MACROECONOMETRICS
IDEA 2013/14
Problem Set1
Time Series
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1. Is the following MA(2) process stationary?

$$y_t = (1 + 2.4L + 0.8L^2)\varepsilon_t$$

with ε_t with variance 1. If so calculate its autocovariances.

Solution
We can write

$$y_t = \varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

and then calculate the **mean**:

$$\begin{aligned} E(y_t) &= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}) \\ &= E(\varepsilon_t) + 2.4E(\varepsilon_{t-1}) + 0.8E(\varepsilon_{t-2}) \\ &= 0 \end{aligned}$$

and calculate the **variance**:

$$\begin{aligned} E(y_t - E(y_t))^2 &= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})^2 \\ &= E(\varepsilon_t^2 + 2.4^2\varepsilon_{t-1}^2 + 0.8^2\varepsilon_{t-2}^2) \\ &= (1 + 2.4^2 + 0.8^2)\sigma^2 \\ &= 7.4 \end{aligned}$$

the **first autocovariance**:

$$\begin{aligned} E(y_t y_{t-1}) &= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})E(\varepsilon_{t-1} + 2.4\varepsilon_{t-2} + 0.8\varepsilon_{t-3}) \\ &= E(2.4\varepsilon_{t-1}^2) + E(0.8 \times 2.4\varepsilon_{t-2}^2) \\ &= (2.4 + 0.8 \times 2.4)\sigma^2 \\ &= 4.32 \end{aligned}$$

the **second autocovariance**:

$$\begin{aligned} E(y_t y_{t-2}) &= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})E(\varepsilon_{t-2} + 2.4\varepsilon_{t-3} + 0.8\varepsilon_{t-4}) \\ &= E(0.8\varepsilon_{t-2}^2) \\ &= 0.8 \end{aligned}$$

Higher autocovariances are all equal to zero. Where we used the properties that $E(\varepsilon_t) = 0\ \forall t; E(\varepsilon_t\varepsilon_{t-j}) = 0\ \forall j \neq t; E(\varepsilon_t^2) = \sigma^2\ \forall t$; and finally as given in the instructions $\sigma^2 = 1$.

Since the mean and the autocovariances do not depend on time, the MA(2) process is covariance-stationary.

2. A covariance stationary AR(p) process

$$(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)y_t=\varepsilon_t$$

has an MA representation given by

$$y_t=\psi(L)\varepsilon_t$$

with

$$\psi(L)=(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)^{-1}$$

or

$$(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)(\psi_0+\psi_1 L+\psi_2 L^2+...)=1 \tag{1}$$

In order for (1) to be true, the implied coefficient on L^0 must be unity and the coefficients on $L, L^2, ...$ must be zero. Write out these conditions explicitly and show that they imply a recursive algorithm for generating the MA weights ψ_0, ψ_1.... Show that this recursion is algebraically equivalent to setting ψ_j equal to the (1,1) element of the matrix

$$F=\begin{pmatrix}\phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}$$

raised to the j-th power.

Solution

From (1) we get that

$$\begin{aligned}(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)(\psi_0+\psi_1 L+\psi_2 L^2+...)&=1\\ (\psi_0-\psi_0\phi_1 L-\psi_0\phi_2 L^2-...-\psi_0\phi_p L^p+\psi_1 L-\psi_1\phi_1 L^2-\psi_1\phi_2 L^3+...)&=1\\ \psi_0+(\psi_1-\psi_0\phi_1)L-(\psi_0\phi_2+\psi_1\phi_1-\psi_2)L^2&=1\end{aligned}$$

and then taking into account the implied coefficients on $L^0, L, L^2, ...$ we have

$$\begin{aligned}\psi_0&=1\\ \psi_1-\psi_0\phi_1&=0\\ \psi_0\phi_2+\psi_1\phi_1-\psi_2&=0\\ &\vdots\end{aligned}$$

which imply that

$$\begin{aligned}\psi_0 &= 1\\ \psi_1 &= \phi_1\\ \psi_2 &= \phi_1^2 + \phi_2\\ &\vdots\end{aligned}$$

In terms of matrices, the companion form is given by

$$\underbrace{\begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1}\end{pmatrix}}_{\boldsymbol{Z}_t} = \underbrace{\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}}_{\boldsymbol{F}} = \underbrace{\begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p}\end{pmatrix}}_{\boldsymbol{Z}_{t-1}} + \underbrace{\begin{pmatrix} \boldsymbol{\epsilon_t} \\ 0 \\ 0 \\ \vdots \\ 0\end{pmatrix}}_{\boldsymbol{E}_t}$$

that is

$$\boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{E}_t$$

and recursively we get

$$\begin{aligned}\boldsymbol{Z}_{t+1} &= \boldsymbol{F}\boldsymbol{Z}_t + \boldsymbol{E}_{t+1}\\ \boldsymbol{Z}_{t+2} &= \boldsymbol{F}\boldsymbol{Z}_{t+1} + \boldsymbol{E}_{t+2}\\ &= \boldsymbol{F}(\boldsymbol{F}\boldsymbol{Z}_t + \boldsymbol{E}_{t+1}) + \boldsymbol{E}_{t+2}\\ &= \boldsymbol{F}^2\boldsymbol{Z}_t + \boldsymbol{F}\boldsymbol{E}_{t+1} + \boldsymbol{E}_{t+2}\end{aligned}$$

So we can write

$$\boldsymbol{F}^2 = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \times \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} \phi_1^2 + \phi_2 & \cdots & \\ \vdots & \ddots & \vdots \end{pmatrix}$$

Notice that now the effect of y_{t-1} on y_{t+2} is given by the element $(1,1)$ in matrix $\boldsymbol{F}^2$, i.e. $\psi_2 = \phi_1^2 + \phi_2$ as previously.

3. Is the process

$$y_t = (1 + 2.4L + 0.8L^2)\varepsilon_t$$

invertible? If not find the invertible representation.

Solution

The MA operator is

$$\theta(z) = 1 + 2.4z + 0.8z^2$$

and the characteristic equation is

$$0.8z^2 + 2.4z + 1 = 0$$

and the roots are given by

$$z_{1,2} = \frac{-2.4 \pm \sqrt{2.4^2 - 4 \times 0.8}}{2 \times 0.8},$$

therefore $z_1 = -0.5$ and $z_2 = -2.5$. According to the definition of invertibility, $y_t = \theta(z)\varepsilon_t$ is invertible iff $\theta(z) \neq 0$ $\forall z \in \mathbb{C}$ such that $|z| \leq 1$, or equivalently iff $\theta(z) = 0$ $\forall z \in \mathbb{C}$ such that $|z| > 1$. Since we have $\theta(z) = 0$ and $|z_1| = 0.5 < 1$, invertibility is violated. Now we need to find the invertible representation.

We can factorize the above polynomial and find the roots λ_1 and λ_2 such that

$$(1 + 2.4z + 0.8z^2) = (1 - \lambda_1 z)(1 - \lambda_2 z),$$

$$\lambda_1 = \frac{1}{z_1} = \frac{1}{-0.5} = -2$$
$$\lambda_2 = \frac{1}{z_2} = \frac{1}{-2.5} = -0.4$$

therefore the invertible operator is

$$(1 + 0.5z)(1 + 0.4z) = (1 + 0.9z + 0.2z^2)$$

and the invertible representation is

$$y_t = (1 + 0.9z + 0.2z^2)\varepsilon_t$$

where we changed the wrong root λ_1 (since its absolute value is greater than one) to its reciprocal.

We can also calculate the mean, variance, and autocovariances of the invertible representation and find that

$$E(y_t) = 0$$
$$E(y_t - E(y_t))^2 = 1.85\sigma^2$$
$$E(y_t y_{t-1}) = 1.08\sigma^2$$
$$E(y_t y_{t-2}) = 0,2\sigma^2$$

Therefore the two representations are equivalent if in the invertible case $E(\varepsilon_t^2) = \sigma^2 = 4$.