CIS 278 (CS1) Programming Methods: C++

Lesson 22: Recursion

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Section 22.1: Introduction to Recursion

Up to now whenever we have needed to repeat actions in a program we have used some kind of a loop. This technique is called "iteration". A second technique for causing actions in a program to repeat is called recursion. Recursion happens when a function calls itself. (Technical note: you could also have recursion if you have two functions that call each other. We will not discuss this technique in this class.)

Before we delve into recursion there are four preliminary observations I would like to make.

First, you should only write a function that calls itself when you specifically want to use recursion and you understand what you are doing. I have had students submit assignments in which they had a function calling itself only because they didn't understand how to use a loop to handle the repetition. Using recursion in this manner is really bad practice.

Second, it is important to realize that you can always use iteration to write any program. In other words, recursion is a programming tool that can simplify your code (in some cases dramatically) but is never required. Any program that you write using recursion could be rewritten using iteration. Not necessarily easily, but it could be done. The reverse is also true: any program written using iteration could be rewritten using recursion.

Third, there are always two ways of looking at a recursive solution to a problem. One is what I like to call the "leap of faith" method. As you will soon see, when writing a recursive function we are allowed to assume (with certain restrictions) that the function we are writing already exists and works as desired. Although this is mathematically provable (don't worry, we won't go there), it is difficult for many students to feel comfortable with. You will find recursion to be much easier to use if you can get yourself to take this "leap of faith".

The second way to look at a recursive solution to a problem is to try to follow through with each recursive function call, looking at the processes in precise detail, to understand what is really going on in the computer when the function is executed. We will see that this can get you buried in details rather quickly. Although we will discuss some techniques for getting to the details like this, it is highly recommended that you avoid this as much as possible when writing your recursive functions. Keeping things at the "leap of faith" level will usually serve our needs much better.

Fourth, often when writing a recursive function it is hard to see how things are going to fit together when you start. The best strategy is to follow the steps I will outline below, gradually building up and filling in the pieces of the recursive function, and (hopefully) before you know it the entire function will be done.

Section 22.2: The pow Function

In order to explain the process of writing a recursive function let's use a simple example. C++ has a pow function that evaluates expressions involving exponents. For example, pow(2,5) equals 32. We will write a very simple version of this function which requires the base to be an integer and the exponent to be a non-negative integer. We will use the name "power" to avoid confusion with C++'s pow function. Here is the header for our power function:

```
int power(int base, int exponent)
```

The first question to ask yourself when writing a recursive function is "what is the base case?" The base case is used to make sure that the recursive function does not recurse infinitely. If the function calls itself every time it is called, then it will never stop. So there must be a situation in which the function does NOT call itself. To get more specific, the base case for a problem should be a case of the problem that can be solved immediately without having to call the function again. Furthermore, it should be the simplest such case. For example, in our power function, the base case could be exponent == 2, because when the exponent is 2 we can immediately return base * base as the return value of the function. However, the case where exponent == 1 is simpler and so is preferable. In fact, there is a simpler case yet: when exponent == 0, we can immediately return 1 as the return value of the function (since anything raised to the 0 power is 1).

Having determined the base case, we can now write part of our function:

```
int power(int base, int exponent)
{
    if (exponent == 0){
        return 1;
    } else {
        [to be continued...]
    }
}
```

See? We are already half way through writing our function! Now on to the second question: what do we want to do in the recursive case? There are several hints that we can use to figure out how to proceed. (1) this is a value-returning function, so each case, including the recursive case, is probably going to involve a return. (2) Since this is a recursive function, we know that there is going to be a call to power somewhere in the return statement. (3) An important rule (perhaps the most important rule) for writing recursive functions is that when you make the recursive call you must use an argument that gets you closer to the base case. If you do this, then you can assume that the function call does what it is supposed to do, even though you haven't finished writing the function yet (this is the leap of faith I mentioned above). In our case, the base case checks to see if exponent is 0, so the argument in our recursive call will have to be something smaller than the current value of exponent. A good first try is to simply subtract 1 from exponent to make it smaller. Putting these observations together, we have

```
int power(int base, int exponent)
{
    if (exponent == 0){
        return 1;
    } else {
        return ... power(base, exponent - 1);
    }
}
```

The only thing left for us to do is to figure out what should go in the place of the ... in the code above. We use the fact that X^n is equal to X times X^{n-1} to determine that we need to multiply our function call by base in order to return the correct answer. Here is the final result, which has been placed in a complete C++ program:

```
#include <iostream>
using namespace std;

int power(int base, int exponent);

int main()
{
    int result;
    result = power(2, 3);
```

```
    cout << result << endl;
}

int power(int base, int exponent)
{
    if (exponent == 0){
        return 1;
    } else {
        return base * power(base, exponent - 1);
    }
}
```

Section 22.3: Analyzing the power Function -- Technique #1

Now we are going to use two different techniques to take a close look at the details behind how this recursive function works. The first technique is a good way to predict the results of a recursive function call. You might want to use this technique if you are writing a recursive function and you aren't getting the results you expect. This technique also might come in handy if you are doing your exercises or taking the final and you are asked to predict the results of a recursive function call.

The technique involves making a table with three columns. In the left column we will write our function call. In the middle column we will write the intermediate result of this function call. It is called an "intermediate result" because with a recursive function call we can't immediately determine the final result. This intermediate result will likely involve a call to the same function with arguments that are closer to the base case of the function. In the third column of our table we will write the final result. As you will see, the first and second column will be filled as we go down the table, and then the third column will be filled as we go back up the table, with the final result of the original function call finally appearing at the top of the third column.

Let's take a look at what happens when we call our power function with arguments 2 and 3. Here is the initial setup of our table:

function call	intermediate result	final result
pow(2, 3)		

Looking at the code of our power function, we see that, because exponent is not equal to 0, the intermediate result of the function call is base * power(base, exponent -1) or, in our case, 2 * power(2, 2). Let's add this to our table:

function call	intermediate result	final result
pow(2, 3)	2 * pow(2, 2)	

We are now faced with the problem that we don't know what power(2, 2) is. So we start a new row in our table, placing power(2, 2) in the first column because that is now what we need to compute:

function call	intermediate result	final result
pow(2, 3)	2 * pow(2, 2)	
pow(2, 2)		

Continuing in this way we eventually end up with power(2, 0) in the left column. Looking at our code again, we see that the intermediate result for this function call is "1":

function call	intermediate result	final result
pow(2, 3)	2 * pow(2, 2)	
pow(2, 2)	2 * pow(2, 1)	
pow(2, 1)	2 * pow(2, 0)	
pow(2, 0)	1	

We have finally reached a point where we can actually write in the final result, since the intermediate result doesn't include any function calls:

function call	intermediate result	final result
pow(2, 3)	2 * pow(2, 2)	
pow(2, 2)	2 * pow(2, 1)	
pow(2, 1)	2 * pow(2, 0)	
pow(2, 0)	1	1

Now that we have a final result, we can start working our way back up the table. The fourth row in our table tells us that power(2, 0) has a final result of "1", so we can now compute the final result in the third row: 2 * power(2, 0) == 2 * 1 == 2:

function call	intermediate result	final result
pow(2, 3)	2 * pow(2, 2)	
pow(2, 2)	2 * pow(2, 1)	
pow(2, 1)	2 * pow(2, 0)	2
pow(2, 0)	1	1

Continuing on up the table in this way we can complete the second row of the table and then the first row of the table:

function call	intermediate result	final result
pow(2, 3)	2 * pow(2, 2)	8
pow(2, 2)	2 * pow(2, 1)	4
pow(2, 1)	2 * pow(2, 0)	2
pow(2, 0)	1	1

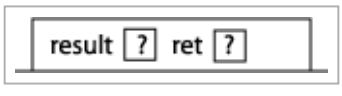
This tells us that the result of the original function call is 8.

This is a good technique for determining the result of a recursive function call. Note that in other cases (for example if the function is not a value-returning function but produces some output) you may have to be creative to get this technique to work, but it has never failed me. I hope that seeing the recursive calls play out as you have seen in this example helps you to understand what is going on behind the scenes when you call a recursive function.

Section 22.4: Analyzing the power Function -- Technique #2

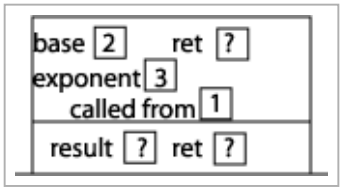
Let's now analyze the power function using a second technique. This technique is more complex than the last one, so you probably won't want to use it in your own analyses of recursive functions. I show it here only to give you a better understanding of how the computer itself actually handles a call to a recursive function. I suggest that you follow the explanation carefully, despite its tediousness.

Recall from our pointers lesson that all automatic variables are stored in a highly structured part of memory called the stack. Each time a function is called, all of the memory that that function will need is allocated on the stack in what is called an **activation record** for that function call. The first activation record on the stack will always be an activation record for the function main. Our main function (see the complete program listed in section 1 above) has one local variable, result, so this first activation record will have to include memory for that variable. In addition (this is a complex but important point!), whenever a function calls a value returning function, its activation record will need some place to store the value that gets returned. In our pictures we will label this memory location with the abbreviation "ret" (short for "returned value"). So here is a picture of the stack with the initial activation record for main:

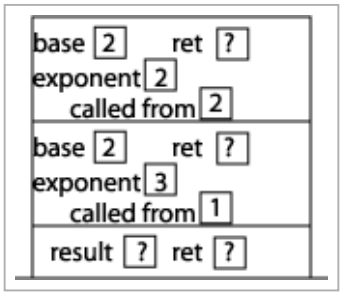


Now let's start executing the statements inside main. The first thing that happens is the function power is called, so we need to add an activation record for this function call. This activation record will include 2 memory locations for the two parameters, an additional memory location to store the return value of the function call that appears in the function, and one additional thing: the computer will have to have some way of knowing where to return to when this function is done executing. In other words, we need to know which of the calls to the power function that we see in our program initiated this activation record. The computer does this by storing in this activation record the memory address of the next machine instruction to be executed when the execution of this function ends. To simplify things, we will number the calls to power: the call from main will be call #1, and the call from inside power will be call #2. Then we just have to store either 1 or 2. Since the

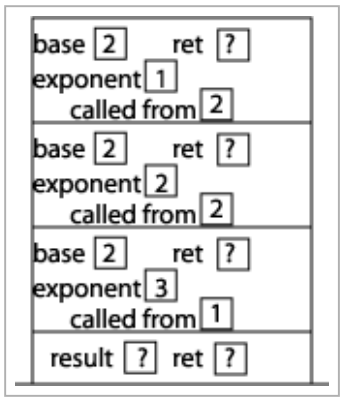
initial call to power is from main, we will store a 1 in a memory location named "called from" as pictured here:



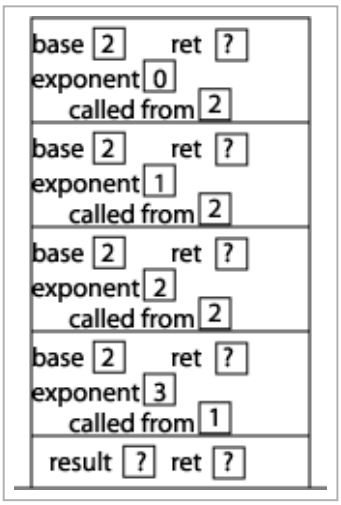
Now we can start executing the statements inside power. Since exponent is not equal to 0, we execute the "else" part of the if statement, which involves a function call to the function power, this time with arguments 2 and 2. We put a "2" in the called from memory location to indicate that the function was called from within power. Setting up that activation record results in this picture of the stack:



Now we can start executing the statements inside power. Since exponent is not equal to 0, we execute the "else" part of the if statement, which involves a function call to the function power, this time with arguments 2 and 1. We put a "2" in the called from memory location to indicate that the function was called from within power. Setting up that activation record results in this picture of the stack:



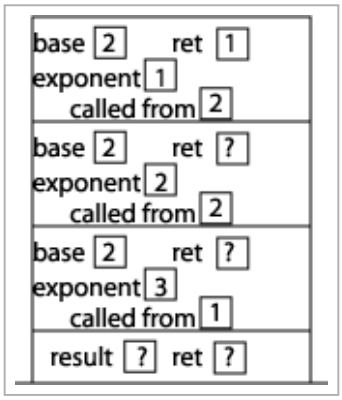
Now we can start executing the statements inside power. Since exponent is not equal to 0, we execute the "else" part of the if statement, which involves a function call to the function power, this time with arguments 2 and 0. We put a "2" in the called from memory location to indicate that the function was called from within power. Setting up that activation record results in this picture of the stack:

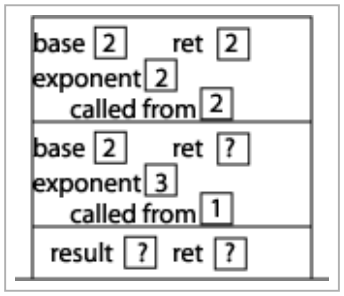


Now we can start executing the statements inside power. Since exponent is now equal to 0, we execute the "if" part of the if statement. We execute the "return 1" statement by placing a 1 in the "ret" memory address of the previous activation record. Since the current power function is now done executing, we check the value stored in the "called from" memory address, which tells us to continue execution from the function call to power inside power. In addition, we erase the current activation record from the stack:

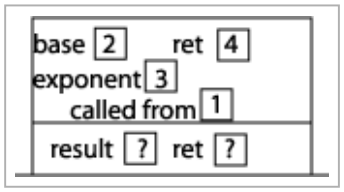
Continuing our execution, we find that we are in the middle of executing the statement "return base * power(base, exponent - 1);". The value of "power(2, 0)" (stored in the "ret" memory location) is 1, so we want to return 2 * 1 which is 2. This result gets stored in the "ret" memory location of the previous activation record. Since the current power function is now done executing, we check the value stored in the "called from" memory address, which tells us to continue execution from the function call to power inside power. In addition, we erase the current activation record from the stack:

Continuing our execution, we find that we are in the middle of executing the statement "return base * power(base, exponent - 1);". The value of "power(2, 1)" (stored in the "ret" memory location) is 2, so we

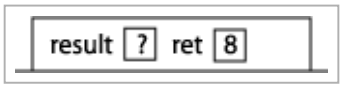




want to return 2 * 2 which is 4. This result gets stored in the "ret" memory location of the previous activation record. Since the current power function is now done executing, we check the value stored in the "called from" memory address, which tells us to continue execution from the function call to power inside power. In addition, we erase the current activation record from the stack:



Continuing our execution, we find that we are in the middle of executing the statement "return base * power(base, exponent - 1);". The value of "power(2, 2)" (stored in the "ret" memory location) is 4, so we want to return 2 * 4 which is 8. This result gets stored in the "ret" memory location of the previous activation record. Since the current power function is now done executing, we check the value stored in the "called from" memory address, which tells us to continue execution from the function call to power inside main. In addition, we erase the current activation record from the stack:



Continuing our execution (back in main now), we find that we are in the middle of executing the statement "result = power(2, 3);". The value of "power(2, 3)" (stored in the "ret" memory location) is 8, so we want to assign result the value 8, which then is output by the cout statement.

Section 22.5: The findSmallest Function

In the rest of this lesson we will basically be developing a series of increasingly complex recursive functions. I strongly suggest that you spend a few minutes trying to write each of these functions on your own before reading the solution. In each section I will present a problem, give a function header for the function we are going to write, and then suggest that you stop reading and try it on your own.

The first function we will attempt will be named "findSmallest". This function will take one argument that is an array of integers and a second argument that is an int that indicates how many elements of the array are being used (i.e. how many items are in the array). The function should return the smallest item in the list. Here is our function header:

int findSmallest(const int list[], int numItems)

Now is where you should stop reading and try this on your own!

What should our base case be? Ask yourself: what is a situation in which we can easily determine result of the function without making a recursive call? When the number of items in the list is 1! When numItems == 1, that single item is the smallest item, and so we simply return it.

It is a good idea to always ask yourself after you have thought of a base case if there is a simpler base case. In this case, we might wonder whether a simpler base case would be when the number of items in the list is 0. But what should this function return if the number of items in the list is 0? It doesn't really make sense to call this function when the number of items is 0. So we need to impose a pre-condition for this function: numItems > 0.

Here is our code so far:

```
int findSmallest(const int list[], int numItems)
```

```
{
    if (numItems == 1){
        return list[0];
    } else {
        [to be continued...]
    }
}
```

Now for our recursive case. Remember that when trying to write the recursive case, we can call the function we are writing as long as we call it with arguments that get us closer to the base case. What might that function call look like? Since the base case occurs when numItems == 1, it makes sense that we would call it with numItems reduced by 1:

findSmallest(list, numItems - 1);

What will this function call give us? The smallest item in the list **not counting the last item**! Can we use this information to figure out what the smallest item in the **entire** list is? Yes. If the last item in the list is smaller than the result of this function call, then we should return the last item in the list. Otherwise, we should return the result of this function call. Translating that last sentence into C++ we get the following function:

```
int findSmallest(const int list[], int numItems)
{
    if (numItems == 1){
        return list[0];
    } else if (list[numItems - 1] < findSmallest(list, numItems - 1)){
        return list[numItems - 1];
    } else {
        return findSmallest(list, numItems - 1);
    }
}
```

This function will work, but there is some ugliness to it. Having multiple recursive function calls in a recursive function is extremely inefficient, because each of those recursive calls may result in a large number of additional function calls. If we can avoid multiple recursvie function calls we should, and in this case there is no reason to calculate the smallest item in the list twice. Let's just do it once and store the result in a local variable:

```
int findSmallest(int list[], int numItems)
{
    if (numItems == 1){
        return list[0];
    } else {
        int smallest = findSmallest(list, numItems - 1);
        if (list[numItems - 1] < smallest){
            return list[numItems - 1];
        } else {
            return smallest;
        }
    }
}
```

This is better, but we can do even better if we take advantage of C++'s min function, which returns the smallest of its two arguments. This function is in the algorithm header file. Here is a complete program that we can use to test the final version of our findSmallest function:

```
#include <iostream>
#include <algorithm>
using namespace std;
```

```
int findSmallest(const int list[], int size);

int main()
{
    int list[] = {4,6,2,9,5,1,7,3};
    int size = 8;

    cout << "The smallest item in the list is "
         << findSmallest(list, size) << endl;
}

int findSmallest(const int list[], int numItems)
{
    if (numItems == 1){
        return list[0];
    } else {
        return min(list[numItems - 1],
                   findSmallest(list, numItems - 1));
    }
}
```

Section 22.6: The printVertical Function

We will now write a function called printVertical that will take a single integer argument and write the integer vertically, one digit per line. We will assume that the arguement is non-negative. So, for example, if we call the function like this:

printVertical(9746);

We should get the following output:

9
7
4
6

This is an appropriate example for us to work on at this point because we have not yet done a recursive function that is void and that produces output. Things may look a little different, but the rules for writing recursive functions outlined so far in this lesson still apply.

You might think for a moment about how you would do this without using recursion. It is possible, but not easy. (If you don't think it would be all that hard, take a few moments and write it!) Here is our function header:

void printVertical(int num)

Now is where you should stop reading and try this on your own!

What should our base case be? If num has only one digit, then we can just print out the number and be done with it. How can we check to see that it has only one digit? Just check to see if it is less than 10. Here is our code so far:

```
void printVertical(int num)
{
    if (num < 10){
        cout << num << endl;
    } else {
        [to be continued...]
    }
}
```

What do we want to do in the recursive case? When writing a recursive function, if the recursive call is not immediately obvious to you, ask yourself: How should I call the function recursively so that (1) the

arguments get me closer to the base case and (2) the function call helps me get to the solution to the problem. In the case of printVertical, my first thought might be to subtract 1 from num for my recursive call, since that gets me closer to the base case of num < 10. However, calling printVertical with num - 1 as the argument doesn't really help me with the problem of printing num vertically. For example, if num is 9746, printing the number 9745 vertically doesn't really help me. What would help me? What if I could print the number 974 vertically? That would help! Then I would just have to print the digit 6 and I'd be done! Can I call printVertical and have it print everything except the last digit? Yes. Call it with an argument of num / 10. 9746 / 10 is 974! So the solution is to call printVertical with an argument of num / 10 to print all of the digits except for the last one, and then print the last digit by printing the expression num % 10. Here is our complete code, including a driver program:

```
#include <iostream>
using namespace std;

void printVertical(int num);

int main()
{
    printVertical(9746);
}


void printVertical(int num)
{
    if (num < 10){
        cout << num << endl;
    } else {
        printVertical(num / 10);
        cout << num % 10 << endl;
    }
}
```

Section 22.7: Towers of Hanoi

In the mountains near the city of Hanoi there is a Buddhist monastery. Outside the monastery there are three tall towers (which we will call LEFT, MIDDLE, and RIGHT) and 64 stone disks. Each of the disks has a hole in its center so that it can be placed on one of the towers (think of the towers as poles). Each disk is a different size. At the beginning of time, all 64 disks were stacked on the LEFT tower, ordered from largest (on the bottom) to smallest (at the top). Day and night the monks who live in the monastery unceasingly move disks one at a time from one tower to another. They never place a disk on a tower that holds a smaller disk. When they have moved all 64 disks from the LEFT tower to the RIGHT tower, it will be the end of time.

I don't know about you, but this has me a bit worried, so I've decided to write a function to simulate this process, and then analyze the function to see how much time I have left.

I suggest that you run a **Towers of Hanoi simulation** a few times to become familiar with the problem.

If we play with this problem for awhile we begin to see a pattern in the solutions. For example, when we move 4 disks from the LEFT tower to the RIGHT tower, we always begin by using a sequence of moves to get 3 disks from the LEFT tower to the MIDDLE tower; then we move the 1 remaining disk from the LEFT tower to the RIGHT tower; finally we use a sequence of moves to get those 3 disks that are sitting on the MIDDLE tower to the RIGHT tower. Let's state this more generally. Assume that there are n disks on the LEFT tower. The problem can be solved by moving n - 1 disks from the LEFT tower to the MIDDLE tower, then moving the remaining disk on the LEFT tower to the RIGHT tower, then moving the n - 1 disks that are sitting on the MIDDLE tower to the RIGHT tower. By now I hope that this sounds to you like a naturally recursive function. We can move the n - 1 disks by using a recursive call.

Let's write our recursive function now. Rather than use fancy graphics, we will write a function that prints out instructions for solving the problem. For example, here is the output that our function should produce if the number of disks is 4:

```
move a disk from the LEFT tower to the MIDDLE tower.
move a disk from the LEFT tower to the RIGHT tower.
move a disk from the MIDDLE tower to the RIGHT tower.
move a disk from the LEFT tower to the MIDDLE tower.
move a disk from the RIGHT tower to the LEFT tower.
move a disk from the RIGHT tower to the MIDDLE tower.
move a disk from the LEFT tower to the MIDDLE tower.
move a disk from the LEFT tower to the RIGHT tower.
move a disk from the MIDDLE tower to the RIGHT tower.
move a disk from the MIDDLE tower to the LEFT tower.
move a disk from the RIGHT tower to the LEFT tower.
move a disk from the MIDDLE tower to the RIGHT tower.
move a disk from the LEFT tower to the MIDDLE tower.
move a disk from the LEFT tower to the RIGHT tower.
move a disk from the MIDDLE tower to the RIGHT tower.
```

Let's first think about what our parameters will need to be. We'll need to know how many disks to move, which tower we are moving the disks from (the source), which tower we are moving the disks to (the target), and which tower we are using for temporary storage (the storage). Here is the function header:

```
void moveDisks(int numDisks,
               string source,
               string target,
               string storage)
```

A function call that would result in the output given above would be:

moveDisks(4, "LEFT", "RIGHT", "MIDDLE");
Now is where you should stop reading and try this on your own!

What should the base case be? We could choose numDisks == 1 for our base case, but there is a simpler base case. We'll go with numDisks == 0. What action do we take if numDisks is 0? None! So one way to structure our solution would be

```
void moveDisks(int numDisks,
               string source,
               string target,
               string storage)
{
    if (numDisks == 0){
        ;    // do nothing!
    } else {
        [to be continued...]
    }
}
```

This, however, is a little clumsy. Let's reverse the condition in the if statement so that we have an if with no else. This will look a little different from the typical recursive function in which you can explicitly see both the base case and the recursive case, but it will be better code. You'll just have to remember that the base case is there implicitly because the function does nothing if the if condition is false. Here is our improved way to structure the solution:

```
void moveDisks(int numDisks,
               string source,
               string target,
               string storage)
{
    if (numDisks > 0){
        [recursive case]
    }
}
```

What do we want to do in the recursive case? This is where the strategy described above comes into play. We will first use a recursive call to move numDisks - 1 disks from the source to the storage (note that the

argument in the recursive call will be numDisks - 1, which means we'll be getting closer to our base case of numDisks == 0). Then we will use a cout statement to print the instruction to move one disk from the source to the the target. Finally we will use a second recursive call to move those numDisks - 1 disks that are sitting on the storage tower to the target. Here is our final program, which will produce the output shown above.

```
#include <iostream>
using namespace std;

const int NUM_DISKS = 4;

void moveDisks(int numDisks,
               string source,
               string target,
               string storage);

int main()
{
    moveDisks(NUM_DISKS, "LEFT", "RIGHT", "MIDDLE");
}


void moveDisks(int numDisks,
               string source,
               string target,
               string storage)
{
    if (numDisks > 0){
        moveDisks(numDisks - 1, source, storage, target);
        cout << "move a disk from the " << source
             << " tower to the " << target << " tower." << endl;
        moveDisks(numDisks - 1, storage, target, source);
    }
}
```

Section 22.8: Analyzing Towers of Hanoi

I hope you can read this section slowly enough to really enjoy it. I think it is a lot of fun.

I still want to find out how long I have to enjoy before the end of time hits. Let's figure out how long it will take those monks to move the 64 disks from the LEFT tower to the RIGHT tower. Let's start by figuring out how many moves it will take. 1 disk takes 1 move. 2 disks take 3 moves. 3 disks take 7 moves. How many moves will 4 disks take? Can I calculate this without actually counting? Sure. To move 4 disks we first move 3 disks (7 moves), then we move 1 disk (1 move), then we move 3 disks again (7 moves) for a total of 15 moves. Using the same technique, we find that 5 disks take 31 moves (15 + 1 + 15) and 6 disks take 63 moves (31 + 1 + 31). Let's try to find a formula that tells us how many moves it takes to move n disks. Here's a table listing our results so far:

number of disks	number of moves
1	1
2	3
3	7
4	15
5	31
6	63
n	??

The key to finding a formula for moving n disks is to notice that you can get each of the numbers in the right column of this table by taking 2 raised to the power of the corresponding number in the left column and then subtracting 1. In other words, if the number of disks is n, the number of moves required is $2^n - 1$.

Where does this leave us? The number of moves required to move 64 disks is $2^{64} - 1$, which is about 1.8×10^{19}. So how much time will this take? Let's assume that the monks can move 1 disk per minute. (I think this is a lot faster than the monks can actually move the disks, since those stone disks are quite heavy, but let's go with this estimate and see what happens.) Based on this assumption, it will take the monks 1.8×10^{19} minutes to move the disks. Dividing this by 60 to find out how many hours it will take them gives us about 3×10^{17} hours. Dividing by 24 to find out how many days gives us about 1.3×10^{16} days. Dividing by 365 to find out how many years gives us about 3.5×10^{13} (about 35 trillion) years. Whew, I'm feeling much better now.

Just for fun, let's talk about how long it would take a computer to move the disks. To give the computer an advantage, let's take out the cout statement so that the instructions don't actually have to be output, just calculated internally. When I did this on my computer with 23 disks it took about 12 seconds. Since it takes about 8,400,000 moves to move 23 disks, I figure that my computer is moving 700,000 disks per second (8,400,000 / 12). (Incidentally, I'm working on a 550Mhz Mac. When I tried this on a 500Mhz PC it came out to about 100,000 moves per second. And yes, this was done a long time ago :) 1.8×10^{19} moves. Dividing by 700,000 to find out how many seconds it would take my computer to move 64 disks we get 2.6×10^{13} seconds. Dividing by 60 to find out how many minutes gives us 4.4×10^{11} minutes. Dividing by 60 to find out how many hours gives us 7.3×10^{9} hours. Dividing by 24 to find out how many days gives us 3×10^{8} days. Dividing by 365 to find out how many years gives us 835,632 years. Still a pretty long time.

Part of the point of this analysis is to give you some experience with a problem that would take even a computer a very long time to solve. In a few applications worrying about a millisecond here and there is important, but in most applications other concerns far outweigh the concern of losing a millisecond or so. Where we start worrying about efficiency is when we get to some algorithms that could potentially take a completely unacceptable amount of time to run. In the case of the Towers of Hanoi problem, there does not seem to be a more efficient way to solve the problem; however, there are some unacceptably inefficient algorithms for which we have alternatives that are much faster.