A Study on Quintessence: An Alternative to Cosmological Constant

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Abstract

This report explores the fundamental challenges of the cosmological constant in a technical level and establishes radiative instability as the main problem with cosmological constant. To address this issue, Quintessence(a dynamic scalar field) is proposed as a candidate for dark energy, responsible for the late-time accelerated expansion of the universe. The report develops a theoretical formulation of a general Quintessence scalar field and examines the cosmological dynamics. A numerical study is done on thawing type of quintessence models and compared with the result of ΛCDM.

Contents

1 Introduction

Observational data from type Ia supernovae provides evidence of the accelerated expansion of the universe. About 70% of the energy density of the universe is made of dark energy which behaves differently then ordinary matter or radiation. A standard proposal is to describe dark energy as a constant energy density given by cosmological constant $\Lambda > 0$. This has led to a concordance cosmological model known as ΛCDM. However, despite its observational success, the ΛCDM model faces deep theoretical challenges.

In the following section of the article we will show that vacuum energy participates in the value of Λ. The renormalized value of the vacuum energy is too large to be compatible with observations. We will show that radiative instability and the need for repeated fine tuning constitute the primary problem underlying its theoretical inconsistency.

So far there is no satisfactory scenario where the small energy scale of the dark energy can be naturally explained by the vacuum energy related to particle physics. An alternative mechanism to explain the origin of dark energy is Quintessence. Quintessence is described by a canonical scalar field ϕ minimally coupled to gravity[1].A dynamical study shows us how the field evolves over time. Provided some initial kinetic energy and a negligible potential, the fields start by rolling up or down the potential: this is kination. The Hubble friction being huge in the early universe, the fields are quickly slowed down. During radiation domination, the kinetic energy continues to get highly reduced, so much that fields appear frozen. This goes on until the potential forces become non-negligible: this happens at some point during the radiation- matter phase. Then the scalar fields get accelerated and their kinetic energy starts rising again. It remains very small until after matter domination, when finally Hubble friction becomes small enough to allow a proper field displacement. Then, the fields roll down the potential, while dark energy is getting dominant. What happens exactly today and in the future is model dependent, but the whole dynamics just described for the past is (almost) not. A numerical study of the models provides data for such observation.

The outline of the article is as follows. In section 2 we discuss Cosmological constant in a technical manner exploring the underlying problem from a classical and quantum mechanical point of view. In section 3 we discuss the origin of Quintessence in the literature. Section 4 reviews the key elements of the dynamics of quintessence. Section 5 gives numerical study of a thawing type of Multi field Quintessence model based of exponential and hilltop potential.

2 The Cosmological Constant Problem

2.1 The Cosmological Constant

The classical action for gravitational field together with the action for matter is given as

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R - 2\Lambda_B) + S_{matter}[g_{\mu\nu}, \Psi] \tag{2.1}$$

where $\kappa \equiv 8\pi G/c^4 \equiv 8\pi/m_{\text{Pl}}^2 \equiv 1/M_{\text{Pl}}^2$, m_{Pl} and M_{Pl} being the Planck mass and the reduced Planck mass respectively. λ_B is the bare cosmological constant and has the dimension of the inverse of square length. Variation of the total action with respect to the matric tensor leads to the Einstein equation of motion which reads:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.2}$$

where the stress-energy tensor is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{2.3}$$

The stress energy tensor of a field placed in the vacuum state must be given by

$$\langle 0 | T_{\mu\nu} | 0 \rangle = -\rho_{\text{vac}} g_{\mu\nu} \tag{2.4}$$

where ρ_{vac} is the constant energy density of the vacuum.

In flat spacetime the only invariant tensor is $\eta_{\mu\nu}$. Since the vacuum state must be same for all observer, in curved space one necessarily has

$$\langle T_{\mu\nu}\rangle = \rho_{\rm vac}(t,\mathbf{x})g_{\mu\nu} \tag{2.5}$$

This means that the form of stress energy tensor must be conserved, $\rho_{\rm vac}$ must be constant. Consider a scalar field Φ. The corresponding action reads

$$S_\Phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right] \tag{2.6}$$

where $V(\Phi)$ is the potential.Using the definition is 2.3 the corresponding stress energy tensor reads

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\alpha\beta}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + V(\Phi)\right] \tag{2.7}$$

The expression shows that the scalar field is infect a perfect fluid. Now, the vacuum energy is the minimum energy state. In this case the kinetic energy vanishes and the fields at the minimum of its potential. Tn this case the stress energy reduces to

$$\langle T_{\mu\nu}\rangle = V(\Phi_{\rm min})g_{\mu\nu} \tag{2.8}$$

where $\rho_{\rm vac} = V(\Phi_{\rm min})$. From equation (2.5) and (2.8) we can see that that there are at least two sources for the vacuum energy. The classical contribution which originates from the value of the potential at its minimum, is given by (2.5) and the quantum contribution originating from the zero point fluctuation, is given by equation (2.8). The zero point fluctuations are a form of energy and all form of energy gravitate. Hence taking qft into account, Einstein equation can be written as,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_B g_{\mu\nu} = \kappa T^{\rm matter}_{\mu\nu} + \kappa\langle T_{\mu\nu}\rangle \tag{2.9}$$

Here the 2nd term on R.H.S is the contribution origianting from the vacuum. Using the form of $\langle T_{\mu\nu} \rangle$ we get,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} + \kappa \rho_{\text{vac}} = \kappa T^{\text{matter}}_{\mu\nu} \tag{2.10}$$

From 2.10 we can conclude effective cosmological constant $\Lambda_{\text{eff}} = \Lambda_B + \kappa \rho_{\text{vac}}$. This is a quantity one can observe and constrain when one carries out the test.[2]

2.2 The Classical Cosmological Constant Problem

In the previous section we showed that the effective cosmological constant receives a contribution from the minimum value of potential. In this section we will show how to minimize this contribution.

Let us consider a scalar field Φ in interaction with another scalar field Ψ such that,

$$V(\Phi, \Psi) = V(\Phi) + \frac{\bar{g}}{2} \Phi^2 \Psi^2 \tag{2.11}$$

where $\bar{g}$ is a dimensionless coupling constant and $V(\Phi)$ is the self interacting potential. If the field Ψ is in thermal equilibrium, one is entitled to replace Ψ^2 with $\langle \Psi^2 \rangle_T$. where the average is taken in a thermal state with temperature T. The effective potential becomes,

$$V_{\text{eff}}(\Phi) = V_0 + \frac{\lambda}{4} (\Phi^2 - \nu^2) + \frac{\bar{g}}{2} \Phi^2 T^2 \tag{2.12}$$

where λ being a coupling constant describing the self interaction and ν the value of the scalar field at the minimum potential in absence of an interaction with Ψ. The quantity V_0 denotes the classical offset. This potential can be expressed as,

$$V_{\text{eff}}(\Phi) = V_0 + \frac{\lambda \nu^4}{4} + \frac{\lambda \nu^2}{2} \left(\frac{T^2}{T^2_{\text{cri}}} - 1 \right) \Phi^2 + \frac{\lambda \Phi^4}{4} \tag{2.13}$$

where we have defined $T_{\rm cri} = \nu\sqrt{\frac{\lambda}{g}}$. The above equation gives the effective mass as

$$m_{\rm eff}^2(T) = \frac{\lambda\nu^2}{2}\left(\frac{T^2}{T_{\rm cri}^2} - 1\right) \tag{2.14}$$

As a consequence,

- When $T > T_{\rm cri}$ (before the transition), the square of the effective mass is positive. The minimum is located at $\Phi = 0$ and the corresponding value of the vacuum energy is $V_0 + \frac{\lambda\nu^4}{4}$.

- When $T < T_{\rm cri}$ (after the transition), the square of the effective mass is negative.The minimum is located at $\Phi = \nu$ and the corresponding value of the vacuum energy is V_0.

The choice of a parameter in the potential allows us to change the vacuum energy at the classical level. Let us calculate the vacuum energy induced by the electroweak phase transition. One can adjust the vacuum energy to be zero today by tuning the parameter V_0. Then the vacuum energy prior to the electroweak phase transition is $\rho_{\rm vac}^{\rm EW} \simeq -1.2 \times 10^8 GeV^4 \simeq -10^{55}\rho_{cri}$ which contradicts observation [2].

2.3 The Quantum-Mechanical Cosmological Constant Problem

In the previous section, we showed that the classical contribution to the cosmological constant can be made zero today by the choice of potential. However, $\rho_{\rm vac}$ still receives contribution from the zero point fluctuations of the quantum fields.

Let us consider a simple real field scalar field with the potential $V(\Phi) = \frac{m^2\Phi^2}{2}$ where m is the mass of the scalar particle. In flat space time the equation of motion is nothing but the Klein-Gordon equation given as,

$$-\ddot{\Phi} + \delta^{ij}\partial_i\partial_j\Phi - m^2\Phi = 0 \tag{2.15}$$

Since the scalar field is free, the equation of motion is linear and one can Fourier expand $\Phi(t, \mathbf{x})$ as,

$$\Phi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{\sqrt{2\omega(k)}} \left(c_k e^{-i\omega t + i\mathbf{k}.\mathbf{x}} + c_k^\dagger e^{i\omega t - i\mathbf{k}.\mathbf{x}} \right) \tag{2.16}$$

with $\omega(k) \equiv \sqrt{k^2 + m^2}$. The field has been quantized by considering c_k and $c_k^\dagger$ as quantum operator satisfying the following commutation relation

$$\left[c_k, c_{k'}^\dagger \right] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{2.17}$$

We can now ccalculate $\langle 0|T_{\mu\nu}|0\rangle$. We need to find mean values various quantities depending on the field operator and its derivatives.

$$\begin{aligned} \langle 0|\dot{\Phi}^2|0\rangle &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\omega(k)} \omega^2(k) \\ \langle 0|\delta^{ij}\partial_i\partial_j\Phi|0\rangle &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\omega(k)} \mathbf{k}^2 \\ \langle 0|\Phi^2|0\rangle &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\omega(k)} \end{aligned} \tag{2.18}$$

These equation can be used to determine the energy density and pressure of the vacuum. From the stress energy tensor expression in (2.7),

$$T_{00} = \mathcal{H} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}\delta^{ij}\partial_i\partial_j\Phi + \frac{1}{2}\Phi^2 \tag{2.19}$$

Where $\mathcal{H}$ denotes the Hamiltonian density. The expression for energy density becomes:

$$\langle \rho \rangle = \langle 0|u^0 u^0 T_{00}|0\rangle = \frac{1}{(2\pi)^3} \frac{1}{2} \int d^3\mathbf{k}\omega(k) \tag{2.20}$$

In the same manner the expression for pressure is

$$\langle p \rangle = \langle 0|\frac{1}{2}\dot{\Phi}^2 - \frac{1}{6}\delta^{ij}\partial_i\partial_j\Phi - \frac{1}{2}m^2\Phi^2|0\rangle = \frac{1}{(2\pi)^3} \frac{1}{6} \int d^3\mathbf{k} \frac{k^2}{\omega(k)} \tag{2.21}$$

From the above consideration, using the definition of effective cosmological constant we get

$$\Lambda_{\text{eff}} = \Lambda_B + \frac{\kappa}{(2\pi)^3} \int d^3\mathbf{k} \frac{1}{2} \omega(k) \tag{2.22}$$

The problem with the calculation rises as the energy density blows up in the ultraviolet regime Thus making the cosmological constant infinity.

In order to regularize the infinities the most common method is introducing a cut-off at $k = M$, where M is the scale at which the effective theory used before breaks down.[2]. In this approach the energy density and pressure can be calculated as follows:

$$\begin{aligned} \langle \rho \rangle &= \frac{1}{4\pi^2} \int_0^M dk k^2 \sqrt{k^2 + m^2} \\ &= \frac{M^4}{16\pi^2} \left[\sqrt{1 + \frac{m^2}{M^2}} \left(1 + \frac{1}{2} \frac{m^2}{M^2} \right) - \frac{1}{2} \frac{m^4}{M^4} ln \left(\frac{m}{M} + \frac{m}{M} \sqrt{1 + \frac{m^2}{M^2}} \right) \right] \end{aligned} \tag{2.23}$$

$$\begin{aligned} \langle p \rangle &= \frac{1}{3} \frac{1}{4\pi^2} \int_0^M dk \frac{k^4}{\sqrt{k^2 + m^2}} \\ &= \frac{1}{3} \frac{M^4}{16\pi^2} \left[\sqrt{1 + \frac{m^2}{M^2}} \left(1 - \frac{3}{2} \frac{m^2}{M^2} \right) + \frac{3}{2} \frac{m^4}{M^4} ln \left(\frac{m}{M} + \frac{m}{M} \sqrt{1 + \frac{m^2}{M^2}} \right) \right] \end{aligned} \tag{2.24}$$

Using a appropriate regularization scheme and preserving Lorentz invariance, we get the following expression for energy density[2]

$$\langle \rho \rangle \simeq -\frac{m^4}{64\pi^2} \left[\frac{2}{\epsilon} + \frac{3}{2} - \gamma - \ln \left(\frac{m^2}{4\pi\mu^2} \right) \right] + ... \tag{2.25}$$

The one loop contribution to the vacuum energy is given by

$$V_{vac}^{\phi,1loop} \sim -\frac{m^4}{64\pi^2} \Big[\frac{2}{\epsilon} + \ln \frac{\mu^2}{m^2} + \text{finite} \Big] \tag{2.26}$$

The divergence term requires us to add the following counter term which depends on an arbitrary subtraction scale, M

$$\Lambda^{1loop} \sim \frac{m^4}{64\pi^2} \Big[\frac{2}{\epsilon} + \ln \frac{\mu^2}{M^2} \Big] \tag{2.27}$$

so that the renormalised vacuum energy (at one loop)is given by

$$\Lambda_{ren}^{1loop} \sim \frac{m^4}{64\pi^2}\left[\ln\frac{m^2}{M^2} - \text{finite}\right] \tag{2.28}$$

With heavy particle contributing more or less upto the TeV scale suggests the finite contributions to the one loop renormalised vacuum energy are canceling to at least an accuracy of one part in 10^{60} in nature. The problem arises when we alter our effective description of matter by going to two loops. For perturbative theories without finely tuned couplings this means the two loop correction is not significantly suppressed with respect to the loop contribution. Therefore, the cancellation we imposed at one loop is spoilt and we must retune the finite contributions in the counter term to more or less same degrees of accuracy. Even if we accept the tuning at two loop at three or higher loops we need more accuracy to our tuning. This means the vacuum energy is sensitive to the details UV physics. This is the cosmological constant problem. [3]

3 Origin of Quintessence

The role of a dynamical scalar field for recent acceleration of the cosmic expansion certainly owes a debt to the use of rolling scalar fields for early universe inflation. A scalar field, and more generally a negative equation of state, were implemented as a substitute for the cosmological constant in a flurry of activity in the 1980s.[4]

4 Dynamics of Quintessence

4.1 Scalar field formalism

The spring analogy for quantum zero-point energy illustrates that even in its lowest energy state, a quantum system must have a minimum, non-zero amount of energy obeying uncertainty principle.If we view the cosmological constant as a quantum zeropoint energy corresponding to the ground state of harmonic modes of a field filling space, we can picture this as an array of identical springs, motionless and each stretched to the same length. By contrast, a scalar field would be a dynamical version of this, with the springs oscillating in time and having different lengths at different points in space[4]. Let us consider quintessence in the presence of non relativistic matter described by a barotropic fluid. the total action is given by The total action is given by,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\rm pl}^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] + S_m. \tag{4.1}$$

where g is the determinant of the metric $g_{\mu\nu}$, $M_{\rm pl}$ is the reduced Planck mass, R is the Ricci scalar, S_m is the matter action. We assume that non-relativistic matter does not have a direct coupling to the quintessence field ϕ.

We study the dynamics of quintessence on the flat Friedmann–Lemaître–Robertson–Walker (FLRW) background with the line element.

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2, \tag{4.2}$$

where $a(t)$ is the scale factor with cosmic time t. The pressure and the energy density of quintessence are given, respectively, by

$$\begin{aligned} P_\phi &= \dot{\phi}^2/2 - V(\phi) \\ \rho_\phi &= \dot{\phi}^2/2 + V(\phi) \end{aligned} \tag{4.3}$$

where a dot represents a derivative with respect to t.

The dark energy equation of state is

$$w \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \tag{4.4}$$

The scalar field satisfies the continuity equation

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0,$$

i.e.,

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0,$$

where $H = \dot{a}/a$ is the Hubble expansion rate and $V_{,\phi} = dV/d\phi$.

for a matter fluid with energy density ρ_m and equation of state w_m. the equation of motion following from (4.1) is

$$\begin{aligned} 3M_{\rm pl}^2 H^2 &= \dot{\phi}^2/2 + V(\phi) + \rho_m, \\ 2M_{\rm pl}^2 \dot{H} &= -\left[\dot{\phi}^2 + (1 + w_m)\rho_m\right]. \end{aligned} \tag{4.5}$$

From the dynamical equation we wish to know how much energy is there in the field , how springy it is (how spacetime curvature reacts to the accelerating component) and how stretchy the field is. The energy density is written in terms of dimensionless quantity $\Omega_w = \frac{\rho_\phi}{\rho_c}$. We can regard w as a measure of springiness. As the universe expands, the springs change their springiness like stretching the coils of a spring. This time variation can be taken as $w' = \frac{dw}{dlna}$.

4.2 General Dynamical Behavior

A canonical quintessence field can exhibit four qualitatively different instantaneous behaviors depending on the balance of kinetic energy, potential energy, Hubble friction and potential curvature.

Fast Roll Fast rollers have kinetic energy exceeding their potential energy and so $w > 0$. A fast roll epoch is a characteristic of tracker models which follow attractor trajectories in their dynamics such that at a certain epoch their equation of state is determined by the dominent energy component of the universe.

Slow Roll When the kinetic energy is much smaller than the potential energy, the equation of state is strongly negative $w \approx -1$.This only leads to acceleration of the expansion if the dark energy also dominates the energy density.

Steady Roll Referring to the original quintessence model of Linde (1987) using a linear potential,the model does have a constant right hand side of the Klein-Gordon equation of motion, and for a long time the dynamics stays reasonably close to the line where the field acceleration.

Osicllation Common potentials in renormalizable field theories include $V(\phi) \sim \phi^n$ which have a minimum for n even. While the field will have a conventional rolling stage, eventually it will reach the minimum and oscillate around it. If the period for oscillation is much smaller than the Hubble time (as is generally the case) then the effective equation of state becomes

$$w = \frac{n-2}{n+2} \tag{4.6}$$

For a quadratic potential, the field acts like non-relativistic matter, and for a quartic potential it acts like radiation.

Fast-roll/ tracker behavior can dilute a very large initial energy down to an observationally acceptable value (helping the fine-tuning problem), slow-roll looks like Λ but reintroduces tuning, and oscillatory potentials eventually behave like matter or radiation.Steady model starts generically from a frozen, cosmological constant-like state due to Hubble friction, then thaws and rolls down the potential. However, because the potential has no minimum, the field rolls into territory where the potential goes negative, which actually leads to a collapsing universe, rather than an accelerating expansion. Realistic models typically mix these modes across cosmic history.

4.3 Fundamental Modes of Dynamics

By examining the physical impact of the three different terms in the Klein-Gordon equation ,we can identify boundaries in the phase space corresponding to different physical conditions as shown in figure 1[4].

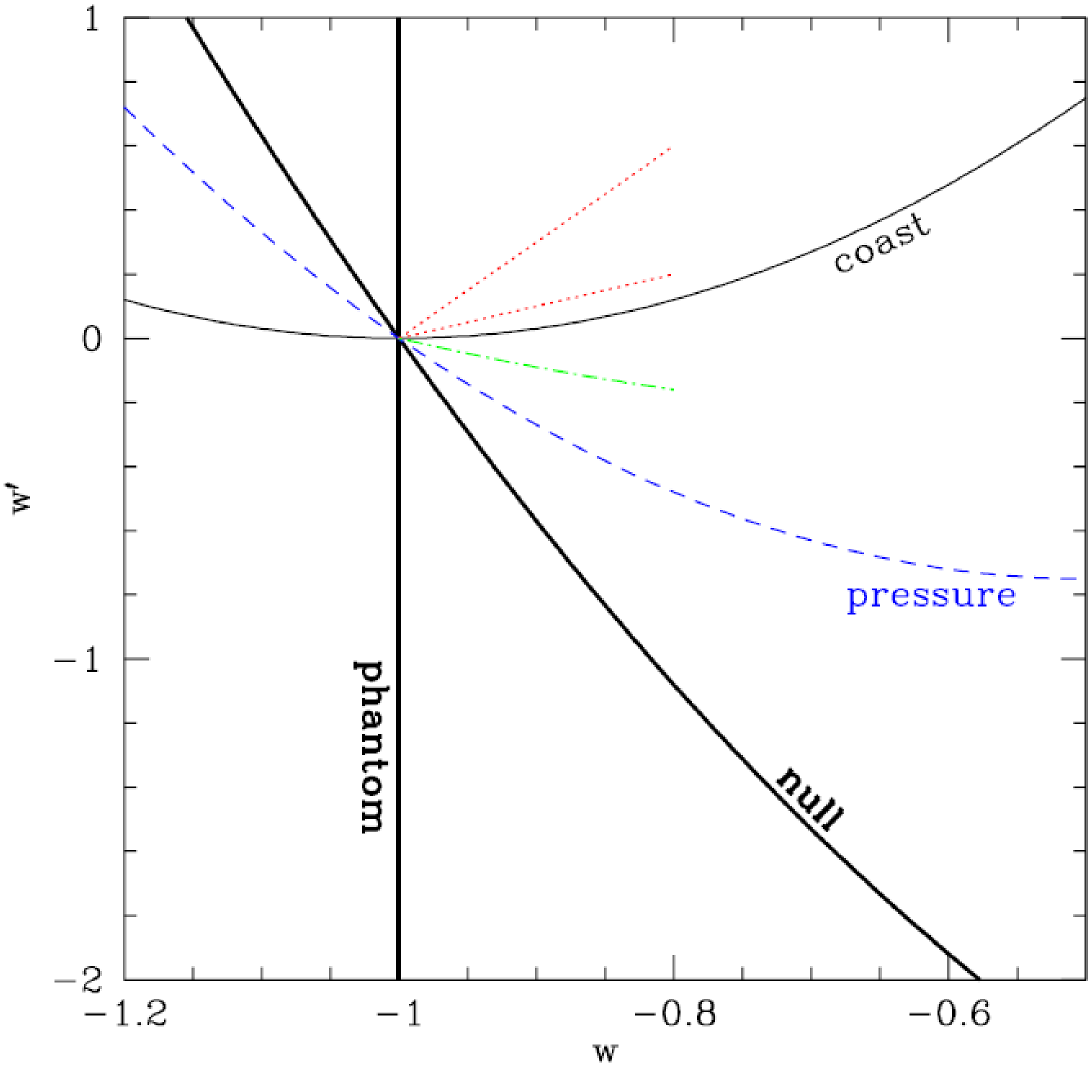

Figure 1: The dynamical phase space $w - w'$ is divided by three curves defined by physical conditions: the phantom line $w = -1$, the null line $w' = -3(1 - w^2)$ following from a flat potential, and the coasting line $w' = 3(1 + w)^2$ following from constant field velocity. These extend across the phase space.

Phantom line This line separates null energy condition $p + \rho \geq 0 (w \geq -1)$ from physics that violates it.

Null line Consider the forcing term of the potential slope. When the field rolls down the potential , $\dot{V} \leq 0$, this corresponds to

$$w' \geq -3(1 - w^2) \tag{4.7}$$

Null line corresponds to $V_{,\phi} = 0$ and smoothly passes through the point $(w, w') = (-1, 0)$.

Coasting line The dividing line between these dynamics, where the field is freely coasting at constant velocity $\dot{\phi}$ is $w' = 3(1+w)^2$.

4.4 Models of Quintessence

The dynamics of quintessence in the presence of non-relativistic matter has been studied in detail for many different potentials.Depending on the evolution of w, we can broadly classify quintessence models into two classes.

4.4.1 Thawing Model

The field is initially frozen by large Hubble friction in the early universe, so that $\dot{\varphi} \simeq 0$ and $w_\varphi \simeq -1$. As $H(t)$ decreases the friction weakens and the field "thaws", beginning to roll slowly away from the initial position. Observationally thawing models typically exhibit small, late departures from $w = -1$ with $w' > 0$ at early departure. The thawing region of phase space is bounded by a dynamical history

$$1 + w \lesssim w' \leq 3(1 + w) \tag{4.8}$$

shown in Figure 1 bounded by the red dotted line.

4.4.2 Freezing Model

The field rolls more actively at earlier times (possibly tracking the dominant background), so w may have been appreciably > -1 in the past; as the field evolves the potential flattens (or friction dominates) and the field slows, asymptotically approaching $w \to -1$ so that $w' \to 0$ at late times. As the field rolls toward the minimum, decelerating in its motion (lying below the coasting line), gradually approaching asymptotically a static cosmological constant state, it is said to be freezing.The freezing region oof the phase space is defined by a dynamical history

$$0 \leq w' \leq 3w(1+w) \tag{4.9}$$

shown by the bounded region by the green dot-dash curve and blue dashed curve. Tracker model is a subclass of freezing model in which the scalar field evolution has an attractor behavior: for a very wide range of initial conditions the field evolution converges onto a common trajectory ("the tracker"). While the background (radiation or matter) dominates, the scalar energy density evolves in a way that tracks the dominant component preventing the field from dominating early

5 Numerical Study of a Quintessence Model

In this section we will analyze a multi field quintessence model of thawing type.We will give an analysis of the realistic solution in a model independent matter.[5]

5.1 Scalar-field Formalism

5.1.1 Cosmological model and equation

The equations of motion, namely the two Friedmann equations and the scalar field equations of motion, are given by

$$F_1 = 0\,, \quad F_2 = 0\,, \quad E^i = 0 \tag{5.1}$$

where

$$F_1 = 3H^2 - \frac{1}{M_p^2}\sum_n \rho_n\,, \tag{5.2}$$

$$F_2 = \dot{H} + \frac{1}{2M_p^2}\sum_n (1+w_n)\rho_n\,, \tag{5.3}$$

$$E^i = \ddot{\varphi}^i + \Gamma^i_{jk}\dot{\varphi}^j\dot{\varphi}^k + 3H\dot{\varphi}^i + g^{ij}\partial_j V\,. \tag{5.4}$$

For a function f, we denote $\dot{f} \equiv \partial_t f$. The Hubble parameter is $H = \dot{a}/a$, and we restrict to $a(t) > 0$ for all times except in the limit to the origin. Γ^i_{jk} is the Christoffel symbol for the field space metric g_{ij}. The energy densities ρ_n are those of perfect fluids with pressure p_n, to which we associate equation of state parameters $w_n = p_n/\rho_n$. We consider in F_1, F_2 the components listed in Table 1, namely radiation, matter, curvature and scalar field. The scalar component ρ_φ will stand for dark energy. In the following, we set $M_p = 1$. Without loss of generality, we normalize $a(t_0) = 1$ for a given time t_0; this time will often correspond to today. We denote by an index 0 quantities at this time t_0. In particular, ρ_{r0} and ρ_{m0} are constant.

component	ρ_n	w_n	ρ_a
radiation	$\rho_{r0}a^{-4}$	$\frac{1}{3}$	
matter	$\rho_{m0}a^{-3}$	0	$\rho_{a0}a^{-3(1+w_m)}$
curvature	$-3k\,a^{-2}$	$-\frac{1}{3}$	
scalar	$\frac{1}{2}g_{ij}\dot{\varphi}^i\dot{\varphi}^j + V$	$\frac{\frac{1}{2}g_{ij}\dot{\varphi}^i\dot{\varphi}^j - V}{\frac{1}{2}g_{ij}\dot{\varphi}^i\dot{\varphi}^j + V}$	

Table 1: Energy density and equation of state parameter of each component entering the cosmological equations.

We introduce notations for the separate scalar components that are kinetic and potential energy as indicated in table 2.

component	ρ_n	w_n
kinetic energy	$\frac{1}{2}g_{ij}\dot{\varphi}^i\dot{\varphi}^j$	1
potential	V	-1

Table 2: Energy density and equation of state parameter for the separate scalar components

Given this settings, the following things are considered.

- We will consider a positive definite g_{ij} and $V \geq 0$.We thus get $\rho_\phi \geq 0$. For $\rho_\phi > 0$, we deduce $-1 \leq w_\phi \leq 1$: This implies $w_\phi < -1$ is not possible in such model.
- For a flat universe we get $\rho_n \geq 0$. Then since $1 + w_n \geq 0$, we deduce from $F_2 = 0$ that $\dot{H} \leq 0$. In addition we will restrict ourselves to expanding universe so $H > 0$. So in the solution H is always large in the early universe.

Finally,

$$\dot{\rho}_\varphi = \dot{\varphi}^i g_{ij}(E^j - 3H\dot{\varphi}^j)\,. \tag{5.5}$$

Therefore, in a solution with some field speed (satisfying $E^j = 0$ and with $H > 0$), we deduce that $\dot{\rho}_\varphi < 0$. In other words, Hubble friction, corresponding to the term $-3H\dot{\varphi}^2$, makes ρ_φ decrease with time. In addition, using $E^j = 0$ gives that

$$\dot{\rho}_\varphi + 3H\rho_\varphi = -3Hw_\varphi\rho_\varphi\,, \tag{5.6}$$

which is the continuity equation for the dark energy or scalar field component. These results will be useful later. Since $H \neq 0$, we can introduce the energy density parameter $\Omega_n \equiv \rho_n/(3H^2)$ for each component. The equations of motion then get rewritten in the convenient form

$$f_1 = 0\,, \quad f_2 = 0\,, \quad e^i = 0 \tag{5.7}$$

where

$$f_1 = 1 - \sum_n \Omega_n\,, \qquad f_2 = \frac{\dot{H}}{3H^2} + \frac{1}{2}\sum_n (1 + w_n)\Omega_n\,, \qquad e^i = \frac{E^i}{3H^2}\,. \tag{5.8}$$

With $\rho_n \geq 0$, we deduce $\Omega_n \geq 0$, so $f_1 = 0$ imposes $\Omega_n \leq 1$: those parameters thus indicate the proportion of the component n in the universe at a given time.

Last but not least, a useful expression for the energy density parameters can be obtained from $F_1/(3H_0^2) = 0$:

$$\frac{H^2}{H_0^2} = \Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{k0}a^{-2} + \Omega_{\varphi 0}\frac{\rho_\varphi}{\rho_{\varphi 0}}\,. \tag{5.9}$$

One deduces the following expressions

$$\Omega_r = \frac{\Omega_{r0}a^{-4}}{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{k0}a^{-2} + \Omega_{\varphi 0}\frac{\rho_\varphi}{\rho_{\varphi 0}}}\,, \tag{5.10}$$

$$\Omega_m = \frac{\Omega_{m0}a^{-3}}{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{k0}a^{-2} + \Omega_{\varphi 0}\frac{\rho_\varphi}{\rho_{\varphi 0}}}\,, \tag{5.11}$$

$$\Omega_\varphi = \frac{\Omega_{\varphi 0}\frac{\rho_\varphi}{\rho_{\varphi 0}}}{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{k0}a^{-2} + \Omega_{\varphi 0}\frac{\rho_\varphi}{\rho_{\varphi 0}}}\,. \tag{5.12}$$

and similarly for Ω_k. As mentioned above, we have here $\Omega_n \geq 0$; the expressions above then make it obvious that $\Omega_n \leq 1$, and that Ω_n gives the proportion of each component.

Let us also introduce the effective equation of state parameter $w_{\rm eff} = \sum_n w_n \Omega_n$. We can then show

$$\frac{F_1 + 2F_2}{3H^2} = \frac{2}{3H^2}\frac{\ddot{a}}{a} + \frac{1}{3} + w_{\rm eff}\,, \tag{5.13}$$

giving the well-known condition for an accelerating solution

$$\ddot{a} > 0 \quad \Leftrightarrow \quad w_{\rm eff} < -\tfrac{1}{3}\,. \tag{5.14}$$

With this formalism at hand, we can now look for solutions. This requires some rewriting of the equations that we now turn to.[5]

5.1.2 Equation system and reformulations

Above 3 equation follow the following relation

$$\dot{F}_1 \;=\; -\,\dot{\phi}^i\, g_{ij}\, E^j \;+\; 6H\, F_2 \tag{5.15}$$

Similarly,

$$\dot{f}_1 \;+\; 2\frac{\dot{H}}{H}\, f_1 \;=\; -\,\dot{\phi}^i\, g_{ij}\, e^j \;+\; 6H\, f_2 \tag{5.16}$$

We turn to reformulations where solutions evolve in terms of $N = \ln a$, the number of e-folds. For a function f, $\partial_N f = \dot{f}/H = f' \times H_0/H$. It is then straightforward to rewrite equations $F_1 = 0, F_2 = 0, E^i = 0$ respectively as follows

$$\boxed{\begin{aligned} \tilde{H} - \sqrt{\Omega_{r0}e^{-4N} + \Omega_{m0}e^{-3N} + \Omega_{k0}e^{-2N} + \tfrac{1}{3}\left(\tfrac{\tilde{H}^2}{2}g_{ij}\partial_N\varphi^i\partial_N\varphi^j + \tilde{V}\right)} = 0 \\ 2\tilde{H}\partial_N\tilde{H} + 4\Omega_{r0}e^{-4N} + 3\Omega_{m0}e^{-3N} + 2\Omega_{k0}e^{-2N} + \tilde{H}^2 g_{ij}\partial_N\varphi^i\partial_N\varphi^j = 0 \\ \tilde{H}^2\partial_N^2\varphi^i + \tilde{H}\partial_N\tilde{H}\partial_N\varphi^i + \tilde{H}^2\Gamma^i_{jk}\partial_N\varphi^j\partial_N\varphi^k + 3\tilde{H}^2\partial_N\varphi^i + g^{ij}\partial_j\varphi^j\tilde{V} = 0 \end{aligned}} \tag{5.17}$$

where $\tilde{V} = \frac{V}{H_0^2}$, $\tilde{H} \equiv \frac{H}{H_0}$. One of the equations is redundant in view of (5.15); we will come back to this point. The system depends on the parameters $\Omega_{r0}, \Omega_{m0}, \Omega_{k0}$ and $g_{ij}, \tilde{V}$, and a solution is now given by the functions $\tilde{H}, \varphi^i$ that depend on N. The initial conditions are set at t_0, namely $a = 1$ or $N = 0$, for which we get $\tilde{H}(0) = 1$. We are once again left with specifying those for $\varphi^i, \partial_N\varphi^i$; since $\dot{\varphi}^i = \tilde{H}\partial_N\varphi^i$, we get $\varphi^i(0) = \partial_N\varphi^i(0)$. We then get from above the relations

$$g_{ij}\partial_N\varphi^i\partial_N\varphi^j(0) = 3\Omega_{\varphi 0}(1 + w_{\varphi 0}), \qquad \tilde{V}(0) = \tfrac{3}{2}\Omega_{\varphi 0}(1 - w_{\varphi 0})\,. \tag{5.18}$$

As before, for a single canonical field, and an invertible potential function, we can trade the field initial conditions for the data $\Omega_{\varphi 0}, w_{\varphi 0}$, with the (typically positive) sign of the field speed. Taking again the fiducial values , we are left to specify $\tilde{V}, w_{\varphi 0}$ to get a solution. If we solve $E^i = 0, F_2 = 0$, the relation (5.15) implies that F_1 is a constant. Evaluating that constant at t_0 using the initial conditions, we get the left-hand side of (5.17a) to be

$$1 - \sqrt{\Omega_{r0} + \Omega_{m0} + \Omega_{k0} + \Omega_{\varphi 0}} \tag{5.19}$$

5.2 Numerical solution examples and domination phases

We numerically obtain examples of solution to the system of equation detailed in the previous sub-section.Obtaining realistic solution from the model means, a solution that describes a universe history which exhibits

- The universe begins in a **kination phase**, an early era where the scalar field's kinetic energy dominates the total energy budget ($\Omega_\phi \approx 1$).

- This is rapidly followed by the standard **radiation-dominated era** ($\Omega_r \approx 1$), during which Ω_ϕ becomes subdominant.

- Next, the universe transitions to the **matter-dominated era** ($\Omega_m \approx 1$).

- Finally, the solution evolves into the current **dark energy-dominated era**, where Ω_ϕ rises again to become the dominant component ($\Omega_\phi \approx 0.7$ today).

Those are captured by respective energy density parameters $\Omega_r, \Omega_m, \Omega_\varphi$, where $0 < \Omega_n < 1$ gives the portion of each component. We do so by restricting to a flat universe i.e $k = 0$ or $\Omega_{k0} = 0$ and a single canonical scalar field. We use fiducial values in agreement to the fiducial values agreeing with the CMB on the flat ΛCDM model.

$$\Omega_{r0} = 0.0001 \ , \Omega_{m0} = 0.3149 \ , \Omega_{\phi0} = 0.685 \ , \Omega_{k0} = 0 \tag{5.20}$$

We are then left to specify V and $w_{\phi0}$.We will consider three different potential.

- A constant potential falling back to ΛCDM with $w_\varphi = -1$.

- A exponential potential with different values of $w_{\varphi0}$.

- A hilltop potential with different values of $w_{\varphi0}$.

ΛCDM According to (5.15), solving $F_1 = 0$ is sufficient to have the complete solution. Let us first note that since $\rho_\phi = \Lambda$ is constant, the expressions (5.10-5.12) of the Ω_n simplify:

$$\begin{aligned}
\Omega_r &= \frac{\Omega_{r0}a^{-4}}{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{\phi0}} \\
\Omega_m &= \frac{\Omega_{m0}a^{-3}}{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{\phi0}} \\
\Omega_\phi &= \frac{\Omega_{\phi0}}{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{\phi0}}.
\end{aligned} \tag{5.21}$$

Figure 2 shows the evolution of Ω_n in terns of the number of e-folds N.[5]

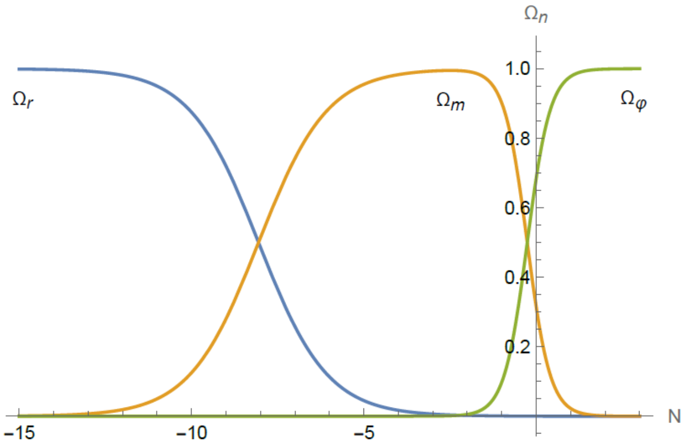

Figure 2: Evolution of the Ω_n for ΛCDM

Exponential Quintessence For a single scalar field, consider an exponential potential,

$$V(\varphi) = V_0 e^{-\lambda\varphi}\,, \qquad \lambda\,, V_0 > 0 \tag{5.22}$$

Redefining the field we can absorb the constant and potential only characterized by λ. We specify $w_{\varphi 0} = -0.51073604885$ allowing the radiation phase to start around $N = -20$. We then numerically solve the dynamical equations. The solution is presented in figure 3.[5]

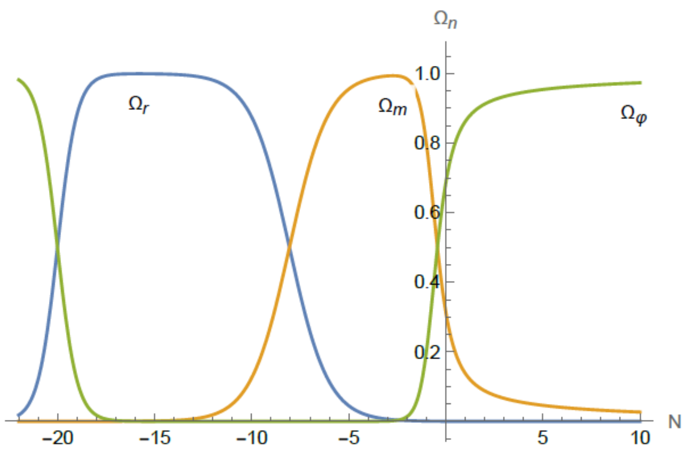

Figure 3: Evolution of the Ω_n for Exponential Quintessence

Hilltop Quintessence Consider the hilltop potential

$$V(\phi) = V_0 \left(1 - \frac{\kappa^2}{2}\phi^2\right), \qquad V_0, \kappa > 0, \tag{5.23}$$

restricted to the field-range where $V > 0$. Set for the numerical example

$$\widetilde{V}_0 = \frac{V_0}{H_0^2} = 5, \qquad \kappa = \frac{1}{2}.$$

Assuming an accelerating solution today implies $\widetilde{V}_0 > 2\Omega_{\phi 0} \approx 1.3700$ and the value chosen can't be lowered much.For this example we tune $w_{\phi 0} = -0.76201230846$ so that radiation domination starts around $N = -20$. Solving the equations we get the following figure.[5]

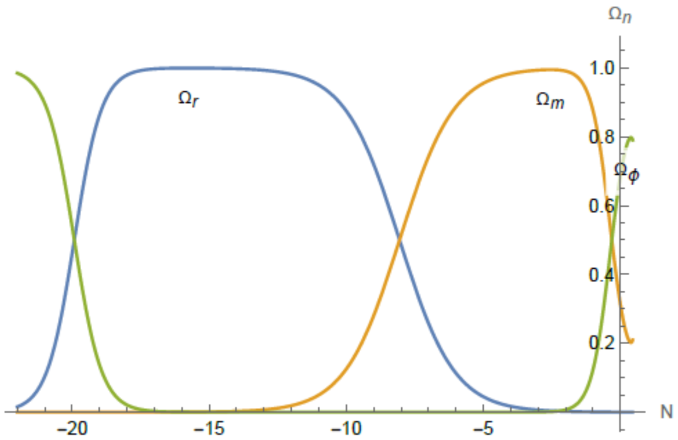

Figure 4: Evolution of the Ω_n for Hilltop Quintessence

A crucial observation from the solution is that the scalar field, ϕ, remains "frozen" at a nearly constant value for the vast majority of cosmic history. During the entire radiation and matter-dominated eras, its value changes very little. The field only begins to rise its potential in the very late universe, near the present day.Both Quintessence model shows the thawing of the field in a similar manner shown the the following figure.[5]

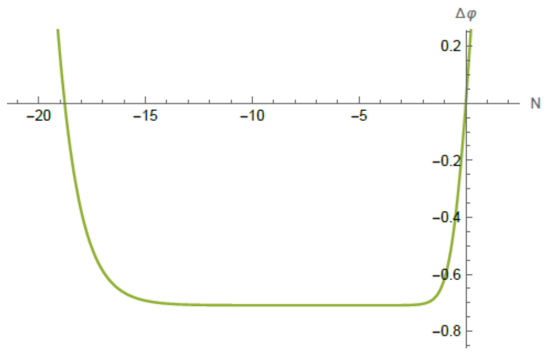

Figure 5: Evolution of φ for Exponential Quintessence

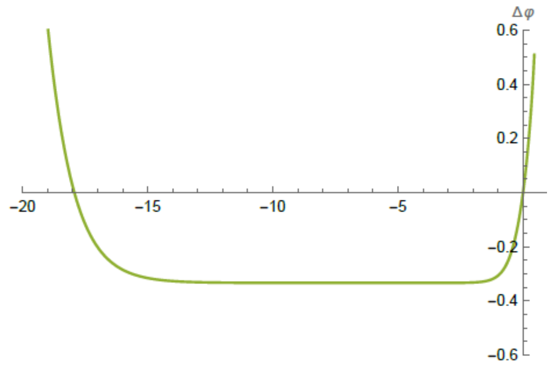

Figure 6: Evolution of φ for Hilltop Quintessence

Hence the Quintessence models gives identical observation to ΛCDM model for most of history (the "frozen" phase) and only begin to show their true, dynamical nature in the recent universe (the "thawing" phase).

6 Conclusion

Cosmological constant is considered to be the possible candidate for dark energy, driving the accelerated expansion of the universe. ΛCDM model established based on the cosmological constant has its theoretical and observational success. However, the value of cosmological constant doesn't overlap with observation and creates radiative instability and repeated loop correction as shown in the study. From our numerical study we have shown that Quintessence models fit the domination phases of the universe correctly as the ΛCDM model.We have found that the scalar field remains "frozen" by Hubble friction for the vast majority of cosmic history, causing its equation of state to perfectly mimic a cosmological constant.Only in the late-time universe, as Hubble friction weakens, does the field "thaw" and begin to evolve, causing w_φ to rise above -1.

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