

BAYESIAN STATISTICS

ASSIGNMENT 1

QUESTION 1: THE GALENSHORE DISTRIBUTION

Point a.

$Y|\theta \sim \text{Galenshore}(a, \theta)$ is such that $p(y|\theta)$ is a density in the exponential family indeed

$$p(y|\theta) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2} \mathbb{1}_{\{y>0\}}, \theta > 0, a > 0.$$

Then defining $\phi \stackrel{\text{def}}{=} \theta^2$ one has

$$p(y|\phi) = h(y)c(\phi)e^{\phi t(y)} \text{ with } h(y) = \frac{2y^{2a-1}}{\Gamma(a)} \mathbb{1}_{\{y>0\}}, c(\phi) = \phi^a, t(y) = -y^2,$$

hence, by the easy shape of a distribution in the exponential family, we can state that a class of conjugate priors for $p(y|\phi)$ is such that

$$p(\phi) \propto c(\phi)^{n_0} e^{\phi n_0 t_0} = \phi^{an_0} e^{\phi n_0 t_0}.$$

If ϕ has density $p(\phi)$ and we want to obtain the density of $\theta = \sqrt{\phi}$ it is sufficient to define the map $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f(x) = \sqrt{x}$ and recall that

$$p_\theta(\theta) = p_\phi(f(\phi)) = p_\phi(f^{-1}(\theta)) \left| \frac{df^{-1}(\theta)}{d\theta} \right|.$$

Observing that $\left| \frac{df^{-1}(\theta)}{d\theta} \right| = \left| \frac{d\theta^2}{d\theta} \right| = 2\theta$

$$p(\theta) \stackrel{\text{def}}{=} p_\theta(\theta) = p_\phi(\theta^2) 2\theta \propto \theta^{2an_0} e^{\theta^2 n_0 t_0} 2\theta.$$

Remark

Observing the three parameters a, n_0 and t_0 we can say

- $a > 0$ by hypothesis;
- $n_0 > 0$ because it represents the *prior sample size* ($p(\theta)$ has the same kernel of $p(y|\theta)$ after n_0 observations);
- $t_0 < 0$ because it is the *prior guess* that we make for t , with $t(y) = -\frac{y^2}{2}, \forall y \in \mathbb{R}^+ : t_0 = \frac{\sum_{i=1}^n t(y_i)}{n} = -\frac{\sum_{i=1}^n y_i^2}{n} < 0$.

Hence we have

$$an_0 + 1 > 0 \text{ and } -n_0 t_0 > 0.$$

So we can rewrite

$$p(\theta) \propto 2\theta^{2an_0+1} e^{-(\sqrt{-n_0 t_0})^2 \theta^2}$$

and recognizing the kernel of a Galenshore distribution we can write explicitly

$$\begin{aligned}
 p(\theta) &= 2\theta^{2(an_0+1)-1} e^{-(\sqrt{-n_0 t_0})^2 \theta^2} \cdot \underbrace{\frac{(\sqrt{-n_0 t_0})^{2(an_0+1)}}{\Gamma(an_0+1)}}_{\text{it does not depends on } \theta} \underbrace{\mathbb{1}_{\theta>0}}_{\text{by hypotesis}} = \\
 &= \frac{2}{\Gamma(an_0+1)} \left(\sqrt{-n_0 t_0}\right)^{2(an_0+1)} \theta^{2(an_0+1)-1} e^{-(\sqrt{-n_0 t_0})^2 \theta^2} \mathbb{1}_{\theta>0} \\
 &\quad \Downarrow \\
 \theta &\sim \text{Galenshore}\left(an_0+1, \sqrt{-n_0 t_0}\right).
 \end{aligned}$$

Finally we plot a few of these densities Galenshore $(an_0+1, \sqrt{-n_0 t_0})$ sampled with the following code:

- $n_0 = 1, t_0 = -1, a = 1 \implies \text{Galenshore}(2, 1);$
- $n_0 = 2, t_0 = -1, a = 1 \implies \text{Galenshore}(3, \sqrt{2});$
- $n_0 = 2, t_0 = -2, a = 1 \implies \text{Galenshore}(3, 2);$
- $n_0 = 2, t_0 = -2, a = 2 \implies \text{Galenshore}(5, 2);$
- $n_0 = 3, t_0 = -3, a = 1 \implies \text{Galenshore}(4, 3);$
- $n_0 = 3, t_0 = -4, a = 1 \implies \text{Galenshore}(4, 4);$

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dgalenshore = function(y, a, theta) {
  (2 / gamma(a)) * theta^(2 * a) * y^(2 * a - 1) * exp(-(theta^2) * y^2)
}

y = seq(0.01, 3.5, length = 1000)
df = rbind(
  data.frame(y = y, gal_y = dgalenshore(y, 2, 1), label = "(2, 1)"),
  data.frame(y = y, gal_y = dgalenshore(y, 3, sqrt(2)), label = "(3, sqrt(2))"),
  data.frame(y = y, gal_y = dgalenshore(y, 3, 2), label = "(3, 2)"),
  data.frame(y = y, gal_y = dgalenshore(y, 5, 2), label = "(5, 2)"),
  data.frame(y = y, gal_y = dgalenshore(y, 4, 3), label = "(4, 3)"),
  data.frame(y = y, gal_y = dgalenshore(y, 4, 4), label = "(4, 4)")
)

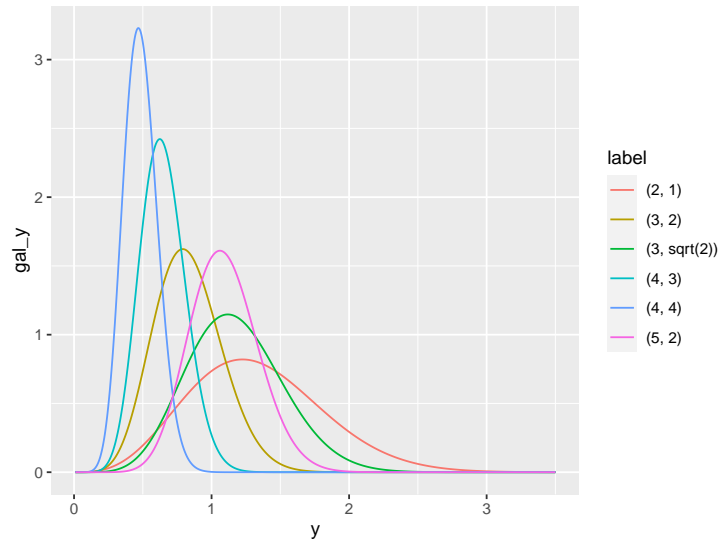
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Then we plot all of them at the same time:

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ggplot(df, aes(y, gal_y, group = label, color = label)) +
  geom_line() + coord_fixed(ratio = 1)

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Point b.

Let's define $b \stackrel{\text{def}}{=} an_0 + 1$ and $c = \sqrt{-n_0 t_0}$ for coinciseness.

Recalling $\theta \sim \text{Galenshore}(b, c)$ and $Y_i | \theta \sim \text{Galenshore}(a, \theta), \forall i \in 1 : n$ and defining $\text{SS}(y_{1:n}) \stackrel{\text{def}}{=} \sum_{i=1}^n y_i^2$ we have

$$\begin{aligned} p(\theta | y_{1:n}) &= p(\theta) p(y_{1:n} | \theta) \propto \\ &\propto (\theta^{2b-1} e^{-c^2 \theta^2}) (\theta^{2na} e^{-\theta^2 \sum_{i=1}^n y_i^2}) \propto \\ &\propto \theta^{2(an+b)-1} e^{-(c^2 + \text{SS}(y_{1:n})) \theta^2}. \end{aligned}$$

Hence we recognize the kernel of a Galenshore $(an + b, \sqrt{c^2 + \text{SS}(y_{1:n})})$

$$\implies \theta | Y_{1:n} \sim \text{Galenshore} \left(a(n + n_0) + 1, \sqrt{\text{SS}(y_{1:n}) - n_0 t_0} \right).$$

Point c.

$$\begin{aligned} \frac{p(\theta_a | y_{1:n})}{p(\theta_b | y_{1:n})} &= \frac{2\Gamma(a(n + n_0) + 1)}{\Gamma(a(n + n_0) + 1)2} (\text{SS}(y_{1:n}) - n_0 t_0)^{(a(n+n_0)+1)(1-1)} \left(\frac{\theta_a}{\theta_b} \right)^{2a(n+n_0)+1} e^{-(\text{SS}(y_{1:n}) - n_0 t_0)(\theta_a^2 - \theta_b^2)} = \\ &= \left(\frac{\theta_a}{\theta_b} \right)^{2a(n+n_0)+1} e^{-(\sum_{i=1}^n y_i^2 - n_0 t_0)(\theta_a^2 - \theta_b^2)}. \end{aligned}$$

Hence

$$\mathbb{P}(\theta \in A | Y_{1:n} = y_{1:n}) = \mathbb{P} \left(\theta \in A \mid \sum_{i=1}^n Y_i^2 = \sum_{i=1}^n y_i^2 \right), \forall A$$

and then, by definition $\text{SS}(Y_{1:n}) \stackrel{\text{def}}{=} \sum_{i=1}^n Y_i^2$ is a sufficient statistic.

Point d.

Recalling that $\theta | Y_{1:n} \sim \text{Galenshore} \left(a(n + n_0) + 1, \sqrt{\text{SS}(y_{1:n}) - n_0 t_0} \right)$ and that if $X \sim \text{Galenshore}(a, \theta) \implies \mathbb{E}[X] = \frac{\Gamma(a + \frac{1}{2})}{\theta \Gamma(a)}$ we have

$$\mathbb{E}[\theta | y_{1:n}] = \frac{\Gamma(a(n + n_0) + \frac{3}{2})}{\left(\sqrt{\text{SS}(y_{1:n}) - n_0 t_0} \right) \Gamma(a(n + n_0) + 1)}$$

Point e.

With the usual notation $b \stackrel{\text{def}}{=} an_0 + 1$ and $c = \sqrt{-n_0 t_0}$:

$$\begin{aligned} p(y_{n+1} | y_{1:n}) &= \int_0^\infty p(y_{n+1} | \theta) p(\theta | y_{1:n}) d\theta = \\ &= \int_0^\infty \frac{2}{\Gamma(a)} \theta^{2a} y_{n+1}^{2a-1} e^{-\theta^2 y_{n+1}^2} \cdot \frac{2}{\Gamma(an+b)} \left(c^2 + \text{SS}(y_{1:n}) \right)^{an+b} \theta^{2(an+b)-1} e^{-(c^2 + \text{SS}(y_{1:n})) \theta^2} d\theta = \\ &= \frac{4}{\Gamma(a) \Gamma(an+b)} y_{n+1}^{2a-1} \left(c^2 + \text{SS}(y_{1:n}) \right)^{an+b} \int_0^\infty \underbrace{\theta^{2(a+an+b)-1} e^{-(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2) \theta^2}}_{\text{kernel of a Galenshore}(a+an+b, \sqrt{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2})} d\theta \end{aligned}$$

$$\begin{aligned}
& \Downarrow \\
& \int_0^\infty \theta^{2(a+an+b)-1} e^{-(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2)\theta^2} = \frac{\Gamma(a+an+b)}{2} \left(\frac{1}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} \right)^{a+an+b} \\
& \Downarrow \\
p(y_{n+1}|y_{1:n}) &= \frac{4}{\Gamma(a)\Gamma(an+b)} y_{n+1}^{2a-1} \left(c^2 + \text{SS}(y_{1:n}) \right)^{an+b} \frac{\Gamma(a+an+b)}{2} \left(\frac{1}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} \right)^{a+an+b} = \\
&= \frac{2}{y_{n+1}} \frac{\Gamma(a+an+b)}{\Gamma(a)\Gamma(an+b)} \left(\frac{y_{n+1}^2}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} \right)^a \left(\frac{c^2 + \text{SS}(y_{1:n})}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} \right)^{an+b} = \\
&= \frac{2y_{n+1} (c^2 + \text{SS}(y_{1:n}))}{(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2)^2} \cdot \\
&\quad \cdot \underbrace{\frac{1}{B(a, an+b)} \left(\frac{y_{n+1}^2}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} \right)^{a-1} \left(1 - \frac{y_{n+1}^2}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} \right)^{(an+b)-1}}_{\text{density of } X \sim B(a, an+b) \text{ evaluated on } \frac{y_{n+1}^2}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2}} = \\
&= \frac{2y_{n+1} (c^2 + \text{SS}(y_{1:n}))}{(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2)^2} p_X\left(\frac{y_{n+1}^2}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2}\right).
\end{aligned}$$

Now one should note that this is a differentiable transformation of \mathbb{R}^+ of an unknown random variable. Indeed if one try to derive, with respect to y_{n+1} , the variable of the density of X in our last expression obtains

$$\begin{aligned}
\frac{d}{dy_{n+1}} \frac{y_{n+1}^2}{c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2} &= \frac{2y_{n+1}(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2) - y_{n+1}^2 2y_{n+1}}{(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2)^2} = \\
&= \frac{2y_{n+1}(c^2 + \text{SS}(y_{1:n}))}{(c^2 + \text{SS}(y_{1:n}) + y_{n+1}^2)^2} \cdot \\
&\quad > 0 \text{ indeed } y_{n+1} \in \mathbb{R}^+ \text{ and the other terms are squared}
\end{aligned}$$

Hence if we define $f^{-1} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f^{-1}(x) = \frac{x^2}{c^2 + \text{SS}(y_{1:n}) + x^2}$ we can state

$$\begin{aligned}
p(y_{n+1}|y_{1:n}) &= \left| \frac{d}{dy_{n+1}} f^{-1}(y_{n+1}) \right| p_X(f^{-1}(y_{n+1})) = \\
&= p_{f(X)}(y_{n+1}).
\end{aligned}$$

Let's compute f explicitly

$$f^{-1}(x) = \frac{x^2}{c^2 + \text{SS}(y_{1:n}) + x^2} = 1 - \frac{c^2 + \text{SS}(y_{1:n})}{c^2 + \text{SS}(y_{1:n}) + x^2}$$

hence

$$\begin{aligned}
x = 1 - \frac{c^2 + \text{SS}(y_{1:n})}{c^2 + \text{SS}(y_{1:n}) + f(x)^2} &\iff 1 - x = \frac{c^2 + \text{SS}(y_{1:n})}{c^2 + \text{SS}(y_{1:n}) + f(x)^2} \iff \\
&\iff \frac{f(x)^2}{c^2 + \text{SS}(y_{1:n})} + 1 = \frac{1}{1-x} \iff \\
&\iff f(x) = \sqrt{\frac{x}{1-x}} \sqrt{c^2 + \text{SS}(y_{1:n})}.
\end{aligned}$$

This leads us to conclude that

$$Y_{n+1}|Y_{1:n} \sim f(B(a, an+b)), \text{ with } f: \mathbb{R}^+ \rightarrow \mathbb{R}^+, f(x) = \sqrt{\frac{x}{1-x}} \sqrt{c^2 + \text{SS}(y_{1:n})}.$$