A peculiar continued fraction

$$f(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

now consider f(x) - 1, in other words

$$f(x) = 1 + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}} \quad \text{for x = 1 we have ...}$$

$$f(1) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

I will show that this continued fraction "converges" and surprisingly "converges" to the golden ratio

the "compact" form of the peculiar continued fraction, is found by...

$$f(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}, \quad y = f(x) \implies y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

Therefore...

$$y = \frac{1}{x+y} \equiv y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

$$y = \frac{1}{x+y} \implies y(x+y) = 1 \implies y^2 + xy = 1$$

A nice compact quadratic polynomial that can be expanded to the pecuilar continued fraction

Following the same algebraic manipulation it is easy to see that...

$$f(x) = 1 + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

when

$$f(x) = y \quad \text{and} \quad x = 1$$

$$\implies y = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \quad \text{is equivalent to} \quad y = 1 + \frac{1}{y}$$

Then be the same algebraic manipulation we arrive at a very similar quadratic equation:

$$0 = y^2 - y - 1$$

From the quadratic equation we can find the roots:

$$y = \frac{1 \pm \sqrt{5}}{2}$$

and strangely ϕ, the golden ratio, is the positive root of the polynomial..

$$\phi = \frac{1 + \sqrt{5}}{2}$$

A doubly peculiar nested radical

Recall the polynomial whose positieve and negative roots are the positive and negative golden ratio, respectively

$$0 = y^2 - y - 1$$

if we add y and 1 to the equation and take the sqaure root we arrrive at a interesting radical equation

$$\implies y = \sqrt{1 + y}$$

Then if we expand we arrive at cool nested radical

$$y = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

since

$$y = \frac{1 + \sqrt{5}}{2}$$

and

$$\phi = \frac{1 + \sqrt{5}}{2}$$

then

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

and also since

$$y = \sqrt{1 + y}$$

then

$$\phi = \sqrt{1 + \phi}$$

A very strange result, it doesn't make sense of a number to be equal to the sqaure root of itself plus 1, $2 \neq \sqrt{1 + 2}$

But, i have arrived at an interesting result...

Why is phi, the golden ratio important?

so far I have shown that only phi is a solution to the following three equations

$$0 = y^2 - y - 1$$
$$y = 1 + \frac{1}{y}$$
$$y = \sqrt{1 + y}$$

Only when $y = \pm\phi$ are the above equations true, since the nested radical and contiued fraction forms flow from the polynomial form and since there are only 2 solutions to the polynomial form $y = \pm\phi$, then $\pm\phi$ is the only solution to the nested radical and coninuted fraction form

Then this means thats the answer to the question that titles this section, ϕ is important because it is the only number that solves that polynomial and subsequently solves those equations.

But why limit ourselves to just this arbitrary polynomial? What's so special about $0 = y^2 - y - 1$ other than phi being its only solution?

I will show that there is another equally peculiar polynomial with an accompanying set of equations with equally irrational solutions, in fact I will show that there are precisely countably infine sets of equations that satisfy the mystical properties of ϕ

Consider the polynomial:

$$0 = y^2 - y - n, \quad n \in \mathbb{N}, \quad n \neq 0$$

(When n is 0 the solutions are not irrational... and they are not "phi-esque", not like the golden ration)

The solution to this quadratic is:

$$y = \frac{1 \pm \sqrt{1 + 4n}}{2}$$

I will show that for any nonzero natural number this solution is always irrational and I will show that this solution is the only soltuion to a certain set of peculiar equations... just like the golden ratio