## Question 1 (15 points): The Galenshore distribution (Hoff 3.9).

An unknown quantity $Y$ has a Galenshore$(a, \theta)$ distribution if its density is given by

$$p(y) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2}$$

for $y > 0, \theta > 0$ and $a > 0$. Assume for now that $a$ is known. For this density,

$$\mathbb{E}[Y] = \frac{\Gamma(a + \frac{1}{2})}{\theta \Gamma(a)}, \quad \mathbb{E}[Y^2] = \frac{a}{\theta^2}$$

a) Identify a class of conjugate prior densities for $\theta$. Plot a few (e.g. 4 or 6) members of this class of densities.

b) Let $Y_1, \ldots, Y_n \mid \theta \overset{iid}{\sim}$ Galenshore$(a, \theta)$. Find the posterior distribution of $\theta$ given $Y_{1:n} = y_{1:n}$, using a prior from your conjugate class.

c) Write down $\frac{p(\theta_a | y_{1:n})}{p(\theta_b | y_{1:n})}$ and simplify. Identify a sufficient statistic.

d) Determine $\mathbb{E}[\theta \mid y_{1:n}]$.

e) Determine the form of the posterior predictive density $p(y_{n+1} \mid y_{1:n})$.

a) $y \sim Gal(a, \theta)$ has a density in the exponential family indeed

$p(y \mid \theta) = \frac{2}{\Gamma(a)} \cdot \theta^{2a} y^{2a-1} e^{-\theta^2 y^2} \mathbb{1}_{\{y>0\}} \quad (=: p_{Gal(a,\theta)}(y))$ , $\theta > 0, a > 0$

$\phi := \theta^2$ maybe prove it (just at the very end).

$p(y \mid \phi) = h(y) \cdot c(\phi) \cdot e^{\phi \cdot t(y)}$ with $h(y) = \frac{2y^{2a-1}}{\Gamma(a)} \mathbb{1}_{\{y>0\}}$, $c(\phi) = (\phi)^a$ and $t(y) = -y^2$.

$\Rightarrow$ we know that a class of conj. prior for a distr. in the exponential family is

$p(\phi) \propto c(\phi)^{m_0} \cdot e^{\phi \cdot m_0 \cdot t_0} = \phi^{a m_0} \cdot e^{\phi m_0 t_0}$ ($\leftarrow$ use $m_0 = c_1, m_0 t_0 = c_2$ as initial params, then change them (1) )

If $\Phi \sim p(\phi)$ and we want to obtain the dist of $\Theta^2 = \Phi \Rightarrow \Theta = \sqrt{\Phi} = \varphi(\Phi)$ $\quad \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto \sqrt{x}$

$\Rightarrow p_\Theta(\theta) \propto p_\Phi(\varphi^{-1}(\theta)) \cdot (\det(\underbrace{J_{\varphi^{-1}}(\theta)}_{d\theta^2/d\theta = 2\theta})) = p(\theta^2) \cdot 2\theta = \theta^{2am_0} e^{\theta^2 m_0 t_0} \cdot 2\theta =$ $\quad \varphi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto x^2$

$= 2\,\theta^{2am_0+1} \cdot e^{\underbrace{\theta^2 m_0 \cdot t_0}}$

ok!

$= -(\sqrt{-m_0 t_0})^2 \theta^2$

one betw. $m_0$ and $t_0$ $\le 0$. $\sqrt{-m_0 t_0}$ will be the parameter $c$ of the distribution: it'll appear as an $(\sqrt{m_0 t_0})^{2a}$

for a 1st param

$2am_0 + 1 = 2k - 1$

recalling $\theta > 0$ by hp.

$k = (2am_0 + 2)/2 = am_0 + 1$

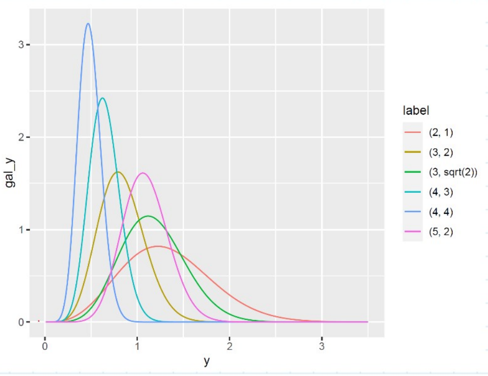
$\Rightarrow \quad = 2(am_0 + 1) - 1$

$\Rightarrow p_\Theta(\theta) = \frac{2}{\Gamma(am_0+1)} \cdot (-m_0 t_0)^{am_0+1} \theta^{2(am_0+1)-1} \cdot e^{-(\sqrt{-m_0 t_0})^2 \theta^2} \mathbb{1}_{\{\theta>0\}} \quad \left( := p_{Gal(am_0+1, \sqrt{-m_0 t_0})}(\theta) \right)$

( remark: $am_0 + 1 > 0$, $-m_0 t_0 > 0$ by construction indeed $a > 0$ by hp. ; $m_0 > 0$ because it represent the "prior sample size", the prior has the same kernel of the likelihood based on $m_0$ observations $(p(\tilde{y}_{1:m} \mid \phi))$ ; $t_0 > 0$ because it is the "prior guess" for $t(y) = -y^2/2 < 0 \ \forall y$. )

Let's plot a few of this densities $Gal(am_0+1, \sqrt{-m_0 t_0})$

- $m_0 = 1$, $t_0 = -1$, $a = 1$ $\Rightarrow$ $Gal(2, 1)$
- $m_0 = 2$, $t_0 = -1$, $a = 1$ $\Rightarrow$ $Gal(3, \sqrt{2})$
- $m_0 = 2$, $t_0 = -2$, $a = 1$ $\Rightarrow$ $Gal(3, 2)$
- $m_0 = 2$, $t_0 = -2$, $a = 2$ $\Rightarrow$ $Gal(5, 2)$
- $m_0 = 3$, $t_0 = -3$, $a = 1$ $\Rightarrow$ $Gal(4, 3)$
- $m_0 = 4$, $t_0 = -4$, $a = 1$ $\Rightarrow$ $Gal(4, 4)$



$$y \sim \mathcal{G}al\,(a, \theta)$$

$$\theta \sim \mathcal{G}al\,(b, c) \qquad \propto e^{-\theta^2 y_i^2} \cdot \theta^{2a}$$

**b)** 
$$p(\theta \mid y_{1:m}) \propto p(\theta) \cdot p(y_{1:m} \mid \theta) =$$
$$\propto \left(\theta^{2b-1} \cdot e^{-c^2\theta^2}\right) \theta^{2a\cdot m} \, e^{-\theta^2 \left(\sum_{i=1}^m y_i^2\right)} \propto$$
$$\propto \theta^{2b-1+2am} \exp\left\{-\theta^2 \left(c^2 + \sum_{i=1}^m y_i^2\right)\right\}$$
$$\Rightarrow p(\theta \mid y_{1:m}) \sim \mathcal{G}al\left(am + b, \sqrt{c^2 + \sum_{i=1}^m y_i^2}\right). \qquad SS := \sum_{i=1}^m y_i^2$$

**c)** 
$$p(\theta_a \mid y_{1:m}) \Big/ p(\theta_b \mid y_{1:m}) = \left(\frac{\theta_a}{\theta_b}\right)^{2(am+b)-1} \exp\left\{-\left(\theta_a^2 - \theta_b^2\right)\left(c^2 + \underbrace{\sum y_i^2}_{T(y_{1:m})}\right)\right\}$$

does not dep on $\theta_a, \theta_b$ $\Leftrightarrow$ 1st question: minimal sufficient w.r.t. what?

**d)** 
$$\mathbb{E}[\theta \mid y_{1:m}] \overset{!}{=} \Gamma\left(am + b + 1/2\right) \Big/ \left(c^2 + \sum^m y_i^2\right) \cdot \Gamma(am + b)$$

$$\mathbb{E}[Y] = \frac{\Gamma(a + \frac{1}{2})}{\theta\Gamma(a)}, \quad \left(y \sim \mathcal{G}al\,(a, \theta)\right)$$

**e)** 
$$p(y_{m+1} \mid y_{1:m}) = \int_0^\infty p(y_{m+1} \mid \theta) \cdot p(\theta \mid y_{1:m}) \, d\theta = \leftarrow \quad p(y) = \frac{2}{\Gamma(a)}\theta^{2a}y^{2a-1}e^{-\theta^2 y^2}$$

$$= \int_0^\infty \frac{2}{\Gamma(a)} \cdot \theta^{2a} y_{m+1}^{2a-1} \cdot \exp\left\{-\theta^2 y_{m+1}^2\right\} \cdot \frac{2}{\Gamma(am+b)} \left(\sqrt{c^2 + SS}\right)^{2(am+b)} \cdot \theta^{2(am+b)-1} \cdot \exp\left\{-(c^2+SS)\,\theta^2\right\} d\theta =$$

$$= \frac{4}{\Gamma(a)\,\Gamma(am+b)} \, y_{m+1}^{2a-1} \cdot (c^2+SS)^{am+b} \underbrace{\int_0^\infty \theta^{2(a+am+b)-1} \cdot \exp\left\{-\theta^2\left(\sum^{m+1} y_i^2 + c^2\right)\right\} d\theta}_{} =$$

$SS_{m+1}$

kernel of $\mathcal{G}al\left(a + am + b, \sqrt{SS_{m+1} + c^2}\right)$

$$= \frac{\Gamma(a + am + b)}{2} \cdot \left(\frac{1}{c^2 + SS_{m+1}}\right)^{a+am+b}$$

$$= 2 \cdot \frac{\Gamma((a) + (am+b))}{\Gamma(a)\,\Gamma(am+b)} \, y_{m+1}^{2a} \cdot \frac{1}{y_{m+1}} \cdot (c^2+SS)^{am+b} \cdot \frac{1}{(c^2+SS_{m+1})^a} \cdot \frac{1}{(c^2+SS_{m+1})^{b+ma}} =$$

$$= \frac{2}{y_{m+1}} \cdot \left(1 \Big/ \mathrm{Beta}(a, b+ma) \left((y_{m+1})^2 \Big/ (c^2+SS+y_{m+1}^2)\right)^a \cdot \left(c^2+SS \Big/ (c^2+SS+y_{m+1}^2)\right)^{b+ma}\right) =$$

$$= \frac{2}{y_{m+1}} \cdot \frac{y_{m+1}^2 (c^2+SS)}{(c^2+SS+y_{m+1}^2)^2} \left(\frac{1}{\mathrm{Beta}(a, b+ma)} \left((y_{m+1})^2 \Big/ (c^2+SS+y_{m+1}^2)\right)^{a-1} \cdot \left(1 - \left(y_{m+1}^2 \Big/ (c^2+SS+y_{m+1})^2\right)\right)^{b+ma}\right)$$

$$= \frac{2(c^2+SS)\,y_{m+1}}{(c^2+SS+y_{m+1}^2)^2} \cdot p_X\left((y_{m+1})^2 \Big/ (c^2+SS+y_{m+1}^2)\right)$$

$\hookrightarrow$ pdf of a $X \sim \mathrm{Beta}(a, am+b)$ evaluated on $(y_{m+1})^2 / (c^2+SS+y_{m+1}^2)$.

$$= \det\left(J_{\varphi^{-1}}(y_{m+1})\right) p_X\left(\varphi^{-1}(y_{m+1})\right) = p_{\varphi(X)}(y_{m+1})$$

where 
$$\left|\frac{d}{dy_{m+1}} \varphi^{-1}(y_{m+1})\right| = \frac{2(c^2+SS)\,y_{m+1}}{(c^2+SS+y_{m+1}^2)^2}$$

$$\hookrightarrow \frac{d}{dy_{m+1}}\left((y_{m+1})^2 \Big/ (c^2+SS+y_{m+1}^2)\right) = \frac{2y_{m+1} \cdot (c^2+SS+y_{m+1}^2) - (y_{m+1})^2 \cdot (2y_{m+1})}{(c^2+SS+y_{m+1}^2)^2} = \frac{2(c^2+SS)\,y_{m+1}}{(c^2+SS+y_{m+1}^2)^2}$$

$$\Rightarrow p_{y_{m+1} \mid y_{1:m}}(y_{m+1}) = p_{\varphi(X)}(y_{m+1}), \quad X \sim \mathrm{Beta}(a, am+b), \quad \varphi : \mathbb{R}^+ \circlearrowleft \text{ s.t. } \varphi^{-1}(x) = \frac{x^2}{(c^2+SS+x^2)},$$
$$= 1 - \frac{c^2+SS}{c^2+SS+x^2}$$
$$\Rightarrow 1 - x = \frac{c^2+SS}{c^2+SS+\varphi(x)}$$
$$\Rightarrow \frac{\varphi(x)}{c^2+SS} + 1 = \frac{1}{1-x}$$
$$\Rightarrow \varphi(x) = \frac{x \cdot (c^2+SS)}{1-x}$$

$$\Rightarrow y_{m+1} \mid y_{1:m} \sim \varphi(\mathrm{Beta}(a, am+b)), \quad \varphi(x) = \frac{x}{1-x} \cdot (c^2+SS)$$

To do: ten en 1.