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# A Kinematic CAD Tool for the Design and Control of a Robot Manipulator

## Abstract

*The correct relationship between two connective joint coordinates of a robot manipulator is defined by four link parameters; one being the joint variable and the other three the geometric values. The basis for all open-loop manipulator control is the relationship between the Cartesian coordinates of the end-effector and the joint coordinates; therefore, the fidelity of the Cartesian position and orientation of the end-effector to the real world depend on the accuracy of the four link parameters of each joint. In this paper, a linear analytic error model describes the six possible Cartesian errors and the four independent kinematic errors from which the Cartesian error envelopes due to any combination of four kinds of kinematic errors can be uniquely determined. From a design standpoint, this error model can be used as a guide to minimize the open-loop kinematic errors of the robot manipulator. Finally, a new calibration technique based on this model has also been developed that can be used to correct the kinematic errors of the robot manipulator.*

## 1. Introduction

As we all know, a robot manipulator is a position-oriented mechanical device. The accuracy of the manipulator's position in the real world depends on the accuracy of the manipulator's kinematics. In order to optimize the accuracy of a robot manipulator, a computer-aided design (CAD) tool for kinematic design has to be built such that the designed manipulator, within its working space in the real world, can maintain the minimum tolerant errors of Cartesian position and orientation. The purpose of this paper is to build such a CAD tool by mathematically formulating the relationship between the kinematic errors and the Cartesian errors of the robot manipulator.

Now let us review how the kinematics of a robot manipulator will affect the Cartesian position and orientation errors. A serial-link manipulator consists of a sequence of mechanical links connected by actuated joints. The relationship between two connective joint coordinates is well defined by a homogeneous transformation matrix (Denavit and Hartenberg 1955) where this matrix is determined by four kinds of link parameters, also called kinematic parameters; one is a joint variable and the others are geometric parameters. At present, all the robot manipulators have open-loop linkage control. The control basis is a relationship between the Cartesian coordinates of the end-effector and the joint coodinates, such that the fidelity of the Cartesian position and orientation to the real world depends on the accuracy of the four link parameters of each joint. Because of the nature of these kinematic parameters, the Cartesian errors can be grouped into two categories: (1) the Cartesian errors due to the position accuracy of the joint variables and (2) the Cartesian errors due to the dimensional errors of the other kinematic parameters. Waldron (1978) has developed a model in a general form for the first category, but he has only described the general case, without any mathematical formulation, for the second category. Kumar and Waldron (1981) have written a computer program, based on the model developed by Waldron (1978), that can generate and plot the surfaces of the positioning accuracy of the manipulator; however, their paper only considers the errors of the first category by assuming that the dimensions of the links are precisely known.

In this paper, an explicit mathematical-error model for the above two categories is developed. By using the error transformation between two coordinates and

Work reported in this paper was done while the author was at Unimation Inc., Shelter Rock Lane, Danbury, CT 06810.

The International Journal of Robotics Research,
Vol. 3, No. 1, Spring 1984,
0278-3649/84/010058-10 $05.00/0,
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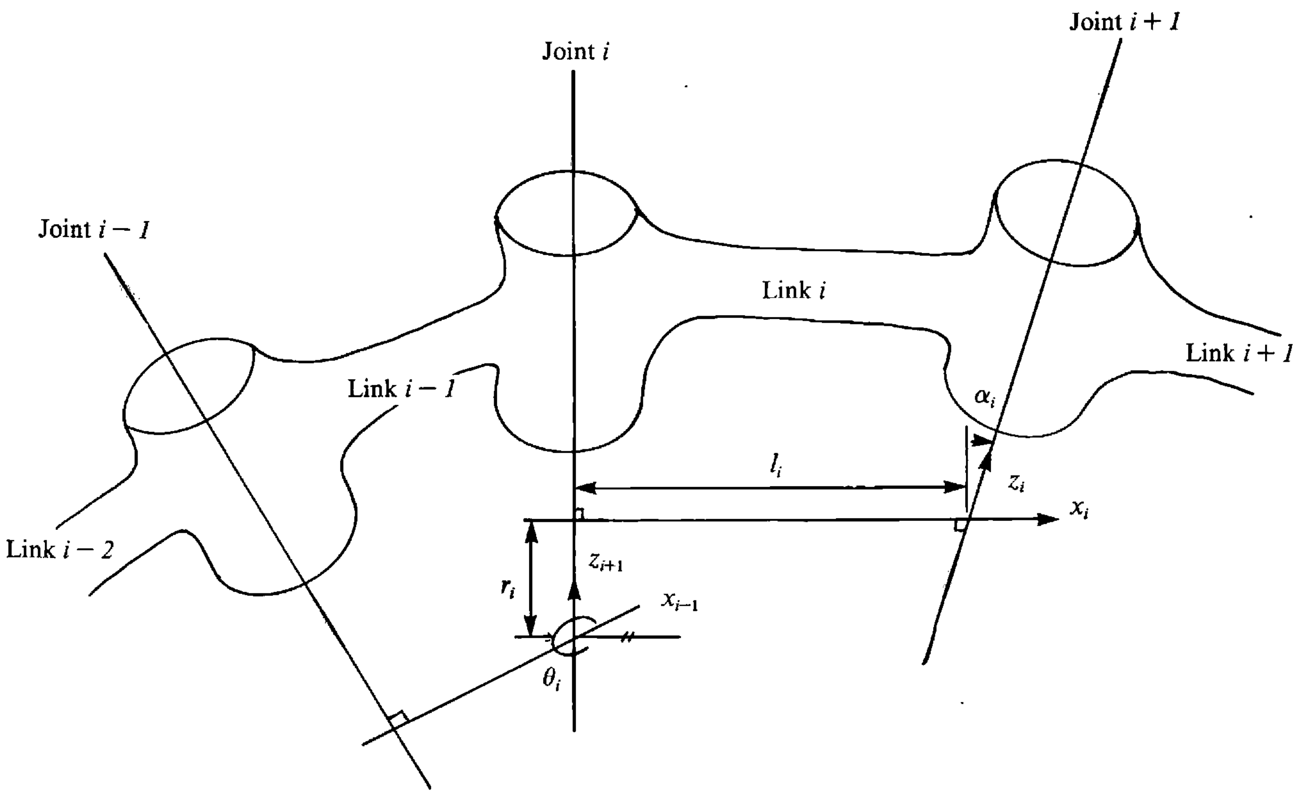
*Fig. 1. Link coordinates and parameters $\theta_i$, $r_i$, $l_i$, and $\alpha_i$.*




ignoring the higher-order terms, the linear error model between the Cartesian errors and the four independent kinematic errors is formulated. For any kind of manipulator, the Cartesian error envelopes caused by any combination of the four kinds of kinematic errors can be easily generated from the developed error model. In addition, this error model can be used as a design guide to minimize the Cartesian errors caused by the kinematic parameters. In addition, a calibration technique has been developed based on this model that can correct the kinematic errors of the robot manipulator.

## 2. Kinematics

For an $N$ degree-of-freedom manipulator, there will be $N$ links and $N$ joints. The relationship between the joint-coordinate frames $i-1$ and $i$ can be represented by a homogeneous transformation matrix $\mathbf{A}_i$ (Denavit and Hartenberg 1955; Paul, Shimano, and Mayer 1981), and

$$\mathbf{A}_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & l_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & l_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & r_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (1)$$

where S and C refer to sine and cosine functions, and $\theta_i$, $r_i$, $l_i$, $\alpha_i$ are the link (or kinematic) parameters of the $i$th joint defined by Denavit and Hartenberg (1955). These parameters are shown in Fig. 1; for a prismatic joint, $l_i = 0$. For convenience, $\mathbf{A}_i$ can be represented by four 3-by-1 vectors as follows.

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{n}_i & \mathbf{o}_i & \mathbf{a}_i & \mathbf{p}_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2)$$

where $\mathbf{n}_i$, $\mathbf{o}_i$, $\mathbf{a}_i$ and $\mathbf{p}_i$ are four 3-by-1 vectors. With the homogeneous transformation matrix $\mathbf{A}_i$, the end of an $N$ degree-of-freedom manipulator can be represented as

$$\mathbf{T}_N = \mathbf{A}_1 * \mathbf{A}_2 * \cdots\cdots\cdots * \mathbf{A}_{N-1} * \mathbf{A}_N, \qquad (3)$$

where $*$ is the matrix multiplication.

Paul, Shimano, and Mayer (1981) have defined a very useful homogeneous transformation matrix $\mathbf{U}_i$, which describes the motion of the end-effector with respect to joint-coordinate frame $i-1$; and

$$\mathbf{U}_i = \mathbf{A}_i * \mathbf{A}_{i+1} * \cdots\cdots * \mathbf{A}_N. \qquad (4)$$

Based on the above definition, $\mathbf{U}_i = \mathbf{T}_N$ and $\mathbf{U}_{N+1} = \mathbf{I}$, the identity matrix.

The matrix $\mathbf{U}_i$ can also be represented by the form

$$\mathbf{U}_i = \begin{bmatrix} \mathbf{u}_i^u & \mathbf{o}_i^u & \mathbf{a}_i^u & \mathbf{p}_i^u \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (5)$$

where $\mathbf{n}_i^u$, $\mathbf{o}_i^u$, $\mathbf{a}_i^u$ and $\mathbf{p}_i^u$ are four 3-by-1 vectors.

## 3. Differential changes between two Coordinate Frames

Given a small change $\delta\mathbf{T}_1$ in position and orientation in coordinate frame 1, there will be a corresponding small change $\delta\mathbf{T}_2$ in coordinate frame 2. If the relationship between two coordinate frames is $\mathbf{T}_1^2$, then the above relationship can be represented as

$$\delta\mathbf{T}_1 * \mathbf{T}_1^2 = \mathbf{T}_1^2 * \delta\mathbf{T}_2 \qquad (6)$$

and

$$\delta\mathbf{T}_2 = \mathbf{T}_1^{2^{-1}} * \delta\mathbf{T}_1 * \mathbf{T}_1^2. \qquad (7)$$

The differential error matrix $\delta\mathbf{T}_i$, $i = 1,2$, can be represented in the following form (Paul 1981) by ignoring the higher-order terms

$$\delta\mathbf{T}_i = \begin{bmatrix} 0 & -\delta z_i & \delta y_i & dx_i \\ \delta z_i & 0 & -\delta x_i & dy_i \\ -\delta y_i & \delta x_i & 0 & dz_i \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (8)$$

where $\mathbf{d}_i = [dx_i \quad dy_i \quad dz_i]^t$ is the small translational changes, and $\boldsymbol{\delta}_i = [\delta x_i \quad \delta y_i \quad \delta z_i]^t$ is the small rotational changes. The superscript $t$ represents the matrix transpose.

If $\mathbf{T}_1^2$ and $\delta\mathbf{T}_1$ are known, the components of $\delta\mathbf{T}_2$ can be solved analytically in the following form (Wu 1980; Paul 1981).

$$\begin{bmatrix} dx_2 \\ dy_2 \\ dz_2 \\ \delta x_2 \\ \delta y_2 \\ \delta z_2 \end{bmatrix} = \begin{bmatrix} \mathbf{n}\cdot\mathbf{d}_1 + (\mathbf{p}\times\mathbf{n})\cdot\boldsymbol{\delta}_1 \\ \mathbf{o}\cdot\mathbf{d}_1 + (\mathbf{p}\times\mathbf{o})\cdot\boldsymbol{\delta}_1 \\ \mathbf{a}\cdot\mathbf{d}_1 + (\mathbf{p}\times\mathbf{a})\cdot\boldsymbol{\delta}_1 \\ \mathbf{n}\cdot\boldsymbol{\delta}_1 \\ \mathbf{o}\cdot\boldsymbol{\delta}_1 \\ \mathbf{a}\cdot\boldsymbol{\delta}_1 \end{bmatrix}, \qquad (9)$$

where $\mathbf{n}$, $\mathbf{o}$, $\mathbf{a}$, and $\mathbf{p}$ are four 3-by-1 vectors of $\mathbf{T}_1^2$.

## 4. Differential Changes Due to the Kinematic Errors

From (Eq. 1), the correct relationship $\mathbf{A}_i$ between joint coordinates $i$ and $i-1$ is determined by its four link parameters $\theta_i$, $r_i$, $l_i$, and $d_i$. For a revolute joint, $\theta_i$ is the joint variable, and the others are fixed dimensional values. For a prismatic joint, $r_i$ is the joint variable, and $l_i = 0$ and the other two are the fixed dimensional values. If there are errors in these link parameters, there will be a differential change $\mathbf{dA}_i$ between the two joint coordinates. Thus the accurate relationship between the two joint coordinates will be equal to $\mathbf{A}_i + \mathbf{dA}_i$. The differential change $\mathbf{dA}_i$ can be estimated by the following linear form

$$\mathbf{dA}_i = \frac{\partial\mathbf{A}_i}{\partial\theta_i}\Delta\theta_i + \frac{\partial\mathbf{A}_i}{\partial r_i}\Delta r_i + \frac{\partial\mathbf{A}_i}{\partial l_i}\Delta l_i + \frac{\partial\mathbf{A}_i}{\partial\alpha_i}\Delta\alpha_i, \qquad (10)$$

where $\Delta\theta_i$, $\Delta r_i$, $\Delta l_i$, and $\Delta\alpha_i$ are the small error changes in the link parameters.

From (Eq. 1),

$$\frac{\partial\mathbf{A}_i}{\partial\theta_i} = \begin{bmatrix} -S\theta_i & -C\theta_i C\alpha_i & C\theta_i S\alpha_i & -l_i S\theta_i \\ C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & l_i C\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (11)$$

$$= \mathbf{Q}'_\theta * \mathbf{A}_i,$$

where

$$\mathbf{Q}'_\theta = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

By representing

$$\frac{\partial \mathbf{A}_i}{\partial \theta_i} = \mathbf{A}_i * \mathbf{Q}_\theta, \quad (13)$$

$\mathbf{Q}_\theta$ can be solved as

$$\mathbf{Q}_\theta = \mathbf{A}_i^{-1} * \mathbf{Q}'_\theta * \mathbf{A}_i$$
$$= \begin{bmatrix} 0 & -C\alpha_i & S\alpha_i & 0 \\ C\alpha_i & 0 & 0 & l_i C\alpha_i \\ -S\alpha_i & 0 & 0 & -l_i S\alpha_i \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (14)$$

Using the same technique as above, the following results are obtained:

$$\frac{\partial \mathbf{A}_i}{\partial r_i} = \mathbf{A}_i * \mathbf{Q}_r, \quad (15)$$

$$\frac{\partial \mathbf{A}_i}{\partial l_i} = \mathbf{A}_i * \mathbf{Q}_l, \quad (16)$$

and

$$\frac{\partial \mathbf{A}_i}{\partial \alpha_i} = \mathbf{A}_i * \mathbf{Q}_\alpha, \quad (17)$$

where

$$\mathbf{Q}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S\alpha_i \\ 0 & 0 & 0 & C\alpha_i \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (18)$$

$$\mathbf{Q}_l = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (19)$$

and

$$\mathbf{Q}_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (20)$$

Based on the above results, (Eq. 10) can be rewritten as

$$\mathbf{dA}_i = \mathbf{A}_i * (\mathbf{Q}_\theta \Delta\theta_i + \mathbf{Q}_r \Delta r_i + \mathbf{Q}_l \Delta l_i + \mathbf{Q}_\alpha \Delta\alpha_i) \quad (21)$$

If an error-matrix transform $\delta\mathbf{A}_i$ is defined with respect to $\mathbf{A}_i$, and

$$\mathbf{dA}_i = \mathbf{A}_i * \delta\mathbf{A}_i, \quad (22)$$

then

$$\delta\mathbf{A}_i = \mathbf{Q}_\theta \Delta\theta_i + \mathbf{Q}_r \Delta r_i + \mathbf{Q}_l \Delta l_i + \mathbf{Q}_\alpha \Delta\alpha_i. \quad (23)$$

Mathematically, $\delta\mathbf{A}_i$ can be solved as

$$\delta\mathbf{A}_i = \begin{bmatrix} 0 & -C\alpha_i\Delta\theta_i & S\alpha_i\Delta\theta_i & \Delta l_i \\ C\alpha_i\Delta\theta_i & 0 & -\Delta\alpha_i & l_i C\alpha_i\Delta\theta_i + S\alpha_i\Delta r_i \\ -S\alpha_i\Delta\theta_i & \Delta\alpha_i & 0 & -l_i S\alpha_i\Delta\theta_i + C\alpha_i\Delta r_i \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

As a result, $\delta\mathbf{A}_i$ has the same form as (Eq. 8), with components

$$\mathbf{d}_i^A = \begin{bmatrix} \Delta l_i \\ l_i C\alpha_i\Delta\theta_i + S\alpha_i\Delta r_i \\ -l_i S\alpha_i\Delta\theta_i + C\alpha_i\Delta r_i \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} 0 \\ l_i C\alpha_i \\ -l_i S\alpha_i \end{bmatrix} \Delta\theta_i + \begin{bmatrix} 0 \\ S\alpha_i \\ C\alpha_i \end{bmatrix} \Delta r_i + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Delta l_i \quad (26)$$

and

$$\delta_i^A = \begin{bmatrix} \Delta\alpha_i \\ S\alpha_i\Delta\theta_i \\ C\alpha_i\Delta\theta_i \end{bmatrix} \quad (27)$$

$$= \begin{bmatrix} 0 \\ S\alpha_i \\ C\alpha_i \end{bmatrix} \Delta\theta_i + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Delta\theta_i. \quad (28)$$

By defining the following three vectors,

$$\mathbf{k}_i^1 = [0 \quad l_i C\alpha_i \quad -l_i S\alpha_i]^t, \tag{29}$$

$$\mathbf{k}_i^2 = [0 \quad s\alpha_i \quad C\alpha_i]^t, \tag{30}$$

and

$$\mathbf{k}_i^3 = [1 \quad 0 \quad 0]^t, \tag{31}$$

the translational and rotational errors at $\mathbf{A}_i$ due to the link parameters' errors can be expressed in the following linear form:

$$\mathbf{d}_i^A = \mathbf{k}_i^1 \Delta\theta_i + \mathbf{k}_i^2 \Delta r_i + \mathbf{k}_i^3 \Delta l_i, \tag{32}$$

$$\boldsymbol{\delta}_i^A = \mathbf{k}_i^2 \Delta\theta_i + \mathbf{k}_i^3 \Delta\alpha_i. \tag{33}$$

The above error expressions are the general form for any type of joint $i$. If joint $i$ is a prismatic joint, the link parameter $l_i = 0$. Thus for a prismatic joint $\mathbf{k}_i^1 = 0$ and $\Delta l_i = 0$, and (Eq. 32) can be reduced to

$$\mathbf{d}_i^A = \mathbf{k}_i^2 \Delta r_i. \tag{34}$$

After determining $\delta\mathbf{A}_i$, a new relation between joint coordinates $i$ and $i - 1$ can be expressed as (Wu 1980; Paul 1981)

$$\mathbf{A}_i + \mathbf{dA}_i = \mathbf{A}_i * (\mathbf{I} + \delta\mathbf{A}_i), \tag{35}$$

where $\mathbf{I}$ is the identity matrix.

## 5. Position and Orientation Errors of an Open-Loop Robot Manipulator

The position accuracy of an open-loop, $N$ degree-of-freedom robot manipulator in the real world depends on the accuracy of four link parameters of every joint. In the previous section, the differential change $\mathbf{dA}_i$ and the error-matrix transform $\delta\mathbf{A}_i$ at joint coordinate $i$ due to four small kinematic errors was determined. Hence, for an $N$ degree-of-freedom manipulator, the accurate position and orientation of the end-effector with respect to the base, due to the $4N$ kinematic errors, can be expressed as

$$\begin{aligned}\mathbf{T}_N + \mathbf{dT}_N &= (\mathbf{A}_1 + \mathbf{dA}_1) * (\mathbf{A}_2 + \mathbf{dA}_2) * \cdots * (\mathbf{A}_N + \mathbf{dA}_N) \\ &= \prod_{i=1}^{N} (\mathbf{A}_i + \mathbf{dA}_i),\end{aligned} \tag{36}$$

where $\mathbf{dT}_N$ represents the total differential change at the end of the manipulator due to the $4N$ kinematic errors.

By expanding (Eq. 36) and ignoring the higher-order differential changes, the following linear result is obtained:

$$\begin{aligned}&\mathbf{T}_N + \mathbf{dT}_N \\ &= \mathbf{T}_N + \sum_{i=1}^{N} (\mathbf{A}_1 * \cdots * \mathbf{A}_{i-1} * \mathbf{dA}_i * \mathbf{A}_{i+1} * \cdots * \mathbf{A}_N).\end{aligned} \tag{37}$$

Due to $\mathbf{dA}_i = \mathbf{A}_i * \delta\mathbf{A}_i$ in (Eq. 22), (Eq. 37) can be rewritten as

$$\mathbf{dT}_N = \sum_{i=1}^{N} (\mathbf{A}_1 * \cdots * \mathbf{A}_i * \delta\mathbf{A}_i * \mathbf{A}_{i+1} * \cdots * \mathbf{A}_N). \tag{38}$$

Using the definition of $\mathbf{T}_N$ in (Eq. 3), (Eq. 38) can be rewritten as

$$\mathbf{dT}_N = \sum_{i=1}^{N} [\mathbf{T}_N * (\mathbf{A}_{i+1} * \cdots * \mathbf{A}_N)^{-1} * \delta\mathbf{A}_i * (\mathbf{A}_{i+1} * \cdots * \mathbf{A}_N)]. \tag{39}$$

Using the definition of $\mathbf{U}_i$ matrix in (Eq. 4), (Eq. 39) becomes

$$\mathbf{dT}_N = \mathbf{T}_N * \left[ \sum_{i=1}^{N} (\mathbf{U}_{i+1}^{-1} * \delta\mathbf{A}_i * \mathbf{U}_{i+1}) \right]. \tag{40}$$

If an error-matrix transform $\delta\mathbf{T}_N$ is defined with respect to $\mathbf{T}_N$, and

$$\mathbf{dT}_N = \mathbf{T}_N * \delta\mathbf{T}_N, \tag{41}$$

then from (Eq. 40),

$$\delta\mathbf{T}_N = \sum_{i=1}^{N} (\mathbf{U}_{i+1}^{-1} * \delta\mathbf{A}_i * \mathbf{U}_{i+1}). \tag{42}$$

Substituting (Eqs. 5, 24, 25, and 27) into (Eqs. 7 and 9), (Eq. 42) can be solved as the following form:

$$\delta\mathbf{T}_N = \begin{bmatrix} 0 & -\delta z^N & \delta y^N & dx^N \\ \delta z^N & 0 & -\delta x^N & dy^N \\ -\delta y^N & \delta x^N & 0 & dz^N \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43}$$

and

$$\begin{bmatrix} dx^N \\ dy^N \\ dz^N \\ \delta x^N \\ \delta y^N \\ \delta z^N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} [\mathbf{n}_{i+1}^u \cdot \mathbf{d}_i^A + (\mathbf{p}_{i+1}^u \times \mathbf{n}_{i+1}^u) \cdot \boldsymbol{\delta}_i^A] \\ \sum_{i=1}^{N} [\mathbf{o}_{i+1}^u \cdot \mathbf{d}_i^A + (\mathbf{p}_{i+1}^u \times \mathbf{o}_{i+1}^u) \cdot \boldsymbol{\delta}_i^A] \\ \sum_{i=1}^{N} [\mathbf{a}_{i+1}^u \cdot \mathbf{d}_i^A + [\mathbf{p}_{i+1}^u \times \mathbf{a}_{i+1}^u) \cdot \boldsymbol{\delta}_i^A] \\ \sum_{i=1}^{N} (\mathbf{n}_{i+1}^u \cdot \boldsymbol{\delta}_i^A) \\ \sum_{i+1}^{N} (\mathbf{o}_{i+1}^u \cdot \boldsymbol{\delta}_i^A) \\ \sum_{i=1}^{N} (\mathbf{a}_{i+1}^u \cdot \boldsymbol{\delta}_i^A) \end{bmatrix}, \quad (44)$$

where $\mathbf{n}_{i+1}^u$, $\mathbf{o}_{i+1}^u$, $\mathbf{a}_{i+1}^u$ and $\mathbf{p}_{i+1}^u$ are four 3-by-1 vectors of $\mathbf{U}_{i+1}$ defined in (Eq. 5); and $\mathbf{d}_i^A$ and $\boldsymbol{\delta}_i^A$ are the six components of $\delta\mathbf{A}_i$ defined in (Eqs. 24, 25, and 27).

By substituting (Eqs. 32 and 33) into (Eq. 44), the six Cartesian error components at $\mathbf{T}_N$ can be solved as the linear function of the $4N$ kinematic errors; and

$$dx^N = \sum_{i=1}^{N} \{[(\mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^1) + (\mathbf{p}_{i+1}^u \times \mathbf{n}_{i+1}^u) \cdot \mathbf{k}_i^2]\Delta\theta_i + (\mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^2)\Delta r_i + (\mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^3)\Delta l_i + [(\mathbf{p}_{i+1}^u \times \mathbf{n}_{i+1}^u) \cdot \mathbf{k}_i^3]\Delta\alpha_i\} \quad (45)$$

$$dy^N = \sum_{i=1}^{N} \{[(\mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^1) + (\mathbf{p}_{i+1}^u \times \mathbf{o}_{i+1}^u) \cdot \mathbf{k}_i^2]\Delta\theta_i + (\mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^2)\Delta r_i + (\mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^3)\Delta l_i + [(\mathbf{p}_{i+1}^u \times \mathbf{o}_{i+1}^u) \cdot \mathbf{k}_i^3]\Delta\alpha_i\} \quad (46)$$

$$dz^N = \sum_{i=1}^{N} \{[(\mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^1) + (\mathbf{p}_{i+1}^u \times \mathbf{a}_{i+1}^u) \cdot \mathbf{k}_i^2]\Delta\theta_i + (\mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^2)\Delta r_i + (\mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^3)\Delta l_i + [(\mathbf{p}_{i+1}^u \times \mathbf{a}_{i+1}^u) \cdot \mathbf{k}_i^3]\Delta\alpha_i\} \quad (47)$$

$$\delta x^N = \sum_{i=1}^{N} [(\mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^2)\Delta\theta_i + (\mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^3)\Delta\alpha_i] \quad (48)$$

$$\delta y^N = \sum_{i=1}^{N} [(\mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^2)\Delta\theta_i + (\mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^3)\Delta\alpha_i] \quad (49)$$

$$\delta z^N = \sum_{i=1}^{N} [(\mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^2)\Delta\theta_i + (\mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^3)\Delta\alpha_i]. \quad (50)$$

The above linear results can be expressed by the following two equations,

$$\mathbf{d}^N = \mathbf{M}_1\Delta\boldsymbol{\theta} + \mathbf{M}_2\Delta\mathbf{r} + \mathbf{M}_3\Delta\mathbf{l} + \mathbf{M}_4\Delta\boldsymbol{\alpha}, \quad (51)$$

and

$$\boldsymbol{\delta}^N = \mathbf{M}_2\Delta\boldsymbol{\theta} + \mathbf{M}_3\Delta\boldsymbol{\alpha}, \quad (52)$$

or by one equation,

$$\begin{bmatrix} \mathbf{d}^N \\ \cdots \\ \boldsymbol{\delta}^N \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 \\ \cdots \\ \mathbf{M}_2 \end{bmatrix}\Delta\boldsymbol{\theta} + \begin{bmatrix} \mathbf{M}_2 \\ \cdots \\ 0 \end{bmatrix}\Delta\mathbf{r} + \begin{bmatrix} \mathbf{M}_3 \\ \cdots \\ 0 \end{bmatrix}\Delta\mathbf{l} + \begin{bmatrix} \mathbf{M}_4 \\ \cdots \\ \mathbf{M}_3 \end{bmatrix}\Delta\boldsymbol{\alpha}, \quad (53)$$

where

$\mathbf{d}^N = [dx^N \quad dy^N \quad dz^N]^t$ are the three translational errors of the end of the manipulator;
$\boldsymbol{\delta}^N = [\delta x^N \quad \delta y^N \quad \delta z^N]^t$ are the three rotational errors of the end of the manipulator;

$$\Delta\boldsymbol{\theta} = [\Delta\theta_1 \; . \; . \; . \; \Delta\theta_N]^t, \; \Delta\mathbf{r} = [\Delta r_1 \; . \; . \; . \; \Delta r_N]^t,$$

$$\Delta\mathbf{l} = [\Delta l_1 \; . \; . \; . \; \Delta l_N]^t, \; \Delta\boldsymbol{\alpha} = [\Delta\alpha_1 \; . \; . \; . \; \Delta\alpha_N]^t.$$

where $\Delta\theta_i$, $\Delta r_i$, $\Delta l_i$, $\Delta\alpha_i$ are the errors in the link parameters of the $i$th joint and $i = 1,2, \ldots, N$; and
$\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{M}_4$ are all 3-by-$N$ matrices whose components are the function of $N$ joint variables, $\mathbf{q} = [q_1 \quad q_2 \; . \; . \; . \; q_N]^t$, where $q_i = \theta_i$ for a revolute joint and $q_i = r_i$ for a prismatic joint.

The $i$th column of $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{M}_4$ can be expressed as follows:

$$\mathbf{M}_1^i = \begin{bmatrix} (\mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^1) + (\mathbf{p}_{i+1}^u \times \mathbf{n}_{i+1}^u) \cdot \mathbf{k}_i^2 \\ (\mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^1 + (\mathbf{p}_{i+1}^u \times \mathbf{o}_{i+1}^u) \cdot \mathbf{k}_i^2 \\ (\mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^1) + (\mathbf{p}_{i+1}^u \times \mathbf{a}_{i+1}^u) \cdot \mathbf{k}_i^2 \end{bmatrix}, \quad (54)$$

$$\mathbf{M}_2^i = \begin{bmatrix} \mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^2 \\ \mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^2 \\ \mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^2 \end{bmatrix}, \quad (55)$$

$$\mathbf{M}_3^i = \begin{bmatrix} \mathbf{n}_{i+1}^u \cdot \mathbf{k}_i^3 \\ \mathbf{o}_{i+1}^u \cdot \mathbf{k}_i^3 \\ \mathbf{a}_{i+1}^u \cdot \mathbf{k}_i^3 \end{bmatrix}, \quad (56)$$

and

$$\mathbf{M}_4^i = \begin{bmatrix} (\mathbf{p}_{i+1}^u \times \mathbf{n}_{i+1}^u) \cdot \mathbf{k}_i^3 \\ (\mathbf{p}_{i+1}^u \times \mathbf{o}_{i+1}^u) \cdot \mathbf{k}_i^3 \\ (\mathbf{p}_{i+1}^u \times \mathbf{a}_{i+1}^u) \cdot \mathbf{k}_i^3 \end{bmatrix}. \tag{57}$$

From the results of (Eqs. 51 and 52), it can be seen that all four kinds of kinematic errors will cause Cartesian translational errors at the end of the manipulator, and only two kinds of kinematic errors will cause Cartesian rotational errors at the end of the manipulator. Since the matrices $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{M}_4$ are the function of joint variables, the six Cartesian errors will vary for different joint positions.

The differential changes $\mathbf{dT}_N$ with respect to the base can be calculated by (Eq. 41) as

$$\begin{aligned} \mathbf{dT}_N &= \mathbf{T}_N * \delta\mathbf{T}_N \\ &= \mathbf{U}_1 * \delta\mathbf{T}_N \\ &= \begin{bmatrix} d\mathbf{n} & d\mathbf{o} & d\mathbf{a} & d\mathbf{p} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{58}$$

where

$$d\mathbf{n} = \mathbf{o}_1^u \delta z^N - \mathbf{a}_1^u \delta y^N, \tag{59}$$

$$d\mathbf{o} = -\mathbf{n}_1^u \delta z^N + \mathbf{a}_1^u \delta x^N, \tag{60}$$

$$d\mathbf{a} = \mathbf{n}_1^u \delta y^N - \mathbf{o}_1^u \delta x^N, \tag{61}$$

$$d\mathbf{p} = \mathbf{n}_1^u dx^N + \mathbf{o}_1^u dy^N + \mathbf{a}_1^u dz^N, \tag{62}$$

and $\mathbf{n}_1^u$, $\mathbf{o}_1^u$, $\mathbf{a}_1^u$, and $\mathbf{p}_1^u$ are four 3-by-1 vectors of $\mathbf{U}_1$ (i.e., $\mathbf{T}_N$).

Hence the correct position and orientation at the end of the manipulator due to the kinematic errors will be equal to

$$\mathbf{T}_N^c = \mathbf{T}_N + \mathbf{dT}_N = \mathbf{T}_N * (\mathbf{I} + \delta\mathbf{T}_N). \tag{63}$$

## 6. Calculation of Position and Orientation Errors

In the previous section, the Cartesian error model of a manipulator due to the kinematic errors (Eq. 53) was derived. Normally, the four link parameters of each joint are three fixed geometric values and one joint variable; hence, the kinematic errors can be separated into two categories:

1. The positional accuracy of a manipulator, where the Cartesian errors at the end of the manipulator are due to the errors in $N$ joint variables
2. The dimensional errors of a manipulator, where the Cartesian errors at the end of the manipulator are due to the errors in the $3N$ geometric parameters

The error model for positional accuracy can be obtained from (Eq. 53) by setting $3N$ geometric parameter errors to zero. The error model for dimensional errors can be obtained by setting the $N$ joint-variable errors to zero. For the purpose of generality, all kinds of kinematic errors in (Eq. 53) will be used for the rest of the paper.

In order to calculate the Cartesian error envelopes at the end of the manipulator, the $4N$ kinematic errors will be considered as random variables. A reasonable assumption is that $\Delta\boldsymbol{\theta}$, $\Delta\mathbf{r}$, $\Delta\mathbf{l}$, and $\Delta\boldsymbol{\alpha}$ are four independent, $N$-variable, zero-mean, normal distributions with the following properties:

$E[\Delta\boldsymbol{\theta}] = E[\Delta\mathbf{r}] = E[\Delta\mathbf{l}] = E[\Delta\boldsymbol{\alpha}] = 0$, where $E[\cdot]$ represents the expected values.

$\mathbf{V}_\theta$ = Variance of $\Delta\boldsymbol{\theta}$ = an $N$-by-$N$ diagonal matrix with components $(\sigma_{\theta 1}^2, \ldots\ldots, \sigma_{\theta N}^2)$, where $\sigma_{\theta i}$ = standard deviation of $\Delta\theta_i$.

$\mathbf{V}_r$ = variance of $\Delta r$ = an $N$-by-$N$ diagonal matrix with components $(\sigma_{r1}^2, \ldots\ldots, \sigma_{rN}^2)$, where $\sigma_{ri}$ = standard deviation of $\Delta r_i$.

$\mathbf{V}_l$ = variance of $\Delta\mathbf{l}$ = an $N$-by-$N$ diagonal matrix with components $(\sigma_{l1}^2, \ldots\ldots, \sigma_{lN}^2)$, where $\sigma_{li}$ = standard deviation of $\Delta l_i$.

$\mathbf{V}_\alpha$ = variance of $\Delta\boldsymbol{\alpha}$ = an $N$-by-$N$ diagonal matrix with components $(\sigma_{\alpha 1}^2, \ldots\ldots, \sigma_{\alpha N}^2)$, where $\sigma_{\alpha i}$ = standard deviation of $\Delta\alpha_i$.

All the covariances between these four random vectors are zero.

By combining the properties of a normal distribution and the relations in (Eqs. 51 and 52), we find that $\mathbf{d}^N$ and $\delta^N$ are also normal distributions with mean

$$\begin{aligned} E[\mathbf{d}^N] &= \mathbf{M}_1 E[\Delta\theta] + \mathbf{M}_2 E[\Delta\mathbf{r}] + \mathbf{M}_3 E[\Delta l] + \mathbf{M}_4 E[\Delta\boldsymbol{\alpha}] \\ &= 0, \end{aligned} \tag{64}$$

$$E[\delta^N] = \mathbf{M}_2 E[\Delta\theta] + \mathbf{M}_3 E[\Delta\alpha] = 0, \tag{65}$$

and variance

$$\begin{aligned}\mathbf{V}_d &= E[(\mathbf{d}^N - E[\mathbf{d}^N])(\mathbf{d}^N - E[\mathbf{d}^N])^t] \\ &= \mathbf{M}_1\mathbf{V}_\theta\mathbf{M}_1^t + \mathbf{M}_2\mathbf{V}_r\mathbf{M}_2^t + \mathbf{M}_3\mathbf{V}_l\mathbf{M}_3^t + \mathbf{M}_4\mathbf{V}_\alpha\mathbf{M}_4^t,\end{aligned} \tag{66}$$

$$\begin{aligned}\mathbf{V}_\delta &= E[(\delta^N - E[\delta^N])(\delta^N - E[\delta^N])^t] \\ &= \mathbf{M}_2\mathbf{V}_\theta\mathbf{M}_2^t + \mathbf{M}_3\mathbf{V}_\alpha\mathbf{M}_3^t),\end{aligned} \tag{67}$$

where $\mathbf{V}_d$ and $\mathbf{V}_\delta$ are 3-by-3 matrices whose components are functions of the joint variables.

The trivariable normal density functions of $\mathbf{d}^N$ and $\delta^N$ have the form

$$\begin{aligned}&f(dx^N, dy^N, dz^N) \\ &\quad = (2\pi)^{-3/2}|\mathbf{V}_d|^{-1/2} \exp\{-.5[(\mathbf{d}^N)^t|\mathbf{V}_d|^{-1}(\mathbf{d}^N)]\}\end{aligned} \tag{68}$$

and

$$\begin{aligned}&f(\delta x^N, \delta y^N, \delta z^N) \\ &\quad = (2\pi)^{-3/2}|\mathbf{V}_\delta|^{-1/2} \exp\{-.5[(\delta^N)^t|\mathbf{V}_\delta|^{-1}(\delta^N)]\}.\end{aligned} \tag{69}$$

With knowledge of the error standard deviation of each link parameters, the Cartesian error envelopes at the end of the manipulator can easily be obtained from (Eqs. 66 and 67). If the error envelopes with respect to the base are desired, they can be obtained from (Eqs. 59–62) by using the property of (Eqs. 68 and 69).

The error envelopes of three independent translational Cartesian errors and three independent rotational Cartesian errors can also be obtained by rotating the axes of $\mathbf{V}_d$ and $\mathbf{V}_\delta$ into their eigenvectors, that is, their principal axes. After this transformation, six independent, zero-mean, normal random variables $\{v_i;\ i = 1, \ldots, 6\}$ with standard deviation $\{\sigma_i;\ i = 1, \ldots, 6\}$ can be obtained. $v_1$, $v_2$, and $v_3$ are on the principal axes of $\mathbf{V}_d$. $v_4$, $v_5$, and $v_6$ are on the principal axes of $\mathbf{V}_\delta$. The density function of $v_i$ is of the form

$$f(v_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\{-v_i^2/2\sigma_i^2\}, \tag{70}$$

and $\sigma_i$ is the function of the joint variables.

The probability of $|v_i| < R_i$ is that

$$\begin{aligned}\text{prob}(|v_i| < R_i) &= \int_{-R_i}^{R_i} f(v_i)\, dv_i \\ &= 2\ \text{erf}\,(R_i/\sigma_i),\end{aligned} \tag{71}$$

where erf ( · ) is the error function of normal distribution and is defined as

$$\text{erf}\,(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\,(-y^2/2)\, dy. \tag{72}$$

By giving the upper bound of error probability, Prob $(|v_i| < R_i) \leq C_i$, the lower bound of the standard deviation $\sigma_i$ can be obtained by using the table of the error function erf ( · ) and

$$\sigma_i \geq R_i/B_i, \tag{73}$$

where $B_i$ is the constant value in the table of erf ( · ) that satisfied (Eq. 71). In order to be within the Cartesian error bound Ci, the joint trajectory at any time must satisfy (Eq. 73).

If the envelopes of the Cartesian error volumes are the condition to be satisfied, the following two new random variables,

$$w_1 = +(v_1^2 + v_2^2 + v_3^2)^{1/2} \tag{74}$$

and

$$w_2 = (v_4^2 + v_5^2 + v_6^2)^{1/2}, \tag{75}$$

will be the test variables for the error bounds.

In general, based on the above technique and the error model, we can always define some variables of interest and generate the result and error envelope. This makes the error model very useful for the kinematic design of a robot manipulator.

## 7. Design of a Robot Manipulator

In order to preserve the fidelity of a robot manipulator in the real world, the design of the kinematic parameters must be optimized. In this section, we will discuss how to minimize the Cartesian errors of an open-loop manipulator.

The results of (Eq. 26) show that if the link parameter $l_i$ is zero for the $i$th joint, the translational errors $\mathbf{d}_i^A$ of the joint coordinates can be reduced to two terms. The results of (Eq. 44) show that the Cartesian translational errors $\mathbf{d}^N$ can also be reduced. These results show that a manipulator with no link-length offset will be more accurate.

It was observed from the results of (Eqs. 45–47) that the Cartesian translational errors $\mathbf{d}^N$ were dominated by the errors in parameters $\theta$ and $\alpha$ because the error terms consisted of the position vector $\mathbf{p}_{i+1}^u$ of the $\mathbf{U}_{i+1}$ matrix. The Cartesian rotational errors were only affected by the errors in $\theta$ and $\alpha$ Thus, if the precision of the parameters $\theta$ and $\alpha$ are very high, then the Cartesian errors of the open-loop manipulator can be reduced to minimum.

Futhermore, the error model can be applied as a CAD tool. By constraining the Cartesian errors to certain error bounds, the maximum manufactured error tolerances of the designed kinematic parameters can be defined. In the previous section, the Cartesian-error envelopes were derived as (Eqs. 66 and 67), which depend on the value of the error tolerance, that is, the error standard deviation of the kinematic parameters. By setting the upper bounds of the error envelopes for the designed manipulator, different sets of manufactured error tolerances can be tested, and the error envelopes of the whole working space of the manipulator can be generated. By this procedure, the maximum manufacturing tolerances of the kinematic parameters can be obtained. If the manufacturing process cannot meet such standards, the kinematic parameters of the manipulator must be carefully calibrated. This will be discussed in the next section.

## 8. Calibration for Kinematic Errors

A calibration of the manipulator is necessary to minimize manufactured errors in the kinematic parameters. First, let us determine how many kinematic errors have to be solved for an open-loop, **N** degree-of-freedom robot manipulator.

In the general case, there are 4**N** kinematic errors. However, as described in the previous section, the Cartesian errors are dominated by the errors in $\theta$ and $\alpha$. In addition, the manufactured accuracy of the translational parameters **r** and **l** are much higher than the angular parameters $\theta$ and $\alpha$. Thus (Eq. 53) can be estimated as follows:

$$\begin{bmatrix} \mathbf{d}^N \\ \delta^N \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix} \Delta\theta + \begin{bmatrix} \mathbf{M}_4 \\ \mathbf{M}_3 \end{bmatrix} \Delta\alpha, \tag{76}$$

and only 2**N** kinematic errors need to be solved.

In the special case that a designed robot manipulator has some precision points on each joint, then the calibration of the joint variables can always be achieved by moving each joint to its precision points. Thus, instead of 4**N** kinematic errors, there are only 3**N** kinematic errors to be solved. In the case of (Eq. 76), there would be only **N** + **K** kinematic errors to be solved, where **K** is the number of the prismatic joint.

As can be seen from the discussion above, the number of kinematic errors that need to be solved depends on the particular case. For generality, **NK** will be used to represent the number of kinematic errors that will be solved, and a calibration scheme will be presented to calibrate these **NK** kinematic parameters.

First, the joint position of the manipulator can be roughly calibrated by moving the joints to their estimated zero position; then the actual Cartesian position $\mathbf{T}_N$ of the manipulator can be calculated from its joint positions and 3**N** geometric parameters. By moving the manipulator to a known Cartesian position $\mathbf{T}_N^c$ in the known real world, one group of the six Cartesian errors can be obtained by comparing $\mathbf{T}_N$ and $\mathbf{T}_N^c$ in (Eq. 63). $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{M}_4$ in (Eq. 76) can also be calculated from joint positions and 3**N** geometric parameters as in (Eqs. 54–57). For one precise Cartesian position $\mathbf{T}_N^c$, six equations for these **NK** error variables will be obtained. By moving the manipulator to **G** precise Cartesian positions, that satisfy **NK** $\leq$ 6**G**, these **NK** error variables can be solved analytically. Then the new accurate values of the **NK** kinematic parameters can be obtained by adding these solved errors to the old values. After this procedures has been repeated several times, the kinematic errors converge to zero. The final values of the **NK** kinematic parameters are the accurate calibrated values.

One important note is that although the kinematic errors can be calibrated by the above procedure, the robot joint solutions will become much more complicated than the normally simple joint solutions. Hence, the best way to improve the accuracy of the designed manipulator is to base it on the developed model to optimize the kinematic data.

## 9. Conclusion

The accuracy of a robot manipulator has long been one of the principal concerns of robot design and control. In the past, there was a lack of an explicit mathematical model to analyze the errors; therefore, the kinematic design of a robot manipulator could not be optimized.

In this paper, a simple, linear, Cartesian-error model of the kinematic errors of a robot manipulator has been developed, in a straightforward manner, from the kinematic equations. The method is based on the forward differential relation between two coordinate frames. By using the error-transformation matrix, the errors at each joint are transformed to the end-effector of the manipulator. The results show that the relationship between Cartesian errors and kinematic errors can be represented by a linear model.

Based on this model, the Cartesian-error envelopes caused by any combination of the four kinds of kinematic errors can be mathematically generated. This error model can then be used to minimize the Cartesian errors of an open-loop manipulator by properly designing the kinematic parameters. For example, with knowledge the manufacturing-error tolerances, the Cartesian-error envelopes for the designed kinematic parameters can be generated by the model. If the Cartesian errors were not acceptable, the designed kinematic parameters would have to be redesigned. Hence, we can always determine whether a robot manipulator with the designed kinematic parameters will meet the accuracy requirements.

In order to improve the accuracy of a current industrial robot, a calibration technique based on this error model has been proposed that will correct the kinematic errors of a robot manipulator. Though a calibration method can correct the kinematic errors of a robot, it complicates the robot's joint solution. Hence, the best way to improve robot accuracy is to design it properly. Since the developed model can describe the problem of robot accuracy, by using this model as a CAD tool, the kinematics of the designed robot can be optimized to meet the requirements of accuracy. As a result, the developed model is a very useful tool for solving the accuracy problem of robot manipulators.

## Acknowledgments

The author wishes to thank Dr. R. Paul and Mr. R. Casler for suggestions and encouragement following their reading of the original manuscript. Special thanks to my friend and colleague, Mr. V. McCarroll, for his patient, consistent help. Thanks also to Miss D. Kennedy and Mrs. B. J. Hull, for their help, and to the referees for their reviews and invaluable comments.

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