A peculiar continued fraction

$$f(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

now consider f(x) - 1, in other words

$$f(x) = 1 + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}} \quad \text{for x = 1 we have ...}$$

$$f(1) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

I will show that this continued fraction "converges" and surprisingly "converges" to the golden ratio

the "compact" form of the peculiar continued fraction, is found by...

$$f(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}, \quad y = f(x) \implies y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

Therefore...

$$y = \frac{1}{x+y} \equiv y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

$$y = \frac{1}{x+y} \implies y(x+y) = 1 \implies y^2 + xy = 1$$

A nice compact quadratic polynomial that can be expanded to the pecuilar continued fraction

Following the same algebraic manipulation it is easy to see that...

$$f(x) = 1 + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}$$

when

$$f(x) = y \quad \text{and} \quad x = 1$$

$$\implies y = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \quad \text{is equivalent to} \quad y = 1 + \frac{1}{y}$$

Then be the same algebraic manipulation we arrive at a very similar quadratic equation:

$$0 = y^2 - y - 1$$

From the quadratic equation we can find the roots:

$$y = \frac{1 \pm \sqrt{5}}{2}$$

and strangely ϕ, the golden ratio, is the positive root of the polynomial..

$$\phi = \frac{1 + \sqrt{5}}{2}$$

A doubly peculiar nested radical

Recall the polynomial whose positieve and negative roots are the positive and negative golden ratio, respectively

$$0 = y^2 - y - 1$$

if we add y and 1 to the equation and take the sqaure root we arrrive at a interesting radical equation

$$\implies y = \sqrt{1 + y}$$

Then if we expand we arrive at cool nested radical

$$y = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

since

$$y = \frac{1+\sqrt{5}}{2}$$

and

$$\phi = \frac{1+\sqrt{5}}{2}$$

then

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

and also since

$$y = \sqrt{1+y}$$

then

$$\phi = \sqrt{1+\phi}$$

A very strange result, it doesn't make sense of a number to be equal to the sqaure root of itself plus 1, $2 \neq \sqrt{1+2}$

But, i have arrived at an interesting result...

Why is phi, the golden ratio important?

so far I have shown that only phi is a solution to the following three equations

$$\begin{aligned} 0 &= y^2 - y - 1 \\ y &= 1 + \frac{1}{y} \\ y &= \sqrt{1+y} \end{aligned}$$

Only when $y = \pm\phi$ are the above equations true, since the nested radical and contiued fraction forms flow from the polynomial form and since there are only 2 solutions to the polynomial form $y = \pm\phi$, then $\pm\phi$ is the only solution to the nested radical and coninuted fraction form

Then this means thats the answer to the question that titles this section, ϕ is important because it is the only number that solves that polynomial and subsequently solves those equations.

But why limit ourselves to just this arbitrary polynomial? What's so special about $0 = y^2 - y - 1$ other than phi being its only solution?

I will show that there is another equally peculiar polynomial with an accompanying set of equations with equally irrational solutions, in fact I will show that there are precisely countably infine sets of equations that satisfy the mystical properties of ϕ

Consider the polynomial:

$$0 = y^2 - ny - 1, \quad n \in \mathbb{N}^+$$

The solution to this quadratic is:

$$y = \frac{n \pm \sqrt{n^2 + 4}}{2}$$

I will show that for any positive natural number this solution is always irrational and I will show that this solution is the only soltuion to a certain set of peculiar equations... just like the golden ratio

Irrationality of solutions to the "golden polynomial"

Let

$$y = \frac{n \pm \sqrt{n^2 + 4}}{2} \quad n \in \mathbb{N}^+$$

If y is irrational then by the defintion of irrational numbers $n \pm \sqrt{n^2 + 4}$ cannot be an integer. This implies that $n^2 + 4$ is not a perfect sqaure. I will proceed by contradiction.

Let

$$\begin{aligned} & m^2 = n^2 + 4 \quad \text{for } n \in \mathbb{N}^+ \quad m \in \mathbb{N} \\ \implies & 4 = m^2 - n^2 \\ \implies & 4 = (m-n)(m+n) \end{aligned}$$

Since $4 = 4 \times 1$ or $4 = 2 \times 2$

$$\implies 4 = m - n \quad \text{and} \quad 1 = m + n$$

But this is logically impossible since $m + n > m - n$ also we have:

$$1 = m - n \quad \text{and} \quad 4 = m + n$$

This system of equations implies $m = \frac{5}{2}$ which contradicts the defintion that $m \in \mathbb{N}$ finnaly we have:

$$2 = m - n \quad \text{and} \quad 2 = m + n$$

This system of equations implies $m = 2$ and $n = 0$ but this contradicts the defintion that $n \in \mathbb{N}^+$

With this, it is shown that $n^2 + 4$ is never a perfect sqaure for all $n \in \mathbb{N}^+$ which implies that all solutions to $y^2 - ny - 1 = 0$ for $n \in \mathbb{N}^+$ are irratioinal. QED

The "golden equations"

I define the golden equations to be the set of equations derived from $y^2 - ny - 1 = 0$ for $n \in \mathbb{N}^+$ particularly these two equations:

$$y = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ddots}}} \quad \text{in compact form:} \quad y = n + \frac{1}{y} \tag{1}$$

$$y = \sqrt{1 + n\sqrt{1 + n\sqrt{1 + \dots}}} \quad \text{in compact form:} \quad y = \sqrt{1 + ny} \tag{2}$$

For $n \in \mathbb{N}^+$ I will show that the solutions to these equations are irrational, and i define the solutions to these equations to be "golden" as the golden ratio is the solution to these equations when $n = 1$, in short I will define a seqeunce of numbers that I call "golden numbers". A number is a golden number if it is an irrational solution to its corresponding golden equations. I am doing this to show that the golden ratio is trivial and there is nothing special about it. And it will be made clear that there are countably infinte golden numbers.

Irrationality of solutions to golden equations

The proof that the solutions to the golden equations are irrational follows by transitivity as I have already shwon that the golden polynomial, and by extension its solutions, is equivalent to the golden equations. This is clear since the golden equations are derived from the golden polynomial.

Let $n \in \mathbb{N}^+$

$$y = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ddots}}} \implies y = n + \frac{1}{y}$$

$$y = n + \frac{1}{y} \implies y^2 - ny - 1 = 0 \implies y = \frac{n \pm \sqrt{n^2 + 4}}{2}$$

And as I have shown above, y is always irratioinal when $y = \frac{n \pm \sqrt{n^2+4}}{2}$

Similarly, it can be easily found that the second golden equation also only has irratioinal solutions:

$$y = \sqrt{1 + n\sqrt{1 + n\sqrt{1 + \dots}}} \implies y = \sqrt{1 + ny}$$

$$y = \sqrt{1 + ny} \implies y^2 - ny - 1 = 0 \implies y = \frac{n \pm \sqrt{n^2 + 4}}{2}$$

And again as I have already shown $y = \frac{n \pm \sqrt{n^2+4}}{2}$ is irrational for nonzero natural number, n.

Golden numbers

In this section I will define a set of numbers I call golden numbers, and I will show that this set is countably infinite.

Let

$$\Phi := \left\{ p_n \mid p_n = \frac{n + \sqrt{n^2 + 4}}{2}, n \in \mathbb{N}^+ \right\}$$
$$p_1 = \phi \quad \text{The golden ratio, } \phi \approx 1.6180$$
$$p_2 \approx 2.4142 \quad \text{The "second" golden ratio}$$

I have now defined Φ, the set of golden numbers. Next I will prove that the set is countably infinte by proving that there exixts a bijection from Φ to the set of natural numbers $\mathbb{N}$.

TODO: prove the bijection... how?????