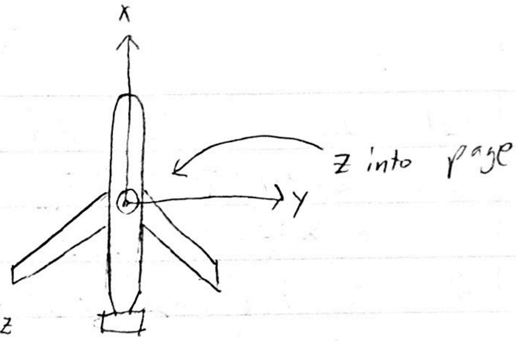
1. $I_{xy} = \int xy\,dm$

Product of inertia I_{xy} is equal to the integral of the mass distribution of the Aircraft multiplied by the x, y position



$$I_{xy} = \int xy\,dm = \iint y \int \rho x\,dx\,dy\,dz = \iint y\left(\int_{-\infty}^{0} \rho(x,y,z)\,x\,dx + \int_{0}^{\infty} \rho(x,y,z)\,x\,dx\right)dy\,dz$$

By Symmetry: $\int_{-\infty}^{0} \rho(x,y,z)\,x\,dx + \int_{0}^{\infty} \rho(x,y,z)\,x\,dx = 0$ $\therefore$ $I_{xy} = I_{yx} = \iint y(0)\,dy\,dz = 0$

Similarly:

$$I_{xz} = I_{zx} = \int xz\,dm = \iint z \int \rho x\,dx\,dy\,dz = \iint z\left(\int_{-\infty}^{0} \rho(x,y,z)\,x\,dx + \int_{0}^{\infty} \rho(x,y,z)\,x\,dx\right)dy\,dz$$

From above $\int_{-\infty}^{0} \rho(x,y,z)\,x\,dx + \int_{0}^{\infty} \rho(x,y,z)\,x\,dx = 0$ $\therefore$ $I_{xz} = I_{zx} = 0$

2. Find eigenvalues and eigenvectors

$$A = \begin{pmatrix} -5 & 100 \\ -10 & 10 \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$$(-5-\lambda)(10-\lambda) - 100(-10) = 0$$
$$-50 + 5\lambda - 10\lambda + \lambda^2 + 1000 = 0$$
$$\lambda^2 - 5\lambda + 950 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(950)}}{2}$$

$$\boxed{\lambda = 2.5 \pm 30.721j}$$

$(A - \lambda I)v = 0$, $\lambda = 2.5 + 30.721j$

$$\begin{pmatrix} -5-(2.5+30.721j) & 100 \\ -10 & 10-(2.5+30.721j) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$(-7.5 - 30.721j)v_1 + 100v_2 = 0$

$-10v_1 + (7.5 - 30.721j)v_2 = 0$

$v_1 = \frac{-100v_2}{-7.5-30.721j}$ $\quad v_2 = \frac{10v_1}{7.5-30.721j}$ ... $\begin{pmatrix} \frac{-100}{-7.5-30.721j} \\ 1 \end{pmatrix} v_2 \rightarrow \begin{pmatrix} .75 - 3.0721j \\ 1 \end{pmatrix}$, $\lambda = 2.5 + 30.721j$

$(A - \lambda I)v = 0$ $\quad \lambda = 2.5 - 30.721j$

$$\begin{pmatrix} -5-(2.5-30.721j) & 100 \\ -10 & 10-(2.5-30.721j) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$(-7.5 + 30.721j)v_1 + 100v_2 = 0$

$-10v_1 + (7.5 + 30.721j)v_2 = 0$

$v_1 = \frac{-100V_2}{-7.5+30.721j}$, $\quad v_2 = \frac{10v_1}{7.5+30.721j}$ $\quad \begin{pmatrix} \frac{-100}{-7.5+30.721j} \\ 1 \end{pmatrix} v_2 \rightarrow \begin{pmatrix} .75 + 3.0721j \\ 1 \end{pmatrix}$ $\lambda = 2.5 - 30.721j$

$$\boxed{\lambda = 2.5 + 30.721j \rightarrow \begin{pmatrix} .75 - 3.0721j \\ 1 \end{pmatrix} \qquad \lambda = 2.5 - 30.721j \rightarrow \begin{pmatrix} .75 + 3.0721j \\ 1 \end{pmatrix}}$$

3i) Using perturbed angle of attack equation:

$$\alpha = \tan^{-1}\left(\frac{w}{U}\right) \approx \frac{w}{U}$$

if $\alpha = 0$, then $\boxed{w=0}$

$\dot{w}$ is the time derivative of w.

$$w = \tan(\alpha)\cdot U$$

$$\frac{d}{dt}(w) = \frac{d}{dt}(\tan(\alpha)U) = 0$$

$$\boxed{\dot{w} = 0}$$

3ii)

From lecture notes:

$$\dot{w} = Z_u u + Z_w w + qU_0 - g\sin\Theta_0 \theta$$

From 3i) $\dot{w} = w = 0$ so

$$0 = Z_u u + qU_0 - g\sin\Theta_0\theta$$

Additionally since there is no roll or yaw motion, $q = \dot{\theta}$

$$0 = Z_u u + \dot{\theta}U_0 - g\sin\Theta_0\theta$$

Assume that θ is very small, so the term tends to 0

Then:

$$0 = Z_u u + \dot{\theta}U_0 \rightarrow \boxed{\dot{\theta} = -\frac{Z_u u}{U_0}}$$

3iii)

From the lecture slides:

$$\dot{u} = X_u u + X_w w - g\cos\Theta_0\theta$$

$$X_u = 0, \quad \dot{\theta} = -\frac{Z_u}{U_0}u$$

$$\frac{d}{dt}(\dot{u}) = \frac{d}{dt}\left((0)u + X_w w - g\cos\Theta_0\theta\right)$$

$$\ddot{u} = X_w\dot{w} - g\cos\Theta_0\dot{\theta}$$

$$\dot{w} = 0$$

$$\ddot{u} = -g\cos\Theta_0\left(-\frac{Z_u u}{U_0}\right) = g\cos\Theta_0 Z_u u\frac{1}{U_0}$$

$$\ddot{u} - \frac{g\cos\Theta_0 Z_u}{U_0}u = 0$$

From slides:

$$\frac{\partial Z}{\partial u} = -\frac{\partial L}{\partial u} - \alpha\cancel{\frac{\partial D}{\partial u}}^{0} \quad (\alpha = 0)$$

$$\frac{\partial Z}{\partial u} = -\frac{\partial}{\partial u}\left(\frac{1}{2}\rho S C_L u^2\right) = -\rho S C_L u$$

$$-Z_u = \rho S C_L u \longrightarrow \text{to top}$$

Assume steady, level flight: $L = W = mg$ and $\Theta_0 = 0$

$$\frac{1}{2}\rho S C_L u^2 = \frac{1}{2}(\rho S C_L u)u = mg$$

$$= \frac{1}{2}(-Z_u)u = mg$$

$$Z_u = \frac{-2mg}{u} \xrightarrow{\text{normalize}} Z_u = \frac{-2g}{u}$$

because of steady flight assumption, $u = U_0$

$$\ddot{u} - \frac{g(1)\left(\frac{-2g}{U_0}\right)}{U_0}u = 0$$

$$\ddot{u} + \frac{2g^2}{U_0^2}u = 0$$

Lanchester formula says $\Omega = \frac{g}{U_0}\sqrt{2}$

$$\ddot{u} + \Omega^2 u = 0$$

$$\Omega^2 = \frac{2g^2}{U_0^2}$$

$$\Omega = \frac{g}{U_0}\sqrt{2}$$

4.a McLean 2.1

$$\dot{X} = \begin{pmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{pmatrix} = Ax + B\delta = \begin{pmatrix} X_u & X_w & 0 & -g\cos\Theta_0 \\ Z_u & Z_w & U_0 & -g\sin\Theta_0 \\ \tilde{M}_u & \tilde{M}_w & \tilde{M}_q & \tilde{M}_\theta \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} + \begin{pmatrix} X_{\delta e} & X_{\delta p} \\ Z_{\delta e} & Z_{\delta p} \\ \tilde{M}_{\delta e} & \tilde{M}_{\delta p} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_e \\ \delta_p \end{pmatrix}$$

From McLean Appendix B, For Charlie 3:

$X_u = -0.0002$, $X_w = 0.026$, $Z_u = -0.09$, $Z_w = -0.624$, $U_0 = 250$, $\Theta_0 = \gamma_0 = 0$

$X_{\delta e} = 0$ $X_{\delta p} = X_{\delta th} = 3.434 \cdot 10^{-6}$, $Z_{\delta e} = -8.05$, $Z_{\delta p} = Z_{\delta th} = -1.5 \cdot 10^{-7}$

$\tilde{M}_u = M_u + M_{\dot{w}} Z_u = -0.0000 + -0.0007 \cdot (-0.09) = -7 \cdot 10^{-6}$

$\tilde{M}_w = M_w + M_{\dot{w}} Z_w = -0.005 + (-0.0007)(-0.624) = -4.56 \cdot 10^{-3}$

$\tilde{M}_q = M_q + U_0 M_{\dot{w}} = -0.668 + 250 \cdot (-0.0007) = -0.843$

$\tilde{M}_\theta = -g M_{\dot{w}} \sin\Theta_0 = 0$

$\tilde{M}_{\delta e} = M_{\delta e} + M_{\dot{w}} Z_{\delta e} = -2.08 + (-0.0007)(-8.05) = -2.074$

$\tilde{M}_{\delta p} = M_{\delta p} + M_{\dot{w}} Z_{\delta p} = 0.67 \cdot 10^{-7} + (-0.0007)(-1.5 \cdot 10^{-7}) = 6.711 \cdot 10^{-8}$

$$\dot{X} = \begin{pmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -0.0002 & 0.026 & 0 & -9.81 \\ -0.09 & -0.624 & 250 & 0 \\ -7 \cdot 10^{-6} & -4.56 \cdot 10^{-3} & -0.843 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} + \begin{pmatrix} 0 & 3.434 \cdot 10^{-6} \\ -8.05 & -1.5 \cdot 10^{-7} \\ -2.074 & 6.711 \cdot 10^{-8} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_e \\ \delta_{th} \end{pmatrix}$$

4 b. (2.2 McLean) By a similar process to part a, the following state space representation for a Charlie-4 aircraft can be found, using only elevator input as the control input

$$\dot{X} = \begin{pmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0.0002 & 0.039 & 0 & -9.81 \\ -0.07 & -0.317 & 250 & 0 \\ 8.8 \cdot 10^{-5} & -2.873 \cdot 10^{-3} & -4.394 \cdot 10^{-1} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} + \begin{pmatrix} 0.44 \\ -5.46 \\ -1.158 \cdot 10^{0} \\ 0 \end{pmatrix} \delta_e$$

To extract the effects of the elevator on vertical velocity, let $C = (0 \ 1 \ 0 \ 0)$ and $y = Cx$ where $x = (u \ w \ q \ \theta)^T$ and let $\dot{x} = Ax + B\delta_e$ as above.

Finally, using $G(s) = C((sI - A)^{-1} B + D$ where D is a zero matrix, the transfer function, $G(s)$ can be found. Over $\rightarrow$

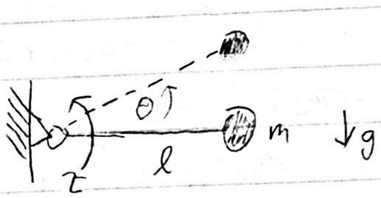
$$G(s) = C(sI-A)^{-1}B$$

$$sI - A = \begin{pmatrix} s-0.0002 & 0.039 & 0 & -9.8 \\ -0.07 & s+0.317 & 250 & 0 \\ 2.7\cdot10^{-4} & -2.873\cdot10^{-3} & s+4.39\cdot10^{-1} & 0 \\ 0 & 0 & 1 & s \end{pmatrix}$$

using MATLAB:

$$G(s) = \left(\frac{-5.46s^3 - 291.93s^2 + 0.0545s - 0.7991}{s^4 + 0.7558s^3 + 0.86s^2 + 0.001s + 0.0022} \right) \text{[deg]}$$

Convert to radians: $G(s) = \dfrac{-0.0953s^3 - 5.0943s^2 + 0.001s^2 - 0.0145}{0.0175s^4 + 0.0132s^3 + 0.015s^2}$ [rad]



5. a.

Find equilibrium torque τ_e at $\theta_e = 0$, $\dot{\theta}_e = 0$:

$\boxed{\tau_e = mgl}$

$\ddot{\theta}\, ml^2 = \tau - mgl\cos\theta$

Linearized Model about $\theta_e = 0$, $\dot{\theta}_e = 0$, $\tau_e = mgl$

$\ddot{\theta} + \frac{g}{l}\cos\theta = u$ where $u = \frac{1}{ml^2}\tau$

Let $x_1 = \theta$, $x_2 = \dot{\theta}$ → $\dot{x}_2 + \frac{g}{l}\cos x_1 = u$, $\dot{x}_1 = x_2$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g/l\cos x_1 + u \end{pmatrix} = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \end{pmatrix}$$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{x_1=0,\, x_2=0,\, u=mgl} = 0 \qquad \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1=0,\, x_2=0,\, u=mgl} = 1 \qquad \left.\frac{\partial f_1}{\partial u}\right|_{\substack{x_1=0\\x_2=0\\u=mgl}} = 0$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0\\u=mgl}} = \left.\frac{g}{l}\sin x_1\right|_{x_1=0} = 0 \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{x_1=0\\x_2=0\\u=mgl}} = 0 \qquad \left.\frac{\partial f_2}{\partial u}\right|_{\substack{x_1=0\\x_2=0\\u=mgl}} = 1$$

$$\boxed{\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}\tilde{u}}$$

Not statically stable. The pendulum will not return to its initial position if disturbed.

5b.

Inverted Case: $\theta = 90°$, $\dot{\theta} = 0$, $\tau_e = 0$

Equilibrium torque $\tau_e = 0$

$$\ddot{\theta}\, ml^2 = -mgl\cos\theta$$

$$\ddot{\theta} + g/l \cos\theta = 0$$

Let $x_1 = \theta$, $x_2 = \dot{\theta}$ $\rightarrow$ $\dot{x}_2 + g/l \cos x_1 = u$ $\quad \dot{x}_1 = x_2$

Linearize about $x_1 = 90° = \pi/2$ rad, $x_2 = 0$, $u = 0$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g/l \cos x_1 + u \end{pmatrix} = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \end{pmatrix}$$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{x_1, x_2, u} = 0 \quad \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1, x_2, u} = 1 \quad \left.\frac{\partial f_1}{\partial u}\right|_{x_1, x_2, u} = 0$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{x_1, x_2, u} = \left.g/l \sin(x_1)\right|_{x_1 = \pi/2} = g/l \quad \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1, x_2, u} = 0 \quad \left.\frac{\partial f_2}{\partial u}\right|_{x_1, x_2, u} = 1$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u$$

$$\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/ml^2 \end{pmatrix} \tilde{u}$$

This system is not statically stable because any perturbation of θ will result in the pendulum moving away from $\theta = 90°$ and not returning.