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Multivariate Statistical Analysis
Problem Set 1

Exercise 2

2.1

Let first note that Σ is invertible since $\det(\Sigma) = -2\rho^3 - 3\rho^2 + 1$ which is greater than 0, $\forall p \in \left[-1, \frac{1}{2}\right]$.
We compute the inverse of Σ by exploiting the identity

$$\Sigma = (1+\rho)I - \rho a a^T \text{ with } a = (1, 1, -1)$$

and applying the following theorem, known as the Neumann Series Theorem:

Theorem (Neumann Series)**.** *Let T be a linear mapping $T : \mathbb{R}^n \to \mathbb{R}^n$. If the series $\sum_{i=0}^{\infty} T^i$ converges, then $I - T$ is invertible and it holds*

$$(I-T)^{-1} = \sum_{i=0}^{\infty} T^i.$$

First we rewrite

$$\Sigma = (1+\rho)(I - caa^T) \text{ with } a = (1, 1, -1) \text{ and } c = \frac{\rho}{1+\rho}.$$

Let $A \stackrel{\text{def}}{=} I - caa^T$, it holds

$$\begin{aligned}
A^{-1} = (I - caa^T)^{-1} &= \sum_{i=0}^{\infty} (caa^T)^i = \\
&= \sum_{i=0}^{\infty} c^i (aa^T)^i = I + \sum_{i=i}^{\infty} c^i (\|a\|^2)^{i-1} aa^T = \\
&= I + caa^T \sum_{i=0}^{\infty} (c\|a\|^2)^{i-1} = I + caa^T \sum_{i=i}^{\infty} (c\|a\|^2)^i = \\
&= I + caa^T \frac{1}{1 - c\|a\|^2} = I + \frac{p}{1+p} \frac{1}{1 - 3\frac{p}{1+p}} aa^T = \\
&= I + \frac{p}{1-2p} aa^T.
\end{aligned}$$

Thus

$$\Sigma^{-1} = (1+\rho)^{-1} A^{-1} = \frac{1}{1+\rho}(I + \frac{p}{1-2p} aa^T).$$

We can compute Σ^{-1} also in many other ways, for example we can suppose that Σ^{-1} is of the same form Σ, i.e.

$$\Sigma^{-1} = yI + kaa^t$$

and than find the values for y and k.

2.2

A faster way to find the spectrum (set of eigenvalues, meant with multiplicity) is reported below. We exploit some basic properties of the spectrum.
We denote $\operatorname{Sp}(\Sigma)$ the spectrum of the matrix Σ (as a linear operator).

$$\begin{aligned}\operatorname{Sp}(\Sigma) &= \operatorname{Sp}\left((1+\rho)(I - \frac{\rho}{1+\rho}aa^t)\right) = \\ &= (1+\rho)\operatorname{Sp}\left(I - \frac{\rho}{1+\rho}aa^t\right) = \\ &= (1+\rho)\left(1 - \frac{\rho}{1+\rho}\operatorname{Sp}\left(aa^t\right)\right).\end{aligned}$$

Observing $(aa^T)a = \|a\|^2 a$ and $\operatorname{rank}\left(aa^T\right) = 1$ it holds

$$\operatorname{Sp}\left(aa^T\right) = \left\{0, 0, \|a\|^2\right\}.$$

Hence

$$\begin{aligned}\operatorname{Sp}(\Sigma) &= (1+\rho)\left(1 - \frac{\rho}{1+\rho}\{0, 0, \|a\|^2\}\right) = \\ &= (1+\rho)\left\{1, 1, 1 - 3\frac{\rho}{1+\rho}\right\} = \\ &= \{1+\rho, 1+\rho, 1+\rho-3\rho\} = \\ &= \{1+\rho, 1+\rho, 1-2\rho\}.\end{aligned}$$

where the multiplications and translations of sets are mean component wise.

2.3

We first write the eigenvalues of Σ in ascending order.
We distinguish the following two cases:

1. if $\rho \in \left[0, \frac{1}{2}\right)$ then $1+\rho \geq 1-2\rho$. This leads to

$$\begin{cases}\lambda_1 = 1+\rho \\ \lambda_2 = 1+\rho \\ \lambda_3 = 1-2\rho\end{cases}, \text{ with } \lambda_1 \geq \lambda_2 \geq \lambda_3.$$

   Now we find ρ such that the first two principal components (PCs) account for more than 80% of the total variation of Z.
   Since λ_i corresponds to the variance of the i-th PC $\forall i \in \{1,2,3\}$ and the variation up to the k-th PC corresponds to the sum of the first k eigenvalues, we just need to find ρ such that

$$\lambda_1 + \lambda_2 > 0.8(\lambda_1 + \lambda_2 + \lambda_3).$$

   By solving the inequality we get

$$2(1+\rho) > \frac{4}{5}3 \iff 1+\rho > \frac{6}{5} \iff \rho > \frac{1}{5}.$$

2. if $\rho \in (-1, 0)$ then $1+\rho \leq 1-2\rho$. This leads to

$$\begin{cases}\lambda_1 = 1-2\rho \\ \lambda_2 = 1+\rho \\ \lambda_3 = 1+\rho\end{cases}, \text{ with } \lambda_1 \geq \lambda_2 \geq \lambda_3.$$

By using the same argument we used in the previous poin we obtain that ρ have to satisfy the following condition:

$$(1-2\rho)+(1+\rho) > \frac{4}{5}3 \iff 2-\rho > \frac{12}{5} \iff \rho < -\frac{2}{5}.$$

Hence for $\rho \in \left[0, \frac{1}{2}\right)$ it must be $\rho > \frac{1}{2}$ and for $\rho \in (-1, 0)$ it must be $\rho < -\frac{2}{5}$.
So $\forall \rho \in \left(-1, -\frac{2}{5}\right) \cup \left(\frac{1}{2}, 1\right)$ PC1 and PC2 account for more than 80% of the total variation of Z.

2.4

2.5

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c = sqrt(qchisq(0.95, df = 2))
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rho = 0.2
mu_y = 1 / 5 * c(1, 9)
sigma_y = 6 / 25 * matrix(c(4, 1, 1, 4), nrow = 2)
eig = eigen(sigma_y, symmetric = T)
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