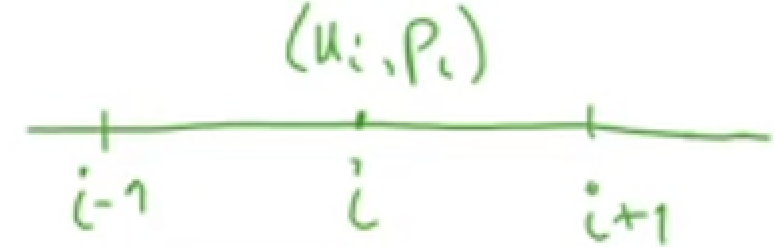
Collocated vs. Staggered grids

- Illustration: consider 1D N-S system

$$\frac{\partial}{\partial x}(u^2) = 2u\,\frac{\partial u}{\partial x}$$



(1) $$\frac{\partial u}{\partial x} = 0$$

(2) $$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2}$$

C.D. discretization

(3) $$\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = 0$$

(4) $$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{2\Delta x} = -\frac{1}{\rho}\,\frac{p_{i+1} - p_{i-1}}{2\Delta x} + \nu\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

- apply the "pressure correction approach": u^* : intermediate velocity
  - split the N-S equations in two parts

$$(5)\quad \frac{u_i^* - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$(6)\quad \frac{u_i^{n+1} - u_i^*}{\Delta t} = -\frac{1}{\rho} \frac{p_{i+1} - p_{i-1}}{2\Delta x}$$

$$\hookrightarrow \quad u_i^{n+1} = -\frac{\Delta t}{\rho} \frac{p_{i+1} - p_{i-1}}{2\Delta x} + u_i^*$$

use this expression in continuity (3)

$$\frac{1}{2\Delta x}\Big(\underbrace{-\frac{\Delta t}{\rho} \frac{p_{i+2} - p_i}{2\Delta x} + u_{i+1}^*}_{u_{i+1}^{n+1}} + \underbrace{\frac{\Delta t}{\rho} \frac{p_i - p_{i-2}}{2\Delta x} - u_{i-1}^*}_{u_{i-1}^{n+1}}\Big) = 0$$

Rewrite

$$(7)\quad \frac{p_{i+2} - 2p_i + p_{i-2}}{4\Delta x^2} = \frac{u_{i+1}^* - u_{i-1}^*}{2\Delta x} \cdot \frac{\rho}{\Delta t}$$

$\frac{\partial^2 p}{\partial x^2}$ on a $2\Delta x$ grid

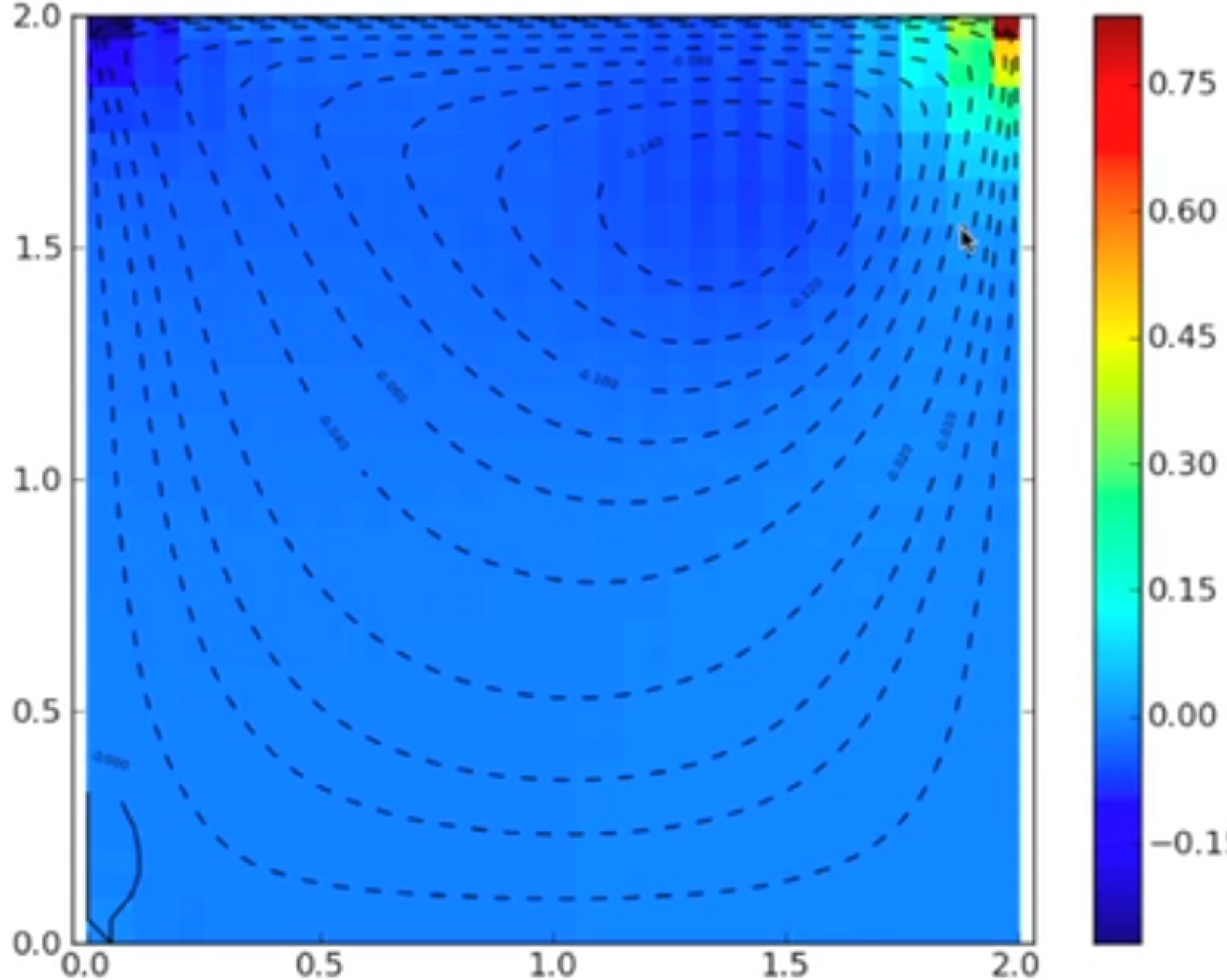
$\hookrightarrow$ Poisson equation for p, ensures that continuity is satisfied

- *Odd-even decoupling:*

stencil for p : $i+2,\ i,\ i-2$
stencil for u^* : $i+1,\ i-1$

  - the pressure at point i is not influenced by the velocity component u_i^n and viceversa $\Rightarrow$ can result in high-frequency oscillations!

Step 11
41x41 mesh
dx=dy=0.05
nu=0.01 giving
Re=200



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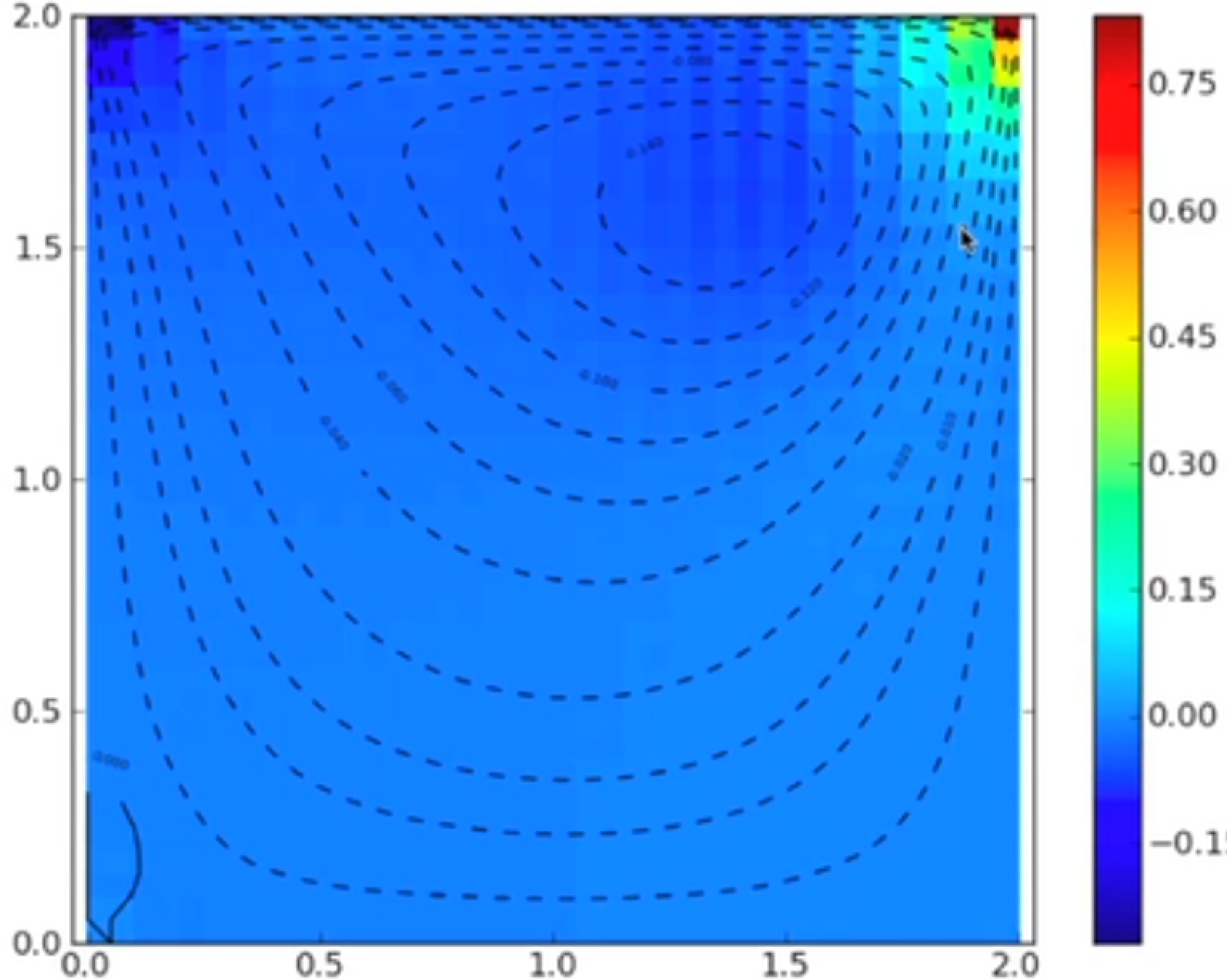

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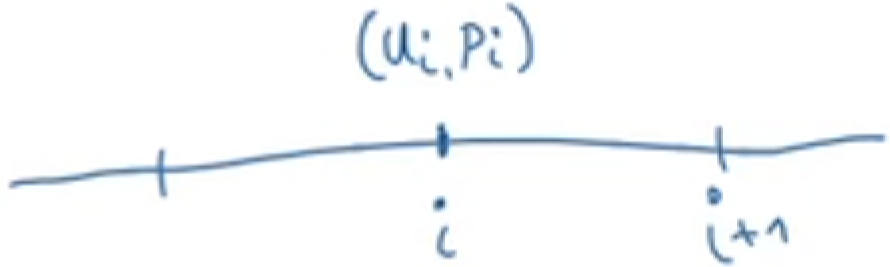
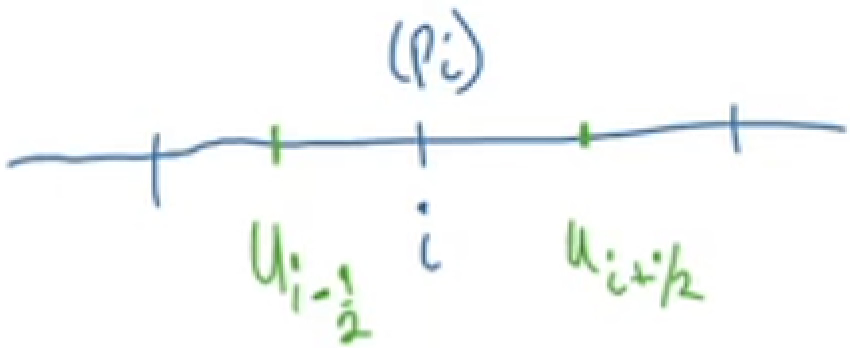
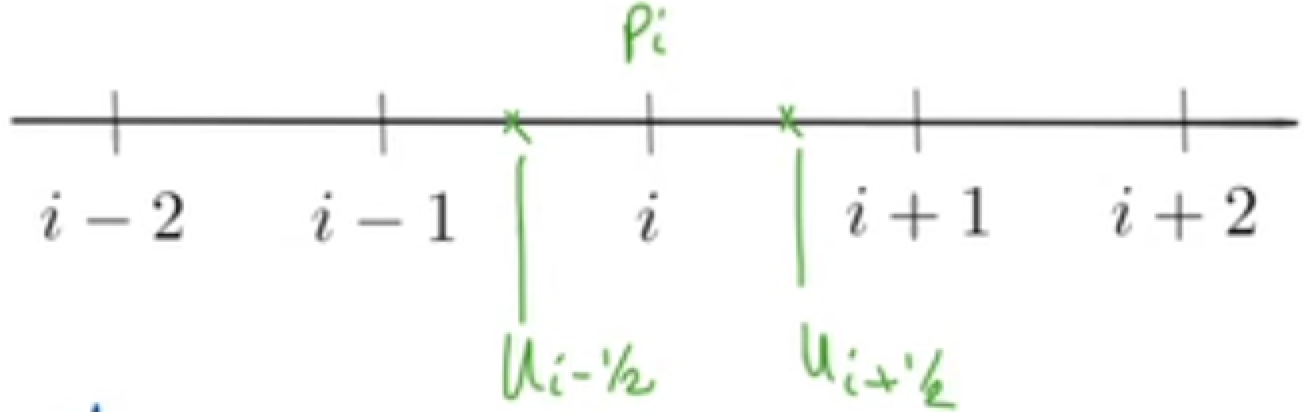


collocated grid



staggered grid

- *Solution* $\Rightarrow$ “staggered grid”
  - *due to Harlow & Welch (1965)*



(8) $$\frac{u^{n+1}_{i+1/2} - u^{n+1}_{i-1/2}}{\Delta x} = 0$$

"fractional step" or "pressure correction"

(9) $$\frac{u^{n+1}_{i+1/2} - u^{*}_{i+1/2}}{\Delta t} = -\frac{1}{\rho}\,\frac{p_{i+1} - p_i}{\Delta x}$$

$\Bigg\{$ As before, subs. in continuity

(2) $$\frac{p_{i+1} - 2p_i + p_{i-1}}{\Delta x^2} = \frac{\rho}{\Delta t}\,\frac{u^{*}_{i+1/2} - u^{*}_{i-1/2}}{\Delta x}$$

if we get values at $i+1/2$, $i-1/2$ by averaging adjacent values $\longrightarrow$ fully coupled,

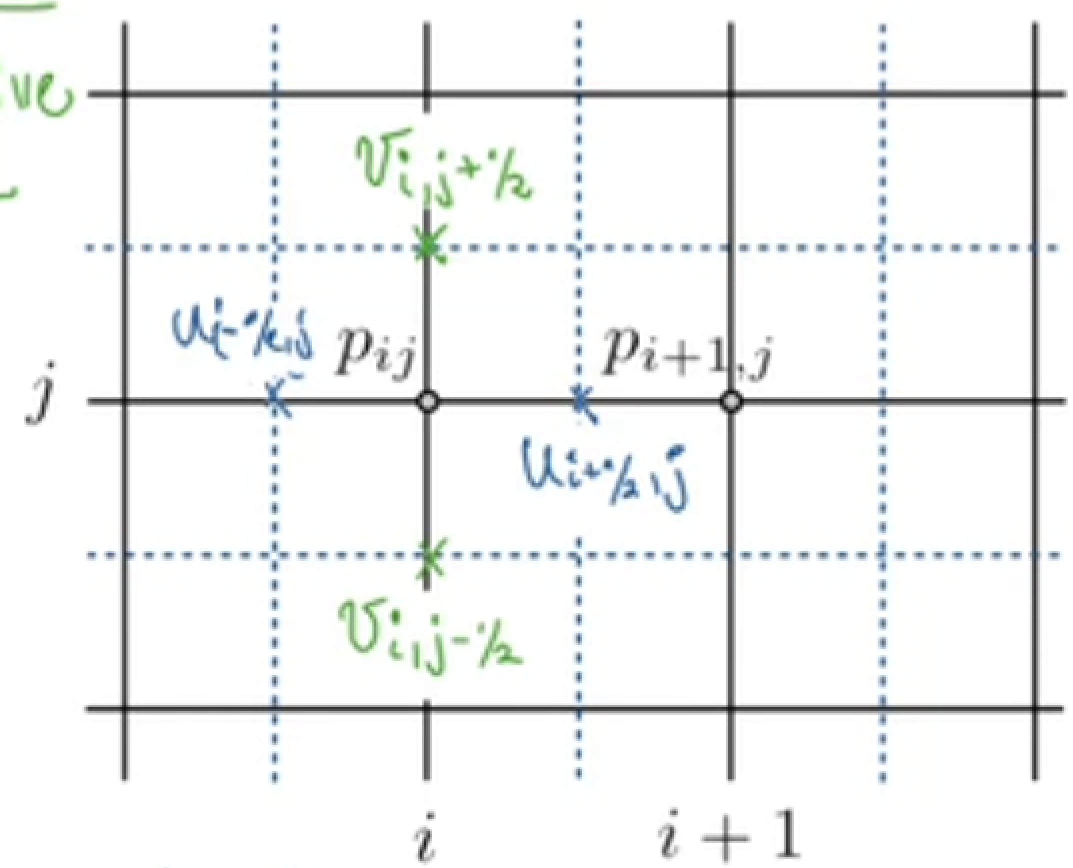
- *Staggered grid in 2D:*

"conservative form"

(11) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(12) $$\frac{\partial u}{\partial t} + \underbrace{\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}}_{\circledast} = -\frac{\partial \varphi}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(13) $$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{\partial \varphi}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$



$\varphi \rightarrow$ pressure divided by ρ

F.D. approximations

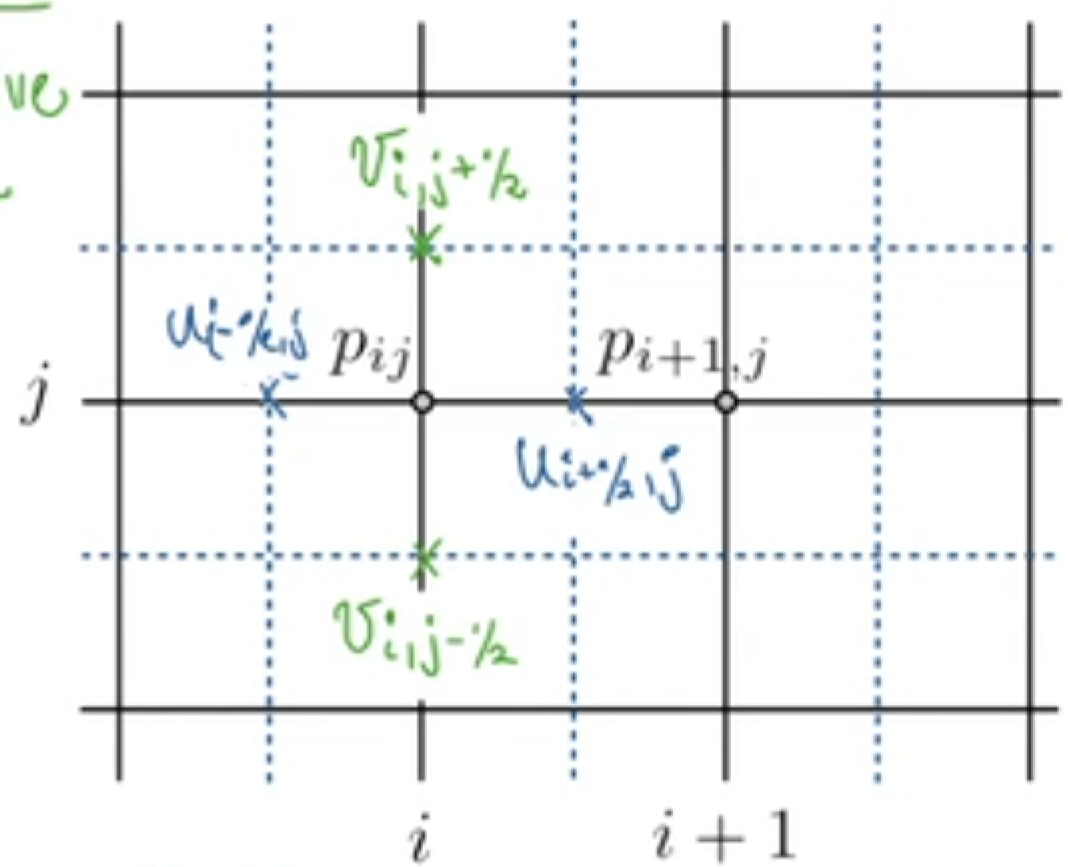
$$\frac{\partial u}{\partial t} \rightarrow \left(u^{n+1}_{i+1/2,j} - u^{n}_{i+1/2,j}\right)\frac{1}{\Delta t}$$

$$\frac{\partial u^2}{\partial x} \rightarrow \frac{1}{\Delta x}\left(\left(u^n_{i,j}\right)^2 - \left(u^n_{i+1,j}\right)^2\right)$$

$$\frac{\partial (uv)}{\partial y} \rightarrow \frac{1}{\Delta y}\left\{\left(u^n_{i+\frac{1}{2},j+1/2}\right)\left(v^n_{i+1/2,j+1/2}\right) - \left(u^n_{i+1/2,j-1/2}\right)\left(v^n_{i+1/2,j-1/2}\right)\right]$$

- *Staggered grid in 2D:*

// conservative form //



(11) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

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$\varphi \to$ pressure divided by ρ

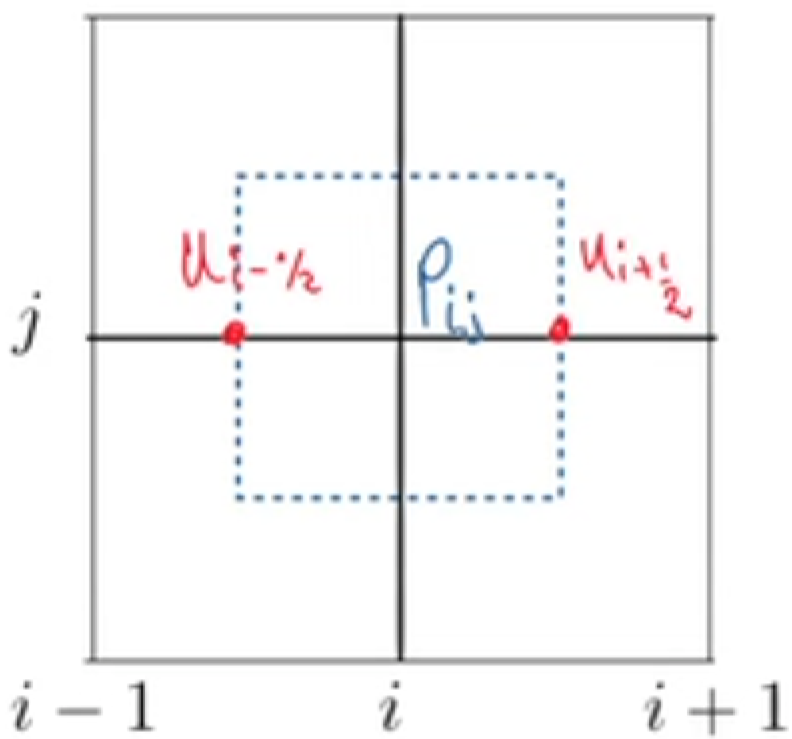
F.D. approximations

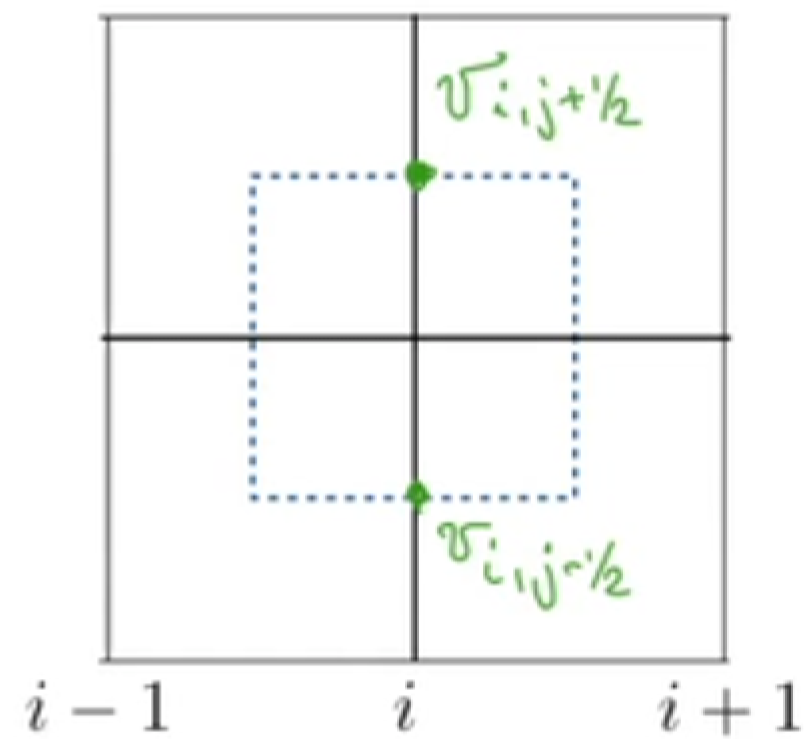
$$\frac{\partial u}{\partial t} \to \left(u^{n+1}_{i+1/2,j} - u^{n}_{i+1/2,j}\right)\frac{1}{\Delta t}$$

$$\frac{\partial u^2}{\partial x} \to \frac{1}{\Delta x}\left(\left(u^n_{i,j}\right)^2 - \left(u^n_{i+1,j}\right)^2 \right)$$

$$\frac{\partial (uv)}{\partial y} \to \frac{1}{\Delta y}\left\{ \left(u^n_{i+\frac{1}{2},j+1/2}\right)\left(v^n_{i+1/2,j+1/2}\right) - \left(u^n_{i+1/2,j-1/2}\right)\left(v^n_{i+1/2,j-1/2}\right) \right]$$

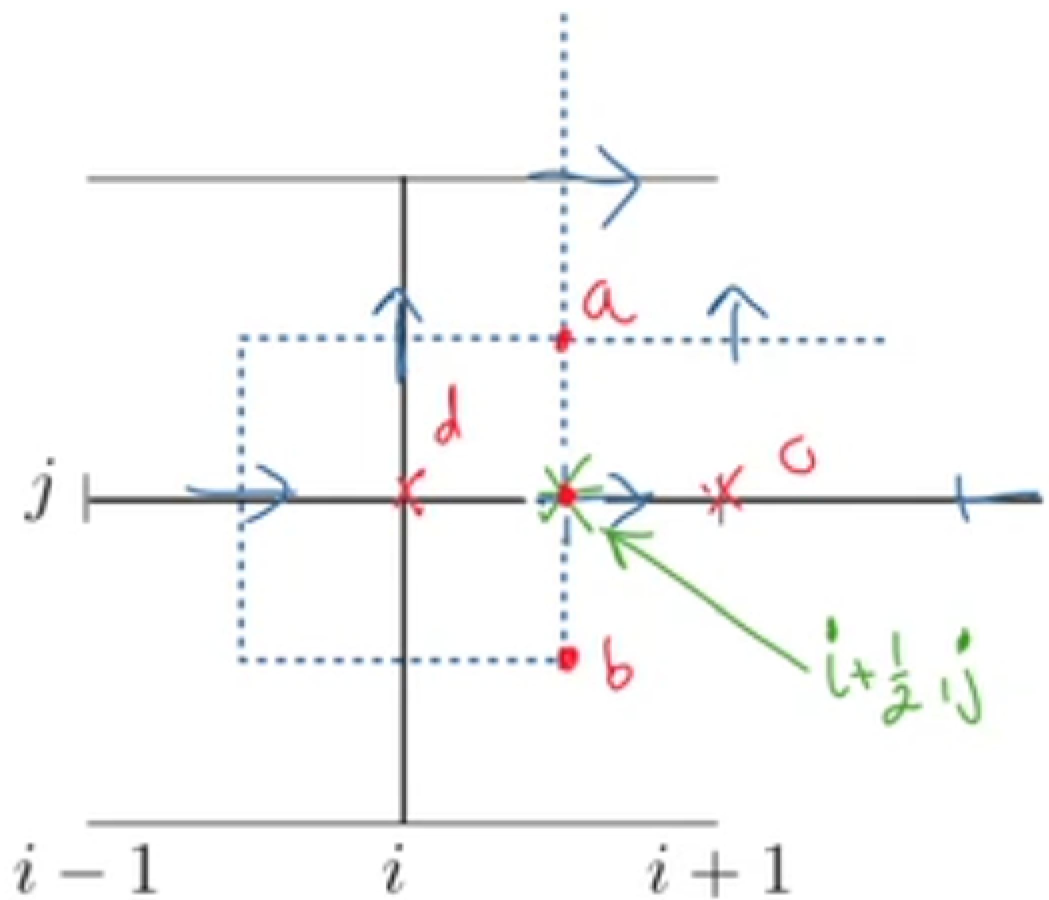
- horizontal velocity located at the mid-points of the vertical "cell edges" and vertical velocity at the midpoints of the horizontal cell edges.





- to get, e.g., convection terms

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{u_a v_a - u_b v_b}{\Delta y}$$



take: $u_a = \frac{1}{2}\left(u_{i+\frac{1}{2},j+1} + u_{i+\frac{1}{2},j}\right)$

$v_a = \frac{1}{2}\left(v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}}\right)$

similarly u_b, v_b

For $\frac{\partial}{\partial x}u^2 \simeq \frac{u_c^2 - u_d^2}{\Delta x}$ where $u_c = u_{i+1,j}$ (not known)

$$= \frac{1}{2}\left(u_{i+\frac{3}{2},j} + u_{i+\frac{1}{2},j}\right)$$

historical notes

- HW'65 introduced "marker and cell method"
  - included a ste of marker particles to follow free surfaces (now obsolete)
  - current usage of "MAC method"
    - *projection method using a staggered grid*

THE PHYSICS OF FLUIDS VOLUME 8, NUMBER 12 DECEMBER 1965

Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with Free Surface

Francis H. Harlow and J. Eddie Welch
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico

(Received 16 April 1965; final manuscript received 3 September 1965)

Boundary conditions for incompressible N-S

(14) $$\nabla \cdot \vec{u} = 0$$

$\underline{x} \in \Omega$
domain

(15) $$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \vec{u}$$

(16) I.C. $\rightarrow$ $\underline{u}(\underline{x}, t_0) = \underline{u}_0(\underline{x})$, with $\nabla \cdot \underline{u}_0 = 0$ in Ω

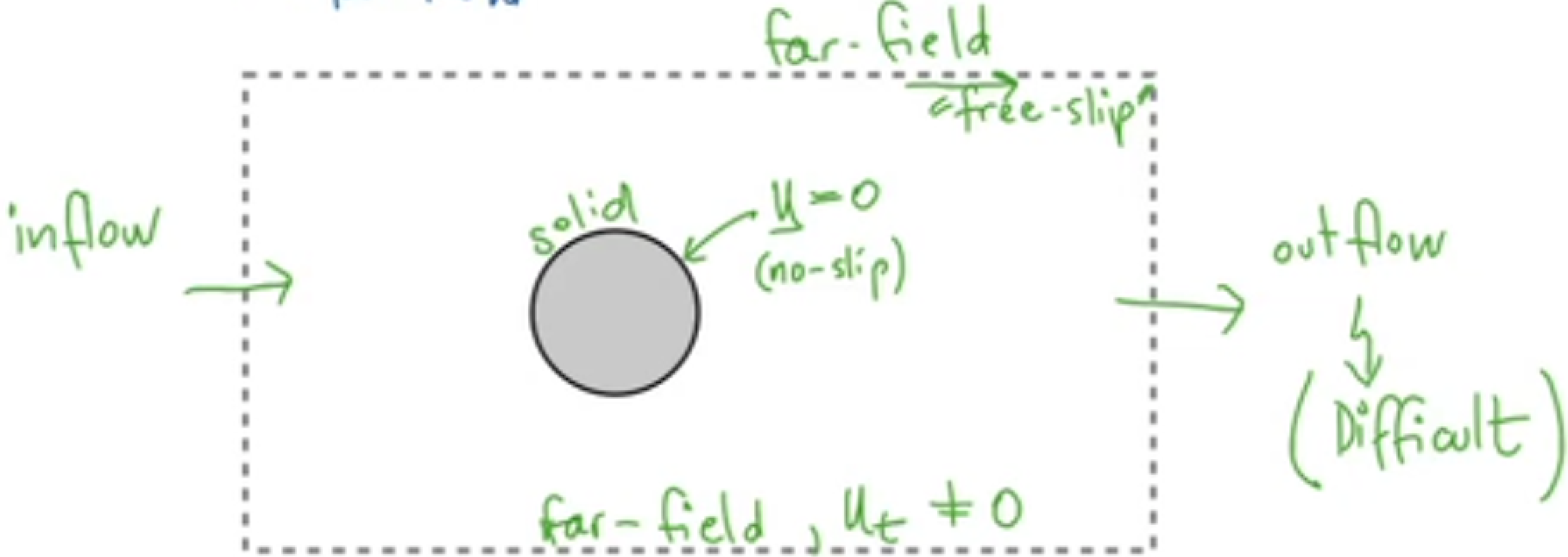
(17) B.C. $\rightarrow$ $\underline{u}(\underline{x}, t) = \underline{u}_\Gamma(\underline{x}, t)$, $\underline{x} \in \Gamma = \partial\Omega$

Global mass conservation

(18) $\oint_\Gamma \hat{n} \cdot \underline{u}_\Gamma \, d\sigma = 0$, $\hat{n}$: normal to bdry. Γ

- in practice, B.C.s are stipulated separately for the different types:

  - inflow
  - outflow
  - solid surface
  - far-field



No-slip at body surface $\rightarrow$ $\underline{u}\big|_{\Gamma} = \underline{u}_{body}$ (= 0 usually)

* pressure at surface is not known
* No equation for p (we have to derive one)

$\hookrightarrow$ PPE (pressure Poisson eqn.) $\quad \nabla^2 p = -\nabla \cdot (\underline{u} \cdot \nabla \underline{u}) \quad (19)$

Applied Mechanics Reviews MAY 2006, Vol. 59 / 107

On Boundary Conditions for Incompressible Navier-Stokes Problems

Dietmar Rempfer
Department of Materials, Mechanical and Aerospace Engineering,
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We revisit the issue of finding proper boundary conditions for the field equations describing incompressible flow problems, for quantities like pressure or vorticity, which often do not have immediately obvious "physical" boundary conditions. Most of the issues are discussed for the example of a primitive-variables formulation of the incompressible Navier-Stokes equations in the form of momentum equations plus the pressure Poisson equation. However, analogous problems also exist in other formulations, some of which are briefly reviewed as well. This review article cites 95 references.
[DOI: 10.1115/1.2177683]

Keywords: flow simulation, numerical methods, Navier-Stokes equations

- "fractional step method" or "projection method"

(20) $$\frac{\underline{u}^* - \underline{u}^n}{\Delta t} = -(\underline{u}^n \cdot \nabla)\underline{u}^n + \frac{1}{Re}\nabla^2 \underline{u}^n \quad , \underline{x} \in \Omega$$

(21) $$\underline{u}^* \big|_\Gamma = \underline{u}_\Gamma(x, t^{n+1}), \underline{x} \in \Gamma$$

$\hookrightarrow$ $\underline{u}^*$ not necessarily divergence-free

$\rightarrow$ restore div-free velocity introducing a "projector", $\tilde{p}$

(22) $$\frac{\underline{u}^{n+1} - \underline{u}^*}{\Delta t} = -\nabla \tilde{p}^{n+1} \quad , x \in \Omega$$

(23) $$\nabla \cdot \underline{u}^{n+1} = 0 \quad , \hat{n} \cdot \underline{u}^{n+1} \big|_\Gamma = \hat{n} \cdot \underline{u}_\Gamma(\underline{x}, t^{n+1})$$

$\tilde{p}$ Not identical with pressure

- the projector is not identical with the pressure

JOURNAL OF COMPUTATIONAL PHYSICS 59, 308–323 (1985)

Application of a Fractional-Step Method to Incompressible Navier–Stokes Equations

J. KIM AND P. MOIN

Computational Fluid Dynamics Branch,
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Received March 15, 1984; revised September 4, 1984

$$\frac{u_i^{n+1} - \hat{u}_i}{\Delta t} = -G(\phi^{n+1}),$$

overall accuracy of the splitting method is still second order. Note that ϕ is different from the original pressure: in fact, $p = \phi + (\Delta t/2\,\mathrm{Re})\nabla^2\phi$. All the spatial derivatives

- equation for $\tilde{p}$

(24) $$\nabla^2 \tilde{p}^{n+1} = \frac{\nabla \cdot u^*}{\Delta t}$$

By considering B.C.s (21) & (23) $\Rightarrow \hat{n} \cdot \nabla \tilde{p}^{n+1} \big|_{\Gamma} = 0$

or $$\frac{\partial \tilde{p}^{n+1}}{\partial n} = 0$$ homogeneous Neumann B.C.

- equation for $\tilde{p}$

Applied Mechanics Reviews

MAY 2006, Vol. 59 / 107

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The exploration of this problem of finding appropriate boundary conditions for the pressure forms the core of the present paper. As mentioned in our introduction, in addressing this question we are dealing with a highly controversial issue. In order to do so,

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 7, 1111–1145 (1987)

ON PRESSURE BOUNDARY CONDITIONS FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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AND

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$$(\partial \mathbf{u}/\partial t) + \nabla P = \nu\,\nabla^2 \mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} \equiv \mathbf{f}$$

and

$$\nabla\cdot(\partial \mathbf{u}/\partial t) = 0 \quad \text{in} \quad \Omega;$$

these imply

$$\nabla^2 P = \nabla\cdot\mathbf{f} \quad \text{in} \quad \Omega;$$

also

$$\partial P/\partial n = \mathbf{n}\cdot(\mathbf{f} - (\partial\mathbf{u}/\partial t)) \quad \text{on} \quad \Gamma.$$

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cal example below, it is illegal to write down the momentum equation (1) taken at the boundary and derive a pressure boundary condition from it by simply projecting the result on the wall-normal coordinate η.

Despite the progress in understanding that had been achieved by the contributions mentioned above, in 1987 Gresho and Sani perpetuated and greatly contributed to the confusion in this area

situation any more settled now? Unfortunately, the answer to that question is negative: Improper pressure boundary conditions were still presented in review articles in the 1990s [12,60], and they found their way into some of the newest textbooks on computational fluid mechanics [13,61].

- Final note …

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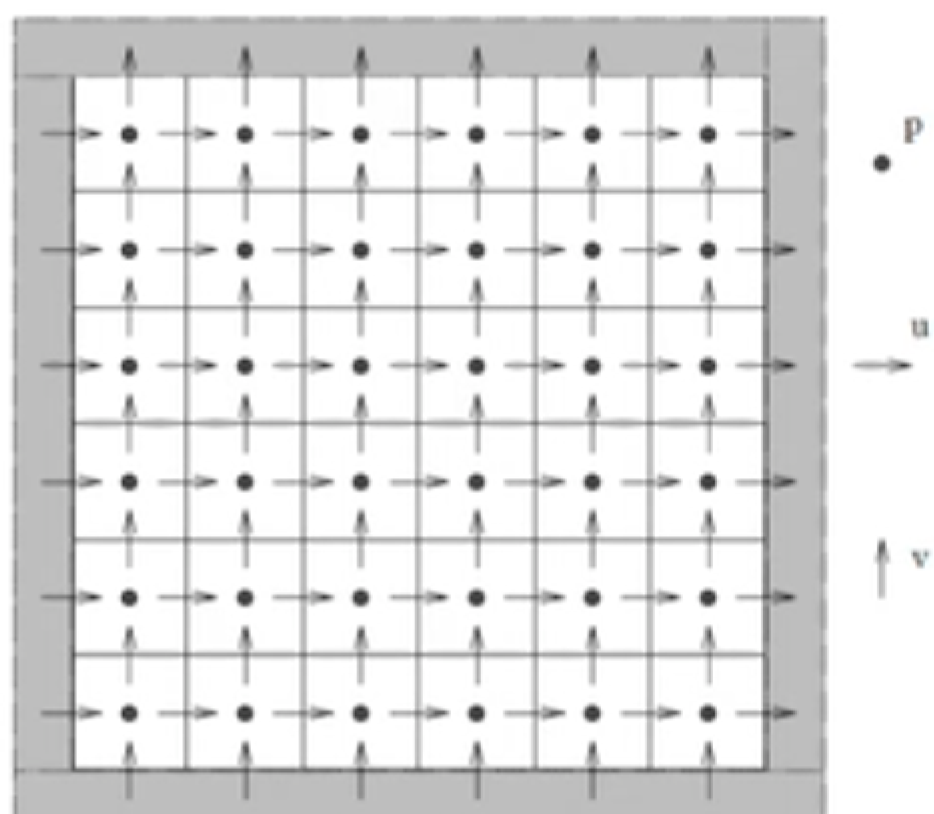
equations are evaluated at velocity nodes, and the continuity equation is enforced for each cell. One important advantage of using the staggered mesh for incompressible flows is that ad hoc pressure boundary conditions are not required.



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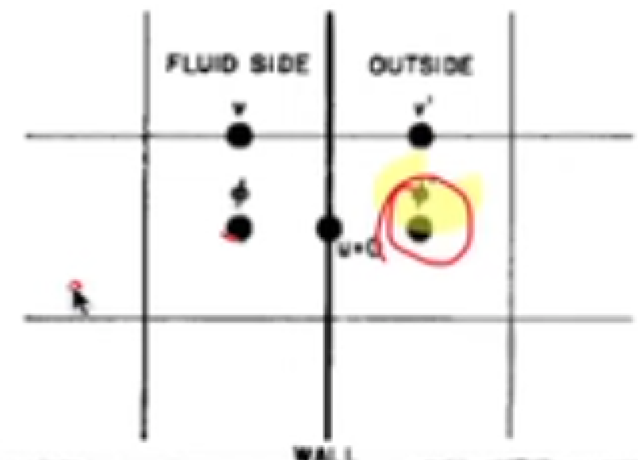
wall as shown in Fig. 4. The vertical velocities are simply reversed across the wall. Since $D = 0$ in the fluid cell, it follows that the vanishing of D' is accomplished only if $u' = +u_1$, in contrast to the requirement for u' for a free-slip wall.

In summary: (a) for a free-slip wall normal velocity reverses while tangential velocity remains the same; (b) for a no-slip wall normal velocity remains the same, while tangential velocity reverses.

For case (a) the pressure condition has been derived and has a simple form. For case (b) the no-slip wall, if vertical

$$\varphi' = \varphi_1 \pm g_x\,\delta x \pm (2\nu u_1/\delta x) \qquad (11)$$



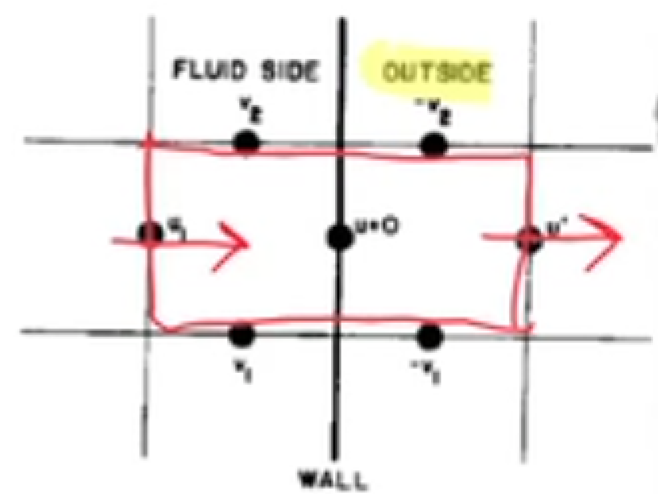


Fig. 4. Reflection of field variables near a wall.