A document for implementing potential correction method into autolens

Substructures finding collaboration[1,2,3]
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ABSTRACT

The invisible mass substructures embedded in the lens galaxy could imprint anomaly flux signals to the gravitationally lensed arc. With dedicated lens modeling, we could recover the total mass and position of those invisible mass substructures in principle. One of the non-parametric ways to do this task is the so-called 'potential correction method' (or 'gravitational imaging technique'). In this team document, we review the basic framework of the potential correction method, with a specific focus on technical aspects. The essential aim of this document is to promote a more efficient collaboration and pave a clear road map to implement the potential correction method into autolens.

Key words: gravitational lensing – galaxies : haloes – galaxies : structure – dark matter

1 BACKGROUND

The N-body simulation predicts there are lots of small (10^7-$10^8 M_\odot$), invisible dark matter subhaloes that reside in galaxies. The abundance function, structure (such as condensation) of those subhaloes are closely related to the nature of dark matter, a very fundamental problem for modern cosmology.

To detect those small masses, invisible subhaloes, the gravitational lensing phenomenon provide a unique prober. The basic idea is the gravitational perturbation induced by the dark matter subhaloes in the galaxy could imprint anomalous flux signals to the lensing image, in a sense either the 'flux-ratio anomaly' (for point source like quasar), or distortions to the extended lensed arc (for extended source like galaxies). Thus in principle, we can start from those anomalous flux signals to recover the mass structure of those potentially-existed subhaloes

One of the non-parametric ways to do this subhalo detection task via gravitational lensing is the so-called 'potential correction method' (or gravitational imaging technique). In essence, this method first fit the lensing image (i.e, the extended arc) with some simple while physically-motivated parametric mass model (such as the Elliptical Power Law mass model, EPL), which could represent the smooth mass distribution of real lenses approximately (for convenience, we term this model fit as the 'macro model'). Because the macro model can not fully capture the complex mass structure in real lenses (such as the small mass perturbation of subhaloes), this typically leads to image-residuals. Those image-residuals can be used to guide how to iteratively add linear corrections to the lensing potential given by the macro model so that eliminate the image-residuals caused by the imperfect (smooth) lens mass model. Eventually, we can reconstruct those small and clumpy mass perturbations if they really exist in the lens galaxy

Now we show the core mathematical formulas that demonstrate how the potential correction method works. For simplicity, let us assume no PSF-blurring (Point Spread Function) effect exist, the image-residuals ($R^L(\theta)$) given by the macro model in lens-plane can be expressed as,

$$R^L(\boldsymbol{\theta}) = I^L_D(\boldsymbol{\theta}) - I^L_M(\boldsymbol{\theta}) = I^S_D(\boldsymbol{\beta_D}) - I^S_M(\boldsymbol{\beta_M}) \,, \tag{1}$$

where $I^L_M(\boldsymbol{\theta})$ and $I^L_D(\boldsymbol{\theta})$ denote the lensed arc image given by macro model and the data itself repectively; $\boldsymbol{\theta}$ represents the arbitrary angular position on the lens plane. When there is no PSF, $I^L_M(\boldsymbol{\theta})$ and $I^L_D(\boldsymbol{\theta})$ are equal to the $I^S_M(\boldsymbol{\beta_M})$ and $I^S_D(\boldsymbol{\beta_D})$ respectively; where $I^S_M(\boldsymbol{\beta_M})$ is the source surface brightness values predicted by the macro model at the source-plane position $\boldsymbol{\beta_M}$, where

$$\boldsymbol{\beta_M} = \boldsymbol{\theta} - \boldsymbol{\alpha_M} = \boldsymbol{\theta} - \frac{\partial \psi_M(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}. \tag{2}$$

Here, $\psi_M(\theta)$ represents the lensing potential given by the macro-model. Similarly, $I^S_D(\boldsymbol{\beta_D})$ represents the true surface brightness value of the source at position $\boldsymbol{\beta_D}$ (mapping the lens-plane position $\boldsymbol{\theta}$ back to the source-plane according to the lens's true potential ψ_D, i.e, $\boldsymbol{\beta_D} = \boldsymbol{\theta} - \boldsymbol{\alpha_D} = \boldsymbol{\theta} - \frac{\partial \psi_D(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$). When the mismatch between $\Psi_M(\boldsymbol{\theta})$ and the $\Psi_D(\boldsymbol{\theta})$ is small enough, the source brightness distribution predicted by the macro model (I^S_M) is almost equal to the true source light distribution (I^S_D), i.e, $I^S_M \approx I^S_D$; Then the $R_L(\boldsymbol{\theta})$ is purely due to $\boldsymbol{\beta_M} \neq \boldsymbol{\beta_D}$, which is induced by the mismatch between $\psi_M(\boldsymbol{\theta})$ and the $\psi_D(\boldsymbol{\theta})$, i.e,

$$\psi_D(\boldsymbol{\theta}) = \psi_M(\boldsymbol{\theta}) + \delta\psi(\boldsymbol{\theta}), \tag{3}$$

via the lens mapping relation,

$$
\begin{aligned}
R^L(\boldsymbol{\theta}) &= I_D^S(\boldsymbol{\beta_D}) - I_M^S(\boldsymbol{\beta_M}) \\
&= I_M^S(\boldsymbol{\beta_D}) - I_M^S(\boldsymbol{\beta_M}) \\
&= \frac{\partial I_M^S}{\partial \boldsymbol{\beta}} \times (\boldsymbol{\beta_D} - \boldsymbol{\beta_M}) \\
&= \frac{\partial I_M^S}{\partial \boldsymbol{\beta}} \times ((\boldsymbol{\theta} - \boldsymbol{\alpha_D}) - (\boldsymbol{\theta} - \boldsymbol{\alpha_M})) \\
&= \frac{\partial I_M^S}{\partial \boldsymbol{\beta}} \times (\boldsymbol{\alpha_M} - \boldsymbol{\alpha_D}) \\
&= \frac{\partial I_M^S}{\partial \boldsymbol{\beta}} \times (-\frac{\partial \delta\psi}{\partial \boldsymbol{\theta}}) \; .
\end{aligned} \tag{4}
$$

The above equation show the image residuals ($R^L(\boldsymbol{\theta})$) is the product of source brightness gradient (on source-plane) and the gradient of lens-plane potential correction ($-\frac{\partial \delta\psi}{\partial \boldsymbol{\theta}}$).

The whole story of the potential correction method is based on equation (4), i.e, given the image residuals (R^L) and source brightness gradient (on the source-plane) predicted by the macro model, how do we solve the potential correction term $\delta\psi$? We expect when we add the $\delta\psi$ term to the initial lensing potential predicted by the macro model, the R^L can be eliminated (or reduced). Naively, equation (4) is a partial differential equation, the $\delta\psi$ can be solved by integrating along the characteristics of the partial differential equation (Suyu & Blandford 2006). In Koopmans (2005), they proposed a novel (maybe faster?) iterative algorithm to solve the partial differential equation (4) by casting it into matrix expression, this method is further refined by Vegetti & Koopmans (2009) to incorporate the ideas of the bayesian source regularization (Suyu et al. 2006) and the adaptive source-plane pixelization (Dye & Warren 2005). We are going to implement the potential correction method described in Vegetti & Koopmans (2009) into autolens.

The derivation of equation (4) assumes $I_M^S \approx I_D^S$, however, this assumption may be broken in practice, in particular for lenses with massively clumpy structures that deviate from the case that a smooth lens mass model can describe significantly. Therefore, we should not anticipate a single step potential correction based on equation (4) can completely eliminate image residuals. Nevertheless, the potential correction method can still work, but in an iterative way. On each iteration, we fit the lensed image given the current lensing potential, the image residuals of this fit are used to guide the potential correction that we need (to reduce residuals). This potential correction is further used to improve the current lensing potential model. At the beginning of this iterative process, neither the lensing potential model nor the source intensity model close to the true values of the lens. However, with the processing of the iteration, both the lensing potential and the source intensity model gradually converge to the true value of the lens coherently.

2 CODE IMPLEMENTATION

We review the formal mathematical framework of the potential correction method in matrix language (Section 2.1). We depict the workflow of the potential correction method (in a more programming-oriented way) in Section 2.2. Important tricks that ensure the convergence of the algorithm are shown in Section 2.3.

2.1 Framework in matrix-language

Following Koopmans (2005); Vegetti & Koopmans (2009), we cast equation (4) into a matrix equation and solving it with the classical linear inversion method. The data (observed image) can be expressed as a column vector $\boldsymbol{d}$, each element d_j in $\boldsymbol{d}$ represents the brightness value at the data pixel j, where j=1, 2, ..., N_d. Similarly, the $\boldsymbol{n}$ represents the column vector of the data noise (with N_d elements). The (unknown) source intensity is also expressed as a column vector $\boldsymbol{s}$, each element s_i in $\boldsymbol{s}$ represents the brightness value at the source-plane pixel i, where i=1, 2, ..., N_s. $\boldsymbol{d}$ and $\boldsymbol{s}$ are connected with the blurred lensing operator ($\boldsymbol{BL}$),

$$\boldsymbol{BL}(\psi)\boldsymbol{s} = \boldsymbol{d} + \boldsymbol{n} \tag{5}$$

where $\boldsymbol{B}$ and $\boldsymbol{L}$ are blurring and lensing operators (matrices), respectively. $\boldsymbol{L}$ is a $N_d \times N_s$ matrix, when acting on the source brightness column vector $\boldsymbol{s}$, generating a column vector representing the ideal lensed-source image (i.e, $\boldsymbol{Ls}$, a $N_d \times 1$ matrix). The blurring operator is a $N_d \times N_d$ matrix, when acting on $\boldsymbol{Ls}$, generating a column vector representing the blurred (or PSF-convolved) lensed-source image (a $N_d \times 1$ matrix). The lensing operator depends on the specific lens-mass model (denoted with the lensing potential column vector $\boldsymbol{\psi}$, which is of dimensions $N_d \times 1$; Each element in $\boldsymbol{\psi}$ represents the lensing potential value on the corresponding data (or image) grid).

Suppose we use a smooth mass model, such as the EPL, to represent the lens' mass. We denote the best-fit macro mass model with its lensing potential as $\boldsymbol{\psi_0}$ (dimensions: $N_d \times 1$), then equation (1) can be cast into the matrix expression, i.e,

$$\delta\boldsymbol{d} = \boldsymbol{d} - \boldsymbol{BL}(\boldsymbol{\psi_0})\boldsymbol{s}_{\mathbf{MP}}. \tag{6}$$

Here, $\delta\boldsymbol{d}$, $\boldsymbol{d}$, and $\boldsymbol{BL}(\psi)\boldsymbol{s}_{\mathbf{MP}}$ are the column vector (or matrix) representation of $R^L(\theta)$, $I_D^L(\theta)$ and $I_M^L(\theta)$ in equation (1), respectively. $\boldsymbol{s}_{\mathbf{MP}}$ is the maximizing-posterior source reconstruction under the lensing potential $\boldsymbol{\psi_0}$ (a series of surface brightness values reconstructed on the source-plane grid, see Suyu et al. (2006)).

Following equation (4), the image residuals ($R^L(\boldsymbol{\theta})$) is also the product of source brightness gradient (on source-plane) and the gradient of lens-plane potential correction ($-\frac{\partial \delta\psi}{\partial \boldsymbol{\theta}}$). We cast this equation into matrix form,

$$\delta\boldsymbol{d} = -\boldsymbol{B}\boldsymbol{C}_\mathrm{f}\boldsymbol{D}_\mathrm{s}\,(\boldsymbol{s}_{\mathbf{MP}})\,\boldsymbol{D}_{\psi}\boldsymbol{\delta\psi} \tag{7}$$

We illustrate the meaning of each matrix term in equation (7) one by one (from right to left order).

(i) $\boldsymbol{\delta\psi}$ is a column vector with dimensions of $N_p \times 1$, which save the potential correction values on the 'potential-correction grid'. The 'potential-correction grid' has N_p pixels, which is typically coarser than the data (or image) grid with N_d pixels, so that $\boldsymbol{\delta\psi}$ is not underconstrained[1] (Suyu et al. 2009).

(ii) $\boldsymbol{D}_\psi$ is a gradient operation matrix (dimensions: $2N_p \times N_p$), when acting on $\boldsymbol{\delta\psi}$, generating a column vector (dimensions: $2N_p \times 1$) representing the gradient (i.e, x and y-directional first derivative) of $\boldsymbol{\delta\psi}$. For explicity, we show the generic strucutre of matrix $\boldsymbol{D}_\psi\boldsymbol{\delta\psi}$

[1] Despite Vegetti & Koopmans (2009) do also use the data (or image) grid as the 'potential-correction grid', so that $N_d = N_p$, We show the generic case (when $N_d \neq N_p$) here.

(dimensions: $2N_p \times 1$),

$$\boldsymbol{D}_{\psi}\boldsymbol{\delta\psi} = \begin{pmatrix} \cdots \\ \frac{\partial\delta\psi\left(\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j}}\right)}{\partial\theta_x} \\ \frac{\partial\delta\psi\left(\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j}}\right)}{\partial\theta_y} \\ \frac{\partial\delta\psi\left(\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j+1}}\right)}{\partial\theta_x} \\ \frac{\partial\delta\psi\left(\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j+1}}\right)}{\partial\theta_y} \\ \cdots \end{pmatrix} . \tag{8}$$

Here, $\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j}}$ is the 2-dimensional position vector of the j-th 'potential correction' pixels (typically coarser than the image pixel). The x-component of the gradient of the potential correction ($\delta\psi$) at $\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j}}$ is then given by $\frac{\partial\delta\psi\left(\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j}}\right)}{\partial\theta_x}$.

(iii) $\boldsymbol{D}_{\rm s}$ is a matrix with dimensions of $N_p \times 2N_p$, which saves the gradient of the source surface brightness. The generic structure of matrix $\boldsymbol{D}_{\rm s}$ is,

$$\boldsymbol{D}_{\rm s} = \begin{pmatrix} \cdots & & & & & \\ & \frac{\partial S\left(\boldsymbol{\beta}^{\boldsymbol{p}}_{\boldsymbol{j}}\right)}{\partial\beta_x} & \frac{\partial S\left(\boldsymbol{\beta}^{\boldsymbol{p}}_{\boldsymbol{j}}\right)}{\partial\beta_y} & & & \\ & & & \frac{\partial S\left(\boldsymbol{\beta}^{\boldsymbol{p}}_{\boldsymbol{j+1}}\right)}{\partial\beta_x} & \frac{\partial S\left(\boldsymbol{\beta}^{\boldsymbol{p}}_{\boldsymbol{j+1}}\right)}{\partial\beta_y} & \\ & & & & & \cdots \end{pmatrix} . \tag{9}$$

Here, $\boldsymbol{\beta}^{\boldsymbol{p}}_{\boldsymbol{j}}$ is the ray-traced position of vector $\boldsymbol{\theta}^{\boldsymbol{p}}_{\boldsymbol{j}}$ on the source-plane under the lens mass model denoted by $\boldsymbol{\psi_0}$. S represents the distribution function of the source's surface brightness, which can be obtained by linearly interpolating the $\boldsymbol{s}_{\mathbf{MP}}$. The x-component of the brightness gradient of the source (on the source-plane) are then denoted by $\frac{\partial S\left(\boldsymbol{\beta}^{\boldsymbol{p}}_{\boldsymbol{j}}\right)}{\partial\beta_x}$.

(iv) $\boldsymbol{C}_{\rm f}$ is a conformation matrix with dimensions of $N_d \times N_p$. This matrix is constructed to map a column vector defined on the coarser 'potential correction grid' (dimensions: $N_p \times 1$) to a new vector (dimensions: $N_d \times 1$) defined on the finer data (or image) grid, via the linear interpolation.

(v) $\boldsymbol{B}$ is just the blurring matrix we introduced before. We should note, although the effect of PSF is ignored when deriving equation (4) for simplicity, PSF-blurring can be easily incorporated by further multiplying $\boldsymbol{C}_{\rm f}\boldsymbol{D}_{\rm s}\left(\boldsymbol{s}_{\mathbf{MP}}\right)\boldsymbol{D}_{\psi}\boldsymbol{\delta\psi}$ by the blurring matrix $\boldsymbol{B}$.

2.2 Algorithm implementation

We describe two slightly different potential correction schemes used by Suyu et al. (2009) and Koopmans (2005); Vegetti & Koopmans (2009) respectively. Although our priority is implementing Koopmans & Vegetti's one.

2.2.1 *Suyu's method*

Combining equations (6) and (7), we have

$$-\boldsymbol{B}\boldsymbol{C}_{\rm f}\boldsymbol{D}_{\rm s}\left(\boldsymbol{s}_{\mathbf{MP}}\right)\boldsymbol{D}_{\psi}\boldsymbol{\delta\psi} = \boldsymbol{d} - \boldsymbol{B}\boldsymbol{L}(\boldsymbol{\psi_0})\boldsymbol{s}_{\mathbf{MP}} \ . \tag{10}$$

Equation (10) show that given a lens mass model $\boldsymbol{\psi_0}$, the relationship between the potential correction vector $\boldsymbol{\delta\psi}$ and the image residuals of the 'best-fit' macro model (the right-hand side of equation (10)) is again a linear response (similar to the pixelized source reconstruction problem), the $\boldsymbol{\delta\psi}$ can be solved via a linear inversion (Suyu et al. 2009). The regularization (i.e, a smoothness prior condition on $\boldsymbol{\delta\psi}$) can also be incorporated easily into this inversion process.

The whole workflow of Suyu's method is listed as follows,

(i) Given $\boldsymbol{\psi_0}$, solving the best-fit source intensity ($\boldsymbol{s}_{\mathbf{MP}}$) via a Bayesian linear inversion. By-products of this process are the source intensity gradient ($\boldsymbol{D}_{\rm s}\left(\boldsymbol{s}_{\mathbf{MP}}\right)$) and the image residuals ($\boldsymbol{d} - \boldsymbol{B}\boldsymbol{L}(\boldsymbol{\psi_0})\boldsymbol{s}_{\mathbf{MP}}$).

(ii) With the source intensity gradient and the image residuals given by step-i, solving the best-fit potential correction ($\boldsymbol{\delta\psi}$), again via a Bayesian linear inversion (equation (10)).

(iii) Add $\boldsymbol{\delta\psi}$ to $\boldsymbol{\psi_0}$ to form the new $\boldsymbol{\psi_0}$[2].

(iv) Repeat steps (i)-(iii) until the code converges. The convergence criterion is defined as the relative potential corrections between all image pairs being less than 0.1%.

2.2.2 *Koopmans & Vegetti's method*

In Suyu's method, the linear inversion of the source intensity and the lens potential correction are performed sequentially. In a single iteration, we either fix the potential field and only optimize the source intensity, or fix the source intensity and only optimize the lens potential correction. For Koopmans & Vegetti's method, given the initial guesses of the source intensity and the lens potential, the linear inversion of the new source intensity and the lens potential correction are performed simultaneously. Thus the source light and the lens mass compete with each other to fit the image data, this feature may improve the convergence performance of the potential correction algorithm.

How does Vegetti's method work? We only need to slightly modify equation (10),

$$-\boldsymbol{B}\boldsymbol{C}_{\rm f}\boldsymbol{D}_{\rm s}\left(\boldsymbol{s}_{\mathbf{P}}\right)\boldsymbol{D}_{\psi}\boldsymbol{\delta\psi} = \boldsymbol{d} - \boldsymbol{B}\boldsymbol{L}(\boldsymbol{\psi}_{\mathbf{P}})\boldsymbol{s} \ . \tag{11}$$

Here, $\boldsymbol{s}_{\mathbf{P}}$ and $\boldsymbol{\psi}_{\mathbf{P}}$ represent the previously found best-fitting source intensity and lens potential, respectively. Re-arrange equation (11), we have,

$$\boldsymbol{B}\left[\boldsymbol{L}(\boldsymbol{\psi}_{\mathbf{P}})\boldsymbol{s} - \boldsymbol{C}_{\rm f}\boldsymbol{D}_{\rm s}\left(\boldsymbol{s}_{\mathbf{P}}\right)\boldsymbol{D}_{\psi}\boldsymbol{\delta\psi}\right] = \boldsymbol{d} \ . \tag{12}$$

Equation (12) can be written into the following more compact form,

$$\boldsymbol{B}\boldsymbol{L}_{\rm c}\left(\boldsymbol{\psi}_{\mathbf{P}}, \boldsymbol{s}_{\mathbf{P}}\right)\boldsymbol{r} \equiv \boldsymbol{M}_{\rm c}\boldsymbol{r} = \boldsymbol{d} \ , \tag{13}$$

if we introduce the block matrices

$$\boldsymbol{L}_{\rm c}\left(\boldsymbol{\psi}_{\mathbf{P}}, \boldsymbol{s}_{\mathbf{P}}\right) \equiv \left(\boldsymbol{L}\left(\boldsymbol{\psi}_{\mathbf{P}}\right) \mid -\boldsymbol{C}_{\rm f}\boldsymbol{D}_{\rm s}\left(\boldsymbol{s}_{\mathbf{P}}\right)\boldsymbol{D}_{\psi}\right) \ , \tag{14}$$

and

$$\boldsymbol{r} \equiv \begin{pmatrix} \boldsymbol{s} \\ \delta\boldsymbol{\psi} \end{pmatrix} . \tag{15}$$

The dimensions of $\boldsymbol{L}_{\rm c}$ (or $\boldsymbol{M}_{\rm c}$) and $\boldsymbol{r}$ are $N_d \times (N_s + N_p)$ and $(N_s + N_p) \times 1$, respectively. We can clearly see that the linear response matrix $\boldsymbol{M}_{\rm c}$ connects the column vector $\boldsymbol{r}$ and $\boldsymbol{d}$, therefore the Bayesian inference of the vector $\boldsymbol{r}$ (includes both $\boldsymbol{s}$ and $\delta\boldsymbol{\psi}$) given the image data $\boldsymbol{d}$ is a linear inversion, and is a solved problem (Suyu et al. 2006).

Formally, the Bayesian linear inversion minimizes the following penalty function[3],

$$P\left(\boldsymbol{s}, \boldsymbol{\delta\psi} \mid \lambda_s, \lambda_{\delta\psi}, \boldsymbol{s}_{\boldsymbol{P}}, \boldsymbol{\psi}_{\boldsymbol{P}}\right) = \chi^2 + \lambda_s^2 \left\|\boldsymbol{H}_s\boldsymbol{s}\right\|_2^2 + \lambda_{\delta\psi}^2 \left\|\boldsymbol{H}_{\delta\psi}\boldsymbol{\delta\psi}\right\|_2^2$$

[2] Linear interpolation may be needed if $\boldsymbol{\delta\psi}$ and $\boldsymbol{\psi_0}$ are defined on grids with different resolutions.

[3] $\|\boldsymbol{H}_s\boldsymbol{s}\|_2^2$ is just an alternative expression of $\boldsymbol{s}^{\boldsymbol{T}}\boldsymbol{H}_s^T\boldsymbol{H}_s\boldsymbol{s}$.

(16)

with

$$\chi^2 = \left[\boldsymbol{M}_c\left(\boldsymbol{\psi_P}, \boldsymbol{s_P}\right)\boldsymbol{r} - \boldsymbol{d}\right]^T \boldsymbol{C}_d^{-1}\left[\boldsymbol{M}_c\left(\boldsymbol{\psi_P}, \boldsymbol{s_P}\right)\boldsymbol{r} - \boldsymbol{d}\right] . \tag{17}$$

The first term in equation (16) is just chi-square of the image model-fit, where $\boldsymbol{C}_d$ is the covariance matrix of the data (dimensions: $N_d \times N_d$, typically diagonal for a good ccd image data). The second and third terms in the penalty function contains our belief on the smoothness of the source intensity ($\boldsymbol{s}$) and lens potential correction ($\delta\boldsymbol{\psi}$), i.e, the prior information. The two constant λ_s and $\lambda_{\delta\psi}$ control the strength of the smoothness of $\boldsymbol{s}$ and $\delta\boldsymbol{\psi}$ respectively; Higher λ_s ($\lambda_{\delta\psi}$) implies a smoother $\boldsymbol{s}$ ($\delta\boldsymbol{\psi}$). $\boldsymbol{H}_{\delta\psi}$ is just the a derivative operator for the lens potential correction $\delta\boldsymbol{\psi}$ that acts on the $\delta\boldsymbol{\psi}$ to produce the gradient of the $\delta\boldsymbol{\psi}$ at each pixel. In practice, a convenient approach to compose $\boldsymbol{H}_{\delta\psi}$, is to write the differential operators in the x and y directions (i.e, $\boldsymbol{H}_{\delta\psi,\theta_x}$ and $\boldsymbol{H}_{\delta\psi,\theta_y}$) separately and then superposition them in the following way

$$\boldsymbol{H}_{\delta\psi}^T\boldsymbol{H}_{\delta\psi} = \boldsymbol{H}_{\delta\psi,\theta_x}^T\boldsymbol{H}_{\delta\psi,\theta_x} + \boldsymbol{H}_{\delta\psi,\theta_y}^T\boldsymbol{H}_{\delta\psi,\theta_y} . \tag{18}$$

Here, both the dimensions of matrices $\boldsymbol{H}_{\delta\psi,\theta_x}^T\boldsymbol{H}_{\delta\psi,\theta_x}$ and $\boldsymbol{H}_{\delta\psi,\theta_y}^T\boldsymbol{H}_{\delta\psi,\theta_y}$ are $N_P \times N_P$. $\boldsymbol{s}^T\boldsymbol{H}_{\delta\psi}^T\boldsymbol{H}_{\delta\psi}\boldsymbol{s}$ return the square of the lens potential correction gradient summed over all the $\delta\boldsymbol{\psi}$ pixels. The differential operators of the source intensity ($\boldsymbol{H}_s$) can be constructed similarly.

The best-fit solution that minimizes the penalty function $P\left(\boldsymbol{s}, \delta\boldsymbol{\psi} \mid \lambda_s, \lambda_{\delta\psi}, \boldsymbol{s_P}, \boldsymbol{\psi_P}\right)$ can be derived by setting

$$\frac{\partial P\left(\boldsymbol{s}, \delta\boldsymbol{\psi} \mid \lambda_s, \lambda_{\delta\psi}, \boldsymbol{s_P}, \boldsymbol{\psi_P}\right)}{\partial \boldsymbol{r}} = \boldsymbol{0} . \tag{19}$$

This results in the following matrix equation:

$$\left(\boldsymbol{M}_{\mathrm{c}}^T\boldsymbol{C}_{\mathrm{d}}^{-1}\boldsymbol{M}_{\mathrm{c}} + \boldsymbol{R}^T\boldsymbol{R}\right)\boldsymbol{r} = \boldsymbol{M}_{\mathrm{c}}^T\boldsymbol{C}_{\mathrm{d}}^{-1}\boldsymbol{d} , \tag{20}$$

with

$$\boldsymbol{R}^T\boldsymbol{R} = \begin{pmatrix} \lambda_{\mathrm{s}}^2\boldsymbol{H}_s^T\boldsymbol{H}_s & \\ & \lambda_{\delta\psi}^2\boldsymbol{H}_{\delta\psi}^T\boldsymbol{H}_{\delta\psi} \end{pmatrix} . \tag{21}$$

The dimensions of the block matrix $\boldsymbol{R}^T\boldsymbol{R}$ is $(N_s + N_P) \times (N_s + N_P)$.

The iterative algorithm of the Koopmans & Vegetti's method can be expressed as follows,

1. $i = 0$, $\boldsymbol{s}_i = \boldsymbol{0}$, $\delta\boldsymbol{\psi}_i = \boldsymbol{0}$, $\boldsymbol{\psi_i}$ = initial model
2. Determine $\boldsymbol{D}_s\left(\boldsymbol{s}_i\right), \boldsymbol{L}_{\mathrm{c}}\left(\boldsymbol{\psi}_i\right)$
3. Solve $\left(\boldsymbol{M}_{\mathrm{c}}^T\boldsymbol{C}_{\mathrm{d}}^{-1}\boldsymbol{M}_{\mathrm{c}} + \boldsymbol{R}^T\boldsymbol{R}\right)\boldsymbol{r} = \boldsymbol{M}_{\mathrm{c}}^T\boldsymbol{C}_{\mathrm{d}}^{-1}\boldsymbol{d}$ using regularization parameters $\lambda_{\delta\psi,i}$ and $\lambda_{s,i}$
4. Extract $\boldsymbol{s}_{i+1}$ and $\delta\boldsymbol{\psi}_{i+1}$ from $\boldsymbol{r}$, $\boldsymbol{\psi}_{i+1} = \boldsymbol{\psi}_i + \delta\boldsymbol{\psi}_{i+1}$, $i = i + 1$
5. Re-scale $\boldsymbol{\psi}_{i+1}$ according to equation (22) (see section 2.3).
6. Has $P\left(\boldsymbol{s}_i, \boldsymbol{\psi}_{i+1}\right)$ (equation (16)) converged? Yes : Exit; No: Goto 2

where the i denotes the index of iterations.

Currently, I am going to take the following optimization workflow to implement our potential correction algorithm (highly motivated by (Koopmans 2005)). The whole workflow will be divided into three phases:

In phase-1, the initial model for the lens potential ($\boldsymbol{\psi_0}$) comes from the macro model fit directly. the regularization strength of the potential correction takes a very large value at iteration-0 ($\lambda_{\delta\psi,0} = 10^9$), this value is further lower by a factor of 0.1 at each iteration. $\lambda_{s,i}$ is fixed during the iterative potential correction and takes a relatively large value (10 times the best-fit value given by the macro model), to ensure a smooth source gradient calculation and good convergence performance of the potential correction algorithm.

In phase-2, we use the corrected lens potential model given by phase-1 (keep fixed) to optimize the source regularization strength.

In phase-3, we use the best source regularization strength given by step-2, keep it fixed, and re-do the potential correction procedure shown in phase-1.

2.3 Important tricks

Since the lens modeling is typically restricted to the image data, and both the lens mass and source light are unknown during the model fit, this implies our linear corrected lens potential at each iteration may be affected by the so-called 'mass-sheet degeneracy'. Suppose the true lens potential of a lens system is denoted by $\psi(\vec{\theta})$, then all the ψ_ν, satisfying following transformation relation

$$\psi_\nu(\vec{\theta}) = \frac{1-\nu}{2}|\vec{\theta}|^2 + \vec{a}\cdot\vec{\theta} + c + \nu\psi(\vec{\theta}), \tag{22}$$

can fit the image data equally well. We illustrate the meaning of each term in this transformation one by one,

(i) $\frac{1-\nu}{2}|\vec{\theta}|^2 + \nu\psi(\vec{\theta})$: this terms are just the classical mass-sheet transformation.

(ii) $\vec{a}\cdot\vec{\theta}$: this is a gradient-sheet term, which change the origin (which is unknowable) on the source-plane (from $\vec{0}$ to $\vec{a}$).

(iii) c: this constant term changes the zero point of the lens potential.

To ensure the potential correction algorithm is stable, i.e, always being able to converge, we need to avoid the random change of the source light model during the iterative procedure, either due to the mass-sheet transformation (shrink or expand the source size), or the gradient-sheet term (the randomly wandering of the source center). Also, we want to avoid the randomness for the zero point of the lens potential. We achieve these goals by,

(i) manually setting the $\nu = 1$.

(ii) selecting three points in the region where the lensed arc locates and requiring the lens potential at those three points to remain unchanged after each potential correction. This allows us to solve the value of $\vec{a}$ and c.

As an example, see the right panel of figure-1 in Suyu et al. (2009).

3 FUTURE WORKS

To be continued...

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DATA AVAILABILITY

The inclusion of a Data Availability Statement is a requirement for articles published in MNRAS. Data Availability Statements provide a standardised format for readers to understand the availability of data underlying the research results described in the article. The statement may refer to original data generated in the course of the study or to third-party data analysed in the article. The statement should describe and provide means of access, where possible, by linking to the data or providing the required accession numbers for the relevant databases or DOIs.

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APPENDIX A: SOME EXTRA MATERIAL

If you want to present additional material which would interrupt the flow of the main paper, it can be placed in an Appendix which appears after the list of references.

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