ME 658 HW2

Monday, April 3, 2023 9:38 AM

1 i For Charlie 1:

$X_u = -0.021$ $\quad \tilde{M}_u = M_u + M_{\dot{w}} Z_u = 0.000036 + (-0.0008)(-0.2) = 0.000196$

$X_w = 0.122$ $\quad \tilde{M}_w = M_w + M_{\dot{w}} Z_w = -0.006 + (-0.0008)(-0.512) = -0.00559$

$Z_u = -0.2$ $\quad \tilde{M}_q = M_q + U_0 M_{\dot{w}} = -0.357 + 67(-0.0008) = -0.4106$

$Z_w = -0.512$ $\quad M_\theta = -g M_{\dot{w}} \sin\Theta_0 = -9.81 \cdot (-0.0008) \sin(0) = 0$

$Z_q = -1.9$ $\quad \Theta_0 = 0$

And from the lecture notes:

$$A = \begin{pmatrix} X_u & X_w & 0 & -g\cos\Theta_0 \\ Z_u & Z_w & U_0 & -mg\sin\Theta_0 \\ \tilde{M}_u & \tilde{M}_w & \tilde{M}_q & \tilde{M}_\theta \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -0.021 & 0.122 & 0 & -9.81 \\ -0.2 & -0.512 & 67 & 0 \\ 0.000196 & -0.00559 & -0.4106 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the Eigenvectors and Eigenvalues of A to find the Short Period and Phugoid Modes:

	Phugoid	Short Period
Eigenvalues	$-0.0033 \pm 0.1408j$	$-0.4645 \pm 0.6188j$
Eigenvectors	$\begin{pmatrix} -.0515 \\ -.9937 \\ -.0008 \\ -.0085 \end{pmatrix} \pm \begin{pmatrix} .09831 \\ 0.0 \\ -.0089 \\ .0077 \end{pmatrix} j$	$\begin{pmatrix} -.9918 \\ .1254 \\ -.0021 \\ .0033 \end{pmatrix} \pm \begin{pmatrix} 0.0 \\ .0209 \\ .0004 \\ .0145 \end{pmatrix} j$
Natural Frequency	$\omega_n = 0.1408$	$\omega_n = .7761$
damping ratio	$\zeta = 0.0234$ rad	$\zeta = 0.648$ rad

ii Comparison of actual values with approximations:

Short Period	Natural Frequency	Damping Ratio
No Approximation	0.7761 rad/s	0.648
Full Approximation	0.765 rad/s	0.603
Coarse Approximation	0.634 rad/s	0.282

Phugoid	Natural Frequency	Damping Ratio
No Approximation	0.1408 rad/s	0.0234
Full Approximation	NA	NA
Coarse Approximation	0.171 rad/s	0.061

Short Period full approximation:

$$2\zeta_{sp}\omega_{sp} = -\left(Z_w + M_q + M_{\dot{w}}U_0\right) = -\left(-.512 + -.357 + (-.0008)(67)\right) = .9226$$

$$\omega_{sp}^2 = Z_w M_q - U_0 M_w = .5849 \rightarrow \omega_{sp} = 0.765$$

$$\zeta_{sp} = 0.603$$

S.P.

Coarse Approximation

$$2\zeta_{sp}\omega_{sp} \approx -M_q = 0.357$$

$$\omega_{sp}^2 \approx -U_0 M_w = -(67)(-0.006) = 0.402 \rightarrow \omega_{sp} = 0.634$$

$$\zeta_{sp} = 0.282$$

Phugoid Coarse:

$$2\zeta_{ph}\omega_{ph} = -X_u = 0.021$$

$$\omega_{ph}^2 = \frac{-gZ_u}{U_0} = \frac{-(9.81)(-0.2)}{67} = 0.029$$

$$\omega_{ph} = 0.171$$

$$\zeta_{ph} = 0.061$$

iii a. Initial Condition to excite only Phugoid mode:

$$x_0 = Re\{V_{ph}\} + Im\{V_{ph}\} = \begin{pmatrix} -0.0515 + 0.09931 \\ -0.9937 + 0.0 \\ -0.0008 - 0.0089 \\ -0.0085 - 0.0077 \end{pmatrix} = \begin{pmatrix} 0.04681 \\ -0.9937 \\ -0.0097 \\ -0.0162 \end{pmatrix}$$

Plot here

b. Initial conditions to excite Short Period mode:

$$X_0 = Re\{V_{sp}\} + Im\{V_{sp}\} = \begin{pmatrix} -.9918 \\ .1254 \\ -.0021 \\ .0033 \end{pmatrix} + \begin{pmatrix} 0.0 \\ .0209 \\ .0004 \\ .0145 \end{pmatrix} = \begin{pmatrix} -0.9918 \\ 0.1463 \\ -0.0017 \\ 0.0178 \end{pmatrix}$$

Plot here

iVa. Find Transfer function from δ_E to vertical velocity, w

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$X_{\delta e} = 0.292$

$Z_{\delta e} = -1.96$

$M_{\delta e} = M_{\delta_E} + M_{\dot{w}} \cdot Z_{\delta e} = -0.374 + -0.0008(-1.96) = -0.376$

Already know A. $B = \begin{pmatrix} X_{\delta e} \\ Z_{\delta e} \\ M_{\delta e} \\ 0 \end{pmatrix} = \begin{pmatrix} 0.292 \\ -1.96 \\ -0.376 \\ 0 \end{pmatrix}$

Already know A. $B = \begin{pmatrix} X_{\delta_E} \\ Z_{\delta_E} \\ M_{\delta_E} \\ 0 \end{pmatrix} = \begin{pmatrix} 0.292 \\ -1.96 \\ -0.376 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -0.021 & 0.122 & 0 & -9.81 \\ 0.2 & -0.512 & 67 & 0 \\ 0.000196 & -0.00559 & -0.4106 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} + \begin{pmatrix} 0.292 \\ -1.96 \\ -0.376 \\ 0 \end{pmatrix}$$

$\left.\begin{array}{l} C = (0\ 1\ 0\ 0) \\ D = 0 \end{array}\right\}$ - Want to find transfer function from δ_E to vertical velocity, w

$$G(s) = C(sI - A)^{-1}B + D$$

MATLAB ss2tf:

$$G(s) = \frac{-1.96s^3 - 25.98s^2 - 0.518s + 0.73}{s^4 + 0.94s^3 + 0.58s^2 + 0.0026s - 0.01}$$

Find Transfer Function from δ_E to normal acceleration of the center of gravity: a_z

b. Already know A and B

$$a_z = Z_u u + Z_w w + Z_{\delta_E}\delta_E = \underbrace{(Z_u \quad Z_w \quad 0 \quad 0)}_{C} \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} + \underbrace{Z_{\delta_E}}_{D} \delta_E$$

MATLAB ss2tf:

$$G(s) = \frac{-1.96s^4 - 0.9s^3 + 12.16s^2 + 0.12s - 0.712}{s^4 + 0.94s^3 + 0.57s^2 + 0.0026s - 0.01}$$

V.

Plot here

Plot here

2.

Not working

3a. Show $w_i^T A = \lambda_i w_i^T$:

If λ_i is an eigenvalue of A, then

$Av_i = \lambda_i v_i$ and $v_i = w_i$

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{pmatrix}$$

$$AV = V\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$V^{-1}AVV^{-1} = V^{-1}V\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}V^{-1} = V^{-1}A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{pmatrix}$$

$$\begin{pmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{pmatrix}A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{pmatrix}$$

$\therefore \; w_i^T A \overset{\checkmark}{=} \lambda_i w_i^T$

b. If λ_i is an eigenvector of A then

$$Av_i = \lambda_i v_i$$

Take complex conjugate of both sides:

$$\overline{Av_i} = \overline{\lambda_i v_i} = \bar{A}\bar{v}_i = \bar{\lambda}_i \bar{v}_i$$

since A is real, $A = \bar{A}$

since ... is real, ...

$\therefore\; A\bar{V}_i = \bar{\lambda}_i \bar{V}_i$ and $\bar{\lambda}_i$ is also an eigenvalue of A

and $\bar{V}_i$ is also an eigenvector of A

Finally, by 3 part a, $\bar{w}_i^T$ must be a left eigenvector of A

c. Show X_0 is real for all c_1, c_2

$$X_0 = c_1 \operatorname{Re}\{V_i\} + c_2 \operatorname{Im}\{V_i\}$$

since V_i is composed of a real and an imaginary part, $\underline{a}$ and $\underline{b}$

$$V_i = \underline{a} + \underline{b}j$$

and

$$X_0 = c_1 \operatorname{Re}\{\underline{a}+\underline{b}j\} + c_2 \operatorname{Im}\{\underline{a}+\underline{b}j\}$$

$$= \frac{c_1}{2}\left(\underline{a}+\underline{b}j + \underline{a} - \underline{b}j\right) + \frac{c_2}{2j}\left(\underline{a}+\underline{b}j - \underline{a} + \underline{b}j\right)$$

$$= c_1 \underline{a} + c_2 \underline{b}$$

$\therefore\; X_0$ is real for any c_1, c_2

Show $W_k^T X_0 = 0$ for all $W_k^T \neq W_i^T, \bar{W}_i^T$

$$W_k^T X_0 = c_1 W_k^T (V_i + \bar{V}_i)\frac{1}{2} + c_2 W_k^T (V_i - \bar{V}_i)\frac{1}{2j}$$

$$= \frac{c_1}{2} W_k^T V_i + \frac{c_1}{2} W_k^T \bar{V}_i + \frac{c_2}{2j} W_k^T V_i - \frac{c_2}{2j} W_k^T \bar{V}_i$$

Since each term contains a dot product of 2 eigenvectors, each term is equal to 0 unless $W_k^T = W_i^T$ or $W_k^T = \bar{W}_i^T$. This is because the eigenvectors of a real matrix are orthogonal.

d.

Show $X(t) = 2 \operatorname{Re}\{ B_i e^{\lambda_i t} V_i \}$

d.

Show $x(t) = 2\,\mathrm{Re}\left\{ \beta_i e^{\lambda_i t} v_i \right\}$

$\dot{x} = Ax \rightarrow \quad x(t) = e^{At} x(0) = e^{At} x_0$

Since A has n eigenvalues and n eigenvectors it is diagonalizable $A = V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} V^{-1}$

where V and V^{-1} are the same V and V^{-1} that are given in the problem statement

$$x(t) = e^{At} x_0 = V \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} V^{-1} x_0$$

Since many of these vectors are orthogonal, $x(t)$ reduces to

$$x(t) = \sum \left(v_i e^{\lambda_i t} w_i^T \right) x_0 = v_i e^{\lambda_i t} w_i^T x_0 + \bar{v}_i e^{\bar{\lambda}_i t} \bar{w}_i^T x_0$$

$$= \beta_i v_i e^{\lambda_i t} + \bar{\beta}_i \bar{v}_i e^{\bar{\lambda}_i t} = 2\,\mathrm{Re}\left\{ \beta_i v_i e^{\lambda_i t} \right\}$$

from Lecture notes 6

$\therefore \quad x(t) = 2\,\mathrm{Re}\left\{ \beta_i v_i e^{\lambda_i t} \right\}$