THE KINEMATIC ERROR MODEL FOR THE DESIGN OF ROBOT MANIPULATOR

TA6 - 9:30

By

Chi-haur Wu, Ph.D.

Unimation Inc.
Shelter Rock Lane
Danbury, CT 06810

ABSTRACT

The correct relationship between two connective joint coordinates of a robot manipulator is defined by four link parameters; one is the joint variable and the others are geomatrical values. Also,the basis for all open-loop manipulator control is the relationship between the Cartesian coordinates of the end effector and the joint coordinates. Hence,the fidelity of the Cartesian position and orientation of the end-effector to the real world depends on the accuracy of the four link parameters of each joint. In this paper, a linear analytical model between the six Cartesian errors and the four independent kinds of kinematic errors has been developed. Based on this model, the Cartesian error envelopes due to any combination of four kinds of kinematic errors can be uniquely determined. From the point of view of design, this error model can be used as a guide to minimize the open-loop kinematic errors of the robot manipulator. Finally, a new calibration technique based on this model has also been developed which can be used to correct the kinematic errors of the robot manipulator.

I. INTRODUCTION

As we all know that a robot manipulator is a position oriented mechanical device. The accuracy of the manipulator's position in the real world depends on the accuracy of the manipulator's kinematics. In order to optimize the accuracy of robot manipulator, a CAD tool for kinematic design has to be built such that the manipulator can maintain the minimum tolerent errors of Cartesian position and orientation within its working space in the real world. The purpose of this paper is to build such a CAD tool by mathematically formulated the relationship between the kinematic errors and the Cartesian errors of the robot manipulator.

Now let's review how the kinematics of a robot manipulator will effect the Cartesian position and orientation errors. A serial link manipulator consists of a sequence of mechanical links connected together by actuated joints. The relationship between two connective joint coordinates is well defined by a homogeneous transformation matrix[1]. The matrix is determined by four kinds of link parameters, also called kinematic parameters; one is joint variable and the others are geometrical parameters. At present, the robot manipulator are the open-loop linkage control. The basis for all the manipulator control is a relationship between the Cartesian coordinates of the end-effector and the joint coordinates. The fidelity of the Cartesian position and orientation to the real world depends on the accuracy of the four link parameters of each joint. Due to the nature of these kinematic parameters, the Cartesian errors can be grouped into two categories: (A) the Cartesian errors due to the position accuracy of the joint variables, (B) the Cartesian errors due to the dimensional errors of the other kinematic parameters. Waldron[3] has developed a model for the first category but the model is in general form. As to the second category, he only has described in general and without any mathmatical formulation.

In this paper, an explict mathmatical error model for above two categories has been developed. By using the error transformation between two coordinates and ignoring the higher order terms, the linear error model between the Cartesian errors and the four independent kinematic errors has been formulated. For any kind of manipulator, the Cartesian error envelopes caused by any combination of the four kinds of kinematic errors can be easily generated from the developed error model. In addtion, this error model can be used as a design guide to -inimize the Cartesian errors caused by the inematic parameters. Finally, a calibration technique has been developed based on this model and which can correct the kinematic errors of the robot manipulator.

II. KINEMATICS

For an N degrees-of-freedom manipulator, there will be N links and N joints. The relationship between the joint coordinate frames i-1 and i can be represented by a homogeneous transformation matrix Ai [1,2] and

$$Ai = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & l_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & l_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & r_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where C and S refer to SINE and COSINE functions, and θ_i , r_i , l_i , α_i are the link (or kinematic) parameters of the ith joint defined in [1]. These parameters are showed in Figure 1; for a prismatic joint, l_i = 0. For convenience, Ai can be representd by four 3 by 1 vectors as follows

$$Ai = \begin{bmatrix} \underline{n} & \underline{o} & \underline{a} & \underline{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} . \quad (2)$$

With the relation matrix Ai, the end of N degrees-of-freedom manipulator can be represented as

$$T_N = A_1 * A_2 * \ldots\ldots . \bullet A_{N-1} * A_N \quad (3)$$

where "*" is the matrix multiplication.

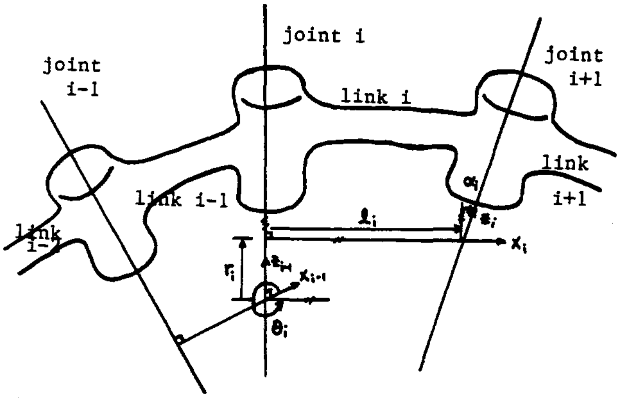

Figure 1. Link coordinates and parameters θ_i , r_i , ℓ_i , α_i .

Paul[2] has defined a very useful homogeneous transformation matrix Ui which described the motion of the end-effector with respect to joint coordinate frame i-1 , and

$$Ui = A_i * A_{i+1} * \ldots * A_N \quad . \tag{4}$$

Based on above definition, $U_1 = T_N$ and $U_{N+1} =$ I , the identity matrix.

The matrix Ui can also be represented by following form

$$Ui = \begin{bmatrix} \underline{n}_i^u & \underline{o}_i^u & \underline{a}_i^u & \underline{p}_i^u \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where $\underline{n}_i^u$, $\underline{o}_i^u$, $\underline{a}_i^u$ and $\underline{p}_i^u$ are 3 by 1 vectors.

III. DIFFERENTIAL CHANGES BETWEEN TWO COORDINATE FRAMES

Given a small change δT_1 in position and orientation in coordinate frame 1 then there will be a corresponding small change δT_2 in coordinate frame 2. If the relationship between two coordinate frames is T_1^2 then above relationship can be represented as

$$\delta T_1 * T_1^2 = T_1^2 * \delta T_2 \tag{6}$$

and
$$\delta T_2 = (T_1^2)^{-1} * \delta T_1 * T_1^2 \quad . \tag{7}$$

The differential error matrix δT_i , i=1,2 ,can be represented in the following form [5] by ignoring the higher order terms

$$\delta T_i = \begin{bmatrix} 0 & -\delta z_i & \delta y_i & dx_i \\ \delta z_i & 0 & -\delta x_i & dy_i \\ -\delta y_i & \delta x_i & 0 & dz_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where $\underline{d} = [dx_i \;\; dy_i \;\; dz_i]^t$ is the small translational changes and $\underline{\delta}_i = [\delta x_i \;\; \delta y_i \;\; \delta z_i]^t$ is the small rotational changes. (where the superscript 't' represents the matrix transpose.)

If T_1^2 and δT_1 were known then the components of δT_2 can be solved analytically as following form[4,5]

$$\begin{bmatrix} dx_2 \\ dy_2 \\ dz_2 \\ \delta x_2 \\ \delta y_2 \\ \delta z_2 \end{bmatrix} = \begin{bmatrix} \underline{n}\cdot\underline{d}_1 + (\underline{p} \times \underline{n})\cdot\underline{\delta}_1 \\ \underline{o}\cdot\underline{d}_1 + (\underline{p} \times \underline{o})\cdot\underline{\delta}_1 \\ \underline{a}\cdot\underline{d}_1 + (\underline{p} \times \underline{a})\cdot\underline{\delta}_1 \\ \underline{n}\cdot\underline{\delta}_1 \\ \underline{o}\cdot\underline{\delta}_1 \\ \underline{a}\cdot\underline{\delta}_1 \end{bmatrix} \tag{9}$$

where $\underline{n}$, $\underline{o}$, $\underline{a}$, and $\underline{p}$, are four 3-vectors of T_1^2 .

IV. DIFFERENTIAL CHANGES DUE TO THE KINEMATIC ERRORS

From EQ(1), the correct relationship Ai between joint coordinates i and i-1 is determined by its four link parameters θ_i , r_i , ℓ_i , and α_i . For a revolute joint, θ_i is the joint variable and the others are fixed dimensional values. As to a prismatic joint, r_i is the joint variable and $\ell_i = 0$ and the other two are the fixed dimensional values. If there are errors in these link parameters then there will be a differential change dA_i between the two joint coordinates. Thus the accurate relationship between the two joint coordinates will be equal to Ai+dAi. The differential change dAi can be estimated as following linear form

$$dAi = \frac{\partial A_i}{\partial \theta_i}\Delta\theta_i + \frac{\partial A_i}{\partial r_i}\Delta r_i + \frac{\partial A_i}{\partial \ell_i}\Delta\ell_i + \frac{\partial A_i}{\partial \alpha_i}\Delta\alpha_i \tag{10}$$

where $\Delta\theta_i$, Δr_i , $\Delta\ell_i$ and $\Delta\alpha_i$ are the small error changes in the link parameters.

From EQ(1),

$$\frac{\partial A_i}{\partial \theta_i} = \begin{bmatrix} -S\theta_i & -C\theta_i\, C\alpha_i & C\theta_i\, S\alpha_i & -\ell_i S\theta_i \\ C\theta_i & -S\theta_i\, C\alpha_i & S\theta_i\, S\alpha_i & \ell_i C\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= Q_\theta' * Ai \tag{11}$$

where

$$Q_\theta' = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{12}$$

By representing

$$\frac{\partial A_i}{\partial \theta_i} = Ai * Q_\theta \ , \tag{13}$$

Q_θ can be solved as

$$Q_\theta = Ai^{-1} * Q_\theta' * Ai$$

$$= \begin{bmatrix} 0 & -C\alpha_i & S\alpha_i & 0 \\ C\alpha_i & 0 & 0 & \ell_i C\alpha_i \\ -S\alpha_i & 0 & 0 & -\ell_i S\alpha_i \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{14}$$

Same technique as above, following results are obtained

$$\frac{\partial A_i}{\partial r_i} = Ai * Q_r \ , \tag{15}$$

$$\frac{\partial A_i}{\partial \ell_i} = Ai * Q_\ell \ , \tag{16}$$

and
$$\frac{\partial A_i}{\partial \alpha_i} = Ai * Q_\alpha \ , \tag{17}$$

where

$$Q_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S\alpha_i \\ 0 & 0 & 0 & C\alpha_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

$$Q_\ell = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

$$Q_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Based on above results, EQ(10) can be rewritten as

$$dA_i = A_i * (Q_\theta \Delta\theta_i + Q_r \Delta r_i + Q_\ell \Delta\ell_i + Q_\alpha \Delta\alpha_i) \tag{21}$$

By defining an error matrix transform δA_i with respect to A_i and

$$dA_i = A_i * \delta A_i \tag{22}$$

then

$$\delta A_i = Q_\theta \Delta\theta_i + Q_r \Delta r_i + Q_\ell \Delta\ell_i + Q_\alpha \Delta\alpha_i \tag{23}$$

Mathematically δA_i can be solved as

$$\delta A_i = \begin{bmatrix} 0 & -C\alpha_i\Delta\theta_i & S\alpha_i\Delta\theta_i & \Delta\ell_i \\ C\alpha_i\Delta\theta_i & 0 & -\Delta\alpha_i & \ell_i C\alpha_i\Delta\theta_i + S\alpha_i\Delta r_i \\ -S\alpha_i\Delta\theta_i & \Delta\alpha_i & 0 & -\ell_i S\alpha_i\Delta\theta_i + C\alpha_i\Delta r_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

As a result, δA_i has the same form as EQ(8) with components

$$\underline{d}_i^A = \begin{bmatrix} \Delta\ell_i \\ \ell_i C\alpha_i\Delta\theta_i + S\alpha_i\Delta r_i \\ -\ell_i S\alpha_i\Delta\theta_i + C\alpha_i\Delta r_i \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 0 \\ \ell_i C\alpha_i \\ -\ell_i S\alpha_i \end{bmatrix} \Delta\theta_i + \begin{bmatrix} 0 \\ S\alpha_i \\ C\alpha_i \end{bmatrix} \Delta r_i + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Delta\ell_i \tag{26}$$

and

$$\underline{\delta}_i^A = \begin{bmatrix} \Delta\alpha_i \\ S\alpha_i\Delta\theta_i \\ C\alpha_i\Delta\theta_i \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} 0 \\ S\alpha_i \\ C\alpha_i \end{bmatrix} \Delta\theta_i + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Delta\alpha_i \tag{28}$$

By defining following three vectors

$$\underline{k}_i^1 = [\,0 \quad \ell_i C\alpha_i \quad -\ell_i S\alpha_i\,]^t , \tag{29}$$

$$\underline{k}_i^2 = [\,0 \quad S\alpha_i \quad C\alpha_i\,]^t , \tag{30}$$

$$\underline{k}_i^3 = [\,1 \quad 0 \quad 0\,]^t , \tag{31}$$

the translational and rotational errors at Ai due to the link parameters' errors can be expressed in the following linear form

$$\underline{d}_i^A = \underline{k}_i^1 \Delta\theta_i + \underline{k}_i^2 \Delta r_i + \underline{k}_i^3 \Delta\ell_i \tag{32}$$

$$\underline{\delta}_i^A = \underline{k}_i^2 \Delta\theta_i + \underline{k}_i^3 \Delta\alpha_i \tag{33}$$

Above error expressions are the general form for any type of joint i. If joint i is a prismatic joint then the link parameter $\ell_i = 0$. Thus for a prismatic joint $\underline{k}_i^1 = 0$ and $\Delta\ell_i = 0$, and EQ(32) can be reduced to

$$\underline{d}_i^A = \underline{k}_i^2 \Delta r_i \tag{34}$$

After determined δA_i , the new relation between joint coordinates i and i-1 can be expressed as

$$Ai + dAi = Ai * (\, I + \delta Ai \,) \tag{35}$$

where I is the identity matrix.

V. POSITION AND ORIENTATION ERRORS OF AN OPEN-LOOP ROBOT MANIPULATOR

The position accuracy of an open-loop , N degrees-of-freedom robot manipulator in the real world depends on the accuracy of four link parameters of every joint. In the previous section, the differential change dAi and the error matrix transform δAi at joint coordinate i due to four small kinematic errors has be determined. Hence, for a N degrees-of-freedom manipulator, the accurate position and orientation of the end-effector with respect to the base due to the 4N kinematic errors can be expressed as

$$T_N + dT_N = (A_1 + dA_1) * (A_2 + dA_2) * \ldots * (A_N + dA_N)$$
$$= \prod_{i=1}^{N} (Ai + dAi) \tag{36}$$

where dT_N represents the total differential changes at the end of manipulator due to the 4N kinematic errors.

By expanding EQ(36) and ignoring the higher order differential changes, following linear result has been obtained

$$T_N + dT_N = T_N + \sum_{i=1}^{N} (A_1 * \ldots * A_{i-1} * dA_i * A_{i+1} * \ldots * A_N) \, . \tag{37}$$

Due to dAi = Ai * δAi in EQ(22), EQ(37) can be rewritten as

$$dT_N = \sum_{i=1}^{N} (A_1 * \ldots * A_i * \delta A_i * A_{i+1} * \ldots * A_N) \tag{38}$$

$$= \sum_{i=1}^{N} [T_N * (A_{i+1} * \ldots * A_N) * \delta A_i * (A_{i+1} * \ldots * A_N)] . \tag{39}$$

Using the definition of U matrix in EQ(4), EQ(39) become

$$dT_N = T_N * [\, \sum_{i=1}^{N} (U_{i+1}^{-1} * \delta A_i * U_{i+1})\,] . \tag{40}$$

By defining an error matrix transform δT_N with respect to T_N and

$$dT_N = T_N * \delta T_N , \tag{41}$$

then from EQ(40) that

$$\delta T_N = \sum_{i=1}^{N} (\, U_{i+1}^{-1} * \delta A_i * U_{i+1} \,) . \tag{42}$$

Substituting EQ(5), EQ(24), EQ(25) and EQ(27) into EQ(7) and EQ(9), EQ(42) can be solved as following form

$$\delta T_N = \begin{bmatrix} 0 & -\delta z^N & \delta y^N & dx^N \\ \delta z^N & 0 & -\delta x^N & dy^N \\ -\delta y^N & \delta x^N & 0 & dz^N \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43}$$

and

$$\begin{bmatrix} dx^N \\ dy^N \\ dz^N \\ \delta x^N \\ \delta y^N \\ \delta z^N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} [\underline{n}_{i+1}^u \cdot \underline{d}_i^A + (\underline{p}_{i+1}^u \times \underline{n}_{i+1}^u) \cdot \underline{\delta}_i^A] \\ \sum_{i=1}^{N} [\underline{o}_{i+1}^u \cdot \underline{d}_i^A + (\underline{p}_{i+1}^u \times \underline{o}_{i+1}^u) \cdot \underline{\delta}_i^A] \\ \sum_{i=1}^{N} [\underline{a}_{i+1}^u \cdot \underline{d}_i^A + (\underline{p}_{i+1}^u \times \underline{a}_{i+1}^u) \cdot \underline{\delta}_i^A] \\ \sum_{i=1}^{N} (\underline{n}_{i+1}^u \cdot \underline{\delta}_i^A) \\ \sum_{i=1}^{N} (\underline{o}_{i+1}^u \cdot \underline{\delta}_i^A) \\ \sum_{i=1}^{N} (\underline{a}_{i+1}^u \cdot \underline{\delta}_i^A) \end{bmatrix} \tag{44}$$

where $\underline{n}^u_{i+1}$, $\underline{o}^u_{i+1}$, $\underline{a}^u_{i+1}$ and $\underline{p}^u_{i+1}$ are four 3-vectors of U_{i+1} defined in EQ(5); $\hat{d}_i$ and $\hat{\delta}_i$ are six components of δAi defined in EQ(24), EQ(25) and EQ(27).

BY substituting EQ(32) and EQ(33) into EQ(44), the six Cartesian error components at T can be solved as the linear function of the 4N kinematic errors; and

$$dx^N = \sum_{i=1}^{N}\{[(\underline{n}^u_{i+1}\cdot\underline{k}^1_i)+(\underline{p}^u_{i+1}\times\underline{n}^u_{i+1})\cdot\underline{k}^2_i]\Delta\theta_i+(\underline{n}^u_{i+1}\cdot\underline{k}^2_i)\Delta r_i$$
$$+(\underline{n}^u_{i+1}\cdot\underline{k}^3_i)\Delta\ell_i+[(\underline{p}^u_{i+1}\times\underline{n}^u_{i+1})\cdot\underline{k}^3_i]\Delta\alpha_i\} \quad (45)$$

$$dy^N = \sum_{i=1}^{N}\{[(\underline{o}^u_{i+1}\cdot\underline{k}^1_i)+(\underline{p}^u_{i+1}\times\underline{o}^u_{i+1})\cdot\underline{k}^2_i]\Delta\theta_i+(\underline{o}^u_{i+1}\cdot\underline{k}^2_i)\Delta r_i$$
$$+(\underline{o}^u_{i+1}\cdot\underline{k}^3_i)\Delta\ell_i+[(\underline{p}^u_{i+1}\times\underline{o}^u_{i+1})\cdot\underline{k}^3_i]\Delta\alpha_i\} \quad (46)$$

$$dz^N = \sum_{i=1}^{N}\{[(\underline{a}^u_{i+1}\cdot\underline{k}^1_i)+(\underline{p}^u_{i+1}\times\underline{a}^u_{i+1})\cdot\underline{k}^2_i]\Delta\theta_i+(\underline{a}^u_{i+1}\cdot\underline{k}^2_i)\Delta r_i$$
$$+(\underline{a}^u_{i+1}\cdot\underline{k}^3_i)\Delta\ell_i+[(\underline{p}^u_{i+1}\times\underline{a}^u_{i+1})\cdot\underline{k}^3_i]\Delta\alpha_i\} \quad (47)$$

$$\delta x^N = \sum_{i=1}^{N}[(\underline{n}^u_{i+1}\cdot\underline{k}^2_i)\Delta\theta_i+(\underline{n}^u_{i+1}\cdot\underline{k}^3_i)\Delta\alpha_i] \quad (48)$$

$$\delta y^N = \sum_{i=1}^{N}[(\underline{o}^u_{i+1}\cdot\underline{k}^2_i)\Delta\theta_i+(\underline{o}^u_{i+1}\cdot\underline{k}^3_i)\Delta\alpha_i] \quad (49)$$

$$\delta z^N = \sum_{i=1}^{N}[(\underline{a}^u_{i+1}\cdot\underline{k}^2_i)\Delta\theta_i+(\underline{a}^u_{i+1}\cdot\underline{k}^3_i)\Delta\alpha_i]\ . \quad (50)$$

For easy expression, above linear results can be expressed by following two equations

$$\underline{d}^N = M_1\Delta\underline{\theta} + M_2\Delta\underline{r} + M_3\Delta\underline{\ell} + M_4\Delta\underline{\alpha} \quad (51)$$

$$\underline{\delta}^N = M_2\Delta\underline{\theta} + M_3\Delta\underline{\alpha} \quad (52)$$

or by one equation

$$\begin{bmatrix}\underline{d}^N\\ \cdots\\ \underline{\delta}^N\end{bmatrix} = \begin{bmatrix}M_1\\ \cdots\\ M_2\end{bmatrix}\Delta\underline{\theta}+\begin{bmatrix}M_2\\ \cdots\\ 0\end{bmatrix}\Delta\underline{r}+\begin{bmatrix}M_3\\ \cdots\\ 0\end{bmatrix}\Delta\underline{\ell}+\begin{bmatrix}M_4\\ \cdots\\ M_3\end{bmatrix}\Delta\underline{\alpha} \quad (53)$$

where

$\underline{d}^N = [dx^N\ dy^N\ dz^N]^t$ are the three translational errors of the end of manipulator;

$\underline{\delta}^N = [\delta x^N\ \delta y^N\ \delta z^N]^t$ are the three rotational errors of the end of manipulator;

$\Delta\underline{\theta} = [\Delta\theta_1\ \ldots\ \Delta\theta_N]^t$, $\Delta\underline{r} = [\Delta r_1\ \ldots\ \Delta r_N]^t$,
$\Delta\underline{\ell} = [\Delta\ell_1\ \ldots\ \Delta\ell_N]^t$, $\Delta\underline{\alpha} = [\Delta\alpha_1\ \ldots\ \Delta\alpha_N]^t$,
where $\Delta\theta_i$, Δr_i, $\Delta\ell_i$, $\Delta\alpha_i$ are the errors in the link parameters of the ith joint and i = 1,2,....,N;

M_1, M_2, M_3 and M_4 are all 3 by N matrix whose components are the function of N joint variables, $\underline{q} = [q_1 \ldots q_i \ldots q_N]^t$ where $q_i = \theta_i$ for a revolute joint and $q_i = r_i$ for a prismatic joint. The ith column of M_1, M_2, M_3 and M_4 can be expressed as follows

$$M^i_1 = \begin{bmatrix}(\underline{n}^u_{i+1}\cdot\underline{k}^1_i)+(\underline{p}^u_{i+1}\times\underline{n}^u_{i+1})\cdot\underline{k}^2_i\\ (\underline{o}^u_{i+1}\cdot\underline{k}^1_i)+(\underline{p}^u_{i+1}\times\underline{o}^u_{i+1})\cdot\underline{k}^2_i\\ (\underline{a}^u_{i+1}\cdot\underline{k}^1_i)+(\underline{p}^u_{i+1}\times\underline{a}^u_{i+1})\cdot\underline{k}^2_i\end{bmatrix}, \quad (54)$$

$$M^i_2 = \begin{bmatrix}\underline{n}^u_{i+1}\cdot\underline{k}^2_i\\ \underline{o}^u_{i+1}\cdot\underline{k}^2_i\\ \underline{a}^u_{i+1}\cdot\underline{k}^2_i\end{bmatrix} \quad (55)$$

$$M^i_3 = \begin{bmatrix}\underline{n}^u_{i+1}\cdot\underline{k}^3_i\\ \underline{o}^u_{i+1}\cdot\underline{k}^3_i\\ \underline{a}^u_{i+1}\cdot\underline{k}^3_i\end{bmatrix}, \quad (56)$$

$$M^i_4 = \begin{bmatrix}(\underline{p}^u_{i+1}\times\underline{n}^u_{i+1})\cdot\underline{k}^3_i\\ (\underline{p}^u_{i+1}\times\underline{o}^u_{i+1})\cdot\underline{k}^3_i\\ (\underline{p}^u_{i+1}\times\underline{a}^u_{i+1})\cdot\underline{k}^3_i\end{bmatrix}, \quad (57)$$

From the results of EQ(51) and EQ(52) that all four kinds of kinematic errors will cause the Cartesian translational errors at the end of manipulator, and only two kinds of kinematic errors will cause the Cartesian rotational errors at the end of manipulator. Since the matrices M_1, M_2, M_3 and M_4 are the function of joint variables so that the six Cartesian errors will be varied for different joint positions.

The differential changes dT_N with respect to the base can be calculated by EQ(41) as

$$\begin{aligned} dT_N &= T_N \bullet \delta T_N\\ &= U_1 * \delta T_N\\ &= \begin{bmatrix} d\underline{n} & d\underline{o} & d\underline{a} & d\underline{p}\\ 0 & 0 & 0 & 0\end{bmatrix} \end{aligned} \quad (58)$$

where

$$d\underline{n} = \underline{o}^u_1\delta z^N - \underline{a}^u_1\delta y^N\ , \quad (59)$$

$$d\underline{o} = -\underline{n}^u_1\delta z^N + \underline{a}^u_1\delta x^N\ , \quad (60)$$

$$d\underline{a} = \underline{n}^u_1\delta y^N - \underline{o}^u_1\delta x^N\ , \quad (61)$$

$$d\underline{p} = \underline{n}^u_1 dx^N + \underline{o}^u_1 dy^N + \underline{a}^u_1 dz^N\ , \quad (62)$$

and $\underline{n}^u_1$, $\underline{o}^u_1$, $\underline{a}^u_1$ and $\underline{p}^u_1$ are four 3-vectors of U_1 (i.e. T_N).

Hence. the correct position and orientation at the end of manipulator due to the kinematic errors will be equal to

$$T^c_N = T_N + dT_N = T_N \bullet (\ I + \delta T_N\ )\ . \quad (63)$$

VI. CALCULATION OF POSITION AND ORIENTATION ERRORS

In the previous section, the Cartesian error model of a manipulator due to the kinematic errors, EQ(53), has been derived. Normally, the four link parameters of each joint are three fixed geomatrical values and one joint variable; hence, the kinematic errors can be seperated into two categories:

(A) Positional accuracy of a manipulator --- the Cartesian errors at the end of manipulator are due to the errors in N joint variables.

(B) Dimensional errors of a manipulator --- the cartesian errors at the end of manipulator are due to the errors in the 3N geomatrical parameters.

The error model for positional accuracy can be obtained from EQ(53) by setting 3N geomatrical parameters' errors to zero. The error model for dimensional errors can be obtained by setting the N joint variables' errors to zero. For the purpose of generality, all kinds of kinematic errors in EQ(53) will be kept for the rest of the paper.

In order to calculate the Cartesian error envelopes at the end of the manipulator, the 4N kinematic errors will be considered as random variables. A reasonable assumption is that $\Delta\underline{\theta}$, $\Delta\underline{r}$, $\Delta\underline{\ell}$ and $\Delta\underline{\alpha}$ are four independent, N-variables, zero-mean normal distributions with following properties

$E[\Delta\underline{\theta}] = E[\Delta\underline{r}] = E[\Delta\underline{\ell}] = E[\Delta\underline{\alpha}] = \underline{0}$, where $E[\cdot]$ represents expect values;

V_θ = Variance of $\Delta\underline{\theta}$ = a N by N diagonal matrix with components $(\sigma^2_{\theta 1},\ldots\ldots,\sigma^2_{\theta N})$, where $\sigma_{\theta i}$ = standard deviation of $\Delta\theta_i$;

V_r = Variance of $\Delta\underline{r}$ = a N by N diagonal matrix with components $(\sigma^2_{r1},\ldots\ldots,\sigma^2_{rN})$, where σ_{ri} = standard deviation of Δr_i ;

V_ℓ = Variance of $\Delta\underline{\ell}$ = a N by N diagonal matrix with components $(\sigma^2_{\ell 1},\ldots\ldots,\sigma^2_{\ell N})$, where $\sigma_{\ell i}$ = standard deviation of $\Delta\ell_i$;

V_α = Variance of $\Delta\underline{\alpha}$ = a N by N diagonal matrix with components $(\sigma^2_{\alpha 1},\ldots\ldots,\sigma^2_{\alpha N})$, where $\sigma_{\alpha i}$ = standard deviation of $\Delta\alpha_i$;

and all the covariances between these four random vectors are zero.

Due to the property of normal distribution and the relations in EQ(51) and EQ(52), $\underline{d}^N$ and $\underline{\delta}^N$ are also normal distributions with mean

$$E[\underline{d}^N] = M_1E[\Delta\underline{\theta}]+M_2E[\Delta\underline{r}]+M_3E[\Delta\underline{\ell}]+M_4E[\Delta\underline{\alpha}] = \underline{0} \qquad (64)$$

$$E[\underline{\delta}^N] = M_2E[\Delta\underline{\theta}] + M_3E[\Delta\underline{\alpha}] = \underline{0} \qquad (65)$$

and variance

$$V_d = E[(\underline{d}^N - E[\underline{d}^N])(\underline{d}^N - E[\underline{d}^N])^t] = M_1V_\theta M_1^t + M_2V_rM_2^t + M_3V_\ell M_3^t + M_4V_\alpha M_4^t \quad ,(66)$$

$$V_\delta = E[(\underline{\delta}^N - E[\underline{\delta}^N])(\underline{\delta}^N - E[\underline{\delta}^N])^t] = M_2V_\theta M_2^t + M_3V_\alpha M_3^t \qquad (67)$$

where V_d and V_δ are 3 by 3 matrix and whose components are functions of the joint variables.

The trivariable normal density function of $\underline{d}^N$ and $\underline{\delta}^N$ are of the form

$$f(dx^N,dy^N,dz^N) = (2\pi)^{-\frac{3}{2}}|V_d|^{-\frac{1}{2}}\exp\{-.5[(\underline{d}^N)^t|V_d|^{-1}(\underline{d}^N)]\} \qquad (68)$$

and

$$f(\delta x^N,\delta y^N,\delta z^N) = (2\pi)^{-\frac{3}{2}}|V_\delta|^{-\frac{1}{2}}\exp\{-.5[(\underline{\delta}^N)^t|V_\delta|^{-1}(\underline{\delta}^N)]\}. \qquad (69)$$

By knowing the error standard deviation of each link parameters, the Cartesian error envelopes at the end of manipulator can be easily obtained from EQ(66) and EQ(67). If the error envelopes with respect to the base is desired then they can be obtained from EQ(59) to EQ(62) by using the property of EQ(68) and EQ(69).

The error envelopes of three independent translational Cartesian errors and three independent rotational Cartesian errors can also be obtained by rotating the axes of V_d and V_δ into their eigenvectors, i.e. their prinaipal axes. After this transformation, six independent, zero-mean, normal random variables $\{v_i;\ i=1,\ldots,6\}$ with standard deviation $\{\sigma_i;\ i=1,\ldots,6\}$ can be obtained. v_1, v_2 and v_3 are on the princlpal axes of V_d. v_4, v_5 and v_6 are on the principal axes of V_δ. The density function of v_i is of the form

$$f(v_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i}\exp\{-v_i^2 / 2\sigma_i^2\} \qquad (70)$$

and σ_i is the function of the joint variables.

The probability of $|v_i| < Ri$ is that

$$\text{Prob}(|v_i| < Ri) = \int_{-R_i}^{R_i} f(v_i)\,dv_i = 2\,\text{erf}(Ri/\sigma_i) \qquad (71)$$

where erf(·) is the error function of normal distribution and which is defined as

$$\text{erf}(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp(-y^2/2)\,dy\ . \qquad (72)$$

By giving the upper bound of error probability that $\text{Prob}(|v_i|<Ri) \leq Ci$, the lower bound of the standard deviation σ_i can be obtained by using the table of the error function erf(·) and

$$\sigma_i \geq Ri/Bi \qquad (73)$$

where Bi is the constant value in the table of erf(·) which satisfied EQ(71). In order to within the Cartesian error bound Ci, the joint trajectory at any time must satisfy EQ(73).

If the envelopes of the Cartesian error volumes were the condition to be satisfied then following two new random variables

$$w_1 = +(v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}} \qquad (74)$$

$$w_2 = +(v_4^2 + v_5^2 + v_6^2)^{\frac{1}{2}} \qquad (75)$$

will be the test variables for the error bounds.

VII. DESIGN OF A ROBOT MANIPULATOR

In order to preserve the fidelity of a robot manipulator in the real world, the design of the kinematic parameters has to be optimized. In this section, we will discuss how to minimize the Cartesian errors of an open-loop manipulator.

From the results of EQ(26) that if the link parameter $\ell_i = 0$ for ith joint then the translational errors $\underline{d}_i^A$ of the joint coordinates can be reduced to two terms. Three Cartesian translational errors $\underline{d}^N$ can also be reduced from the results of EQ(44).

This result concludes that a manipulator with no link length offset will be more accurate.

Observed from the results of EQ(45) to EQ(47) that three Cartesian translational errors d^N were dominated by the errors in parameters $\underline{\theta}$ and $\underline{\alpha}$ due to the error terms consisted of the position vector $\underline{p}^N_{i+1}$ of the U_{i+1} matrix. As to the Cartesian rotational errors, which only effected by the errors in $\underline{\theta}$ and $\underline{\alpha}$. Thus, if the precision of the parameters and were very high then the Cartesian errors of the open-loop manipulator can be reduced to minimum.

In order to decide the manufactural error tolerences of the designed kinematic parameters such that the Cartesian errors will within the error bound, our error model can be applied as a CAD tool. In the previous section, the Cartesian error envelopes have been derived, EQ(66) and EQ(67), which depend on the value and the error tolerence, i.e. the error standard deviation, of the kinematic parameters. By setting the upperbound of the error envelopes for the designed manipulator, different sets of manufactural error tolerences can be tested and the the error envelopes of the whole working space of the manipulator can be generated. By this procedure, the maximum manufactural tolerences of the kinematic parameters can be obtained. If the manufacturing process cannot meet such standards then the kinematic parameters of the manipulator need to be carefully calibrated and which will be discussed in the next section.

VIII. CALIBRATION FOR KINEMATIC ERRORS

When there are manufactural errors in the kinematic parameters, the calibration of the manipulator is necessary such that the kinematic errors can be minimized. Although the Cartesian errors depend on four kinds of kinematic errors, they were dominated by the errors in $\underline{\theta}$ and $\underline{\alpha}$ as described in previous section. In addtion that the manufactural accuracy of the translational parameters $\underline{r}$ and $\underline{\ell}$ are muah higher than the angular parameters $\underline{\theta}$ and $\underline{\alpha}$. Thus EQ(53) can be estimated as follows

$$\begin{bmatrix} \underline{d}^N \\ \underline{\delta}^N \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta\underline{\theta} + \begin{bmatrix} M_4 \\ M_3 \end{bmatrix} \Delta\underline{\alpha} \tag{76}$$

The calibration of the joint variables can always be achieved by designing some precision points on each joint, then from EQ(76) that the Cartesian errors are dominated by N+K kinematic errors, where K is the number of the prismatic joint. Unfortunately, most of the existing manipulator donot have such precision point on each joint; hence, from EQ(76) that there are 2N dominant kinematic errors. For generality, a calibration scheme will be presented to calibrate these 2N kinematic parameters.

First, the joint position of the manipulator can be roughly calibrated by moving the joints to their estimated zero position; then the actual Cartesian position T_N of the manipulator can be calculated from its joint positions and 3N geomatrical parameters. By moving the manipulator to a known Cartesian position T^c_N in the known rel world, one group of the six Cartesian errors can be obtained by comparing T_N and T^c_N in EQ(63). M_1, M_2, M_3 and M_4 in EQ(76) can also be calculated from joint positions and 3N geomatrical parameters as in EQ(54) to EQ(57). For one precise Cartesian position T^c_N, six equations for 2N error variables will be obtained. By moving the manipulator to G precise Cartesian positions and which satisfy $2N \leq 6G$, the 2N error variables $\Delta\underline{\theta}$ and $\Delta\underline{\alpha}$ can be solved analytically. Then the new accurate values of $\underline{\theta}$ and $\underline{\alpha}$ can be obtained by adding these solved errors to the old values of $\underline{\theta}$ and $\underline{\alpha}$. After repeat above procedure several times, the kinematic errors will converge to zero. The final values of $\underline{\theta}$ and $\underline{\alpha}$ will be the accurate calibrated values of the kinematic parameters $\underline{\theta}$ and $\underline{\alpha}$.

Above calibration technique has assumed that the errors in the parameters $\underline{\ell}$ and $\underline{r}$ are negligible due to their insignificant effects. However, all 4N kinematic parameters can be calibrated by the same technique except using EQ(53) instead of EQ(76) and the condition is $4N \leq 6G$.

IX. CONCLUSION

In this paper, a simple linear Cartesian error model of the kinematic errors of robot manipulator has been developed in a straight forward manner from the kinematic equations. From this model, the Cartesian error envelopes for any combination of the four kinds of kinematic errors can be generated. In addition that this model can be used as a CAD tool to minimize the Cartesian errors of an open-loop manipulator. Finally, a calibration technique based on this error model has also been developed which can correct the kinematic errors of a robot manipulator.

X. ACKNOWLEDGEMENT

The author wish to thank Dr. R. Paul and Mr. R. Casler for suggestions and encouragement after reading the original manuscript.

REFERENCES

[1] Denavit,J. and R. S. Hartenberg," A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices," ASME Journal of Applied Mechanics, pp215-221,June 1955.

[2] Paul,R.,Shimano,B. and G. Mayer,"Kinematic Control Equations for Simple Manipulator,"IEEE Trans. on Syst.,Man, and Cybernetics, pp 449-455, June 1981.

[3] Waldron,K.J.," Positioning Accuracy of Manipulators,"Proceeding of N.S.F., Held at University of Florida,pp 111-141, Feb. 1978.

[4] Wu,C.H.,"Force and Position Control of Robot Manipulator," Ph.D. Thesis, Purdue University, Dec. 1980.

[5] Paul, R.,"Robot Manipulators:Mathematics, Programming, and Control," The MIT Press, 1981.