

A peculiar continued fraction

$$f(x) = \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}$$

now consider $f(x) - 1$, in other words

$$f(x) = 1 + \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}} \quad \text{for } x = 1 \text{ we have ...}$$

$$f(1) = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}}$$

I will show that this continued fraction "converges" and surprisingly "converges" to the golden ratio

the "compact" form of the peculiar continued fraction, is found by...

$$f(x) = \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}, \quad y = f(x) \implies y = \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}$$

Therefore...

$$y = \frac{1}{x + y} \equiv y = \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}$$

$$y = \frac{1}{x + y} \implies y(x + y) = 1 \implies y^2 + xy = 1$$

A nice compact quadratic polynomial that can be expanded to the peculiar continued fraction

Following the same algebraic manipulation it is easy to see that...

$$f(x) = 1 + \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}$$

when

$$\begin{aligned} f(x) = y \quad \text{and} \quad x = 1 \\ \implies y = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}} \quad \text{is equivalent to} \quad y = 1 + \frac{1}{y} \end{aligned}$$

Then be the same algebraic manipulation we arrive at a very similar quadratic equation:

$$0 = y^2 - y - 1$$

From the quadratic equation we can find the roots:

$$y = \frac{1 \pm \sqrt{5}}{2}$$

and strangely ϕ , the golden ratio, is the positive root of the polynomial..

$$\phi = \frac{1 + \sqrt{5}}{2}$$

A doubly peculiar nested radical

Recall the polynomial whose positive and negative roots are the positive and negative golden ratio, respectively

$$0 = y^2 - y - 1$$

if we add y and 1 to the equation and take the square root we arrive at a interesting radical equation

$$\implies y = \sqrt{1 + y}$$

Then if we expand we arrive at cool nested radical

$$y = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

since

$$y = \frac{1 + \sqrt{5}}{2}$$

and

$$\phi = \frac{1 + \sqrt{5}}{2}$$

then

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

and also since

$$y = \sqrt{1 + y}$$

then

$$\phi = \sqrt{1 + \phi}$$

A very strange result, it doesn't make sense of a number to be equal to the square root of itself plus 1, $2 \neq \sqrt{1 + 2}$

But, i have arrived at an interesting result...

Why is phi, the golden ratio important?

so far I have shown that only phi is a solution to the following three equations

$$\begin{aligned} 0 &= y^2 - y - 1 \\ y &= 1 + \frac{1}{y} \\ y &= \sqrt{1 + y} \end{aligned}$$

Only when $y = \pm\phi$ are the above equations true, since the nested radical and continued fraction forms flow from the polynomial form and since there are only 2 solutions to the polynomial form $y = \pm\phi$, then $\pm\phi$ is the only solution to the nested radical and continued fraction form

Then this means that the answer to the question that titles this section, ϕ is important because it is the only number that solves that polynomial and subsequently solves those equations.

But why limit ourselves to just this arbitrary polynomial? What's so special about $0 = y^2 - y - 1$ other than ϕ being its only solution?

I will show that there is another equally peculiar polynomial with an accompanying set of equations with equally irrational solutions, in fact I will show that there are precisely countably infinite sets of equations that satisfy the mystical properties of ϕ

Consider the polynomial:

$$0 = y^2 - ny - 1, \quad n \in \mathbb{N}^+$$

The solution to this quadratic is:

$$y = \frac{n \pm \sqrt{n^2 + 4}}{2}$$

I will show that for any positive natural number this solution is always irrational and I will show that this solution is the only solution to a certain set of peculiar equations... just like the golden ratio

Irrationality of solutions to the "golden polynomial"

Let

$$y = \frac{n \pm \sqrt{n^2 + 4}}{2} \quad n \in \mathbb{N}^+$$

If y is irrational then by the definition of irrational numbers $n \pm \sqrt{n^2 + 4}$ cannot be an integer. This implies that $n^2 + 4$ is not a perfect square. I will proceed by contradiction.

Let

$$\begin{aligned} m^2 &= n^2 + 4 \quad \text{for } n \in \mathbb{N}^+ \quad m \in \mathbb{N} \\ \implies 4 &= m^2 - n^2 \\ \implies 4 &= (m - n)(m + n) \end{aligned}$$

Since $4 = 4 \times 1$ or $4 = 2 \times 2$

$$\implies 4 = m - n \quad \text{and} \quad 1 = m + n$$

But this is logically impossible since $m + n > m - n$ also we have:

$$1 = m - n \quad \text{and} \quad 4 = m + n$$

This system of equations implies $m = \frac{5}{2}$ which contradicts the definition that $m \in \mathbb{N}$ finally we have:

$$2 = m - n \quad \text{and} \quad 2 = m + n$$

This system of equations implies $m = 2$ and $n = 0$ but this contradicts the definition that $n \in \mathbb{N}^+$

With this, it is shown that $n^2 + 4$ is never a perfect square for all $n \in \mathbb{N}^+$ which implies that all solutions to $y^2 - ny - 1 = 0$ for $n \in \mathbb{N}^+$ are irrational. QED