Solution to Introduction to the Theory of Computation

Lwins_Lights

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1 Regular Languages

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4 Decidability

4.10 *Refer to the textbook.*

4.11 By the pumping lemma, $L(M)$ is an infinite language $\iff$ $L(M)$ includes a string of at least length p. Since $R = \Sigma^p\Sigma^*$ is a regular language, it is easy to design a PDA P recognizing $L(M) \cap R$. Then use E_{PDA}'s decider to decide whether $L(M) \cap R = \varnothing$.

4.12 *Refer to the textbook.*

4.13 $L(R) \subseteq L(S) \iff L(R) \cup L(S) = L(S)$, which can be decided by EQ_{DFA}'s decider.

4.14 *Refer to the textbook.*

*4.15 By the pumping lemma, $\mathtt{1}^k \in L(G) \implies \mathtt{1}^{k+p!} \in L(G)$ for every $k \geq p$. Therefore $\mathtt{1}^* \subseteq L(G) \iff \{\mathtt{1}^k \mid k \leq p + p!\} \subseteq L(G)$, which can be easily checked by a TM in finite time.

4.16 Since $A = \{\langle R\rangle \mid R \text{ is a regular expression}, R \cap \Sigma^*\mathtt{111}\Sigma^* \neq \varnothing\}$, we only need an E_{DFA}'s decider.

4.17 Suppose we have two DFAs D_1 and D_2. Let D be a DFA recognizing $L(D_1) \oplus L(D_2)$, where $\oplus$ means symmetric difference. By the pumping lemma, $L(D) \cap (\Sigma \cup \epsilon)^p = \varnothing \implies L(D) = \varnothing$, thus p can be the required length.

*4.18 $\Leftarrow$: It is enough to design a TM, which recognizes C by checking whether $\langle x, y\rangle \in D$ for all possible y one by one.

$\Rightarrow$: Suppose TM M recognizes C. Let $D = \{\langle x, y\rangle \mid x \in C \text{ and } y \text{ is the computation history of } M \text{ on input } x\}$, which is obviously decidable.

*4.19 Let C be a recognizable but undecidable language, e.g., A_{TM}. Construct D provided by Problem 4.18. Letting homomorphism f satisfy $f(\langle x, y\rangle) = x$ for every x, y we obtain $f(D) = C$.

4.20 Let M be a TM which runs both $\overline{A}$'s recognizer and $\overline{B}$'s recognizer on its own input. M accepts when $\overline{B}$'s recognizer accepts and rejects when $\overline{A}$'s recognizer accepts. Clearly $C = L(M)$ separates A and B.

4.21 M is a DFA that accepts $w^{\mathcal{R}}$ whenever it accepts w $\iff$ $L(M) = L(M)^{\mathcal{R}}$, which can be decided by EQ_{DFA}'s decider.

4.22 In order to determine whether $L(R)$ is prefix-free, it suffices to check whether the DFA recognizing $L(R)$ has the property that from a reachable accept state we could arrive an accept state again by several transitions.

*4.23 *Refer to the textbook.*

4.24 A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ has an useless state $q \in Q$ if and only if PDA $P' = (Q, \Sigma, \Gamma, \delta, q_0, \{q\})$ recognizes $\varnothing$. Therefore we can use E_{PDA}'s decider to solve it.

*4.25 *Refer to the textbook.*

*4.26 M is a DFA that accepts some palindrome $\iff$ CFL $\{w \mid w = w^{\mathcal{R}}\} \cap L(M)$ is not empty, which can be decided by E_{PDA}'s decider.

*4.27 Refer to the solution to Problem 4.26.

4.28 x is a substring of some $y \in L(G) \iff L(G) \cap \Sigma^* x \Sigma^* \neq \varnothing$, which can be decided by E_{PDA}'s decider.

4.29 By the pumping lemma, $L(G)$ is an infinite language $\iff$ $L(G)$ includes a string of at least length p. So we can first use $\mathit{INFINITE}_{\mathsf{PDA}}$ from Problem 4.11 to check whether $|L(G)| = \infty$, and then check whether $x \in L(G)$ for all x of less-than-p length, one by one.

4.30 Let E be A's enumerator which generates $\langle M_1\rangle, \langle M_2\rangle, \ldots$ in order. Construct $D = \{\langle n\rangle \mid \langle n\rangle \notin L(M_n)\}$.

4.31 First, recursively determine whether $V \stackrel{*}{\Rightarrow} w$ with some $w \in \Sigma^*$, for all variables V. Then, recursively determine whether $V \stackrel{*}{\Rightarrow} xAy$ with some $x, y \in \Sigma^*$, for all variables V.

4.32 Construct DPDA $P'_{(q,x)}$ by modifying P, such that $L(P'_{(q,x)}) = \varnothing$ if and only if (q, x) is a looping situation for P. Then we only need E_{PDA}.

5 Reducibility

5.9 Reduce from A_{TM}. To determine whether TM M accepts w, construct TM N which always accepts `01` but accepts `10` if and only if M accepts w.

5.10 *Refer to the textbook.*

5.11 *Refer to the textbook.*

5.12 Reduce from E_{TM}. To determine whether TM M accepts nothing, construct TM N which simulates M on N's own input w but never writes a blank symbol over a nonblank symbol unless when M accepts.

5.13 Reduce from E_{TM}. To determine whether TM M accepts nothing, construct TM N which simulates M on N's own input w and obviously has no useless state except the one q_{accept}.

5.14 Reduce from A_{TM}. To determine whether TM M accepts w, construct TM N which simulates M on N's own input w but never attempts to move its head left when its head is on the left-most tape cell unless when M accepts.

5.15 Let M' be M after modifying all its transitions $\delta(q_i, a) = (q_{\text{accept}}, b, X)$ to $\delta(q_i, a) = (q_{\text{reject}}, b, X)$, and then modifying all $\delta(q_i, a) = (q_j, b, \mathrm{L})$ to $\delta(q_i, a) = (q_{\text{accept}}, b, \mathrm{R})$. The problem is now reduced to checking whether M', as a *TM with stay put instead of left* described in Problem 3.13, accepts w. It is easy since $L(M')$ is regular by the solution to Problem 3.13.

5.16 Suppose for the sake of contradiction that BB is computable. Then obviously there exists a TM M having k states (let k be sufficiently large), which writes $BB(n) + 1$ `1`s on the tape, when given input $\langle n \rangle$. Further, using M we can construct a series of TMs M_n having exactly $k + n/2$ states for large n, which writes $BB(n) + 1$ `1`s on the tape when started with a blank tape. However, then M_{2k}, as a $2k$-state TM, would write $BB(2k) + 1$ `1`s. Absurd.

5.17 Obviously, in this case a PCP instance P always has a match unless all dominos in P have longer top strings, or they all have longer bottom strings.

5.18 Reduce from *PCP*, since any string with finite alphabet can be encoded to a binary one.

5.19 Trivial.

5.20 We can encode any language to the one over the unary alphabet.

5.21 The hint in the textbook is sufficient.

5.22 $\Leftarrow$: Trivial.

$\Rightarrow$: Let $f(x) = \langle M, x \rangle$, where TM M is A's recognizer.

5.23 $\Leftarrow$: Trivial, since $\mathtt{0}^*\mathtt{1}^*$ is surely decidable.

$\Rightarrow$: Suppose there is an A's decider, then f defined as follows is computable.

$$f(x) = \begin{cases} \mathtt{01}, & x \in A \\ \mathtt{10}, & x \notin A \end{cases}$$

5.24 It immediately follows from $A_{\mathsf{TM}}, \overline{A_{\mathsf{TM}}} \leq_{\mathrm{m}} J$.

5.25 $\overline{E_{\mathsf{TM}}} \leq_{\mathrm{m}} A_{\mathsf{TM}}$ by Problem 5.22 and it is well known that $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$. By the way, it is not difficult to construct an undecidable language B such that $B \equiv_{\mathrm{m}} \overline{B}$.

5.26 The idea is the same as how we deal with A_{LBA} and E_{LBA}.

5.27 Prove that $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{2DIM\text{-}DFA}} \leq_{\mathrm{m}} EQ_{\mathsf{2DIM\text{-}DFA}}$, in which the former reduction can be done by computation history method.

*5.28 *Refer to the textbook.*

5.29 The case P is not nontrivial is trivial. As for the second condition, let $P = \{\langle M\rangle \mid \mathsf{TM}\ M \text{ has 100 states}\}$.

5.30 Trivial.

5.31 Let M be a TM which on input $\langle x\rangle$ ($x \in \mathbb{Z}^+$) calculates $x, f(x), f(f(x)), \ldots$ until it finds some $f^{(n)}(x) = 1$, and then accepts. Let N be a TM uses H to calculate whether $\langle M, x\rangle \in A_{\mathsf{TM}}$ for $x = 1, 2, \ldots$ in order, until it finds some $\langle M, x\rangle \notin A_{\mathsf{TM}}$, and then accepts. We have the positive answer to the $3x+1$ problem if and only if $\langle N, \mathsf{0}\rangle \notin A_{\mathsf{TM}}$.

5.32
   a. As hinted, reduce from *PCP*.
   b. Reduce from $OVERLAP_{\mathsf{CFG}}$ by constructing a grammar whose rules are G's and H's rules and $S \to S_G\$ \mid S_H\$\$$, where S_G and S_H are G's and H's start variables.

5.33 Refer to the proof of undecidability of ALL_{CFG}. Let $w = \#C_1\#C_3\#C_5\#\cdots\#C_6^{\mathcal{R}}\#C_4^{\mathcal{R}}\#C_2^{\mathcal{R}}\#$.

5.34 Reduce from A_{TM}. To determine whether TM N accepts w, construct TM M which simulates N on M's own input w but never modifies the portion of the tape that contains the input w unless when N accepts.

5.35
   a. Just enumerate w.
   b. Reduce from ALL_{CFG}. In order to determine wether $L(G) = \Sigma^*$, construct a grammar whose rules are G's rules and $S \to S_G \mid T;\ T \to aT \mid \epsilon$ $(a \in \Sigma)$, where S_G are G's start variable.

*5.36 See `https://cstheory.stackexchange.com/q/39407/46760` for two different solutions.

6 Advanced Topics in Computability Theory

6.6 Let $M = P_{\langle N \rangle}$ and N print $q(\langle N \rangle) = \langle M \rangle$.

6.7 A TM that always loops.

*6.8 Suppose for the sake of contradiction that f is a reduction from EQ_{TM} to $\overline{EQ_{\mathsf{TM}}}$. It is easy to generalize the fixed-point version of the recursion theorem to find $f(\langle M, N \rangle) = \langle M', N' \rangle$ such that M, N simulate M', N' respectively. Then $\langle M, N \rangle \in EQ_{\mathsf{TM}} \iff \langle M', N' \rangle \in \overline{EQ_{\mathsf{TM}}} \iff \langle M, N \rangle \in \overline{EQ_{\mathsf{TM}}}$. Absurd.

6.9 *Refer to the textbook.*

6.10 *Refer to the textbook.*

*6.11 $(\mathbb{R}, =, <)$.

6.12 *Refer to the textbook.*

6.13 Since $\mathbb{Z}_m$ is finite, any sentence in the language of $\mathcal{F}_m$ can be decided by brute-force checking.

6.14 Let $J = \mathtt{0}A \cup \mathtt{1}B$.

6.15 Let $B = A_{\mathsf{TM}^A} = \{\langle M^A, w \rangle \mid M^A \text{ accepts } w\}$. Then apply any classical method used in proving undecidablity of A_{TM}.

*6.16 **(Kleene–Post)** For convenience let any language L be a subset of $\mathbb{N}$ instead of Σ^*. Denote $\{0, 1, \ldots, m\}$ by $[m]$. Define

$$\mathcal{L}_m(A) = \{L \subseteq \mathbb{N} \mid L \cap [m] = A\} \quad (A \subseteq [m])$$

Let $M_0, M_1, M_2, \ldots$ be all possible oracle TMs. We will give two series of families of languages $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots$ and $\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \cdots$ such that

$$\forall A \in \mathcal{A}_n, B \in \mathcal{B}_n,\ M_n^A \text{ is not } B\text{'s decider and } M_n^B \text{ is not } A\text{'s decider.}$$

Then taking arbitrary $A \in \mathcal{A} = \bigcap_{n \in \mathbb{N}} \mathcal{A}_n$ and $B \in \mathcal{B} = \bigcap_{n \in \mathbb{N}} \mathcal{B}_n$ we have $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$. We build them by induction. Given $\mathcal{A}_{n-1} = \mathcal{L}_m(X)$ and $\mathcal{B}_{n-1} = \mathcal{L}_m(Y)$, we first find $\mathcal{A}'_n \subseteq \mathcal{A}_{n-1}$ and $\mathcal{B}'_n \subseteq \mathcal{B}_{n-1}$ such that $\forall A \in \mathcal{A}'_n, B \in \mathcal{B}'_n,\ M_n^A$ is not B's decider.

- If there is no $A \in \mathcal{A}_{n-1}$ such that M_n^A is a decider, let $\mathcal{A}'_n = \mathcal{A}_{n-1}$ and $\mathcal{B}'_n = \mathcal{B}_{n-1}$.
- Suppose M_n^A is a decider with some $A \in \mathcal{A}_{n-1}$, then there exists an $m' > m$ such that

  $$\forall A' \in \mathcal{L}_{m'}(A \cap [m']),\ m+1 \in L(M_n^A) \iff m+1 \in L(M_n^{A'}).$$

  Then, let $\mathcal{A}'_n = \mathcal{L}_{m'}(A \cap [m'])$, $\mathcal{B}'_n = \mathcal{L}_{m'}(Y)$ or $\mathcal{L}_{m'}(Y \cup \{m+1\})$, depending on whether $m+1 \in L(M_n^A)$.

The same method can be also used to find $\mathcal{A}_n \subseteq \mathcal{A}'_n$ and $\mathcal{B}_n \subseteq \mathcal{B}'_n$ such that $\forall A \in \mathcal{A}_n, B \in \mathcal{B}_n,\ M_n^B$ is not A's decider.

*6.17 Let

$$A = \{\langle M, w \rangle \mid \mathsf{TM}\ M \text{ on input } w \text{ halts with } \mathtt{0} \text{ on its tape}\}$$
$$B = \{\langle M, w \rangle \mid \mathsf{TM}\ M \text{ on input } w \text{ halts with } \mathtt{1} \text{ on its tape}\}$$

If there is a C's decider N, we can construct TM M which on input w first run N on $\langle M, w \rangle$ to know that M would not halt with $x \in \{\mathtt{0}, \mathtt{1}\}$ on M's tape, and then violates it.

6.18 Suppose $L(M) \neq L(N)$, we can enumerate x to find one such that $\langle M, x \rangle \in A_{\mathsf{TM}} \oplus \langle N, x \rangle \in A_{\mathsf{TM}}$ holds, where $\oplus$ means exclusive or.

6.19 $|\{L(M^A) \mid M^A \text{ is an oracle } \mathsf{TM}\}| \leq |\{M^A \mid M^A \text{ is an oracle } \mathsf{TM}\}| \leq \aleph_0 < 2^{\aleph_0} = |\{L \mid L \text{ is a language}\}|$.

6.20 Let M be PCP's recognizer. Then check whether $\langle M, \langle P \rangle \rangle \in A_{\mathsf{TM}}$ to know if instance P has a match.

6.21 Since $\mathrm{K}(x) \le |x| + c$, we can check all possible minimal description $\langle M, w \rangle$ to see if M on input w halts with x on its tape by simulating M, where "possible" means that $|\langle M, w \rangle| < |x| + c$ and $\langle M, w \rangle \in A_{\mathsf{TM}}$.

6.22 Trivial.

6.23 Reduce from Problem 6.24.

6.24 Reduce from Problem 6.25.

6.25 If not, there is an enumerator E which would print infinite many incompressible strings one by one: $s_1, s_2, \ldots$ By using E we can construct TM M, which prints an incompressible string s_i, such that $|s_i| > |\langle M, 0 \rangle|$, on its tape. Then we have $\mathrm{K}(s_i) \le |\langle M, 0 \rangle| < |s_i|$. Absurd.

*6.26 **Solution 1.** Suppose for the sake of contradiction that $\mathrm{K}(xy) \le \mathrm{K}(x) + \mathrm{K}(y) + c$ always holds. Define

$$f_n = \sum_{|x|=n} 2^{-\mathrm{K}(x)}.$$

Then $\mathrm{K}(xy) \le \mathrm{K}(x) + \mathrm{K}(y) + c \implies f_{n+m} \ge 2^{-c} f_n f_m \implies f_{kn} \ge (2^{-c} f_n)^k$. On the other hand, Corollary 6.30 implies $f_n \le n + 1$. Therefore,

$$f_n \le 2^c \sqrt[k]{f_{kn}} \le 2^c \sqrt[k]{kn+1}.$$

Letting $k \to +\infty$ we obtain that $f_n \le 2^c$ for all n. However, there is a TM M which on input $\langle p, q \rangle$ ($p, q \in \mathbb{N}$ and $q < 2^{2^p}$) halts with $r(p, q)$, a 2^p-bits binary representation of q, on its tape. So $\mathrm{K}(r(p, q)) \le 2 \log_2 p + \log_2 q + d$ with some constant d. Then, if $n = 2^p$ for some large p,

$$f_n = \sum_{|x|=n} 2^{-\mathrm{K}(x)} \ge \sum_{q<2^n} 2^{-\mathrm{K}(r(p,q))} \ge 2^{-2\log_2 p - d} \sum_{q<2^n} \frac{1}{q} \ge \frac{n}{2^d (\log_2 n)^2},$$

which apparently contradicts with $f_n \le 2^c$.

Solution 2. $\forall c$, the following process of choosing x and y constructs the inequality directly.

First, select an incompressible string w with length n. Denote its prefix string of length $\frac{1}{2} \log n$ as u. Let number p equal the value of $1u$ when perceived as a binary number.

Then, divide w into two substrings $w = xy$, where $|x| = \frac{1}{2} \log n + p$ and $|y| = n - |x|$ (since $p < 2^{\frac{1}{2} \log n + 1} = 2\sqrt{n}$, $|y| > 0$ is a valid string as long as n is large enough).

We observe that since w is incompressible, $K(xy) \ge n$. Also, $K(y) \le |y| + c_1$ for some constant c_1 independent of n.

We claim that $K(x) \le p + c_2$ for some constant c_2 independent of n. In fact, the following Turing machine M generates x when the input string is the tail string of x with length p:

M="With input string z:

Calculate $q = |z|$ and write q in binary representation on the tape;

Remove the leading character 1 of q, append q with z, and output them together."

Now, with all the above claims, we get $K(xy) - (K(x) + K(y) + c) \ge \frac{1}{2} \log n - (c_1 + c_2 + c) > 0$ as long as n is large enough.

6.27 Show that $\overline{HALT_{\mathsf{TM}}} \le_{\mathrm{m}} S, \overline{S}$.

6.28
a. $x = 0 \iff \forall y,\ x + y = y$
b. $x = 1 \iff \forall y,\ y = 0 \land x + y = 1$
c. $x = y \iff \forall z,\ z = 0 \land x + z = y$
d. $x < y \iff \exists z,\ \neg(z = 0) \land x + z = y$

7 Time Complexity

7.13 $a^b = (a^{\lfloor b/2 \rfloor})^2 \cdot a^{b \bmod 2}$, where $a, b \in \mathbb{N}$.

7.14 $q^t = (q^{\lfloor t/2 \rfloor})^2 \cdot q^{t \bmod 2}$, where $q \in S_k$ and $t \in \mathbb{N}$.

7.15 The hint in the textbook is sufficient.

7.16 *Refer to the textbook.*

7.17 Use dynamic programming. Denote $dp[i][j] = \mathbf{1}\{\langle \{x_1, \dots, x_i\}, j \rangle \in \mathit{SUBSET\text{-}SUM}\}$, where $\alpha \implies \mathbf{1}\{\alpha\} = 1$ and $\neg\alpha \implies \mathbf{1}\{\alpha\} = 0$.

7.18 First $A \in \mathrm{P} = \mathrm{NP}$. Then since there exist $x \in A$ and $y \notin A$, for an arbitrary language $B \in \mathrm{NP} = \mathrm{P}$, f defined as follows is polynomial time computable.

$$f(w) = \begin{cases} x, & w \in B \\ y, & w \notin B \end{cases}$$

*7.19 Let the certificate of $q \in \mathbb{P}$ consist of

- $g \in \mathbb{Z}_m^*$ such that $g^{m-1} = 1$ and $g^{(m-1)/q} \neq 1$ for all prime $q \mid m-1$,
- the standard factorization of $m - 1 = \prod q_i^{r_i}$,
- certificates of $q_i \in \mathbb{P}$.

7.20 It follows from the result of Problem 7.18.

7.21 a. Modify the proof of $\mathit{PATH} \in \mathrm{P}$.

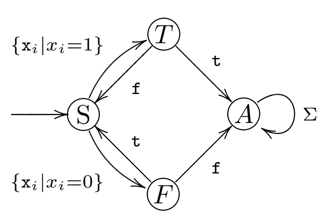
b. It is in NP evidently. For the other side, reduce from *UHAMPATH* by setting k to the amount of nodes in G minus 1.

7.22 It is in NP evidently. For the other side, reduce from SAT. In order to determine whether $\langle\phi\rangle \in \mathit{SAT}$, construct $\phi' = \phi \wedge (z \vee \overline{z})$.

7.23 *Refer to the textbook.*

7.25 If there is no unitary negated clause in ϕ, then ϕ is satisfiable since we can just assign value 1 to all variables. However, if ϕ contains some clauses in the form $(\overline{x_i})$, then this implies that the only possible way to make ϕ true is to let those x_i equal 0. So ϕ can be reduced to some shorter formula repeatedly.

*7.36 It is in NP evidently. For the other side, reduce from *3SAT*. If there is no limitation on Σ, let the following DFA correspond to an assignment to variables $x_1, x_2, \dots, x_n$ in 3cnf-formula ϕ. You may need some extra states and transitions.



Once you solved the problem without regard to the limitation on Σ, based on your solution, consider how to build a reduction where $\Sigma = \{0, 1\}$.

7.37 Using the computation history as certificate we easily obtain $U \in \mathrm{NP}$. And it is easy to show that $\mathit{3SAT} \leq_{\mathrm{P}} U$, by designing an NTM M, which accepts $\langle\phi\rangle$ in polynomial time on at least one branch if and only if $\langle\phi\rangle \in \mathit{3SAT}$.

*7.38 Let $\phi(x/t)$ denote the Boolean formula ϕ after replacing every existance of x in ϕ with t. Suppose there are n variables $x_1, \dots, x_n$ in ϕ. $\langle\phi\rangle \in SAT \implies \langle\phi(x_1/0)\rangle \in SAT$ or $\langle\phi(x_1/1)\rangle \in SAT$, so we can directly assign 0 or 1 to x_1. Recursively assigning $x_2, \dots, x_n$ in this way we have done.

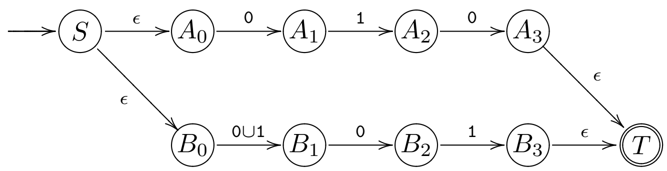
*7.39 Construct language $L = \{\langle n, x, y\rangle \mid n$ has a nontrivial factor in interval $[x, y]\}$, which is apparently in NP = P. Then refer to the solution to Problem 7.38.

*7.40 *Refer to the textbook.*

7.41 Trivial.

*7.42 For b), note that the complement graph of G induces an equivalence relation $\sim$ on Q ($[q]$ is exactly the equivalence class under $\sim$ including q), which has much to do with $\equiv_{L(M)}$ defined in Problem 1.51.

7.43 Here is a sample for $\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3})$.



And $\langle\phi\rangle \notin SAT \iff$ the equivalent minimal NFA is a trivial one.

*7.44 It is a classical problem. See `https://en.wikipedia.org/wiki/2-satisfiability` or refer to Problem 7.51.

7.45 Trivial.

7.46 Since P is closed under complement, we only need to show that $\overline{\textit{MIN-FORMULA}} \in \text{NP} = \text{P}$. It is easy to do because $SAT \in \text{NP} = \text{P}$, and then the certificate for $\langle\phi\rangle \in \overline{\textit{MIN-FORMULA}}$ can be $\langle\phi'\rangle$ such that $|\phi'| < |\phi|$ and they are equivalent.

7.47 $Z = \{\langle G_1, k_1, G_2, k_2\rangle \mid \langle G_1, k_1\rangle \in \textit{CLIQUE}\} - \{\langle G_1, k_1, G_2, k_2\rangle \mid \langle G_2, k_2\rangle \in \textit{CLIQUE}\}$.

*7.48 Obviously $\textit{MAX-CLIQUE} \in \text{DP}$. Suppose there is a graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$. Denote $[n] = \{1, 2, \dots, n\}$. Let $G_+ = (V_+, E_+)$, where

$$\begin{cases} V_+ = \{(i, v_j) \mid i \in [n+1],\ v_j \in V\} \cup \{(i, \alpha) \mid i \in [n+1] - [k]\} \\ E_+ = \{\{(i, u), (j, v)\} \mid i \neq j,\ \{u, v\} \in E\} \cup \{\{(i, \alpha), (j, v)\} \mid i \neq j,\ v \in V \cup \{\alpha\}\} \end{cases}$$

Then $\langle G, k\rangle \in \textit{CLIQUE} \iff \langle G_+, n+1\rangle \in \textit{MAX-CLIQUE}$. Let G_- consist of G_+ and a n-clique (i.e., complete graph K_n). Then $\langle G, k\rangle \notin \textit{CLIQUE} \iff \langle G_-, n\rangle \in \textit{MAX-CLIQUE}$. Try to build a reduction by taking advantage of G_+ and G_-.

*7.49 *Need a solution.*

*7.50 $\overline{EQ_{\textsf{SF-REX}}} = \{\langle R, S\rangle \mid \exists c,\ c \in L(R) \oplus c \in L(S)\}$. Determining whether $c \in L(R)$ can be achieved in (deterministic) polynomial time by constructing a corresponding NFA. Note that any $c \in L(R)$ for a star-free REX satisfies $|c| \leq \text{Poly}(|R|)$.

*7.51 Trivial.

*7.52 *Need a solution.*

*7.53 *Need a solution.*

7.54 *Need a solution.*

8 Space Complexity

8.8 Refer to Example 8.4.

8.9 Build an NTM which nondeterministically guesses a ladder $s, s_2, \ldots, s_k$ and verifies whether $s_k = t$, where k is obviously bounded in $2^{\mathcal{O}(|s|)}$. Then $LADDER_{\mathsf{DFA}} \in \text{NPSPACE} = \text{PSPACE}$ follows.

8.10 It can be reduced to *FORMULA-GAME* directly.

8.11 If so, *SAT* is PSPACE-hard, thus it is PSPACE-complete.

8.12 It is because $\phi_{c_1,c_2,1}$ in the proof of Theorem 8.9 can be written in conjunctive normal form, as we see in the proof of Theorem 7.37 (Cook–Levin theorem).

8.13 Reduce from *TQBF*. Clearly we can construct a $g(n) = n^{100} + 10^{100}$ space TM M deciding *TQBF*. Build mapping $f(\langle\phi\rangle) = \langle M', \langle\phi\rangle\$0^{g(|\langle\phi\rangle|)}\rangle$, where LBA M' does almost the same as M does. On the other side, obviously $A_{\mathsf{LBA}} \in \text{PSPACE}$.

*8.14 For any $\langle G, c, m, h\rangle$, here is a polynomial time algorithm. Let $C[i][j] = +1$ or -1 stand for that we can determine $\langle G, i, j, h\rangle \in$ *HAPPY-CAT* or $\langle G, i, j, h\rangle \in$ *HAPPY-MOUSE*. Similarly let $M[i][j] = +1$ or -1 stand for that we can determine $\langle G, i, j, h\rangle \in$ *HAPPY-MOUSE'* or $\langle G, i, j, h\rangle \in$ *HAPPY-CAT'*, where *HAPPY-CAT'* is defined like *HAPPY-CAT*, except that Mouse moves first. Now we set $M[i][i] = -1$ and $C[i][h] = -1$ for all $i \in G$. Then, repeatedly assign new value to M and C according to the following rules until we get nothing more from the rules.

$$\begin{cases} C[i][j] = -1, & \forall i' \in N(i),\ M[i'][j] = +1 \\ C[i][j] = +1, & \exists i' \in N(i),\ M[i'][j] = -1 \\ M[i][j] = -1, & \forall j' \in N(j),\ C[i][j'] = +1 \\ M[i][j] = +1, & \exists j' \in N(j),\ C[i][j'] = -1 \end{cases}$$

where $N(v)$ stands for the collection of all nodes adjacent to v in G. After this calculation, we claim that $\langle G, c, m, h\rangle \in$ *HAPPY-CAT* $\iff C[c][m] = +1$. The proof of correctness of the algorithm is not evident but also not hard.

8.15 *Need a solution.*

8.16 *Need a solution.*

8.17 A left-to-right scan with memorizing the amount of non-matched `(`s is enough.

*8.18 Let h be a homomorphism that maps brackets to parentheses, e.g., $h($`([)][]`$) =$ `(())()`. Denote the substring of w from the i-th character to the j-th character as $w_{[i,j]}$ and $w_i = w_{[i,i]}$. Then, $w \in B$ if and only if

- $h(w) \in A$, where A is defined in Problem 8.17.
- w_i matches w_j whenever $h(w_{[i+1,j-1]}) \in A$, for all $1 \le i \le j \le |w|$.

*8.19 It is a classical problem. See `https://en.wikipedia.org/wiki/Nim`.

8.20 *Need a solution.*

8.21 *Need a solution.*

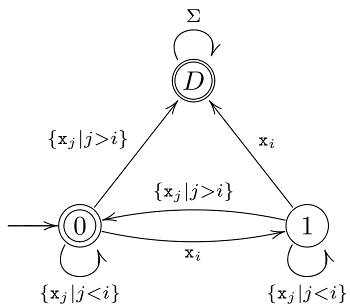
8.22 *Need a solution.*

*8.23 Suppose $G = (V, E)$ has n nodes named $v_0, v_1, \ldots, v_{n-1}$ and define $v_k = v_{k \bmod n}$. Consider the sequence consisting of *directed* edges in G, by seeing an undirected edge as two directed ones: $w_0, w_1, w_2, \ldots$, where

$$w_{m+1} = (v_j, v_k), \text{ if } w_m = (v_i, v_j) \text{ and } \{v_j, v_{i+1}\}, \{v_j, v_{i+2}\}, \ldots, \{v_j, v_{k-1}\} \notin E.$$

G contains no cycle if and only if for every $w_0 = (u, v)$, the first w_r in the sequence such that $w_r = (\cdot, u)$ is (v, u). You may assume G is connected without loss of generality and then use mathematical induction on n to prove this. Note that the sequence $w_0, w_1, w_2, \ldots$ would traverse every edges in G, i.e., perform depth-first search, if G is a tree.

*8.24 **Solution 1.** Let $N_i^{(n)}$ be an NFA given as follows, with $\Sigma = \{\mathtt{x}_1, \ldots, \mathtt{x}_n\}$.

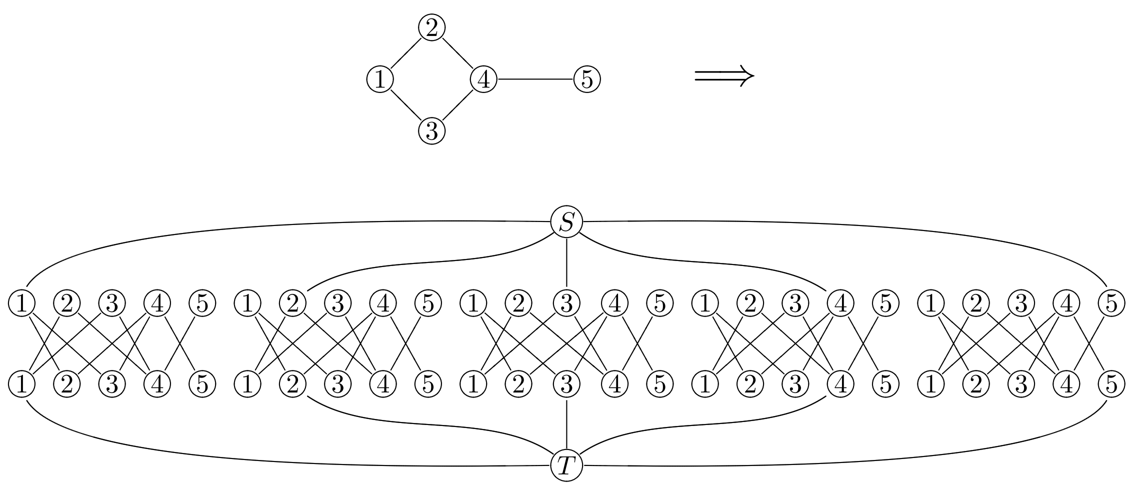


Denote its corresponding regular expression as $R_i^{(n)}$. Then the shortest string which is not in $R_1^{(n)} \cup R_2^{(n)} \cup \cdots \cup R_n^{(n)}$ is $\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1\mathtt{x}_3\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1\mathtt{x}_4\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1\mathtt{x}_3\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1 \cdots \mathtt{x}_n \cdots \mathtt{x}_1\mathtt{x}_2\mathtt{x}_1\mathtt{x}_3\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1$, which is of length $2^n - 1$.

Solution 2. See `https://math.stackexchange.com/a/79031/134950` for a simpler solution involving number theory.

8.25 Surely *BIPARTITE* $\in$ coNL = NL, since we can guess an odd cycle, whose length is no more than n, to verify $\langle G \rangle \notin$ *BIPARTITE*.

8.26 Here is a sample which shows how to do reduction.



8.27 Reduce from *PATH*. In order to determine whether a path from s to t exists in graph $G = (V, E)$, we add some edges (v, s) and (t, v) for all $v \in V$ to obtain a modified graph G'. Then a path exists if and only if G' is strongly connected. On the other side, it is obviously in NL.

8.28 *Need a solution.*

8.29 *Need a solution.*

8.30 *Need a solution.*

*8.31 Note that $\neg x \vee y = x \to y$. Then there is a path from s to t in $G = (V, E)$ if and only if

$$\phi = s \wedge (t \to \neg s) \wedge \bigwedge_{(u,v) \in E} (u \to v)$$

is not satisfiable. Hence $\overline{PATH} \leq_{\mathrm{L}} \mathit{2SAT}$. And $\overline{PATH}$ is NL-complete by NL = coNL. On the other side, you can use the similar approach to show that $\mathit{2SAT} \in \mathrm{NL}$.

8.32 *Need a solution.*

*8.33 In other words, you are asked to find an NL-complete language A and a context-free language B such that $A \equiv_{\mathrm{L}} B$. We choose $A = \mathit{PATH}$ and $\{\mathtt{0}, \mathtt{1}, \mathtt{\#}\}^* \supseteq B = f(A)$, in which $f : A \to B$ is defined as follows.

$$f(\langle G, s, t\rangle) = \langle s\rangle^{\mathcal{R}}(\mathtt{\#}\langle e_1\rangle\langle e_2\rangle \cdots \langle e_m\rangle\mathtt{\#})^n\langle t\rangle,$$

where $G = (V, E)$, $|V| = n$, $E = \{e_1, \ldots, e_m\}$, $\langle e_i\rangle = \mathtt{\#}\langle u_i\rangle\mathtt{\#}\langle v_i\rangle^{\mathcal{R}}\mathtt{\#}$ if $e_i = (u_i, v_i)$.

*8.34 *Refer to the textbook.*

9 Intractability

9.12 $SAT \in \text{TIME}(n^k)$ cannot implies $\text{NP} \subseteq \text{TIME}(n^k)$.

9.13 Trivial.

9.14 Note that $A \in \text{NTIME}(2^{n^k}) \implies pad(A, 2^{n^k}) \in \text{NP}$, and $pad(A, 2^{n^k}) \in \text{P} \implies A \in \text{TIME}(2^{n^{k+1}})$.

9.15 *Refer to the textbook.*

9.16 The proof of Theorem 8.9 shows that a language $L \in \text{SPACE}(f(n))$ can be reduced to the $TQBF$ problem with a log space reduction h such that $|h(w)| = \mathcal{O}(f(|w|)^2)$. Letting $f(n) = n^3$ we will get $L \in \text{SPACE}(((n^3)^2)^{1/3}) = \text{SPACE}(n^2)$, thus $\text{SPACE}(n^3) \subseteq \text{SPACE}(n^2)$ if $TQBF \in \text{SPACE}(n^{1/3})$. Absurd.

*9.17 Let n denote the length of input string for a certain 2DFA D. There is a TM using $\mathcal{O}(n^2)$ time to decide the language $L(D)$ since D has at most $\mathcal{O}(n^2)$ configurations.

9.18 Let $E(R)$ stand for $\langle R \rangle \in E_{\mathsf{REX}\uparrow}$. Then,

- $E(RS) \iff E(R) \wedge E(S)$,
- $E(R \cup S) \iff E(R) \wedge E(S)$,
- $E(R^*) \iff E(R)$.

9.19 Refer to the solution to Problem 7.38.

9.20 *Need a solution.*

9.21
a. Trivial, since SAT is NP-complete and $\overline{SAT}$ is coNP-complete.
b. If so, let $J = \mathtt{0}SAT \cup \mathtt{1}\overline{SAT}$. We have $SAT, \overline{SAT} \leq_{\text{P}} J \in \text{P}^{SAT,1}$.

9.22 We query A and B for the value of $\phi = \exists x_1 \forall x_2 \exists x_3 \ldots [\psi]$. If they do not agree with each other, let them play the formula game on ϕ, after which we adopt the winner's opinion.

9.23 *Need a solution.*

9.24 Refer to Problem 9.25.

*9.25 You already have an $\mathcal{O}(n)$ one, if following the hint given in Problem 9.24b.

10 Advanced Topics in Complexity Theory

10.8 Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A. Denote the substring of w from the a-th character to the b-th character as $w_{[a,b]}$. Recursively, i.e., using divide and conquer solve that

$$\text{is } \delta(q_i, w_{[a,b]}) = q_j? \quad (q_i, q_j \in Q)$$

for a series of (a, b)s.

*10.9 Suppose A has size–depth complexity $(f(n), \mathcal{O}(\log n))$, directly write a Boolean formula according to A's circuit. It already has polynomial size due to the $\mathcal{O}(\log n)$ depth. On the other side, note that

$$\phi(\psi, x_1, \ldots, x_n) = (\neg\psi \wedge \phi(0, x_1, \ldots, x_n)) \vee (\psi \wedge \phi(1, x_1, \ldots, x_n))$$

where ϕ, ψ are Boolean formulas, which can be used appropriately (choose ψ carefully and use the above transformation in a recursive manner) to rewrite a $f(n)$ size Boolean formula within $\mathcal{O}(\log f(n))$ depth.

*10.10 Denote the length of input by n as usual. For a language decided by a k-PDA P, we can use dynamic programming to decide it in polynomial time, thereby proving $\bigcup_k \mathrm{PDA}_k \subseteq \mathrm{P}$. To be specific, let

$$dp[(q, x, p_1, \ldots, p_k)][(q', x', p'_1, \ldots, p'_k)] \in \{0, 1\} \quad (q, q' \in Q,\ x, x' \in \Gamma,\ p_i, p'_i \in \{0, 1, \ldots, n\})$$

stand for whether when P is started with configuration $(q, x, p_1, \ldots, p_k)$, it can reach $(q', x', p'_1, \ldots, p'_k)$, and these two configuration have the same stack except the difference between x and x', and meanwhile P never pops x.

On the other side, we can use a k-PDA P for some k to simulate an arbitrary $\mathcal{O}(\log n)$ space alternating TM, thereby proving $\bigcup_k \mathrm{PDA}_k \supseteq \mathrm{AL} = \mathrm{P}$. Nondeterminism or alternation will not be an issue since a PDA can do depth-first search due to its having a stack. The following facts can help you simulate an $\mathcal{O}(\log n)$ work tape. Let $p_1, \ldots, p_k$ ($\in \{0, 1, \ldots, n\}$) denote the positions of k input heads of a k-PDA P. There exists $m = k + \mathcal{O}(1)$ such that P can manipulate $p_1, \ldots, p_m$ in these manners:

- Test whether $x = 0$, or set $x := 0$, or set $x := x \pm 1$ (abbreviated as x_{+1} or x_{-1}) .
- $y := x$, which can be implemented by
  $y := 0$; $z := 0$; **while** $x \neq 0$ **do** $\{x_{-1};\ y_{+1};\ z_{+1}\}$; **while** $z \neq 0$ **do** $\{z_{-1};\ x_{+1}\}$;
- **repeat** x **times doing** W, which can be implemented by
  $y := x$; **while** $y \neq 0$ **do** $\{y_{-1};\ W\}$;
- $z = x \pm y$, $z = xy$, $z = \lfloor x/y \rfloor$, and so on.

Implement those functions necessary for you to simulate an $\mathcal{O}(\log n)$ read/write work tape.

10.11 Trivial

10.12 $L \in \prod_1 P \iff \overline{L} \in \Sigma_1 P = NP$, so if $P = NP$ then $P = \prod_1 P$.

$L \in \Sigma_2 P \iff L = \{w | \exists x \forall y \langle x, y, w\rangle \in C\}$ where $C \in P \iff L = \{w | \exists x \langle x, w\rangle \in C'\}$ where $C' \in \prod_1 P$, so if $P = NP$ then $P = \Sigma_2 P$.

Using the same method can prove that if $P = NP$ then $\forall k \;\; P = \Sigma_k P = \prod_k P$.

10.13 If $PH = PSPACE$, then $\forall L \in PH,\ L \leq_P TQBF$. So, if $TQBF \in \Sigma_k P$, then all the languages in $\Sigma_i P$ or $\prod_i P$ where $i > k$ can be reduced to $\Sigma_k P$. Therefore, there is no difference between these hierarchies.

10.14 $L \in \Sigma_2 P$

$\iff L = \{w | \exists x \forall y \langle x, y, w\rangle \in C\}$ where C is decidable in polynomial time

$\iff L = \{w | \exists x \langle x, w\rangle \in C'\}$ where $C' = \{\langle x, w\rangle | \forall y \langle x, y, w\rangle \in C\}$

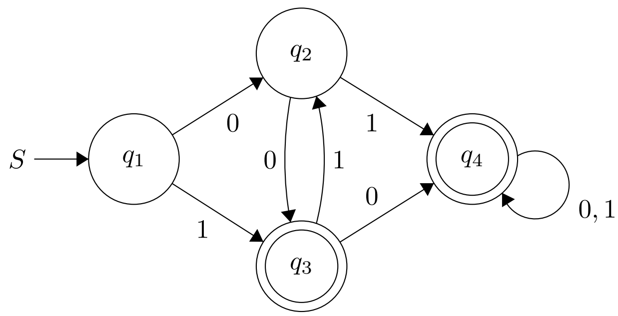
Because $\overline{C'} = \left\{\langle x, w\rangle | \exists y \langle x, y, w\rangle \in \overline{C}\right\}$ and $\overline{C}$ is decidable in polynomial time, we know that $\overline{C'} \in NP$. Therefore $\overline{C'}$ is decidable with oracle SAT in polynomial time. So C' is also decidable with oracle SAT in polynomial time and this should tell us that $L = \{w | \exists x \langle x, w\rangle \in C'\} \iff L \in NP^{SAT}$

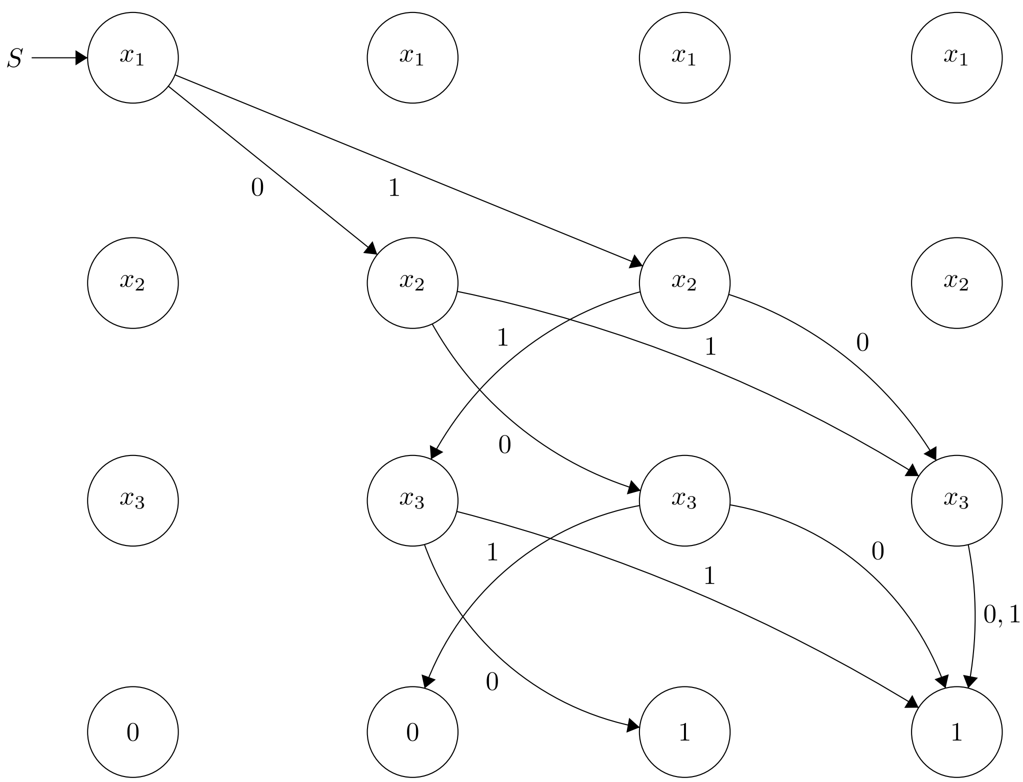
*10.15 See `https://en.wikipedia.org/wiki/Proofs_of_Fermat\%27s_little_theorem`.

10.16 *Refer to the textbook.*

10.17 Use similar method in *Problem 10.22*.

10.18 Since A is a regular language, there exists a DFA M recognizing A. Just constructing a branching program of depth n along the possible paths within n transitions in M gives the right proof. The following conversion from a DFA to a branching program of $n(= 3)$ variables should explain the way to do that. So, the size is at most $|Q_M| \times n$.





10.19 Obviously RP $\subseteq$ NP. We only need to show that if NP $\subseteq$ BPP then NP $\subseteq$ RP. Since SAT is NP complete, we only need to show SAT $\in$ RP. Given a formula ϕ with m variables, find a BPP algorithm B with error probability 2^{-m-1} (the assumption that SAT $\in$ BPP guarantees this). We can run B on ϕ, if it rejects then the RP algorithm R rejects. Otherwise we randomly choose an assignment to the variable x_1 and run B on the reduced formula. If it rejects then we change the value of x_1. Then we perform similar steps for x_2, x_3, ..., x_m, and in the end test whether the resulting formula is true and we accept or reject accordingly.

10.20 See `https://en.wikipedia.org/wiki/ZPP_(complexity)#Intersection_definition`

One note for tackling the difference between worst-time and averaged-time: take the time consumed on a random input as a random variable, and apply Markov's Inequality to it.

10.21 Show that $3SAT \leq_P \overline{EQ_{BP}}$: $\phi \in 3SAT$ iff $L(B1) \neq L(B2)$, where B_1 is a branching program which unconditionally outputs 0 and B_2 is a branching program constructed from ϕ so that ϕ is satisfiable iff B_2 can reach the output node 1 with the corresponding assignment of variables.

10.22 Suppose A is a language that is decided by a probabilistic logarithmic space TM M with error probability $\frac{1}{3}$, we can construct a polynomial time algorithm that calculates the probability that string w of length n is accepted by M. Then we compare it with $\frac{2}{3}$. The algorithm is described as follows.

First, build a graph G according to M and w. The vertices of G are possible configurations of M under input string w (see *Definition 8.20* for the definition of configuration under current context) and the directed edges of G show whether a configuration can be transited into another within one deterministic step or one flip-coin step of M.

We claim that G is a DAG, because otherwise M is not a decider for string w (it can fall into loops and never terminate). Also, since M runs in logarithmic space, the size of G is bounded by some polynomial of n.

Then, we can calculate the acceptance probability on G in polynomial time using techniques of topological sorting and dynamic programming.