Surface Integral Equation

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1 Integral Equation Formulation

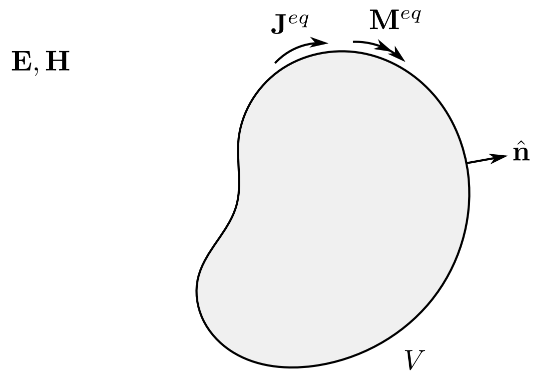

Figure 1: A depiction of scatterer occupying the region V. The equivalent surface currents on the surface S.

In this document, we present a brief discussion on the formulation of SIEs for the problem described in Figure 1. Through this derivation, we utilize the duality of EM fields to speed up the derivation. The objective is to illustrate the basics of applying the Method of Moments (Matrix Method) for the 2D PEC scattering problem using point-match technique for simplicity. The formulation starts with considering sources defined in volume V radiating in unbounded medium. The resulting fields are

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}) &= -jk\eta \int_V \underline{\boldsymbol{G}}(\boldsymbol{r}|\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') d\boldsymbol{r}' - \int_V \nabla \times \underline{\boldsymbol{G}}(\boldsymbol{r}|\boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}') d\boldsymbol{r}' \\
\boldsymbol{H}(\boldsymbol{r}) &= -jk\frac{1}{\eta} \int_V \underline{\boldsymbol{G}}(\boldsymbol{r}|\boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}') d\boldsymbol{r}' + \int_V \nabla \times \underline{\boldsymbol{G}}(\boldsymbol{r}|\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') d\boldsymbol{r}',
\end{aligned} \tag{1}$$

where $k = k_0\sqrt{\mu\epsilon}$ and $k_0 = \omega/c_0 = \omega\sqrt{\mu_0\epsilon_0}$, $\eta = \eta_0\sqrt{\mu/\epsilon}$, and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$. Also,

$$\underline{\boldsymbol{G}}(\boldsymbol{r}|\boldsymbol{r}') = \left[\underline{\boldsymbol{I}} + \frac{\nabla\nabla}{k^2}\right] g(\boldsymbol{r}|\boldsymbol{r}') \tag{2}$$

and

$$g(\boldsymbol{r}|\boldsymbol{r}') = \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \quad |\boldsymbol{r}-\boldsymbol{r}'| \neq 0. \tag{3}$$

Considering the electric field representation, we can write

$$\begin{aligned}\int_V \nabla\nabla g(\boldsymbol{r}|\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' &= -\nabla\int_V \nabla' g(\boldsymbol{r}|\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' \\ &= -\nabla\oint_S g(\boldsymbol{r}|\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')\cdot\hat{\boldsymbol{n}}'ds' + \nabla\int_V g(\boldsymbol{r}|\boldsymbol{r}')\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' \\ &= \int_V \nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')\nabla g(\boldsymbol{r}|\boldsymbol{r}')d\boldsymbol{r}',\end{aligned} \tag{4}$$

since

$$\oint_S g(\boldsymbol{r}|\boldsymbol{r}')\big[\boldsymbol{J}(\boldsymbol{r}')\cdot\hat{\boldsymbol{n}}'\big]ds' = 0. \tag{5}$$

Also, we can write

$$\int_V \Big[\nabla\times g(\boldsymbol{r}|\boldsymbol{r}')\underline{\boldsymbol{I}}\Big]\cdot\boldsymbol{M}(\boldsymbol{r}')d\boldsymbol{r}' = \int_V \nabla g(\boldsymbol{r}|\boldsymbol{r}')\times\boldsymbol{M}(\boldsymbol{r}')d\boldsymbol{r}'. \tag{6}$$

We can combine these expressions into (1) and obtain

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) = -jk\eta\int_V g(\boldsymbol{r}|\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' - j\frac{\eta}{k}\int_V \nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')\nabla g(\boldsymbol{r}|\boldsymbol{r}')d\boldsymbol{r}' \\ + \int_V \boldsymbol{M}(\boldsymbol{r}')\times\nabla g(\boldsymbol{r}|\boldsymbol{r}')d\boldsymbol{r}'\end{aligned} \tag{7}$$

and

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r}) = -jk\frac{1}{\eta}\int_V g(\boldsymbol{r}|\boldsymbol{r}')\boldsymbol{M}(\boldsymbol{r}')d\boldsymbol{r}' - j\frac{1}{k\eta}\int_V \nabla'\cdot\boldsymbol{M}(\boldsymbol{r}')\nabla g(\boldsymbol{r}|\boldsymbol{r}')d\boldsymbol{r}' \\ - \int_V \boldsymbol{J}(\boldsymbol{r}')\times\nabla g(\boldsymbol{r}|\boldsymbol{r}')d\boldsymbol{r}'.\end{aligned} \tag{8}$$

From Love's surface equivalence principle, it is possible to define the equivalent currents on the surface S as follows

$$\begin{aligned}\boldsymbol{E}\times\hat{\boldsymbol{n}} &= \boldsymbol{M}^{eq} \\ \hat{\boldsymbol{n}}\times\boldsymbol{H} &= \boldsymbol{J}^{eq},\end{aligned} \tag{9}$$

and the scattered fields become

$$\boldsymbol{E}^s(\boldsymbol{r}) = -jk\eta \oint_S g(\boldsymbol{r}|\boldsymbol{r}')\boldsymbol{J}^{eq}(\boldsymbol{r}')ds' - j\frac{\eta}{k}\oint_S \nabla' \cdot \boldsymbol{J}^{eq}(\boldsymbol{r}')\nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' + \oint_S \boldsymbol{M}^{eq}(\boldsymbol{r}') \times \nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' \tag{10}$$

and

$$\boldsymbol{H}^s(\boldsymbol{r}) = -jk\frac{1}{\eta} \oint_S g(\boldsymbol{r}|\boldsymbol{r}')\boldsymbol{M}^{eq}(\boldsymbol{r}')ds' - j\frac{1}{k\eta}\oint_S \nabla' \cdot \boldsymbol{M}^{eq}(\boldsymbol{r}')\nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' - \oint_S \boldsymbol{J}^{eq}(\boldsymbol{r}') \times \nabla g(\boldsymbol{r}|\boldsymbol{r}')ds'. \tag{11}$$

Considering the total fields

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}^i + \boldsymbol{E}^s \\ \boldsymbol{H} &= \boldsymbol{H}^i + \boldsymbol{H}^s, \end{aligned} \tag{12}$$

we can obtain the Stratton-Chu EFIE

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}^i(\boldsymbol{r}) &- jk\eta \oint_S [\hat{\boldsymbol{n}}' \times \boldsymbol{H}(\boldsymbol{r}')]g(\boldsymbol{r}|\boldsymbol{r}')ds' \\ &+ \oint_S [\boldsymbol{E}(\boldsymbol{r}') \times \hat{\boldsymbol{n}}'] \times \nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' \\ &- \oint_S [\hat{\boldsymbol{n}}' \cdot \boldsymbol{E}(\boldsymbol{r}')]\nabla g(\boldsymbol{r}|\boldsymbol{r}')ds', \end{aligned} \tag{13}$$

and the Stratton-Chu MFIE is

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}^i(\boldsymbol{r}) &- jk\frac{1}{\eta} \oint_S [\boldsymbol{E}(\boldsymbol{r}') \times \hat{\boldsymbol{n}}']g(\boldsymbol{r}|\boldsymbol{r}')ds' \\ &- \oint_S [\hat{\boldsymbol{n}}' \times \boldsymbol{H}(\boldsymbol{r}')] \times \nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' \\ &- \oint_S [\hat{\boldsymbol{n}}' \cdot \boldsymbol{H}(\boldsymbol{r}')]\nabla g(\boldsymbol{r}|\boldsymbol{r}')ds'. \end{aligned} \tag{14}$$

In the above, we used

$$\begin{aligned}\nabla \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{H}] &= -j\frac{k}{\eta}[\hat{\boldsymbol{n}} \cdot \boldsymbol{E}] \\ \nabla \cdot [\boldsymbol{E} \hat{\times} \boldsymbol{n}] &= -jk\eta[\hat{\boldsymbol{n}} \cdot \boldsymbol{H}].\end{aligned} \tag{15}$$

Also, it is possible to find the identity

$$[\boldsymbol{E} \times \hat{\boldsymbol{n}}'] \times \nabla g(\boldsymbol{r}|\boldsymbol{r}') - [\hat{\boldsymbol{n}}' \cdot \boldsymbol{E}]\nabla g(\boldsymbol{r}|\boldsymbol{r}') = \boldsymbol{E} \times [\hat{\boldsymbol{n}}' \times \nabla g(\boldsymbol{r}|\boldsymbol{r}')] - \boldsymbol{E}[\hat{\boldsymbol{n}}' \cdot \nabla g(\boldsymbol{r}|\boldsymbol{r}')] \tag{16}$$

In order to evaluate the integral over S at the singularity, we considered a hemisphere and simplify the evaluation by setting $\boldsymbol{r} = 0$ and consider

$$\frac{\partial g(0|\boldsymbol{r}')}{\partial n'} = -\hat{\boldsymbol{r}}' \cdot \frac{\partial}{\partial r'} \frac{e^{-jkr'}}{4\pi r'} \hat{\boldsymbol{r}}' = \left[jk + \frac{1}{r'}\right] \frac{e^{-jkr'}}{4\pi r'}. \tag{17}$$

Hence, using the identity (16) in (13) and (14) we obtain

$$\begin{aligned}&\text{P.V.} \oint_S \psi(\boldsymbol{r}') \frac{\partial g(0|\boldsymbol{r}')}{\partial n'} ds' = \\ &\qquad \lim_{r' \to 0} \int_0^{2\pi} \int_0^{\pi/2} \psi(\boldsymbol{r}') \left[jk + \frac{1}{r'}\right] \frac{e^{-jkr'}}{4\pi r'} (r')^2 \sin(\theta') d\theta' d\varphi' = \frac{\psi(0)}{2}.\end{aligned} \tag{18}$$

Considering the principle value integral to extract the singularity from the surface when the observation point is on S, we can obtain

$$\begin{aligned}&\boldsymbol{E}^i(\boldsymbol{r}) - jk\eta \text{P.V.} \oint_S \underbrace{[\hat{\boldsymbol{n}}' \times \boldsymbol{H}(\boldsymbol{r}')]}_{\boldsymbol{J}^{eq}} g(\boldsymbol{r}|\boldsymbol{r}') ds' \\ &\quad + \text{P.V.} \oint_S \underbrace{[\boldsymbol{E}(\boldsymbol{r}') \times \hat{\boldsymbol{n}}']}_{\boldsymbol{M}^{eq}} \times \nabla g(\boldsymbol{r}|\boldsymbol{r}') ds' \\ &\quad - \text{P.V.} \oint_S \underbrace{[\hat{\boldsymbol{n}}' \cdot \boldsymbol{E}(\boldsymbol{r}')]}_{q_e^{eq}/\epsilon_0\epsilon} \nabla g(\boldsymbol{r}|\boldsymbol{r}') ds' \\ &\quad = \begin{cases} \boldsymbol{E}(\boldsymbol{r}), & \boldsymbol{r} \notin V \\ \frac{1}{2}\boldsymbol{E}(\boldsymbol{r}), & \boldsymbol{r} \in S \\ 0, & \boldsymbol{r} \in V \end{cases}\end{aligned} \tag{19}$$

and

$$
\begin{aligned}
&\boldsymbol{H}^i(\boldsymbol{r}) - jk\frac{1}{\eta}\text{P.V.}\oint_S [\underbrace{\boldsymbol{E}(\boldsymbol{r}') \times \hat{\boldsymbol{n}}'}_{\boldsymbol{M}^{eq}}] g(\boldsymbol{r}|\boldsymbol{r}')ds' \\
&\quad - \text{P.V.}\oint_S [\underbrace{\hat{\boldsymbol{n}}' \times \boldsymbol{H}(\boldsymbol{r}')}_{\boldsymbol{J}^{eq}}] \times \nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' \\
&\quad - \text{P.V.}\oint_S [\underbrace{\hat{\boldsymbol{n}}' \cdot \boldsymbol{H}(\boldsymbol{r}')}_{q_m^{eq}/\mu_0\mu}] \nabla g(\boldsymbol{r}|\boldsymbol{r}')ds' \\
&\quad = \begin{cases} \boldsymbol{H}(\boldsymbol{r}), & \boldsymbol{r} \notin V \\ \frac{1}{2}\boldsymbol{H}(\boldsymbol{r}), & \boldsymbol{r} \in S \\ 0, & \boldsymbol{r} \in V \end{cases}
\end{aligned}
\tag{20}
$$

For PEC scatterer, we have the following boundary conditions on S

$$
\begin{aligned}
\hat{\boldsymbol{n}} \times \boldsymbol{E} &= 0 \\
\hat{\boldsymbol{n}} \times \boldsymbol{H} &= \boldsymbol{J}_s \\
\hat{\boldsymbol{n}} \cdot \boldsymbol{D} &= q_{es} \\
\hat{\boldsymbol{n}} \cdot \boldsymbol{B} &= 0.
\end{aligned}
\tag{21}
$$

2 Matrix Method

Here, we quickly review the MM which will be applied later to solve the SIE. Considering the linear integro-differential operator $\mathcal{L}$, forcing function f, and unknown function ϕ, the following integral equation:

$$\mathcal{L}\{\phi\} = f. \tag{22}$$

Expanding the unknown function using N orthonormal basis functions ϕ_n as follows

$$\phi \approx \sum_{n=1}^{N} \alpha_n \phi_n. \tag{23}$$

Hence, the problem becomes

$$\sum_{n=1}^{N} \alpha_n \mathcal{L}\{\phi_n\} = f. \tag{24}$$

This equation is insufficient to solve for the coefficients α_n, hence we introduce the weighting function w_m and apply the convolution integral

$$\sum_{n=1}^{N} \alpha_n \langle \mathcal{L}\{\phi_n\}, w_m \rangle = \langle f, w_m \rangle, \quad m = 1, 2, \ldots, N. \tag{25}$$

The final matrix formulation becomes

$$\begin{bmatrix} \langle \mathcal{L}\{\phi_1\}, w_1 \rangle & \ldots & \langle \mathcal{L}\{\phi_N\}, w_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathcal{L}\{\phi_1\}, w_N \rangle & \ldots & \langle \mathcal{L}\{\phi_N\}, w_N \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} \langle f, w_1 \rangle \\ \vdots \\ \langle f, w_N \rangle \end{bmatrix}. \tag{26}$$

3 2D Problem

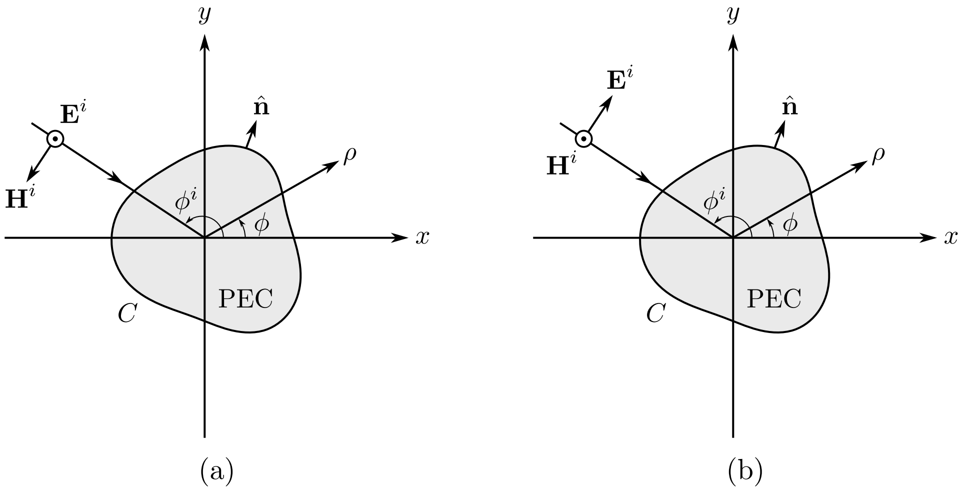

Figure 2: A depiction of scattering from PEC occupying the region S enclosed by the contour C. (a) TM (b) TE

In order to apply the MM for solving the SIEs in (19) and (20), we reduce the 3D problem into 2D as shown in Figure 2. Note that, this problem can be decomposed into TM and TE problems. The 2D Green function is

$$g(\boldsymbol{\rho}|\boldsymbol{\rho}') = \frac{1}{4j} H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|). \tag{27}$$

Here, we use pulse expansion functions and point matching for simplicity. We discretize the scatterer contour C into N line segments as shown in Figure 3. Here, we define the following local coordinate parameters for each segment as follows

$$\boldsymbol{l}_n = l_n \hat{\boldsymbol{l}}_n = (x_{n+1} - x_n)\hat{\boldsymbol{x}} + (y_{n+1} - y_n)\hat{\boldsymbol{y}}, \tag{28}$$

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{l}}_n \times \hat{\boldsymbol{z}} = \frac{1}{l_n}\left[(y_{n+1} - y_n)\hat{\boldsymbol{x}} - (x_{n+1} - x_n)\hat{\boldsymbol{y}}\right], \tag{29}$$

and

$$l_n = \sqrt{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2}, \quad \varphi_n = \tan^{-1}\left[\frac{y_{n+1} - y_n}{x_{n+1} - x_n}\right]. \tag{30}$$

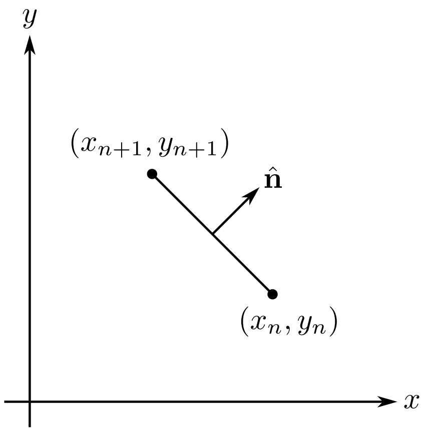

Figure 3: An example of segment n with the normal unit vector $\hat{\boldsymbol{n}}$.

The coordinates inside the segment are

$$\begin{aligned} x &= x_n + l\cos(\varphi_n) \\ y &= y_n + l\sin(\varphi_n), \end{aligned} \tag{31}$$

where $0 \le l \le l_n$ is the local variable. The observation points are the center of each segment are given by

$$\begin{aligned} \bar{x}_m &= x_m + \frac{l_m}{2}\cos(\varphi_m) \\ \bar{y}_m &= y_m + \frac{l_m}{2}\sin(\varphi_m), \end{aligned} \tag{32}$$

for $m = 1, \ldots, N$. Also, we may define

$$\begin{aligned} \boldsymbol{R}_{mn}(l') &= R_{mn}(l')\hat{\boldsymbol{R}}_{mn}(l') \\ &= \big[\bar{x}_m - (x_n + l'\cos(\varphi_n))\big]\hat{\boldsymbol{x}} + \big[\bar{y}_m - (y_n + l'\sin(\varphi_n))\big]\hat{\boldsymbol{y}}, \end{aligned} \tag{33}$$

and

$$R_{mn}(l') = \Big[\big[\bar{x}_m - (x_n + l'\cos(\varphi_n))\big]^2 + \big[\bar{y}_m - (y_n + l'\sin(\varphi_n))\big]^2\Big]^{1/2}. \tag{34}$$

The far-field approximation follows $|\boldsymbol{\rho}-\boldsymbol{\rho}'| \sim \rho - \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}' = \rho - [x'\cos(\varphi) + y'\sin(\varphi)]$ and $\nabla \sim -jk\hat{\boldsymbol{\rho}}$. Thus, we obtain[1]

$$g(\boldsymbol{\rho}|\boldsymbol{\rho}') \sim -j\sqrt{\frac{j}{8\pi k\rho}}e^{-jk\rho}e^{jk[x'\cos(\varphi)+y'\sin(\varphi)]}, \tag{35}$$

and the far-fields become

$$\begin{aligned}\boldsymbol{E}(\varphi) \sim -\eta\sqrt{\frac{jk}{8\pi\rho}}e^{-jk\rho}\Big[\big[\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\big]\cdot\int_S \boldsymbol{J}(\boldsymbol{r}')e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'}d\boldsymbol{\rho}' \\ - \frac{1}{\eta}\hat{\boldsymbol{\rho}}\times\int_S \boldsymbol{M}(\boldsymbol{r}')e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'}d\boldsymbol{\rho}'\Big],\end{aligned} \tag{36}$$

$$\begin{aligned}\boldsymbol{H}(\varphi) \sim -\frac{1}{\eta}\sqrt{\frac{jk}{8\pi\rho}}e^{-jk\rho}\Big[\big[\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\big]\cdot\int_S \boldsymbol{M}(\boldsymbol{r}')e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'}d\boldsymbol{\rho}' \\ + \eta\hat{\boldsymbol{\rho}}\times\int_S \boldsymbol{J}(\boldsymbol{r}')e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'}d\boldsymbol{\rho}'\Big].\end{aligned} \tag{37}$$

To compare the results, we use the scattering cross-section definition

$$\sigma = \lim_{\rho\to\infty} 2\pi\rho\frac{|E^s|^2}{|E^i|^2} = \lim_{\rho\to\infty} 2\pi\rho\frac{|H^s|^2}{|H^i|^2}. \tag{38}$$

[1] Here we used $H_0^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-j(z-\pi/4)}$ for $|z| \longrightarrow \infty$.

3.1 2D TM Problem

For a TM problem, the fields are

$$\begin{aligned} \boldsymbol{E} &= E_z\hat{\boldsymbol{z}} \\ \boldsymbol{H} &= H_x\hat{\boldsymbol{x}} + H_y\hat{\boldsymbol{y}}, \end{aligned} \tag{39}$$

and the incident field is defined as

$$\boldsymbol{E}^i = \hat{\boldsymbol{z}}E_0 e^{jk[x\cos(\varphi_i)+y\sin(\varphi_i)]}. \tag{40}$$

It is possible to show that the EFIE becomes[2]

$$\boldsymbol{E}_z^i = jk\eta \text{P.V.} \oint_C J_z(\boldsymbol{\rho}')g(\boldsymbol{\rho}|\boldsymbol{\rho}')d\boldsymbol{\rho}'. \tag{41}$$

After discretization, we obtain the final MM equation

$$E_0 e^{jk[\bar{x}_m\cos(\varphi_i)+\bar{y}_m\sin(\varphi_i)]} = \frac{k\eta}{4}\sum_{n=1}^{N} J_{zn}\int_0^{l_n} H_0^{(2)}(kR_{mn}(l'))dl'. \tag{42}$$

If $m \neq n$, we use the previous expression. However, if $m = n$ we use

$$E_0 e^{jk[\bar{x}_m\cos(\varphi_i)+\bar{y}_m\sin(\varphi_i)]} = \frac{k\eta}{4}\sum_{n=1}^{N} J_{zn}\left[\int_0^{l_n}\left(H_0^{(2)}(k|l'-l_n/2|) + j\frac{2}{\pi}\ln(l')\right)dl' - j\frac{2}{\pi}l_n\big[\ln(l_n)-1\big]\right], \tag{43}$$

by considering the singularity subtraction. The scattered far-field becomes

$$E_z^s \sim -\eta\sqrt{\frac{jk}{8\pi\rho}}e^{-jk\rho}\sum_{n=1}^{N} J_{zn}l_n e^{jk[\bar{x}_n\cos(\varphi)+\bar{y}_n\sin(\varphi)]}, \tag{44}$$

and the RCS can be written as

$$\sigma/\lambda = 2\pi\left|\frac{\eta}{2}\sum_{n=1}^{N} J_{zn}l_n e^{jk[\bar{x}_n\cos(\varphi)+\bar{y}_n\sin(\varphi)]}\right|^2. \tag{45}$$

[2]Note that $\hat{\boldsymbol{n}}\cdot\boldsymbol{E} = 0$ since $\boldsymbol{E} = E_z\hat{\boldsymbol{z}}$.

An example of the solution for a circular cylinder with radius a is given in Figure 4 compared to the exact solution

$$\sigma/\lambda = \frac{2}{\pi}\Big|\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(ka)}{H_n^{(2)}(ka)} \cos(n\phi)\Big|^2, \tag{46}$$

where

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases}. \tag{47}$$

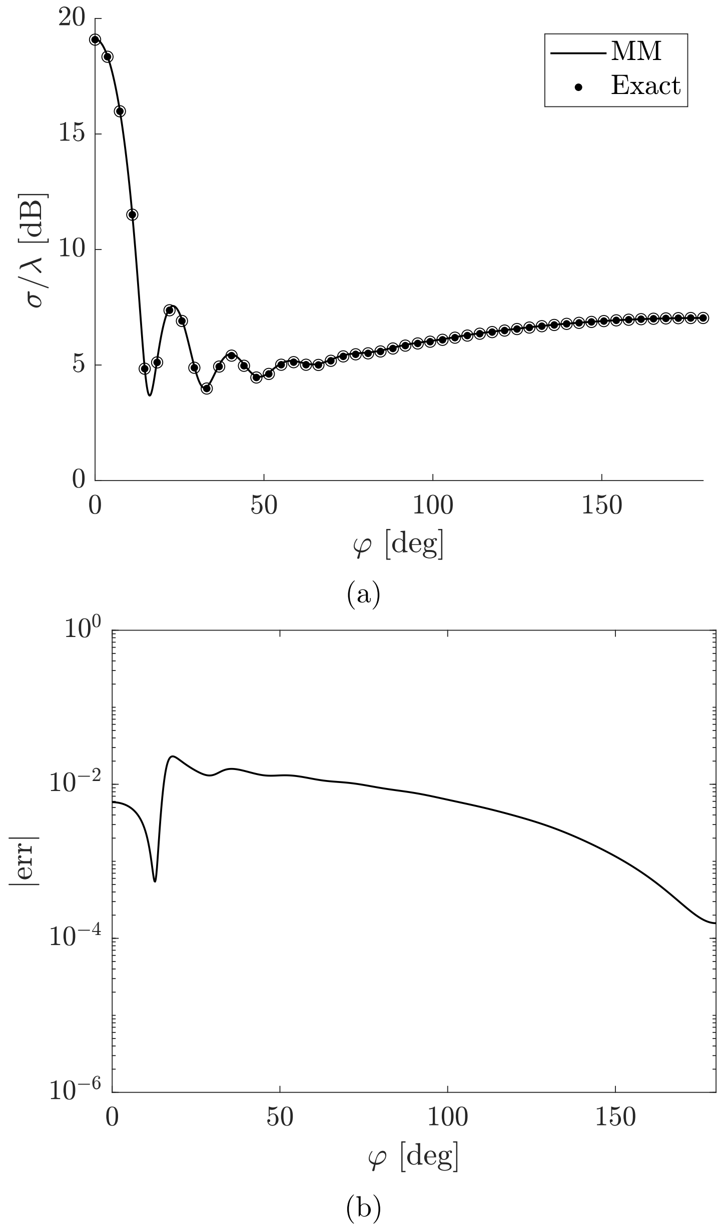

Figure 4: Radar cross-section for circular cylinder of radius $a = 1.6\lambda$ for TM incident field using $N = 150$. (a) RCS (b) Relative error.

3.2 2D TE Problem

For a TE problem, the fields are

$$\begin{aligned} \boldsymbol{H} &= H_z \hat{\boldsymbol{z}} \\ \boldsymbol{E} &= E_x \hat{\boldsymbol{x}} + E_y \hat{\boldsymbol{y}}. \end{aligned} \tag{48}$$

The incident field for MFIE is defined as

$$\boldsymbol{H}^i = \hat{\boldsymbol{z}} H_0 e^{jk[x\cos(\varphi_i)+y\sin(\varphi_i)]}. \tag{49}$$

In the above, we used

$$(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}) \times \nabla g = \nabla' g \times (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}) = (\nabla' g \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{z}}, \tag{50}$$

and the MFIE becomes[3]

$$\boldsymbol{H}_z^i = -\text{P.V.} \oint_C J_l(\boldsymbol{\rho}')(\hat{\boldsymbol{n}}' \cdot \nabla g(\boldsymbol{\rho}|\boldsymbol{\rho}'))d\boldsymbol{\rho}' + \frac{J_l}{2}. \tag{51}$$

After discretization, we obtain the final MM equation

$$H_0 e^{jk[\bar{x}_m\cos(\varphi_i)+\bar{y}_m\sin(\varphi_i)]} = \frac{\delta_{mn}}{2} + \frac{jk}{4}\sum_{n=1}^{N} J_{ln} \int_0^{l_n} \hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{R}}_{mn}(l') H_1^{(2)}(kR_{mn}(l'))dl' \tag{52}$$

More explicitly, if $m \neq n$ we use the expression

$$H_0 e^{jk[\bar{x}_m\cos(\varphi_i)+\bar{y}_m\sin(\varphi_i)]} = \frac{jk}{4}\sum_{n=1}^{N} J_{ln} \int_0^{l_n} \hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{R}}_{mn}(l') H_1^{(2)}(kR_{mn}(l'))dl'. \tag{53}$$

If $m = n$, we use the expression

$$H_0 e^{jk[\bar{x}_m\cos(\varphi_i)+\bar{y}_m\sin(\varphi_i)]} = \frac{1}{2}. \tag{54}$$

Similarly, the scattered far-field becomes

$$H_z^s \sim -\sqrt{\frac{jk}{8\pi\rho}} e^{-jk\rho} \sum_{n=1}^{N} J_{ln} l_n (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\rho}}) e^{jk[\bar{x}_n\cos(\varphi)+\bar{y}_n\sin(\varphi)]}. \tag{55}$$

[3]Note that $\hat{\boldsymbol{n}} \cdot \boldsymbol{H} = 0$ since $\boldsymbol{H} = H_z \hat{\boldsymbol{z}}$..

Note that

$$\hat{\boldsymbol{l}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{l}} \times \hat{\boldsymbol{\rho}}) \times \hat{\boldsymbol{n}} = [(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\rho}})\hat{\boldsymbol{l}} - (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{l}})\hat{\boldsymbol{\rho}}] \times \hat{\boldsymbol{n}} = (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\rho}})\hat{\boldsymbol{z}}. \tag{56}$$

Hence, the RCS can be written as

$$\sigma/\lambda = 2\pi \Big| \frac{1}{2} \sum_{n=1}^{N} J_{ln} l_n (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\rho}}) e^{jk[\bar{x}_n \cos(\varphi) + \bar{y}_n \sin(\varphi)]} \Big|^2. \tag{57}$$

An example of the solution for a circular cylinder with radius a is given in Figure 5 compared to the exact solution[4]

$$\sigma/\lambda = \frac{2}{\pi} \Big| \sum_{n=0}^{\infty} \varepsilon_n \frac{J_n'(ka)}{H_n^{(2)\prime}(ka)} \cos(n\phi) \Big|^2. \tag{58}$$

[4]We may use the identity $\mathcal{J}_n'(z) = -\mathcal{J}_{n+1}(z) + \frac{n}{z}\mathcal{J}_n(z)$ for $\mathcal{J}$ denotes J, Y, $H^{(1)}$, or $H^{(2)}$.

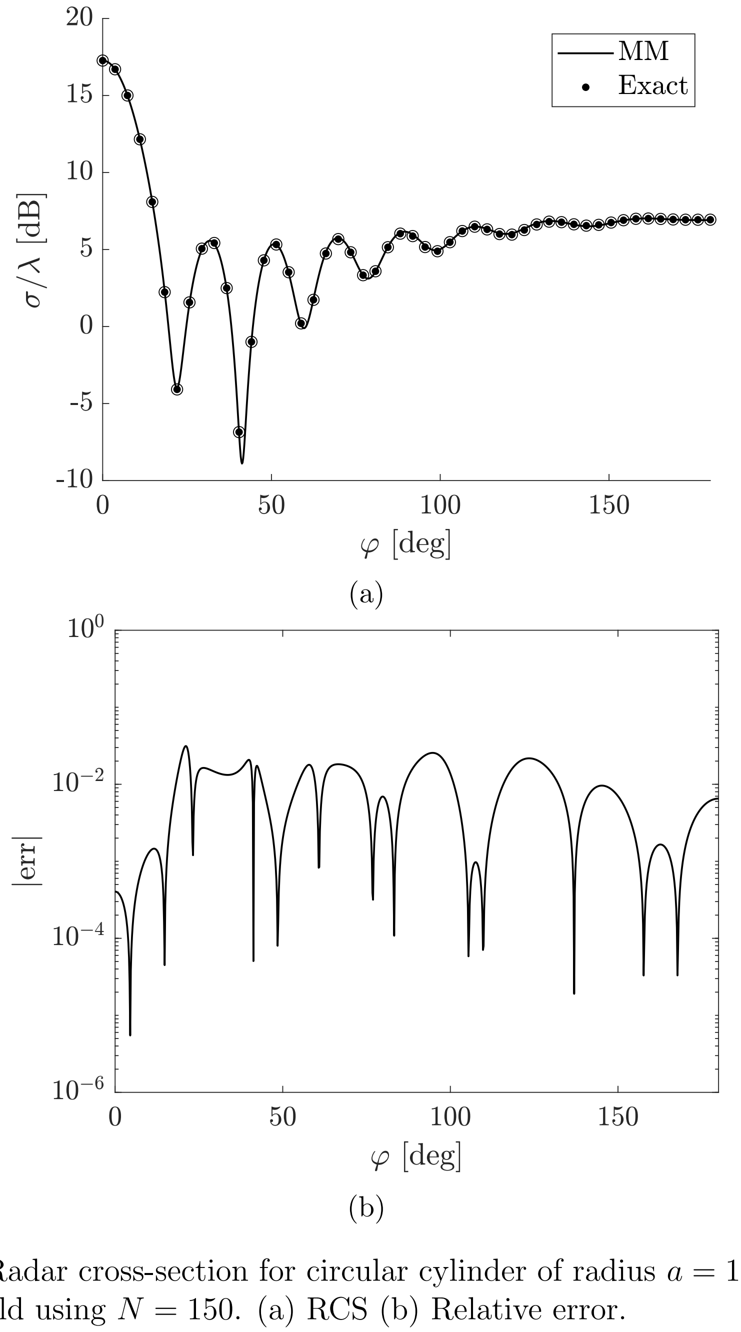

Figure 5: Radar cross-section for circular cylinder of radius $a = 1.6\lambda$ for TE incident field using $N = 150$. (a) RCS (b) Relative error.