Lecture 15: Section 3.4

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In this section we will use **parametric and vector methods** to study general systems of linear equations. This work will enable us to interpret **solution sets of linear systems** with n unknowns as **geometric objects in** $\mathbb{R}^n$**.**

Vector and Parametric equations of lines in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem. Let L be the line in $\mathbb{R}^2$ or $\mathbb{R}^3$ that contains the point x_0 and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through x_0 that is parallel to $\mathbf{v}$ is

$$\mathbf{x} = \mathbf{x_0} + t\mathbf{v}. \tag{1}$$

If $x_0 = \mathbf{0}$, then the line passing through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v}.$$

Example.

Let $x_0 = (1, 2, 3)$ and $\mathbf{v} = (1, 1, 1)$. Then the equation passing through x_0 and $\mathbf{v}$ is

$$x = x_0 + t\mathbf{v},$$

that is to say,

$$x = (1, 2, 3) + t(1, 1, 1).$$

If writing $x = (x_1, x_2, x_3)$, we have

$$x_1 = 1 + t,\ x_2 = 2 + t,\ x_3 = 3 + t.$$

This is the parametric form of the line L.

Example.

Find a vector equation and parametric equations of the line in $\mathbb{R}^2$ that passes through the origin and is parallel to the vector $\mathbf{v} = (2, 3)$.

Solution. Since the line passes through the origin, $x_0 = \mathbf{0}$, and the equation can be expressible in the vector from

$$(x, y) = t(-2, 3).$$

In the parametric from,

$$x = -2t,\ y = 3t.$$

Find a vector equation and parametric equations of the line in $\mathbb{R}^3$ that passes through the point $P_0(1, 2, -3)$ and is parallel to the vector $\mathbf{v} = (4, -5, 1)$.

Solution. In the vector form,

$$x = x_0 = t\mathbf{v} = (1, 2, -3) + t(4, -5, 1).$$

In the parametric form,

$$\begin{aligned} x &= 1 + 4t, \\ y &= 2 - 5t, \\ z &= -3 + t. \end{aligned}$$

Vector and Parametric equations of planes in $\mathbb{R}^3$.

Let $\mathbf{W}$ be the plane in $\mathbb{R}^3$ that contains the point x_0 and is parallel to the non-collinear vectors $\mathbf{v}_1$ and $\mathbf{v_2}$. Then an equation of the plane through x_0 that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v_1} + t_2\mathbf{v_2}. \tag{2}$$

If $x_0 = 0$, then the plane passes through the origin and the equation has the form

$$x = t_1\mathbf{v_1} + t_2\mathbf{v_2}.$$

Example.

Let $x_0 = (2, 1, 3)$, $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (0, 1, 1)$. Find the plane that passes through x_0 and is parallel to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution. The vectors $\mathbf{v_1}, \mathbf{v}_2$ are not collinear because both of them are starting from the origin, and one of them is not a multiple of the other.

By using the theorem above,

$$\begin{aligned} x &= x_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \\ &= (2, 1, 3) + t_1(1, 0, 1) + t_2(0, 1, 1) \\ &= (2 + t_1, 1 + 2t_2, 3 + t_1 + t_2). \end{aligned}$$

Example.

If writing $x = (x_1, x_2, x_3)$ as any point in $\mathbb{R}^3$, then we have the following parametric form

$$\begin{aligned} x_1 &= 2 + t_1, \\ x_2 &= 1 + 2t_2, \\ x_3 &= 3 + t_1 + t_2. \end{aligned}$$

Example.

Find the vector and parametric equations of the plane in $\mathbb{R}^4$ that passes through the point $x_0 = (2, -1, 0, 3)$ and is parallel to both $\mathbf{v}_1 = (1, 5, 2, -4)$ and $\mathbf{v}_2 = (0, 7, -8, 6)$.

Solution. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear. In the vector form,

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

In the parametric form,

$$\begin{aligned} x_1 &= 2 + t_1, \\ x_2 &= -1 + 5t_1 + 7t_2, \\ x_3 &= 2t_1 - 8t_2, \\ x_4 &= 3 - 4t_1 + 6t_2. \end{aligned}$$

Lines through two points in $\mathbb{R}^2$.

Find vector and parametric equations for the line in $\mathbb{R}^2$ that passes through the points $P(0,7)$ and $Q(5,0)$.

Solution. The two points form a vector in $\mathbb{R}^2$:

$$\mathbf{v} = (5,0) - (0,7) = (5,-7).$$

Thus in the vector form

$$\mathbf{x} = (0,7) + t\mathbf{v} = (0,7) + (5t,-7t) = (5t, 7-7t).$$

In the parametric form

$$\begin{aligned} x_1 &= 5t, \\ x_2 &= 7-7t. \end{aligned}$$

Line Segment.

Definition. If $\mathbf{x}_0$ and $\mathbf{x}_1$ are vectors in $\mathbb{R}^n$, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0),\ (0 \leq t \leq 1)$$

defines **the line segment** from $\mathbf{x}_0$ to $\mathbf{x}_1$. When convenient, this equation can be also written as

$$\mathbf{x} = (1-t)\mathbf{x}_0 + t\mathbf{x}_1,\ 0 \leq t \leq 1.$$

Line Segment from one point to another point in $\mathbb{R}^2$.

Find the equation of the line segment in $\mathbb{R}^2$ from $\mathbf{x}_0 = (1, -3)$ and $\mathbf{x}_1 = (5, 6)$.

Solution. The line segment, for $0 \leq t \leq 1$,

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \\ &= (1, -3) + t(4, 9). \end{aligned}$$

or it can be written as

$$(1 - t)\mathbf{x}_0 + t\mathbf{x}_1 = (1 - t)(1, -3) + t(5, 6),\ 0 \leq t \leq 1.$$

Dot product form of a linear system.

A linear equation in the variables $x_1, x_2, \cdots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

for the coefficients $a_1, a_2, \cdots, a_n$ not all zero. This equation can be written as

$$\mathbf{a} \cdot \mathbf{x} = b.$$

The corresponding homogeneous equation is

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0,$$

for the coefficients $a_1, a_2, \cdots, a_n$ not all zero. Similarly

$$\mathbf{a} \cdot \mathbf{x} = 0.$$

Dot product form of a linear system.

Consider the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = 0, \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = 0. \end{cases}$$

If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1$, $\mathbf{r}_2, \cdots, \mathbf{r}_m$, then we can rewrite this system in dot product as

$$\begin{cases} \mathbf{r_1} \cdot x & = 0, \\ \mathbf{r_2} \cdot x & = 0, \\ \vdots & \vdots \\ \mathbf{r}_m \cdot x & = 0. \end{cases}$$

This shows that every solution vector $\mathbf{x}$ is orthogonal to every row vector of the coefficients matrix.

Conclusion. If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}$$

consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.

Orthogonality of row vectors and solution vectors.

Consider the homogeneous linear system

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 0.$$

In the vector from,

$$\mathbf{x} = (-3r - 4s - 2t, r, -2s, s, t, 0).$$

This solution is orthogonal to every row vector in the matrix A.

The relationship between $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.

Theorem. The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.

Proof.

Proof. Let $\mathbf{x}_0$ be any specific solution of $A\mathbf{x} = \mathbf{b}$. Let W be the solution set of $A\mathbf{x} = \mathbf{0}$ and V be the solution set of $A\mathbf{x} = \mathbf{b}$. We first prove that $x_0 + \mathbf{W} \subset \mathbf{V}$.

For any $w \in W$, we show that $x_0 + w$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$:

$$A\left(\mathbf{x}_0 + w\right) = A\mathbf{x}_0 + Aw = \mathbf{0} + \mathbf{b} = \mathbf{b}.$$

Thus $\mathbf{x}_0 + w$ is a solution to the inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$.

Secondly we prove that $\mathbf{V} \subset x_0 + \mathbf{W}$. On the other hand, let $y \in \mathbf{V}$. It is known that $\mathbf{x}_0 \in \mathbf{V}$. Then

$$A(\mathbf{y} - \mathbf{x}_0) = A\mathbf{y} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

which shows that $\mathbf{y} - \mathbf{x}_0 \in \mathbf{W}$, i.e., $y \in x_0 + \mathbf{W}$.

To conclude the proof, we have

$$\mathbf{V} = x_0 + \mathbf{W}.$$

Homework and Reading.

Homework. Exercise. #2, #4, #6, #8,#10,#12,#15,#25. True or false questions on page 160.

Reading. Section 4.1.