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Macroeconometrics
Problem Set 2

**1** Is the following MA(2) process stationary?

$$y_t = (1 + 2.4L + 0.8L^2)\epsilon_t$$

with $\epsilon_t$ with variance 1. If so calculate its autocovariances.

To establish if the process is weakly stationary I checked if the mean and the autocovariances are independent of time. The moments are:

- Mean:

$$\begin{aligned} E(y_t) &= E(\epsilon_t) + 2.4E(\epsilon_{t-1}) + 0.8E(\epsilon_{t-2}) \\ &= 0 \end{aligned}$$

  the later is true assuming that $E(\epsilon_t) = 0, \forall t$, i.e. zero mean

- Variance of $\gamma(0)$

$$\begin{aligned} E(y_t)^2 &= E((\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})(\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})) \\ &= E((\epsilon_t)^2 + (2.4\epsilon_{t-1})^2 + (0.8\epsilon_{t-2})^2) \\ &= 1 + 5.76 + 0.64 \\ &= 7.4 \end{aligned}$$

  the later is true assuming that $E(\epsilon_r\epsilon_s) = 0, \forall r \neq s$

- First autocovariance or $\gamma(1)$

$$\begin{aligned} E(y_t y_{t-1}) &= E((\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})(\epsilon_{t-1} + 2.4\epsilon_{t-2} + 0.8\epsilon_{t-3})) \\ &= E(2.4\epsilon_{t-1}^2 + 2.4 \cdot 0.8\epsilon_{t-2}^2) \\ &= 2.4 + 1.92 \\ &= 4.32 \end{aligned}$$

  again, the later is true assuming that $E(\epsilon_r\epsilon_s) = 0, \forall r \neq s$

- Second autocovariance or $\gamma(1)$

$$\begin{aligned} E(y_t y_{t-2}) &= E((\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})(\epsilon_{t-2} + 2.4\epsilon_{t-3} + 0.8\epsilon_{t-4})) \\ &= E(0.8\epsilon_{t-2}) \\ &= 0.8 \end{aligned}$$

  again, the later is true assuming that $E(\epsilon_r\epsilon_s) = 0, \forall r \neq s$

- Third autocovariance or $\gamma(3)$

$$\begin{aligned} E(y_t y_{t-3}) &= E((\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})(\epsilon_{t-3} + 2.4\epsilon_{t-4} + 0.8\epsilon_{t-5})) \\ &= 0 \end{aligned}$$

  again, the later is true assuming that $E(\epsilon_r \epsilon_s) = 0, \forall r \neq s$

From the later, since all the moments are finite and do not depends on time, the process is stationary (at least weakly). ■

**2** A covariance stationary AR(p) process

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) y_t = \epsilon_t$$

has an MA representation given by

$$y_t = \psi(L)\epsilon_t$$

with $\psi(L) = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1}$ or

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)(\psi_0 + \psi_1 L + \psi_2 L^2 + \dots) = 1 \tag{1}$$

In order for (1) to be true, the implied coefficient on $L^0$ must be unity and the coefficients on $L, L^2, \dots$ must be zero. Write out these conditions explicitly and show that they imply a recursive algorithm for generating the MA weights $\psi_0, \psi_1, \dots$. Show that this recursion is algebraically equivalent to setting $\psi_j$ equal to the (1,1) element of the matrix

$$F = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

raised to the j-th power

solving the multiplication we have that:

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)(\psi_0 + \psi_1 L + \psi_2 L^2 + \dots) = 1$$

$$\psi_0 - \psi_0\phi_1 L - \psi_0\phi_2 L^2 - \cdots - \psi_0\phi_p L^p + \psi_1 L - \psi_1\phi_1 L^2 - \cdots = 1$$

$$\psi_0 + (\psi_1 - \psi_0\phi_1)L - (\psi_0\phi_2 + \psi_1\phi_1 - \psi_2)L^2 \cdots = 1$$

from where we obtain that the conditions for the fist initial weights are:

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 - \psi_0\phi_1 &= 0 \\
\psi_1 &= \psi_0\phi_1 \\
&= (1)\phi_1 \\
\psi_1 &= \phi_1 \\
\psi_0\phi_2 + \psi_1\phi_1 - \psi_2 &= 0 \\
\psi_2 &= \psi_0\phi_2 + \psi_1\phi_1 \\
&= \phi_2 + (\phi_1)\phi_1 \\
\psi_2 &= \phi_2 + \phi_1^2
\end{aligned}$$

Now, in matrixes, the sister will looks like :

$$\underbrace{\begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{pmatrix}}_{\mathbf{Z}_t} = \underbrace{\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}}_{\mathbf{F}} \underbrace{\begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{pmatrix}}_{\mathbf{Z}_{t-1}} + \underbrace{\begin{pmatrix} \epsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{E}_t}$$

that is the companion form for the above system. With the later we have a AR(a) process defined by:

$$\mathbf{Z}_t = \mathbf{F}\mathbf{Z}_{t-1} + \mathbf{E}_t$$

with the later we know the following:

$$\begin{aligned}
\mathbf{Z}_{t+1} &= \mathbf{F}\mathbf{Z}_t + \mathbf{E}_{t+1} \\
\mathbf{Z}_{t+2} &= \mathbf{F}\mathbf{Z}_{t+1} + \mathbf{E}_{t+2} \\
&= \mathbf{F}(\mathbf{F}\mathbf{Z}_t + \mathbf{E}_{t+1}) + \mathbf{E}_{t+2} \\
&= \mathbf{F}^2\mathbf{Z}_t + \mathbf{F}\mathbf{E}_{t+1} + \mathbf{E}_{t+2}
\end{aligned}$$

so that the effect of $y_{t-1}$ on $y_{t+2}$ (i.e $\psi_2$) is given by the element (1,1) from the matrix $\mathbf{F}^2$,

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \times \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} \phi_1^2 + \phi_2 & \cdots & \\ \vdots & \ddots & \vdots \end{pmatrix}$$

and in general the weight $\psi_j$ will be given by the element (1,1) of the matrix $\mathbf{F}^j$

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**3** Is the process
$$y_t = (1 + 2.4L + 0.8L^2)\epsilon_t$$
invertible? If not find the invertible representation.

The process is not invertible since. it is easy to compute the eigenvalues for the matrix

$$F = \begin{pmatrix} 2.4 & 0.8 \\ 1 & 0 \end{pmatrix}$$

The idea is to find the reciprocals of the roots of the process and determine if those are $|z_i| < 1$ for $i = 1, 2$. Then:

$$\begin{pmatrix} -2.4 & -0.8 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times z = 0$$

that is the same as

$$\begin{pmatrix} -2.4 - z & -0.8 \\ 1 & -z \end{pmatrix}$$

and its determinant is given by $z^2 + 2.4z + 0.8 = 0$. to calculate the values of $z_i$ it is used the quadratic formula as follows:

- $z_1 = \frac{-2.4+\sqrt{2.4^2-4\cdot 0.8}}{2\cdot 0.8} = |-0.5| < 1$
- $z_2 = \frac{-2.4-\sqrt{2.4^2-4\cdot 0.8}}{2\cdot 0.8} = |-2.5| > 1$

So, since one of the roots is larger than one, the process is not invertible.

With this, the reciprocal of the roots are given by $\lambda_1 = 1/(-0.5) = -2$ and $\lambda_2 = 1/(-2.5) = -0.4$, so that possible invertible representation can be to change $\lambda_2$ by $\lambda_2^{-1}$, such that the new model is:

$$y_t = (1 + 0.5L)(1 + 0.4L)\epsilon_t = (1 + 0.9L + 0.2L^2)\epsilon_t$$

■

**4** Reproduce the results obtained with the models $AR(AIC)$, $AO$, $PC - u$ and $PC - \Delta y$ and presented in Table 1 of "Why has U.S. inflation become harder to forecast?" By Stock, J and M., Watson.

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