(c) For an optimal point x_0, we show that for $\bar{x_0}$:

$$f_0(\bar{x_0}) = f_0\left((1/k)\sum_{i=1}^{k} Q_i x_0\right) \leq (1/k)\sum_{i=1}^{k} f_0(x_0) = f_0(x_0)$$

$$f_i(\bar{x_0}) \leq (1/k)\sum_{i=1}^{k} f_i(x_0) \leq 0, \quad i = 1, \cdots, m$$

So $\bar{x_0}$ is also optimal. As we have shown in (a), $\bar{x_0} \in \mathcal{F}$.

(d) We know that for permutation $P = P_1, \cdots, P_m$,

$$Px = (1/m)\sum_{i=1}^{m} P_i x = (\mathbf{E}x)\mathbf{1}$$

For an optimal point x_0, we show that for $\bar{x_0}$:

$$f(Px_0) = f((\mathbf{E}x_0)\mathbf{1}) \leq f(x_0)$$

So $(\mathbf{E}x_0)\mathbf{1}$ is also optimal.

4.9.

Since A is square and nonsingular, A^{-1} exists. The problem could be written as:

$$\begin{array}{ll} \text{minimize} & c^T A^{-1} A x \\ \text{subject to} & Ax \preceq b \end{array}$$

If $A^{-T}c \preceq 0$, in order to minimize the objective function, we want to maximize Ax. So the optimal solution is $Ax = b$. The value of the objective function is $c^T A^{-1} b$.

If $A^{-T}c \npreceq 0$, which means at least one of the entries of $A^{-T}c$ is greater than zero, so we can let the corresponding entry of Ax goes to $-\infty$, and then the objective function goes to $-\infty$.

4.11.

(a)

$$\|x\|_\infty = \max |x_i|$$

We want to find the largest entry of $Ax - b$, name it t. It is equivalent to find the smallest value of t that is greater or equal to every entry of $Ax - b$.

Thus, the ℓ_∞-norm approximation problem is equivalent to the following LP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -t\mathbf{1} \preceq Ax - b \preceq t\mathbf{1} \end{array}$$

where $t \in \mathbf{R}$.

(b)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

For each entry of $Ax - b$, which is $a_i^T x - b_i$, let $t_i = |a_i^T x - b_i|$. We can see that t_i is the smallest value satisfies $-t_i < a_i^T x - b_i < t_i$

Thus, the ℓ_1-norm approximation problem is equivalent to the following LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & -t \preceq Ax - b \preceq t \end{array}$$

where $t \in \mathbf{R}^m$.

(c) Similar to what we have shown in (a), the restriction function $\|x\|_\infty \leq 1$ is equivalent to $-\mathbf{1} \preceq x \preceq \mathbf{1}$.

Combining the LP expression of the ℓ_1-norm approximation problem that we have shown in (b), the original problem is equivalent to the following LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & -t \preceq Ax - b \preceq t \\ & -\mathbf{1} \preceq x \preceq \mathbf{1} \end{array}$$

where $t \in \mathbf{R}^m$.

4.13.

If $x_i > 0$, then $a_i^T x \leq b_i$ is equivalent to

$$(\bar{a}_i + v_i)^T x \leq b_i$$

If $x_i < 0$, then $a_i^T x \leq b_i$ is equivalent to

$$(\bar{a}_i - v_i)^T x \leq b_i$$

So the original problem is equivalent to;

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x + V|x| \preceq b \end{array}$$

which is equivalent to;

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x + Vy \preceq b \\ & -y \preceq x \preceq y \end{array}$$

Bonus questions:
3.28.

(a) Since $f(x)$ is convex, for an arbitrary point $(a, f(a))$, there exist an affine function $g_a(x) = a_a x + b_a$ that passes through $(a, f(a))$ and lies beneath $f(x)$.

Therefore,

$$\sup_a g_a(x) \leq f(x)$$

because all $g_a(x)$ lies beneath $f(x)$.

$$\sup_a g_a(x) \geq f(x)$$

because $g_a(x)$ intersect $f(x)$ at point $(a, f(a))$.

Also,

$$\sup\{g(x) \mid g \text{ affine }, g(z) \leq f(z)\} = \sup_a g_a(x)$$

We can conclude that $\tilde{f}(x) = f(x)$.

3. conjugate function for the convex quadratic.

$$f^*(y) = sup_x(y^T x - \frac{1}{2}x^T Qx)$$

If $Qx = b$ has a solution,

$$\frac{\partial}{\partial x}(y^T x - \frac{1}{2}x^T Qx) = y - Qx = 0 \Rightarrow Qx = b$$
$$f^*(y) = \frac{1}{2}y^T Qy$$

If $Qx = b$ has no solution, $y^T x - \frac{1}{2}x^T Qx$ is unbounded.

$$f^*(y, u) = \begin{cases} \frac{1}{2}y^T Qy & \text{Qx=b has a solution} \\ \infty & \text{otherwise} \end{cases}$$