Exercise 2: Modelling Free fall with fixed or varying Drag

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(Dated: December 2, 2019)

I. Introduction

The aim of the exercise is to create a program that plots values of height (y) and vertical speed (v_y) from an initial starting point of y_0. This will be done in 2 ways, one using a formula for y and v and another using Eulers method of differentiation. These will be done for a fixed drag as well as a varying drag that changes with height.

II. Part A - The Analytical Approach

For the first part of the experiment, the free fall was modelled using an analytical approach. This was done through these 2 formulas:

$$y = y_0 - \frac{m}{k} \ln\left[\cosh\left(\frac{t}{\sqrt{\frac{m}{kg}}}\right)\right] \quad (1)$$

$$v_y = -\sqrt{\frac{mg}{k}} \tanh\left(\frac{t}{\sqrt{\frac{m}{kg}}}\right) \quad (2)$$

where y_0 is the initial height, m is the mass of the object, g is the graviational field or $9.81ms^{-2}$). and k is a constant the correlates with air resistance and is defined by:

$$k = \frac{C_d p_0 A}{2} \quad (3)$$

with C_d being the drag coefficient, p_0 being the air density ($1.2kgm^{-3}$) and A being the cross sectional area for the object falling.

For this exercise, all the variables will be based off Felix Baumgartner who is a skydiver. Assuming felix is of average size and weight, the mass I'll be using will be 82kg, the C_d will be 1.2 and the area assuming he was in the 'eagle' positon with his body parallel to the ground, the area would be $0.7m^-2$. (for all of the models these value will be the same)

The program for this section worked by asking the use for the initial height,velocity and the number of points they wanted the graph over. Then 2 arrays were created , one for the height and the other for the velocity. These arrays were then plotted against the time values which spanned from 0 to the time when the height equalled zero in intervals defined by the number of points selected.

Both of these graphs show what I expected, with the velocity at a constant value, but then decreasing unitl terminal velocity

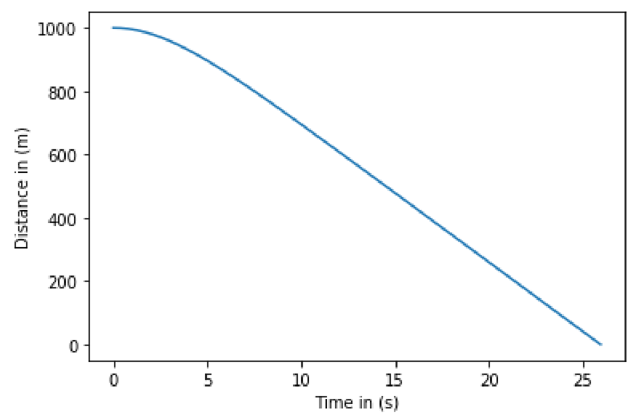

FIG. 1. A graph showing the distance against time for the skydiver falling 1000m

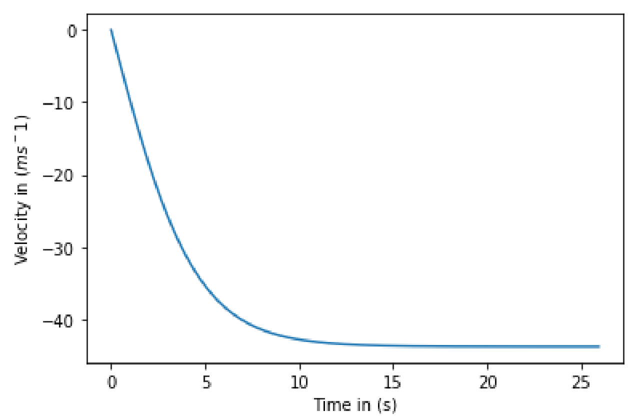

FIG. 2. A graph showing the velcoity against time for the skydiver falling 1000m

III. Part B - Using Euler's Method

A. Understanding Euler's method

Euler's Method is a way of approximating a differential equation and is the simplest range-kutta method. For an equation of (y_n, t_n), Euler's method allows us to find (y_{n+1}, t_{n+1}) for any first order differential equation

Consider an Equation $f(x, t)$ with a staring point of x_0 and t_0. To go to another value of this function, the gradient or rate of change can be found at this starting point and multiplied by δt then added to the staring point.

This can be done for subsequent points to obtain an approximation of the function, depending on how big or small δt is. This can be represented by:

$$x_{n+1} = x_n + \delta t f(x_n, t_n) \quad (4)$$

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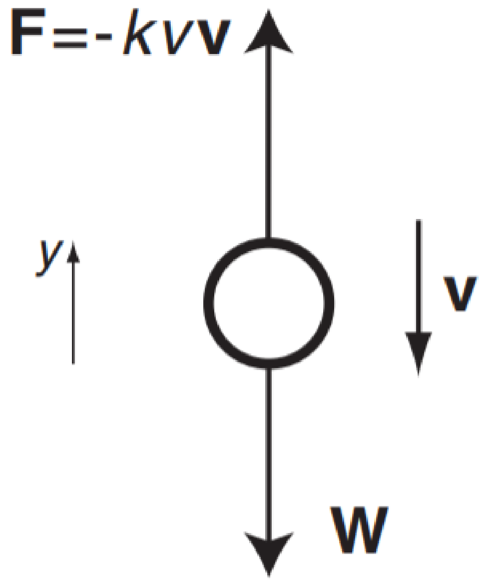

FIG. 3. A figure showing the forces that act on the skydiver as he falls

$$t_{n+1} = t_n + \delta t \tag{5}$$

B. Applications in free fall

For the case of the exercise, there is a problem. Because the equation of motion looks like:

$$m\frac{d^2y}{dx^2} = -mg - k|\frac{dy}{dt}|\frac{dy}{dt} \tag{6}$$

We cant use eulers theorem yet as this is a second order differential equation. However we can use euler's theorem by splitting it into two first order differential equations, one for the velocity and the other for the height.

$$m\frac{dv_y}{dt} = -mg - k|v_y|v_y \tag{7}$$

$$\frac{dy}{dt} = v_y \tag{8}$$

So for y_{n+1} and t_{n+1}, the result is the same as in the general case, with v_y instead of the function:

$$y_{n+1} = y_n + \delta t v_{y,n} \tag{9}$$

$$t_{n+1} = t_n + \delta t \tag{10}$$

for v_{n+1}, we divide equation7 by the mass to just det $\frac{dv_y}{dt}$, because we've taken y to be positive, the velocity will be negative and increasing in the negative y-direction. therefore:

$$v_{n+1} = v_n - \delta(g + \frac{k}{m}|v_{y,n}|v_y, n) \tag{11}$$

C. Results

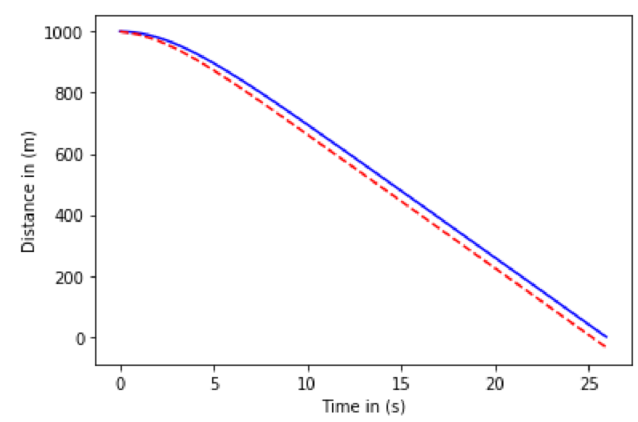

FIG. 4. A graph of the height against time for the skydiver falling from 1000m at rest. The whole blue line is the analytical method and the dotted red line is eulers method for a δt of 1 second

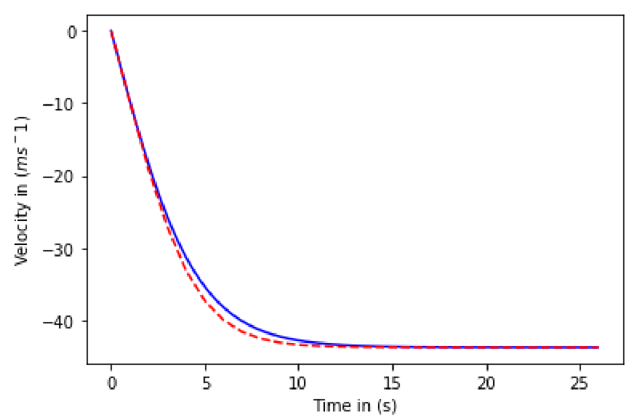

FIG. 5. A graph of the velocity against time for the skydiver falling from 1000m at rest. The whole blue line is the analytical method and the dotted red line is eulers method for a δt of 1 second

Comparing the results, eulers method generally follows the analytical result however not exactly. These differences can be attributed to truncation error. truncation error. This can be limited by reducing the step size, δt. Any δt value less than 0.25 becomes nearly indistinguishable from the analytical method.

When δt gets larger in size, the prediction completely breaks down, and no longer can no longer predict the model.

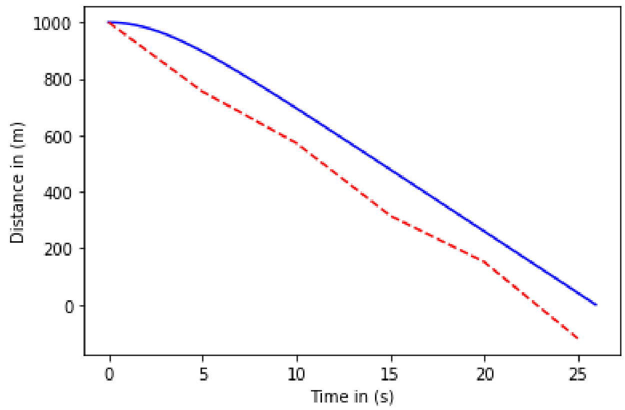

FIG. 6. A graph of the height against time for the skydiver falling from 1000m at rest. The δt is 5 seconds

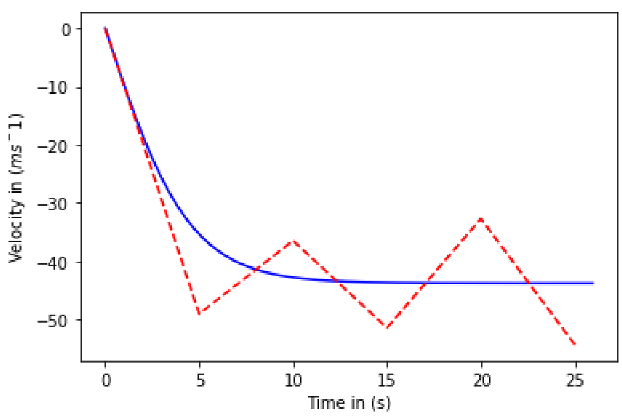

FIG. 7. A graph of the velocity against time for the skydiver falling from 1000m at rest. The δt is 5 seconds

By varying the ratio k/m, it can be seen that the fall time increases as the ration increases. By making the ration smaller and thereby increasing the mass, the fall time is shorter and the terminal velocity is greater.

IV. Part C - Varying density

To simulate more accurately the fall of Felix Baumgartner, the model needs to have a density that changes with height. The air density doesn't change much in say, 1000 metres but felix jumped from approximately 39000m, which will change the shape of the curve of the model significantly.

By using a scale height for the atmosphere and model the change in density as an exponential function:

$$\rho(y) = \rho_0 \exp\left(\frac{-y}{h}\right) \tag{12}$$

where h is equal to $7.64km^{-2}$

Implementing this into the code was straightforward, using a varying density for the analytical method isn't possible so Euler's method was used instead. The Function for the density was simply inserted into the while loop for euler's method.

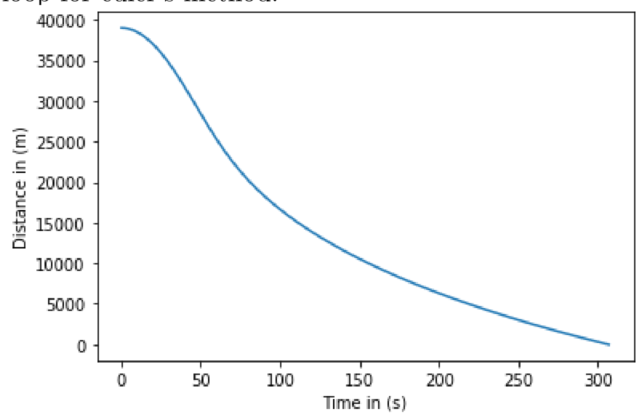

FIG. 8. A graph of height against time for a height of 39000m starting at rest. Here we can see that the skyiver falls quickly for the first 50 seconds then as the atmosphere gets thicker, the skydiver slows down to a more gradual speed

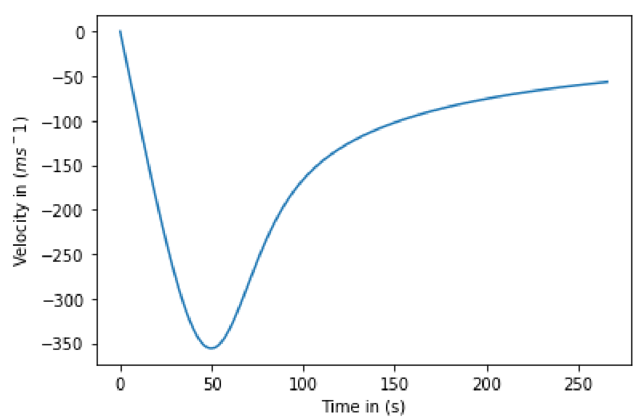

FIG. 9. A graph of velocity against time for a height of 39000m at rest. This correlates to the distance time graph, as the velocity increases rapidly up to 50 seconds, then air resistance has larger effect and slows the skydiver down. From this simulation, felix does break the sound barrier however the mass has been increase to 112kg to account for the 70 pound suit felix baumgautner wore

By varying the height of the jump, the maximum speed increases with the height of the jump. By increasing the ratio of $\frac{C_d A}{m}$ the maximum velocity decreases and the time taken to reach terminal velocity.