1 Definitions

$\boldsymbol{d}$	displacement
$\boldsymbol{u}$	velocity
$\boldsymbol{\nabla}$	del operator
ρ	density
μ	first viscosity constant
λ	second viscosity constant

dyadic product

$$\boldsymbol{ab} = \boldsymbol{ab}^{\mathsf{T}} = (a_i b_j)$$

Frobenius product

$$\boldsymbol{\alpha} : \boldsymbol{\beta} = \sum_{i,j} \alpha_{ij}\beta_{ij}$$

derivatives

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \mathrm{sym}(\boldsymbol{\nabla}\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}})$$

$$\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u})) = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

inner products

$$(a, b) = \int_\Omega ab \, \mathrm{d}\Omega$$

$$(\boldsymbol{a}, \boldsymbol{b}) = \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}\Omega$$

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_\Omega \boldsymbol{\alpha} : \boldsymbol{\beta} \, \mathrm{d}\Omega$$

2 Linear Elasticity

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon}(\boldsymbol{d}) + \lambda\,\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{d}))\boldsymbol{I}$$

strong form

$$\rho\ddot{\boldsymbol{d}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{d}) = \boldsymbol{f}$$

weak form

$$(\rho\ddot{\boldsymbol{d}}, \boldsymbol{b}) + (\boldsymbol{\sigma}(\boldsymbol{d}), \boldsymbol{\epsilon}(\boldsymbol{b})) = (\boldsymbol{f}, \boldsymbol{b}) \qquad \forall \boldsymbol{b} \in \boldsymbol{V}$$

strong form

$$\dot{\boldsymbol{d}} - \boldsymbol{u} = 0$$

$$\rho\dot{\boldsymbol{u}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{d}) = \boldsymbol{f}$$

weak form

$$
\begin{aligned}
&(\dot{\boldsymbol{d}}, \boldsymbol{b}) - (\boldsymbol{u}, \boldsymbol{b}) = 0 && \forall \boldsymbol{b} \in \boldsymbol{V} \\
&(\rho \dot{\boldsymbol{u}}, \boldsymbol{v}) + (\boldsymbol{\sigma}(\boldsymbol{d}), \boldsymbol{\epsilon}(\boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{V}
\end{aligned}
$$

time discretization

$$
\begin{aligned}
&\left(\frac{\boldsymbol{d}}{\Delta t}, \boldsymbol{b}\right) - (\boldsymbol{u}, \boldsymbol{b}) = \left(\frac{\boldsymbol{d}_0}{\Delta t}, \boldsymbol{b}\right) && \forall \boldsymbol{b} \in \boldsymbol{V} \\
&\left(\frac{\rho \boldsymbol{u}}{\Delta t}, \boldsymbol{v}\right) + (\boldsymbol{\sigma}(\boldsymbol{d}), \boldsymbol{\epsilon}(\boldsymbol{v})) = \left(\frac{\boldsymbol{u}_0}{\Delta t} + \boldsymbol{f}, \boldsymbol{v}\right) && \forall \boldsymbol{v} \in \boldsymbol{V}
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{d}, \boldsymbol{b}) - \Delta t(\boldsymbol{u}, \boldsymbol{b}) = (\boldsymbol{d}_0, \boldsymbol{b}) && \forall \boldsymbol{b} \in \boldsymbol{V} \\
&(\boldsymbol{u}, \boldsymbol{v}) + \Delta t(\boldsymbol{\sigma}(\boldsymbol{d}), \boldsymbol{\epsilon}(\boldsymbol{v})) = (\boldsymbol{u}_0 + \Delta t \boldsymbol{f}, \boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{V}
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{d}, \boldsymbol{b}) - \Delta t(\boldsymbol{u}, \boldsymbol{b}) = (\boldsymbol{d}_0, \boldsymbol{b}) && \forall \boldsymbol{b} \in \boldsymbol{V} \\
&(\rho \boldsymbol{u}, \boldsymbol{v}) + \Delta t^2(\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v})) = (\boldsymbol{u}_0 + \Delta t \boldsymbol{f}, \boldsymbol{v}) - \Delta t(\boldsymbol{\sigma}(\boldsymbol{d}_{\boldsymbol{0}}), \boldsymbol{\epsilon}(\boldsymbol{v})) && \forall \boldsymbol{v} \in \boldsymbol{V}
\end{aligned}
$$

$$
(\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v})) = 2\mu(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v})) + \lambda(\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u})), \mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{v})))
$$

2.1 2D Cartesian coordinates

$$
\boldsymbol{\nabla} \boldsymbol{u} = \begin{pmatrix} u_{0,0} & u_{1,0} \\ u_{0,1} & u_{1,1} \end{pmatrix}
$$

$$
\boldsymbol{\epsilon}(\boldsymbol{u}) = \begin{pmatrix} u_{0,0} & \frac{1}{2}(u_{1,0} + u_{0,1}) \\ \frac{1}{2}(u_{1,0} + u_{0,1}) & u_{1,1} \end{pmatrix}
$$

$$
\boldsymbol{\nabla} \cdot \boldsymbol{u} = u_{0,0} + u_{1,1}
$$

$$
\boldsymbol{\sigma} = \begin{pmatrix} 2\mu u_{0,0} + \lambda(u_{0,0} + u_{1,1}) & \mu(u_{1,0} + u_{0,1}) \\ \mu(u_{1,0} + u_{0,1}) & 2\mu u_{1,1} + \lambda(u_{0,0} + u_{1,1}) \end{pmatrix}
$$

$$
\begin{aligned}
2(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v})) = \\
&2u_{0,0}v_{0,0} + u_{0,1}v_{0,1} \\
&+ u_{1,0}v_{0,1} \\
&+ u_{0,1}v_{1,0} \\
&+ u_{1,0}v_{1,0} + 2u_{1,1}v_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) = \\
& u_{0,0} v_{0,0} \\
& + u_{1,1} v_{0,0} \\
& + u_{0,0} v_{1,1} \\
& + u_{1,1} v_{1,1}
\end{aligned}
$$

2.2 Cylindrical coordinates

coordinates

$$
r = x_0 \qquad \theta = x_1 \qquad z = x_2
$$

time derivative

$$
\dot{\boldsymbol{u}} = (\dot{u}_0 - u_1 \dot{x}_1, \dot{u}_1 + u_0 \dot{x}_1, \dot{u}_2)
$$

strain tensor

$$
\boldsymbol{\epsilon}(\boldsymbol{u}) = \begin{pmatrix} u_{0,0} & \frac{1}{2}(\frac{1}{x_0} u_{0,1} + u_{1,0} + \frac{1}{x_0} u_1) & \frac{1}{2}(u_{0,2} + u_{2,0}) \\ \frac{1}{2}(\frac{1}{x_0} u_{0,1} + u_{1,0} + \frac{1}{x_0} u_1) & \frac{1}{x_0}(u_{1,1} + u_0) & \frac{1}{2}(u_{1,2} + \frac{1}{x_0} u_{2,1}) \\ \frac{1}{2}(u_{0,2} + u_{2,0}) & \frac{1}{2}(u_{1,2} + \frac{1}{x_0} u_{2,1}) & u_{2,2} \end{pmatrix}
$$

divergence

$$
\nabla \cdot \boldsymbol{u} = u_{0,0} + \frac{1}{x_0} u_0 + \frac{1}{x_0} u_{1,1} + u_{2,2}
$$

2.2.1 Axisymmetry

$$
\theta = 0 \qquad f_{,\theta} = 0 \qquad z = x_1
$$

time derivative

$$
\dot{\boldsymbol{u}} = (\dot{u}_0, \dot{u}_1)
$$

strain tensor

$$
\boldsymbol{\epsilon}(\boldsymbol{u}) = \begin{pmatrix} u_{0,0} & 0 & \frac{1}{2}(u_{0,1} + u_{1,0}) \\ 0 & \frac{1}{x_0} u_0 & 0 \\ \frac{1}{2}(u_{0,1} + u_{1,0}) & 0 & u_{1,1} \end{pmatrix}
$$

divergence

$$
\nabla \cdot \boldsymbol{u} = u_{0,0} + \frac{1}{x_0} u_0 + u_{1,1}
$$

$$
\begin{aligned}
2(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v})) = &\\
&\frac{2}{x_0^2} u_0 v_0 + 2u_{0,0} v_{0,0} + u_{0,1} v_{0,1}\\
&+ u_{1,0} v_{0,1}\\
&+ u_{0,1} v_{1,0}\\
&+ u_{1,0} v_{1,0} + 2u_{1,1} v_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{u}, \boldsymbol{\nabla} \cdot \boldsymbol{v}) = &\\
&u_{0,0} v_{0,0} + \frac{1}{x_0} u_0 v_{0,0} + \frac{1}{x_0} u_{0,0} v_0 + \frac{1}{x_0^2} u_0 v_0\\
&+ u_{1,1} v_{0,0} + \frac{1}{x_0} u_{1,1} v_0\\
&+ u_{0,0} v_{1,1} + \frac{1}{x_0} u_0 v_{1,1}\\
&+ u_{1,1} v_{1,1}
\end{aligned}
$$