Derivative of a Continued Fraction

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Definitions

$$g(x) = \cfrac{1}{f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ddots}}} \tag{1}$$

$$g'(x) = ? \tag{2}$$

if y = g(x) and x = f(x) then:

$$g(x) = \cfrac{1}{f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ddots}}} \equiv y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}} \tag{3}$$

$$\therefore g'(x) = y' \tag{4}$$

since

$$y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}} \tag{5}$$

then

$$y = \frac{1}{x + y} \tag{6}$$

Algebra

since

$$y = \frac{1}{x + y} \tag{7}$$

$$\therefore 1 = y(x + y) \tag{8}$$

$$0 = y^2 + xy - 1 \tag{9}$$

by the quadratic formula we find y to be:

$$y = \frac{-x \pm \sqrt{x^2 + 4}}{2} \tag{10}$$

we will define g to be the negative part of y and h to be the positive part:

$$g = \frac{-\sqrt{x^2 + 4} - x}{2} \quad \text{and} \quad h = \frac{\sqrt{x^2 + 4} - x}{2} \tag{11}$$

Calculus

$$g' = \frac{-1 + \frac{x}{\sqrt{x^2+4}}}{2} \quad \text{and} \quad h' = \frac{-1 - \frac{x}{\sqrt{x^2+4}}}{2}$$

since g is just the - part of y and h is just the + part. We can see that for some point y = g or y = h

$$(y' - g')(y' - h') = 0 \tag{12}$$
$$(y')^2 - h'y' - g'y' + g'h' = 0 \tag{13}$$
$$\therefore (y')^2 - (h' + g')y' + g'h' = 0 \tag{14}$$
$$\text{since} \quad h' + g' = -1 \quad \text{and} \quad g'h' = \frac{1}{4} - \frac{x^2}{4(x^2+4)} \tag{15}$$
$$\tag{16}$$
$$\text{we arrive at} \qquad (y')^2 + y' - \left(\frac{x^2}{4(x^2+4)} - \frac{1}{4}\right) = 0 \tag{17}$$

we can simplify the last term:

$$\left(\frac{x^2}{4(x^2+4)} - \frac{1}{4}\right) = \frac{-1}{x^2+4} \tag{18}$$

giving us

$$(y')^2 + y' + \frac{1}{x^2+4} = 0 \tag{19}$$

we will introduce a new variable u and set it to the last term:

$$u = \frac{1}{x^2+4} \tag{20}$$

we can immediately see:

$$(y')^2 + y' + u = 0 \tag{21}$$
$$(y')^2 + y' = -u \tag{22}$$
$$y'(y' + 1) = -u \tag{23}$$

Finnaly we arrive at:

$$y' = \frac{u}{1 + y'} \tag{24}$$

in expanded form we see:

$$y' = \cfrac{u}{1 + \cfrac{u}{1 + \cfrac{u}{1 + \ddots}}} \tag{25}$$

Are u important... yes :)

From the initial definitions of g(x) and f(x) I think I can infer that if:

$$g(x) = \cfrac{1}{f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ddots}}} \tag{26}$$

then (27)

$$g'(x) = \cfrac{u}{f'(x) + \cfrac{u}{f'(x) + \cfrac{u}{f'(x) + \ddots}}} \tag{28}$$

for some u (29)

My question is what determines u? Is there some way to calculate u? Some algorithm we can follow that gives us u? I tried lots of strange and suspicious math, but my tactics could not find a system for determining u. On my quest for u I found some strange results that I have written in my notebook, but I suspect they are unworthy to be seen by anyone Anyway thats all I have for u... I might try integrating a continued fraction(c.f.) or a backward continued fraction(b.c.f.) Or maybe a c.f. of derivatives Or what about a c.f. of integrals!? Also I think there might be strange connection between c.f's and generating functions, but I have done no math on the matter, just a hunch I have