TABLE III
EXAMPLE 2: REDUCED-ORDER MODELS

	Eigenvalues of the reduced model F-matrix	
Order	Balanced Matrix	Aggregation
1	-14.193	-0.91
2	$-8.677 \times 10^{-3}, -14.199$	$-0.91, -4.449$
3	$-0.9525, -15.729 \pm 12.345j$	$-0.91, -15.214 \pm 11.622j$
4(a)	$-1.0568, -9.1344, -16.576 \pm 12.23j$	$-0.91, -10.987, -15.214 \pm 11.622j$
4(b)	$-1.0568, -9.1344, -16.576 \pm 12.23j$	$-0.91, -4.449, -10.262 \pm 571.749j$
5	$-0.9600, -9.236 \pm 571.777j, -15.6515 \pm 12.301j$	$-0.91, -10.262 \pm 571.749j, -15.214 \pm 11.622j$

TABLE IV
EXAMPLE 2: max $\{\|\hat{y}(t) - y(t)\|\}$ FOR $0 \leq t \leq 1$

Order of the reduced model (see Table III)	Balanced Matrix	Aggregation, (2.3), (2.8) with $\bar{H}$ given by (2.12)
1	1.16	11.0
2	1.16	10.6
3	0.0174	0.800
4(a)	0.158	0.551
4(b)	0.158	10.6
5	0.0172	0.796

VII. CONCLUDING REMARKS

We have examined the balanced matrix method and the aggregation method of model reduction. Both methods are obtained from a change-of-variables transformation; hence, there is a similarity in their derivations. Our computational examples indicate that a properly chosen aggregation model can compare favorably with an internally dominant balanced matrix model of the same order, provided the eigenvalues of the aggregated model are truly dominant. In some cases the dominance of certain eigenvalues is apparent (Example 1), while in others it is not obvious (Example 2). In the latter case, the balanced matrix models may be better.

Our studies also illustrate the problem of selecting a reduced order model based upon some *a priori* error criterion. Use of such criteria may be no indication of the actual performance of a model, especially when the model is driven by inputs that differ from those used in the derivation of the *a priori* error criteria. Recent work of the authors has been to obtain error criteria (measures of approximation) valid for a large class of inputs.

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Estimation of the Accuracy of a Robot Manipulator

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Abstract—**In order to select high-accuracy robots for precise assembly tasks, a method for estimating the robot's accuracy is needed. Based on the developed error model [1]-[3] describing the accuracy of a robot manipulator resulting from the kinematic errors, a statistical estimation procedure for robot accuracy is developed. By calculating the Cartesian errors at many different arm positions in the known world, the robot accuracy can be estimated by using the maximum likelihood estimate for the covariance matrix of these Cartesian errors.**

I. INTRODUCTION

In the 1960's, the industrial robot was introduced. It was a universal transfer machine and could be programmed to perform many different tasks. Since computers were added to robots, their flexibility and productivity have been increased. Although a greal deal of sophistication has been achieved in specifying positions, in control, and in programming, the robot accuracy problem remains to be solved. Especially for some CAD/CAM applications, the requirement for the positional accuracy is about ± 0.1 mm and the orientational accuracy is about $\pm 0.1°$. These standards are very difficult for present robots to achieve, especially during the Cartesian motion control. Although robot vendors have developed their own calibration schemes to increase their robots'

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accuracy, the accuracy improvement is limited. In order to select the right robots to perform precision tasks, a simple method for estimation of robot accuracy is needed.

The accuracy of robots in the real world depends on the accuracy of its kinematics. The reason is that a robot manipulator is essentially a movable open chain of bodies with one end fixed and the other end free which contains a specialized tool. The control basis is the open-loop linkage control. A serial link manipulator consists of a sequence of mechanical links connected together by actuated joints. At present, the robotic control depends on the relationship between the Cartesian coordinates of the end effector and the joint coordinates. The joint coordinates are well defined by a homogeneous transformation [4] which is the most popular kinematic formulation used in the current robot industry. This transformation is determined by four link parameters, also called kinematic parameters; one is the joint variable, and others are geometrical parameters. Hence, the precision of robot position and orientation with respect to the real world depends on the accuracy of these four kinds of kinematic parameters. The impact of these kinematic errors on robot accuracy is studied in [1]–[3] where a useful mathematical model is established. In addition, Hayati [6] did a similiar analysis for the calibration of a robot manipulator.

In this error model, the relationship between kinematic errors and each of the six errors in the Cartesian coordinates of robot manipulator's end effector or of the six errors in the World coordinates of robot manipulator's base can be represented by an explicit linear equation. The model provides a solid base for solving kinematic accuracy problems of robot manipulators. In order to find a simple method for estimating robot accuracy, a statistical analysis is applied to the error model in this paper. By assuming that the kinematic errors are independent random variables, an estimation of a robot's accuracy is performed by means of a maximum likelihood principle. The results show that the robot's accuracy can be estimated based on observational data only, without knowing the values of individual kinematic errors. The presented estimation method is advantageous for its simplicity and ease of use in testing robots.

II. The Kinematic Error Model

For any N degree-of-freedom manipulator, there are N joints and $N + 1$ links. The relationship between the joint coordinate frames $i - 1$ and i can be well represented by a homogeneous transformation matrix A_i [4]:

$$A_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & l_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & l_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & r_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where S and C represent sine and cosine functions and θ_i, r_i, l_i, α_i are kinematic parameters of the ith joint. The origins of the joint coordinates $i - 1$ and i are located on the axes of joint i and $i + 1$, respectively. Specifically, θ_i is the angle between link i and $i - 1$, r_i is the axial offset along the axis of joint $i - 1$, and l_i and α_i are the common normal distance and the twist angle between joint i and $i - 1$. For a revolute joint, θ_i is the joint variable. As for a prismatic joint, r_i is the joint variable and $l_i = 0$. The kinematic parameters other than the joint variable are fixed dimensional values. Hence, for any N degree-of-freedom manipulator, there are N joint variables and $3N$ geometrical constants.

With the representation of A_i, the motion of the robot manipulator's end effector can be described by the multiplication of N A-matrices as [4]

$$T = A_1 A_2 \cdots A_N. \tag{2}$$

However, in the above representation, it is assumed that there is no kinematic error. The correct relationship A_i between the joint coordinate frames i and $i - 1$ is determined by its four kinematic parameters θ_i, r_i, l_i, and α_i. If errors exist in these kinematic parameters, there will be a differential change dA_i between these two coordinates. Thus, the accurate relationship between these two coordinates could no longer be correctly defined by the transformation A_i. Instead, the true relationship will be $A_i + dA_i$. Using this true relationship between two connective joint coordinate frames, the accurate position and orientation of the end effector of an open-loop, N degree-of-freedom manipulator will be

$$T + dT = (A_1 + dA_1)(A_2 + dA_2) \cdots (A_N + dA_N) \tag{3}$$

where dT represents the total differential changes at the end of the robot manipulator due to the $4N$ kinematic errors. It was shown in [1] and [3] that dT can be expressed as

$$dT = T\delta T \tag{4}$$

where δT is an error transformation matrix with the following form:

$$\delta T = \begin{bmatrix} 0 & -\delta x_3 & \delta x_2 & dx_1 \\ \delta x_3 & 0 & -\delta x_1 & dx_2 \\ -\delta x_2 & \delta x_1 & 0 & dx_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

where dx_1, dx_2, and dx_3 are the three translational errors and δx_1, δx_2, and δx_3 are the three orientational errors, respectively, at the end of the robot manipulator caused by those $4N$ kinematic errors. Let $\boldsymbol{d} = [dx_1, dx_2, dx_3]^t$ and $\boldsymbol{\delta} = [\delta x_1, \delta x_2, \delta x_3]^t$ where the superscript t represents the matrix transpose; then it is shown in [3] that

$$\begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{\delta} \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta\boldsymbol{\theta} + \begin{bmatrix} M_2 \\ 0 \end{bmatrix} \Delta\boldsymbol{r} + \begin{bmatrix} M_3 \\ 0 \end{bmatrix} \Delta\boldsymbol{l} + \begin{bmatrix} M_4 \\ M_3 \end{bmatrix} \Delta\boldsymbol{\alpha} \tag{6}$$

where

$$\Delta\boldsymbol{\theta} = [\Delta\theta_1 \cdots \Delta\theta_N]^t, \quad \Delta\boldsymbol{r} = [\Delta r_1 \cdots \Delta r_N]^t,$$

$$\Delta\boldsymbol{l} = [\Delta l_1 \cdots \Delta l_N]^t, \text{ and } \Delta\boldsymbol{\alpha} = [\Delta\alpha_1 \cdots \Delta\alpha_N]^t$$

are $4N$ kinematic errors; M_1, M_2, M_3, and M_4 are $3 \times N$ matrices whose components are functions of N joint variables.

III. Estimation of Robot Accuracy

It is observed from the results of the previous section that the accuracy of a robot manipulator is determined by the six Cartesian errors. In order to obtain a general description of robot accuracy, a statistical approach is applied. It is reasonable to assume that the components of $\Delta\boldsymbol{\theta}$, $\Delta\boldsymbol{r}$, $\Delta\boldsymbol{l}$, and $\Delta\boldsymbol{\alpha}$ are mutually independent random variables with zero mean. Then it follows from (6) that each component of the translational error vector $\boldsymbol{d}$ is the sum of $4N$ independent zero-mean random variables, and each component of the rotational error vector $\boldsymbol{\delta}$ is the sum of $2N$ independent zero-mean random variables. Therefore, $\boldsymbol{d}$ and $\boldsymbol{\delta}$ are both zero-mean Gaussian vectors. Let the covariance matrices of $\Delta\boldsymbol{\theta}$, $\Delta\boldsymbol{r}$, $\Delta\boldsymbol{l}$, and $\Delta\boldsymbol{\alpha}$ be V_θ, V_r, V_l, and V_α, respectively; then, these are all $N \times N$ diagonal matrices by our assumption. Thus, the covariance matrices of $\boldsymbol{d}$ and $\boldsymbol{\delta}$ are given, respectively, by

$$V_d = M_1 V_\theta M_1^t + M_2 V_r M_2^t + M_3 V_l M_3^t + M_4 V_\alpha M_4^t \tag{7}$$

and

$$V_\delta = M_2 V_\theta M_2^t + M_3 V_\alpha M_3^t. \tag{8}$$

The results of above two equations represent the positional and orientational accuracy of a robot manipulator.

It is clear that both V_d and V_δ are not diagonal. In [3], it was assumed that V_θ, V_r, V_l, and V_α are known, and thus V_d and V_δ can be calculated. As a matter of fact, however, it is very difficult to obtain V_θ, V_r, V_l, and V_α from a robot manipulator. On the other hand, for any given arm position in the known world, the Cartesian errors at the end of a robot manipulator can be obtained by comparing the actual and the desired Cartesian position. Based on this idea, we will present a simple estimation method to obtain the accuracy of a given robot from the observational data only, without knowing the values of kinematic errors.

Let $\boldsymbol{d}_k = (dx_{1k}, dx_{2k}, dx_{3k})$ and $\boldsymbol{\delta}_k = (\delta x_{1k}, \delta x_{2k}, \delta x_{3k})$, $k = 1, 2, \cdots, n$, be n independent realizations of $\boldsymbol{d}$ and $\boldsymbol{\delta}$, respectively, obtained at n different arm positions in the known world. Based on these available data, the 3×3 covariance matrix V_d is to be estimated. Let v_{ij} denote the ijth element of V_d. Since the covariance matrix is always symmetric, we have

$v_{ij} = v_{ji}$. Therefore, there are only six parameters to be estimated. For convenience, let u_{ij} denote the ijth element of V_d^{-1} which denotes the inverse matrix of V_d. In the following, we derive the maximum likelihood estimate (MLE) of V_d.

The joint density of $d_1, d_2, \cdots, d_n$ given V_d is

$$p(d_1, d_2, \cdots, d_n | V_d) = \prod_{k=1}^{n} p(d_k | V_d)$$

$$= (2\pi)^{-3n/2} |V_d|^{-n/2} \exp\left[-\frac{1}{2}\sum_{k=1}^{n} d_k' V_d^{-1} d_k\right] \tag{9}$$

where $|V|$ represents the determinant of matrix V. Using the fact that $|V_d|^{-1} = |V_d^{-1}|$, the likelihood function of V_d can be written as

$$L(V_d) = \log p(d, \cdots, d_n | V_d)$$

$$= -\frac{3n}{2} \log (2\pi) + \frac{n}{2} \log |V_d^{-1}| - \frac{1}{2} \sum_{k=1}^{n} d_k' V_d^{-1} d_k. \tag{10}$$

The above equation can also be regarded as the likelihood function of V_d^{-1} since V_d and V_d^{-1} are uniquely determined by each other.

By setting $\partial L(V_d)/\partial u_{ij} = 0$, we obtain the likelihood equation for estimating u_{ij}:

$$\frac{n}{2} \frac{1}{|V_d^{-1}|} \frac{\partial |V_d^{-1}|}{\partial u_{ij}} - \frac{1}{2} \sum_{k=1}^{n} \left(d_k' \frac{\partial V_d^{-1}}{\partial u_{ij}} d_k \right) = 0 \qquad 1 \le i, j \le 3. \tag{11}$$

It is easy to verify that

$$\partial |V_d^{-1}| / \partial u_{ij} = \text{cofactor of } u_{ij} \text{ and } v_{ij} = (\text{cofactor of } u_{ij}) / |V_d^{-1}|.$$

Consequently, (11) reduces to

$$\frac{n}{2} v_{ij} - \frac{1}{2} \sum_{k=1}^{n} dx_{ik} dx_{jk} = 0 \qquad 1 \le i, j \le 3. \tag{12}$$

In principle, these equations should be solved to obtain the MLE of u_{ij}'s since the likelihood equations were formulated in terms of u_{ij}'s. However, since V_d and V_d^{-1} are uniquely determined by each other, the invariance property of maximum likelihood estimation [5] enables us to solve (12) for MLE of v_{ij}'s. We thus conclude that the MLE of the ijth element of V_d is given by

$$\hat{v}_{ij} = \left[\sum_{k=1}^{n} dx_{ik} dx_{jk} \right] / n \qquad 1 \le i, j \le 3. \tag{13}$$

Similarly, let $\delta_k = (\delta x_{1k}, \delta x_{2k}, \delta x_{3k})$, $k = 1, 2, \cdots, n$ be n independent realizations of δ obtained at n different arm positions in the known world. Then the MLE of the ijth element of the covariance matrix V_δ can be derived:

$$\hat{w}_{ij} = \left[\sum_{k=1}^{n} \delta x_{ik} \delta x_{jk} \right] / n \qquad 1 \le i, j \le 3 \tag{14}$$

where w_{ij} denotes the ijth element of V_δ.

IV. Conclusion

In this paper, a simple algorithm to estimate the accuracy of a given robot is developed. The results show that the robot accuracy can be estimated from the observational data and without knowing the values of kinematic errors. The method is advantageous for its simplicity and ease of use in testing the accuracy of a robot manipulator.

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On the State Estimation for Pseudoparabolic Systems with Stochastic Coefficients

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Abstract—**The purpose of this note is to derive the least-square state estimator for a class of stochastic pseudoparabolic systems under noisy observations. Within the framework of function spaces, properties of the solution with regard to the state equation are studied. By showing the results of digital simulation experiments, the dynamics of the least-square estimator is given under noisy distributed observations.**

I. Introduction

Our main concern is to solve the problem of the state estimation for a class of distributed parameter systems of the so-called pseudoparabolic type, motivated by the desire to estimate the liquid flow in fissured rocks [1], [2]. One of the most common mathematical models is that of a stochastic partial differential equation with a stochastic coefficient, i.e.,

$$\frac{\partial u(t, x)}{\partial t} - a_1 \frac{\partial^3 u(t, x)}{\partial t \partial x^2} - \alpha(t, \omega) \frac{\partial^2 u(t, x)}{\partial x^2}$$

$$= 0 \text{ in } T \times G =]0, t_f[\times]0, 1[\tag{1}$$

with the initial and boundary conditions

$$u(0, x) = u_0(x) \qquad \text{on } G, \tag{2}$$

$$u(t, 0) = u(t, 1) = 0 \qquad \text{on } T. \tag{3}$$

In (1), $u(t, x)$ is the state variable, a_1 and α are, respectively, the positive deterministic constant and the randomly varying parameter. Equation (1) specifies the pressure transformation of flow taking into account the higher order correction term with respect to the viscosity $-a_1 \partial^3 u(t, x)/\partial t \partial x^2$ and the diffusion term $\alpha(t, \omega) \partial^2 u(t, x)/\partial x^2$ which expresses its random influence on the system.

The optimal control problem of the deterministic version of pseudoparabolic systems has been studied by White [3], [4]. However, the question of the state estimate for more general models considering uncertainties appears to be of some interest. We shall assume that

$$\alpha(t, \omega) = a_2 + a_3 \frac{dw(t)}{dt} \tag{4}$$

where a_2, a_3 are known constants and $w(t)$ is a standard Brownian motion process in R^1.

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