Question 1 (15 points): The Galenshore distribution (Hoff 3.9).

An unknown quantity Y has a Galenshore(a, θ) distribution if its density is given by

$$p(y) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2}$$

for $y > 0, \theta > 0$ and $a > 0$. Assume for now that a is known. For this density,

$$\mathbb{E}[Y] = \frac{\Gamma(a + \frac{1}{2})}{\theta \Gamma(a)}, \quad \mathbb{E}[Y^2] = \frac{a}{\theta^2}$$

a) Identify a class of conjugate prior densities for θ. Plot a few (e.g. 4 or 6) members of this class of densities.

b) Let $Y_1, \dots, Y_n \mid \theta \overset{iid}{\sim}$ Galenshore(a, θ). Find the posterior distribution of θ given $Y_{1:n} = y_{1:n}$, using a prior from your conjugate class.

c) Write down $\frac{p(\theta_a \mid y_{1:n})}{p(\theta_b \mid y_{1:n})}$ and simplify. Identify a sufficient statistic.

d) Determine $\mathbb{E}[\theta \mid y_{1:n}]$.

e) Determine the form of the posterior predictive density $p(y_{n+1} \mid y_{1:n})$.

a) $y \sim Gal(a, \theta)$ has a density in the exponential family indeed

$p(y \mid \theta) = \frac{2}{\Gamma(a)} \cdot \theta^{2a} y^{2a-1} e^{-\theta^2 y^2} \mathbb{1}_{\{y>0\}} \left(=: p_{Gal(a,\theta)}(y) \right)$, $\theta > 0, a > 0$

$\downarrow \phi := \theta^2$

$p(y \mid \phi) = h(y) \cdot c(\phi) \cdot e^{\phi \cdot t(y)}$ with $h(y) = \frac{2 y^{2a-1}}{\Gamma(a)} \mathbb{1}_{\{y>0\}}$, $c(\phi) = (\phi)^a$ and $t(y) = -y^2$.

$\Rightarrow$ we know that a class of conj. prior for a distr. in the exponential family is

$p(\phi) \propto c(\phi)^{m_0} \cdot e^{\phi \cdot m_0 \cdot t_0} = \phi^{a m_0} \cdot e^{\phi m_0 t_0}$

If $\Phi \sim p(\phi)$ and we want to obtain the dist of $\Theta^2 = \Phi \Rightarrow \Theta = \sqrt{\Phi} = \varphi(\Phi)$ $\quad \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto \sqrt{x}$

$\Rightarrow p_\Theta(\theta) \propto p_\Phi(\varphi^{-1}(\theta)) \cdot (\det(\underbrace{J_{\varphi^{-1}}(\theta)}_{d\theta^2/d\theta = 2\theta})) = p(\theta^2) \cdot 2\theta = \theta^{2am_0} e^{\theta^2 m_0 t_0} \cdot 2\theta =$ $\quad \varphi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto x^2$

$= 2 \, \theta^{2am_0+1} \cdot e^{\theta^2 m_0 t_0}$

ok!

$= -(\sqrt{-m_0 t_0})^2 \theta^2$

one betw. m_0 and $t_0 < 0$. $\sqrt{-m_0 t_0}$ will be the parameter θ of the distribution: it'll appear as an $(\sqrt{-m_0 t_0})^{2a}$ for a 1st param

$2am_0 + 1 = 2k - 1$

recalling $\theta > 0$ by hp.

$k = (2am_0 + 2)/2 = am_0 + 1$

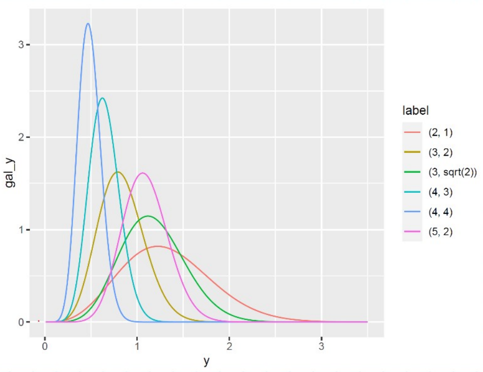
$\Rightarrow = 2(am_0 + 1) - 1$

$\Rightarrow p_\Theta(\theta) = \frac{2}{\Gamma(am_0+1)} \cdot (-m_0 t_0)^{am_0+1} \theta^{2(am_0+1)-1} \cdot e^{-(\sqrt{-m_0 t_0})^2 \theta^2} \mathbb{1}_{\{\theta>0\}} \left(:= p_{Gal(am_0+1, \sqrt{-m_0 t_0})}(\theta) \right)$

(remark: $am_0 + 1 > 0$, $-m_0 t_0 > 0$ by construction indeed $a > 0$ by hp. ; $m_0 > 0$ because it represent the "prior sample size", the prior has the same kernel of the likelyhood based on m_0 observations ($p(\tilde{y}_{1:m} \mid \phi)$) ; $t_0 > 0$ because it is the "prior guess" for $t(y) = -y^2/2 < 0 \; \forall y$.)

Let's plot a few of this densities $Gal(am_0 + 1, \sqrt{-m_0 t_0})$

- $m_0 = 1$, $t_0 = -1$, $a = 1$ $\Rightarrow$ $Gal(2, 1)$
- $m_0 = 2$, $t_0 = -1$, $a = 1$ $\Rightarrow$ $Gal(3, \sqrt{2})$
- $m_0 = 2$, $t_0 = -2$, $a = 1$ $\Rightarrow$ $Gal(3, 2)$
- $m_0 = 2$, $t_0 = -2$, $a = 2$ $\Rightarrow$ $Gal(5, 2)$
- $m_0 = 3$, $t_0 = -3$, $a = 1$ $\Rightarrow$ $Gal(4, 3)$
- $m_0 = 4$, $t_0 = -4$, $a = 1$ $\Rightarrow$ $Gal(4, 4)$



b)