BASIC STATISTICS III

CHI-SQUARED VALUE VS. DISTRIBUTION

One value of "Chi squared" compares observed minus expected values (residuals from a model) for one set of N data points with uncertainties σ (expected residuals)

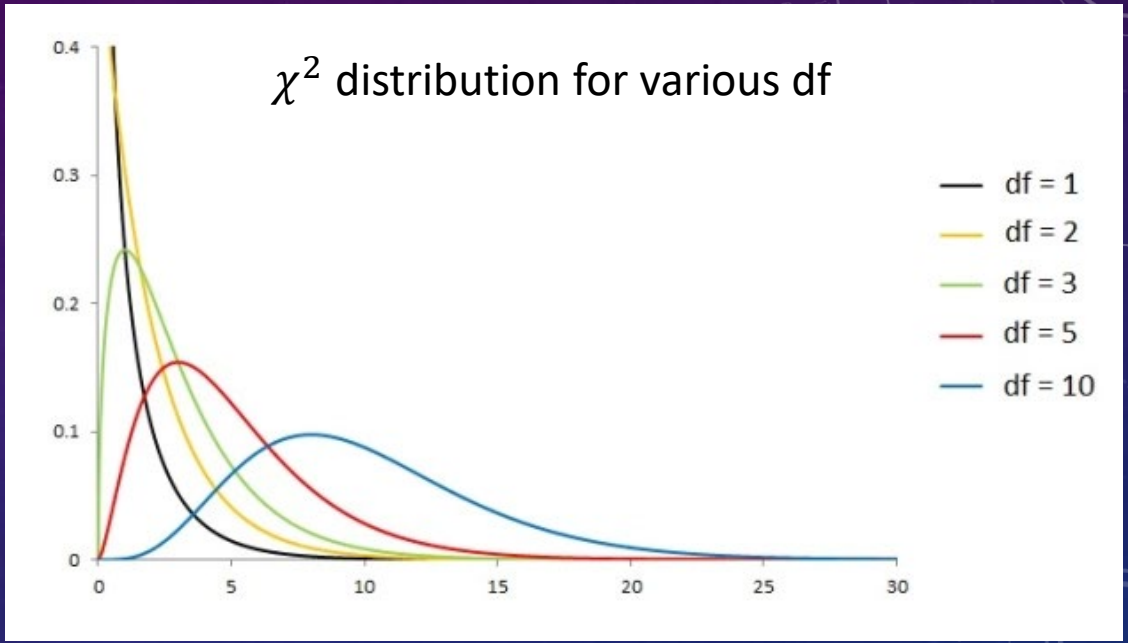
$$\chi^2 = \sum_{i=0,N} \frac{(O_i - E_i)^2}{\sigma_i^2}$$

Different values of χ^2 for different sets of N data points $\{O_i\}$ are drawn from the Chi squared probability distribution, assuming that for a single data point i, the probability of measuring a value O_i is drawn from a normal distribution $\frac{1}{\sqrt{2\pi}\sigma_i} e^{-(O_i - Ei)^2/2\sigma_i^2}$.

CHI-SQUARED AND REDUCED CHI-SQUARED

The equation for the χ^2 distribution is ugly and depends on the # of degrees of freedom:

$df = N - k$ for N data points and k model parameters.



For random χ^2, $\langle\chi^2\rangle \sim df$, so people often define a "reduced" $\widehat{\chi^2} = \chi^2/df$ and expect it to be ~1 for a "good model" (but this is a somewhat risky oversimplification, as you will see in the Interpreting χ^2 tutorial). Notice that for large df, the χ^2 distribution starts to look Gaussian (the central limit theorem!).

CHI-SQUARED AND LIKELIHOOD

Assume that for a given data point i, the probability of measuring a value O_i is $\frac{1}{\sqrt{2\pi}\sigma_i} e^{-(O_i - Ei)^2/2\sigma_i^2}$, and verify for yourself that multiplying the individual data point probability distributions gives an overall probability distribution for the data set $\{O_i\}$ that is proportional to $e^{-\chi^2/2}$ where $\chi^2 = \sum_i \frac{(O_i - E_i)^2}{\sigma_i^2}$.

Based on this, we say that the "likelihood" of a model being correct for a given data set is *proportional to* $e^{-\chi^2/2}$ if the residuals are normally distributed around the data points.

TRADITIONAL MAXIMUM LIKELIHOOD

- seek “best fit” models/parameters
- typically assume likelihood *L* proportional to $e^{-\chi^2/2}$
- min χ^2 ➔ max likelihood
- “maximum likelihood estimators” (MLEs) of parameters α_i of a model are usually found by $\frac{\partial L}{\partial \alpha_i} = 0$ or equivalently $\frac{\partial \ln(L)}{\partial \alpha_i} = 0$
- assuming all residuals follow same normal distribution (i.e. have same σ), called “ordinary least-squares” (OLS) fitting or “minimizing the rms” (root mean square deviations)

TRADITIONAL MAXIMUM LIKELIHOOD

- Example: model $y = \alpha X + \beta$ with equal Gaussian errors σ
- $\chi^2 = \sum_i \frac{(Y_i - (\alpha X_i + \beta))^2}{\sigma^2}$ ➔ max likelihood
- $\frac{\partial \ln(L)}{\partial \alpha} = 0 \rightarrow \frac{\partial \ln\left(e^{-\frac{\chi^2}{2}}\right)}{\partial \alpha} = 0 \rightarrow \sum_i \frac{(Y_i - (\alpha X_i + \beta))X_i}{\sigma^2} = 0$
- $\frac{\partial \ln(L)}{\partial \beta} = 0 \rightarrow \frac{\partial \ln\left(e^{-\frac{\chi^2}{2}}\right)}{\partial \beta} = 0 \rightarrow \sum_i \frac{(Y_i - (\alpha X_i + \beta))}{\sigma^2} = 0$
- two eqns, two unknowns – solve to get result in tutorial:

  $\alpha = \frac{\bar{X}\bar{Y} - \overline{XY}}{(\bar{X})^2 - \overline{X^2}}$ and $\beta = \bar{Y} - \bar{X}\alpha$

  (so for this simple case, no numerical χ^2 minimization is needed; but harder for more parameters or different σ_i)

TRADITIONAL MAXIMUM LIKELIHOOD

- uncertainties on MLE params estimated by 1/E(-H) = inverse of expectation of negative "Hessian matrix"
- Hessian matrix example: $y = \alpha X + \beta$

$$\text{Hessian}(\alpha, \beta) = \begin{bmatrix} \frac{\partial^2}{\partial \alpha^2} \log L(\alpha, \beta) & \frac{\partial^2}{\partial \alpha \partial \beta} \log L(\alpha, \beta) \\ \frac{\partial^2}{\partial \alpha \partial \beta} \log L(\alpha, \beta) & \frac{\partial^2}{\partial \beta^2} \log L(\alpha, \beta) \end{bmatrix}$$

note covariance terms!

- complicated to compute Hessians, often done numerically
- fully worked Hessian for least squares case at http://mathworld.wolfram.com/LeastSquaresFitting.html - note errors on parameters generally decrease as $\frac{1}{\sqrt{N}}$

THE BAYESIAN APPROACH: ~~MAXIMUM~~ LIKELIHOOD DISTRIBUTIONS

- construct space of models with expected ranges of parameters
- prob(model|data) $\propto$ prob(data|model) $\times$ prob(model)
  or Posterior $\propto$ Likelihood $\times$ Prior
- for flat priors on model params, posterior probability of each model is proportional to its likelihood, $e^{-\chi^2/2}$
  (math matches frequentist, but Bayesian would keep full posterior distribution and not make a "point estimate")
- integrate over "nuisance parameters" ("marginalize" over them) to get probability distribution for one parameter

THE BAYESIAN APPROACH: ~~MAXIMUM~~ LIKELIHOOD DISTRIBUTIONS

More uses of marginalization:

- Compare whole classes of models A and B (e.g. straight line or curved?) by integrating over parameters:

$$\text{“Bayes Factor”} = \frac{\int_\alpha d\alpha\, p_A \mathcal{L}(X_i|\alpha,A) p(\alpha|A)}{\int_\alpha d\alpha\, p_B \mathcal{L}(X_i|\alpha,B) p(\alpha|B)}$$

- Get probability distributions for model quantities other than model parameters, e.g., galaxy stellar masses from SPS fits to galaxy SEDs (stellar mass is not an input to the model, rather it is a scale factor determined in the fit)

THE BAYESIAN APPROACH: CHOICE OF PRIORS

- Priors can be “uninformative” in different ways – e.g., uniform in linear parameter, “scale-free” (uniform in log parameter), uniform in a transformed parameter
- Given lots of data and/or small error bars, posterior results should be insensitive to the choice of prior within the spread in the posterior
- Given crappy data or few data points, need to test sensitivity of posterior to substituting reasonable priors – and full Bayesian posterior is crucially more informative in such a case than a frequentist point estimate