Question 1 (15 points): The Galenshore distribution (Hoff 3.9).

An unknown quantity Y has a Galenshore(a, θ) distribution if its density is given by

$$p(y) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2}$$

for $y > 0, \theta > 0$ and $a > 0$. Assume for now that a is known. For this density,

$$\mathbb{E}[Y] = \frac{\Gamma(a + \frac{1}{2})}{\theta \Gamma(a)}, \quad \mathbb{E}[Y^2] = \frac{a}{\theta^2}$$

a) Identify a class of conjugate prior densities for θ. Plot a few (e.g. 4 or 6) members of this class of densities.

b) Let $Y_1, \ldots, Y_n \mid \theta \overset{iid}{\sim}$ Galenshore(a, θ). Find the posterior distribution of θ given $Y_{1:n} = y_{1:n}$, using a prior from your conjugate class.

c) Write down $\frac{p(\theta_a | y_{1:n})}{p(\theta_b | y_{1:n})}$ and simplify. Identify a sufficient statistic.

d) Determine $\mathbb{E}[\theta \mid y_{1:n}]$.

e) Determine the form of the posterior predictive density $p(y_{n+1} \mid y_{1:n})$.

a) $y \sim Gal(a, \theta)$ has a density in the exponential family indeed

$p(y \mid \theta) = \frac{2}{\Gamma(a)} \cdot \theta^{2a} y^{2a-1} e^{-\theta^2 y^2} \mathbb{1}_{\{y>0\}} \quad (=: p_{Gal(a,\theta)}(y)) \quad , \theta > 0, a > 0$

$\phi := \theta^2$ maybe prove it (just at the very end).

$p(y \mid \phi) = h(y) \cdot c(\phi) \cdot e^{\phi \cdot t(y)}$ with $h(y) = \frac{2 y^{2a-1}}{\Gamma(a)} \mathbb{1}_{\{y>0\}}$, $c(\phi) = (\phi)^a$ and $t(y) = -y^2$.

$\Rightarrow$ we know that a class of conj. prior for a distr. in the exponential family is

$p(\phi) \propto c(\phi)^{m_0} \cdot e^{\phi \cdot m_0 \cdot t_0} = \phi^{a m_0} \cdot e^{\phi m_0 t_0}$ (← use $m_0 = c_1$, $m_0 t_0 = c_2$ as initial params, then change them (1))

If $\Phi \sim p(\phi)$ and we want to obtain the dist of $\Theta^2 = \Phi \Rightarrow \Theta = \sqrt{\Phi} = \varphi(\Phi)$ $\quad \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto \sqrt{x}$

$\Rightarrow p_\Theta(\theta) \propto p_\Phi(\varphi^{-1}(\theta)) \cdot (\det(\underbrace{J_{\varphi^{-1}}(\theta)}_{d\theta^2/d\theta = 2\theta})) = p(\theta^2) \cdot 2\theta = \theta^{2am_0} e^{\theta^2 m_0 t_0} \cdot 2\theta =$ $\quad \varphi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto x^2$

$= 2 \, \theta^{2am_0+1} \cdot e^{\theta^2 m_0 \cdot t_0}$

ok!

$e^{\theta^2 m_0 t_0}$: $= -(\sqrt{-m_0 t_0})^2 \theta^2$

one betw. m_0 and t_0 < 0. $\sqrt{-m_0 t_0}$ will be the parameter c of the distribution: it'll appear as an $(\sqrt{m_0 t_0})^{2a}$

for a 1st param

$2am_0 + 1 = 2k - 1$

$k = (2am_0 + 2)/2 = am_0 + 1$

$\Rightarrow = 2(am_0 + 1) - 1$

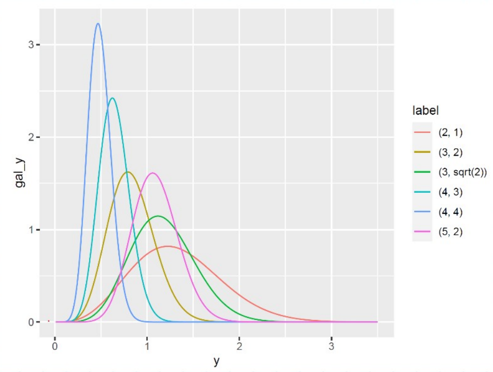
recalling $\theta > 0$ by hp.

$\Rightarrow p_\Theta(\theta) = \frac{2}{\Gamma(am_0+1)} \cdot (-m_0 t_0)^{am_0+1} \theta^{2(am_0+1)-1} \cdot e^{-(\sqrt{-m_0 t_0})^2 \theta^2} \mathbb{1}_{\{\theta>0\}} \quad \left(:= p_{Gal(am_0+1, \sqrt{-m_0 t_0})}(\theta) \right)$

(remark: $am_0 + 1 > 0$, $-m_0 t_0 > 0$ by construction indeed $a > 0$ by hp.; $m_0 > 0$ because it represent the "prior sample size", the prior has the same kernel of the likelihood based on m_0 observations $(p(\tilde{y}_{1:m} \mid \phi))$; $t_0 > 0$ because it is the "prior guess" for $t(y) = -y^2/2 < 0 \;\; \forall y$.)

Let's plot a few of this densities $Gal(am_0 + 1, \sqrt{-m_0 t_0})$

- $m_0 = 1$, $t_0 = -1$, $a = 1$ $\Rightarrow$ $Gal(2, 1)$
- $m_0 = 2$, $t_0 = -1$, $a = 1$ $\Rightarrow$ $Gal(3, \sqrt{2})$
- $m_0 = 2$, $t_0 = -2$, $a = 1$ $\Rightarrow$ $Gal(3, 2)$
- $m_0 = 2$, $t_0 = -2$, $a = 2$ $\Rightarrow$ $Gal(5, 2)$
- $m_0 = 3$, $t_0 = -3$, $a = 1$ $\Rightarrow$ $Gal(4, 3)$
- $m_0 = 4$, $t_0 = -4$, $a = 1$ $\Rightarrow$ $Gal(4, 4)$



$$y \sim \mathcal{G}al\,(a,\theta)$$

$$\theta \sim \mathcal{G}al\,(b,c) \qquad \propto e^{-\theta^2 y_i^2}\cdot\theta^{2a}$$

b) $$p(\theta \mid y_{1:m}) \propto p(\theta)\cdot p(y_{1:m}\mid\theta) =$$
$$\propto \left(\theta^{2b-1}\cdot e^{-c^2\theta^2}\right)\theta^{2a\cdot m}\, e^{-\theta^2\left(\sum_{i=1}^m y_i^2\right)} \propto$$
$$\propto \theta^{2b-1+2am}\exp\left\{-\theta^2\left(c^2+\sum_{i=1}^m y_i^2\right)\right\}$$
$$\Rightarrow p(\theta\mid y_{1:m}) \sim \mathcal{G}al\left(am+b,\ \sqrt{c^2+\sum_{i=1}^m y_i^2}\right).$$

$SS := \sum_{i=1}^m y_i^2$

c) $$p(\theta_a\mid y_{1:m}) \Big/ p(\theta_b\mid y_{1:m}) = \underbrace{\left(\frac{\theta_a}{\theta_b}\right)}^{2(am+b)-1}\exp\left\{-\underbrace{(\theta_a^2-\theta_b^2)}\left(c^2+\underbrace{\sum^m y_i^2}_{T(y_{1:m})}\right)\right\}$$

does not dep on θ_a, θ_b $\Leftarrow\Rightarrow$ 1st question: minimal sufficient w.r.t. what?

d) $$\mathbb{E}[\theta\mid y_{1:m}] \overset{!}{=} \Gamma\left(am+b+1/2\right)\Big/\left(c^2+\sum^m y_i^2\right)\cdot\Gamma(am+b)$$

$$\mathbb{E}[Y] = \frac{\Gamma(a+\frac{1}{2})}{\theta\Gamma(a)}, \quad (y\sim \mathcal{G}al\,(a,\theta))$$

e) $$p(y_{m+1}\mid y_{1:m}) = \int_0^\infty p(y_{m+1}\mid\theta)\cdot p(\theta\mid y_{1:m})\,d\theta = \leftarrow \quad p(y) = \frac{2}{\Gamma(a)}\theta^{2a}y^{2a-1}e^{-\theta^2y^2}$$

$$= \int_0^\infty \frac{2}{\Gamma(a)}\cdot\theta^{2a}y_{m+1}^{2a-1}\cdot\exp\left\{-\theta^2 y_{m+1}^2\right\}\cdot\frac{2}{\Gamma(am+b)}\left(\sqrt{c^2+SS}\right)^{2(am+b)}\cdot\theta^{2(am+b)-1}\cdot\exp\left\{-(c^2+SS)\,\theta^2\right\} =$$

$$= \frac{4}{\Gamma(a)\,\Gamma(am+b)}\,y_{m+1}^{2a-1}\cdot(c^2+SS)^{am+b}\underbrace{\int_0^\infty \theta^{2(a+am+b)-1}\cdot\exp\left\{-\theta^2\Big(\overset{\uparrow\ SS_{m+1}}{\sum^{m+1} y_i^2}+c^2\Big)\right\}d\theta}_{} =$$

kernel of $\mathcal{G}al\left(a+am+b,\ \sqrt{SS_{m+1}+c^2}\right)$

$$= \frac{\Gamma(a+am+b)}{2}\cdot\left(\frac{1}{c^2+SS_{m+1}}\right)^{a+am+b}$$

$$= 2\cdot\frac{\Gamma((a)+(am+b))}{\Gamma(a)\,\Gamma(am+b)}\;y_{m+1}^{2a}\cdot\frac{1}{y_{m+1}}\cdot(c^2+SS)^{am+b}\cdot\frac{1}{(c^2+SS_{m+1})^a}\cdot\frac{1}{(c^2+SS_{m+1})^{b+ma}} =$$

$$= \frac{2}{y_{m+1}}\cdot\left(\underbrace{1\big/\mathrm{Beta}(a,\,b+ma)\left((y_{m+1})^2\big/(c^2+SS+y_{m+1}^2)\right)^a\cdot\left(c^2+SS\big/(c^2+SS+y_{m+1}^2)\right)^{b+ma}}\right)$$

$$= p_X\left(y_{m+1}^2\big/c^2+SS_{m+1}\right),\quad X\sim \mathrm{Beta}\,(a,\,am+b)$$

2nd question: do we need a closed form for the density?

To do: tex ex 1.