ME 702 – COMPUTATIONAL FLUID DYNAMICS

(L Barba)

Practical Module – Burgers Equation, 5 ways

Classic nonlinear 1st order hyperbolic equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \rightarrow \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \quad \text{in Conservative form}$$

OR: $\partial u/\partial t = -\partial E/\partial x$ with $E = \frac{u^2}{2}$

STEP ① : Lax–Friedrichs (explicit, 1st order)

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$$

STEP ② : Lax – Wendroff

with $A = \frac{\partial E}{\partial u}$, the Jacobian (A = u for Burgers)

From the Taylor expansion $u^{n+1} = u^n + u_t \cdot \Delta t + \frac{(\Delta t)^2}{2} u_{tt} + \ldots$

and substituting time derivatives by spatial derivatives
$E_t = -A E_x$ and $u_t = -E_x$ (from the equation)

Get:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \frac{\Delta t}{2}\left(\frac{\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n - \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n}{\Delta x}\right)$$

then approximate:

$$\left(\frac{\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n - \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n}{\Delta x}\right) = \frac{A_{i+1/2}^n\left(\frac{E_{i+1}^n - E_i^n}{\Delta x}\right) - A_{i-1/2}^n\left(\frac{E_i^n - E_{i-1}^n}{\Delta x}\right)}{\Delta x}$$

$$= \frac{\frac{1}{2\Delta x}\left(A_{i+1}^n + A_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \frac{1}{2\Delta x}\left(A_i^n + A_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)}{\Delta x}$$

after evaluating the Jacobian at the midpoints.

Finally, the LW scheme: (with $A = u$)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\Big[\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right)$$

$$- \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\Big].$$

STEP (3) – MacCormack

$$u_i^* = u_i^n - \frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right) \qquad \text{(predictor)}$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{\Delta t}{\Delta x}\left(E_i^* - E_{i-1}^*\right)\right] \qquad \text{(corrector)}$$

STEP (4) – Beam & Warming Implicit

Starting from Taylor expansions + manipulations, we get:

$$u_i^{n+1} = u_i^n + \frac{1}{2}\left[\frac{\partial u}{\partial t}\Big|_i^n + \frac{\partial u}{\partial t}\Big|_i^{n+1}\right]\Delta t + O(\Delta t^3)$$

For Burgers: $\partial u/\partial t = -\partial E/\partial x$, so:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left[\frac{\partial E}{\partial x}\Big|_i^n + \frac{\partial E}{\partial x}\Big|_i^{n+1}\right] + O(\Delta t^2)$$

$$= -\frac{1}{2}\left\{\frac{\partial E}{\partial x}\Big|_i^n + \frac{\partial E}{\partial x}\Big|_i^n + \frac{\partial}{\partial x}\left[A\left(u_i^{n+1} - u_i^n\right)\right]\right.$$

after introducing Taylor expansions for E.

Use 2nd order CD for the term with the Jacobian A:

$$\frac{\partial}{\partial x}\left[A\left(u_i^{n+1} - u_i^n\right)\right] = \frac{1}{2\Delta x}\left(A_{i+1}^n u_{i+1}^{n+1} - A_{i-1}^n u_{i-1}^{n+1}\right)$$

$$- \frac{1}{2\Delta x}\left(A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n\right).$$

Which leads to the tri-diagonal system:

$$-\frac{\Delta t}{4\Delta x}\left(A_{i-1}^{n}\, u_{i-1}^{n+1}\right) + u_i^{n+1} + \frac{\Delta t}{4\Delta x}\left(A_{i+1}^{n}\, u_{i+1}^{n+1}\right) =$$

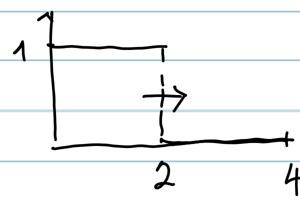
$$= u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t}{4\Delta x}\left(A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n\right).$$

STEP 15 — Add 4th order damping to BW-implicit

add to the RHS:

$$D = -\epsilon_e\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$$

TEST PROBLEM



I.C. $u(x,0) = \begin{cases} 1 & 0 \le x < 2 \\ 0 & 2 \le x \le 4 \end{cases}$

* Investigate the propagation of this discontinuous function with the 5 schemes listed
* Start with $\Delta t = \Delta x = 1$, then change $\Delta t/\Delta x = 1$ to $\Delta t/\Delta x = 0.5$ to see the effect of Courant number
* Change to finer step sizes to see the effect of the mesh resolution.
* for STEP 15, experiment with values $0 < \epsilon_e < 0.125$
  With $\epsilon_e = 0.1$, experiment with different step sizes and different Courant numbers $\Delta t/\Delta x$
* Try some values $\epsilon_e > 0.125$
* Consider the expected advantage of implicit methods (stability at large Δt) and the observations here.

Q — Which scheme gives the best results for the inviscid Burgers equation?