EE290S Final Project Report

Super-Resolution Spectral Detection for FMCW Lidar

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I. Introduction

As it was mentioned in the project proposal, development of integrated ranging system has become a hot topic, as compact and affordable ranging sensor is crucial for applications like self-driving cars and autonomous robots. Especially, FMCW lidar is considered as one of the most promising solutions due to its high sensitivity and ambient noise immunity, which is making it extremely hard to use traditional time-of-flight sensors for outdoor appilcations.

FMCW lidar creates continuous light whose center frequency (or wavelength) is modulated by certain waveform (e.g. triangular wave). Then, this “chirped” light is splitted and certain portion of the light is transmitted towards the target. The rest of modulated light is directly forwarded to the receiver. The receiver captures both the light that is reflected back from the target and direct-forwarded light from the modulator, and beat those two signals using a photodetector to produce a current signal whose frequency is equal to the frequency difference of two light signals. The larger the target distance, the higher the frequency difference or the current signal frequency. In sum, FMCW lidar projects the distance information of the remote target into the frequency domain of the final current signal at the photodetector output.

This implies that the depth resolution of the FMCW lidar is directly linked to the spectral resolution of the detector. Ideally, the lidar signal is just single sinusoidal tone whose frequency is a function of the distance $f_{target}(d)$, and let’s assume this is the case for the scope of this study. Given that, we can express the photocurrent signal in the frequency domain as follows:

$$X(f) = \frac{1}{2}A\delta(f - f_{target}) + \frac{1}{2}A^*\delta(f + f_{target}),\ A \in \mathbb{C} \tag{1}$$

We need to estimate f_{target} from the time domain signal $x(t) = |A|\cos(2\pi f_{target}t + \angle A)$. This is a classic problem of support-detection for line spectra.

However, in actual cases, we can only observe this signal within finite observation window T_{obs}. If the samling period of the analog-to-digital converter is T_{sample} and $N_{obs} = T_{obs}/T_{sample}$, the actual discrete-time signal we have as an input to the spectrum sensor is as follows.

$$x[k] = x(kT_{sample}),\ k \in \{0, 1, \ldots, N_{obs} - 1\} \tag{2}$$

Meanwhile, the bin size of the discrete Fourier transform is determined to be $1/T_{obs}$. In other words, the signal is band-limited in the time domain by T_{obs}.

The problem of super-resolution is to achieve better resolving power beyond this fundamental limit in the case where we can make certain special assumptions for the nature of the signal. Especially, super-resolution of point sources is relevant not only to the line-spectral estimation but also to various real-world applicaions including medical imaging [1], microscopy [2] and wireless network [3]. Among numerous works tried to address this issue, one of the most popular approaches is parametric methods, such as MUSIC [4]. This method is based on eigendecomposition of the sample covariance matrix, and its accuracy relies heavily on a priori knowledge of the signal structure and the quality of covariance matrix estimation. Thus, it does not work very well except for the case of only a few tones with moderate, additive white Gaussian noise.

Recently, super-resolution signal recovery method based on convex optimization is proposed [5]. Even though this method imposes another constraint for minimum separation between point sources, it does not require any other specific knowledge of the signal, and gives well-defined, localized error bound for support recovery in the presence of noise. Especially, the paper also provides practical implementation of the algorithm by formulating the equivalent optimization problem which can be solved by standard semidefinite programming.

In this project, I tried to examine the feasibility of using convex optimization-based spectrum estimation technique in the context of lidar receiver. Particularly, I tried to address following two questions:

- Assuming certain signal-to-noise ratio, can super-resolution detector actually achieve better estimation error compared to general DFT-based detector?
- What is the actual size of the problem? Is there a way to simplify the original optimization problem leveraging the nature of lidar signal?

The remainder of the report is organized as follows. In Section II, a brief review of the formulation of convex optimization problem for super-resolution and its error bound for support detection is presented. Section III begins with the definition of estimation accuracy, and then derives the accuracy of frequency estimation of both DFT and super-resolution detector for given signal-to-noise ratio with corresponding numerical examples. Section IV discusses the size of the problem in lidar systems, and potential strategy to reduce the computational complexity of the algorithm by making certain assumptions to the lidar signal.

II. Super-Resolution Spectral Support Detection Using Convex Optimization

This section provides a brief overview of the results from the works on convex optimization-based super-resolution detection [5] [6] [7]. First, the signal of interest is superpositions of point sources modeled as follows.

$$x(f) = \sum_j a_j \delta(f - f_j),\ F = \{f_j\} \subset [-0.5, 0.5],\ a_j \in \mathbb{C} \tag{3}$$

Point sources are infinite-bandwidth in time domain (here, I model the point sources in the frequency domain different from the original paper so that it fits our application). However, in the measured data y, we only acquire Fourier coefficients within the bandwidth $T_c = T_{obs}/2$, due to low-pass nature of the measurement.

$$x[k] = \int_{-0.5}^{0.5} x(f) e^{-i2\pi k f} df = \sum_j a_j e^{-i2\pi k f_j},\ k \in \mathbb{Z} \tag{4}$$

$$y(f) = \mathcal{F}_n x(f),\ y[k] = \begin{cases} x[k], & |k| \leq T_c \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$\mathcal{F}_n$ is the linear map capturing the coefficients within the bandwidth, and $n = 2T_c + 1$. Note that the signal is discrete in Fourier domain as the original signal is limited length. The main result from the paper is that by solving following convex problem, one can recover the original signal with infinite resolution as long as the minimum distance between sources is larger than $1.38/T_c$.

$$\min_{\tilde{x}} \|\tilde{x}\|_{TV} \quad \text{subject to} \quad \mathcal{F}_n \tilde{x} = y \tag{6}$$

$\|\cdot\|_{TV}$ is total variance, which is 1-norm equivalent in continuous domain. In the noisy setting where the measurement $y(f)$ is corrupted by additive noise $z(f)$ whose 2-norm is bounded by δ, we solve similar problem given as follows.

$$\min_{\tilde{x}} \|\tilde{x}\|_{TV} \quad \text{subject to} \quad \|\mathcal{F}_n \tilde{x} - y\|_2 \leq \delta \tag{7}$$

If you focus on the support recovery problem, one can guarantee that the estimated support $\widetilde{F}$ from this convex problem satisfies following error bound.

Theorem 1. *For any $f_i \in F$, if $|a_i| > C_1\delta$ there exists $\tilde{f}_i \in \widetilde{F}$ such that*

$$\epsilon_i = |f_i - \tilde{f}_i| \leq \frac{1}{T_c}\sqrt{\frac{C_2\delta}{|a_i| - C_1\delta}} \tag{8}$$

C_1 and C_2 are numerical constants.

However, those optimization problems are formulated on continuous domain, and it is not really possible to solve them numerically unless one discretize the support space. The paper also proves that solving following dual problem formulated as a semidefinite program is equivalent to solving the original problem, and presents several numerical examples to actually demonstrate the scheme.

$$\max_{\tilde{u}\in\mathbb{C}^n} \Re(y^*\mathcal{F}_n^*\tilde{u}) \quad \text{subject to} \quad \begin{bmatrix} X & \tilde{u} \\ \tilde{u}^* & 1 \end{bmatrix} \succeq 0, \quad \sum_{i=1}^{n-j} X_{i,i+j} = \begin{cases} 1, & j=0 \\ 0, & j=1,2,\ldots,n-1 \end{cases} \tag{9}$$

III. Performance Evaluation of Super-Resolution Detector in Actual Lidar Systems

In this section, we evaluate the performance of super-resolution support recovery scheme presented in the previous section in presence of noise, and compare it with standard DFT-based detectors. To that end, we first derive closed-form expressions of the mean-squared error of the frequency estimation for both DFT and super-resolution detector. Theoretical discussions are verified by numerical examples from Matlab implementation. Keep in mind that we are dealing with the case of only one harmonic tone in the frequency domain in the scope of this study.

A. DFT-based Detector

DFT-based detector simply performs DFT on the time domain data of length $2N$ and picks the frequency bin with the highest magnitude to estimate the frequency of incoming signal. As we have assumes that there are only one tone in the signal, the goal is essentially the same as N-ary incoherent MFSK receiver with matched filters, which is to locate the frequency bin where the incoming signal resides among N possible bins. To calculate the symbol error rate of MFSK receiver (SER), we use the fact that the probability distribution of the envelope of narrow-band white Gaussian noise is Rayleigh distribution, and also it is Ricean when it is a sinusoid in narrow-band Gaussian noise. Therefore, if the frequency of the incoming signal f_{target} is located in the kth bin F_k, the distribution of the magnitude of ith bin is expressed as follows:

$$P(y_i|f_{target}\in F_k) = \begin{cases} \frac{y_i}{\sigma^2} e^{-(y_i^2+a^2)/2\sigma^2} I_0\left(\frac{y_i a}{\sigma^2}\right), & i=k \\ \frac{y_i}{\sigma^2} e^{-y_i^2/2\sigma^2}, & \text{otherwise} \end{cases} \tag{10}$$

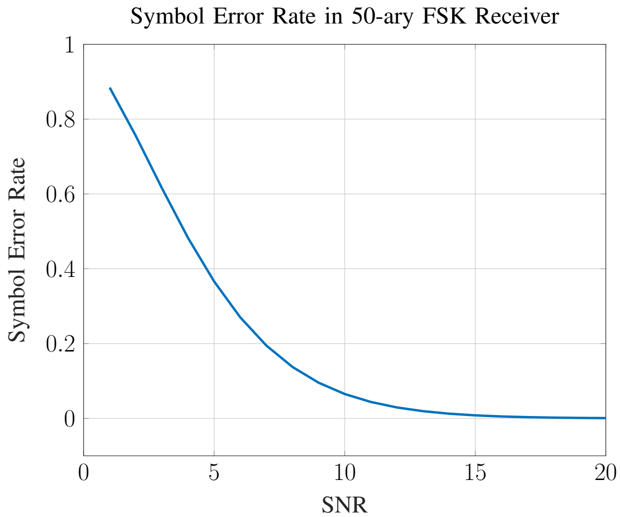
where $a = \sqrt{\frac{T_{obs}}{2}}|A|$, σ^2 is the power spectral density of the noise, and $I_0(\cdot)$ is the 0-th order modified Bessel function of the first kind. With this expression in hand, we can calculate the probability of corretly picking the kth bin, which is $P_{i=k} = 1 - SER$. Also, leveraging the fact that the distribution is symmetric for all other bins, the probability of picking $i(\neq k)$th bin is simply $P_{i\neq k} = SER/(N-1)$. SER drived from the distribution (10) is well known in the literature [8].

$$SER = \sum_{n=1}^{N-1} \frac{(-1)^{n+1}}{n+1}\binom{N-1}{n}\exp\left(-\frac{n}{n+1}SNR\right) \tag{11}$$

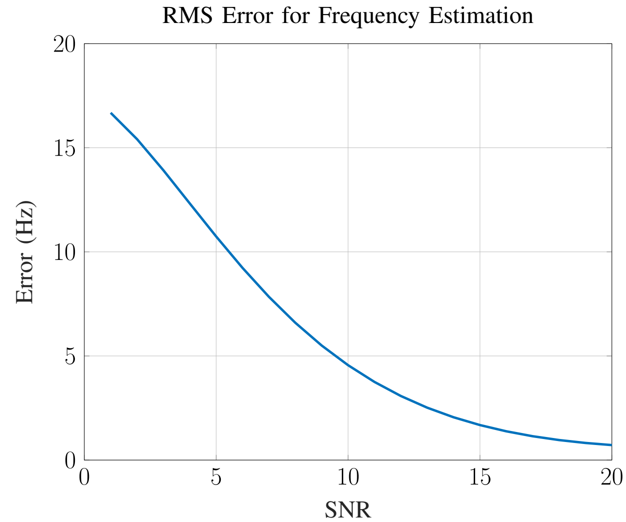
SNR is the signal-to-noise ratio within the kth frequency bin. Now the rms frequency estimation error is easily calculated from this result.

$$RMSE(f_{target}) = \sqrt{\sum_{i=1}^{N}(f_i - f_{target})^2 P_i}, \quad f_i = \frac{i}{2N} f_{sample} \tag{12}$$

Fig. 1. shows the SER and $RMSE$ values of DFT-based detector for different SNR level, when $N = 50$ and $f_{target} = 15.5Hz$. We can see that the rms error increases as the SNR goes down. Due to the numerical precision issues in Matlab, SER calculation was limited to the case of $N \leq 50$.



(a)



(b)

Fig. 1. (a) Symbol error rate for 50-ary FSK communication link (b) Root-mean-squared error of the frequency estimation by modeling DFT detector as a MFSK link, when $f_{target} = 15.5Hz$

B. *Super-Resolution Detector*

The upper bound for the support detection error ϵ for convex optimization-based spectral estimation framework, given that the upper bound for the second norm of the additive noise is δ, is previously stated in Theorem 1. Assuming that we have white Gaussian noise $S(\omega) = PSD$ in the freuqnecy domain, the magnitude of the noise on each time domain discrete sample $z[k]$ is modeled as an independent random variable with distribution $\mathcal{N}(0, \sigma^2)$ and $\sigma^2 = (PSD)(\omega_{sample}/2)/4$. As the second norm of the additive noise is rms sum of independent normal variables, the square of the second norm follows the Chi-squared distribution with degree of freedom N.

$$\|z\|_2^2 = \sigma^2 w, \quad w \sim \chi^2(N) \tag{13}$$

This implies that

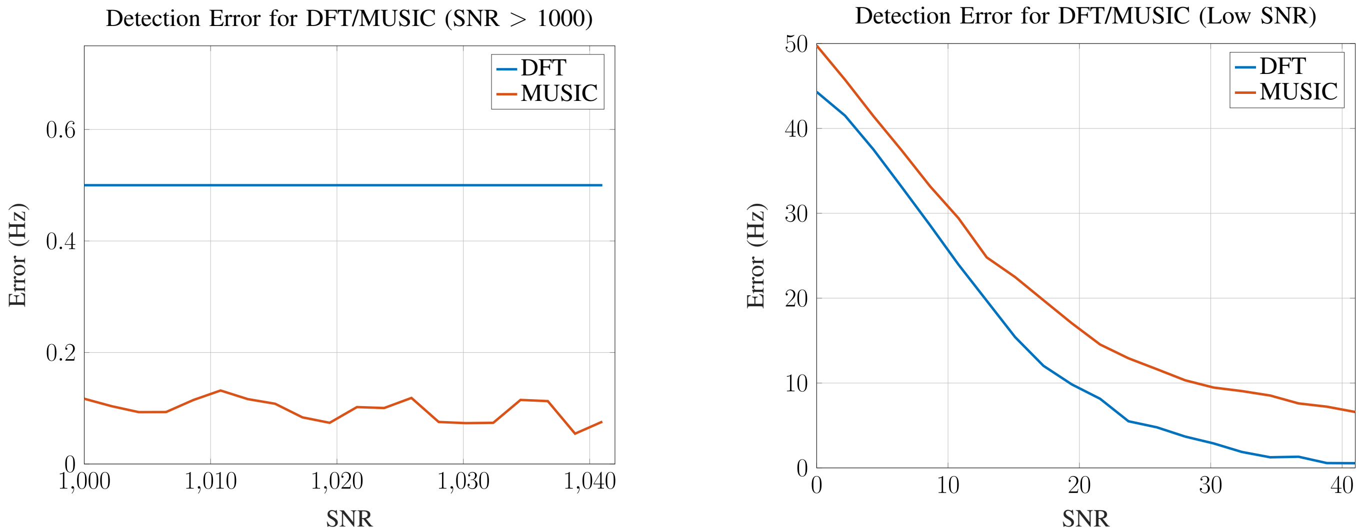
$$P(\delta = \Delta) = P(\|z\|_2^2 \leq \Delta^2) = P(x \leq (\Delta/\sigma)^2) \tag{14}$$

Due to (8), this is also the probability of ϵ being bounded by $\frac{2}{T_{obs}}\sqrt{\frac{C_2\Delta}{|a|-C_1\Delta}}$. It is not really possible to precisely determine the probability of particular ϵ and therefore the rms error since we do not have the information of direct relationship between ϵ and $\|z\|_2$. However, we can at least introduce a pessimistic assumption by replacing inequality in (8) with equality.

$$\epsilon(w) = \frac{2}{T_{obs}}\sqrt{\frac{C_2\|z\|_2}{|a_i| - C_1\|z\|_2}} = \frac{2}{T_{obs}}\sqrt{\frac{C_2\sigma\sqrt{w}}{|a_i| - C_1\sigma\sqrt{w}}} \tag{15}$$

This gives the upper bound of the error, and we can calculate the rms error as follows, and it can be computed numerically.

$$RMSE = \int_0^\infty \epsilon(w)^2 P(w)dw = \int_0^\infty \frac{4}{T_{obs}^2}\frac{C_2\sigma\sqrt{w}}{|a_i| - C_1\sigma\sqrt{w}}P(w)dw \tag{16}$$



(a) (b)

Fig. 2. Root-mean-squared error of the frequency estimation by DFT and MUSIC detector for (a) high SNR and (b) low SNR condition

C. Numerical Example

Based on discussions so far, I implemented both algorithms in Matlab to examine the estimation error versus the noise level. First, DFT-based detector was implemented for the following signal.

- Target frequency (f_{target}): $15.5Hz$
- Sampling rate: $200Hz$
- Number of points: 200, $T_{obs} = 1$

Also, support detection with MUSIC algorithm was exaluated using pmusic function in Matlab's toolbax. Fig. 2 shows the rms error of the support detection estimated from $10,000$ trials for both schemes. In high SNR case (Fig. 2a), DFT always picks the right bin that includes $f_{target} = 15.5Hz$, but due to the finite bin size $1/T_{obs} = 1Hz$, its estimation error is $0.5Hz \neq 0$. On the other hand, MUSIC algorithm can estimate the covariance matrix very precisely as the noise is almost negligible, so it indeed achieved super-resolution support detection ($RMSE \sim 0.1Hz$). However, in the low-SNR regime (Fig. 2b), the performance of MUSIC detector significantly degrades and actually is worse than DFT detector. Also note that the trend between error rate vs. SNR is very similar to what was predicted by the theory (Fig. 1b). It shows higher absolute error because there are 100 possible frequency bins in this example, instead of 50.

Convex optimization-based super-resolution detector was also implemented in Matlab. The dual semidefinite program in (9) is solved by cvx package [9] to detect the location of the support. The SNR level for the signal sample was 50, and the rms error of DFT/MUSIC for this level was $0.5004/5.2863Hz$, respectively from $10,000$ trials. In contrast, the rms error from super-resolution detector was $0.0283Hz$ from 500 trials, which is significant improvement over DFT/MUSIC and way below the resolution limit of $0.5Hz$.

One of the nost notable issues from super-resolution detector was the run-time of the algorithm. In order to solve SDP of size 200 for 500 times, a laptop with a 2.8GHz Intel Core i7 and 16GB memory spent more than 6 hours. Also, it sometimes detected not only the desired tone but also other spurious tones (144 out of 500 trials output two or more tones). Even though those spurious tone has rather small magnitude and can be filtered with thresholding, amplitude estimation requires another convex optimization step after the support recovery step. Thus, supplementary DFT run to recognize the right bin appears to be necessary. Finally, as it was stated in the original theorem (by $|a| > C_1\delta$ constraint), the algorithm failed to converge when the SNR is too low. The SNR of 50 was empirical lower bound where all trials succeed. Note that the SNR here is the signal-to-noise ratio within one DFT bin. Thus

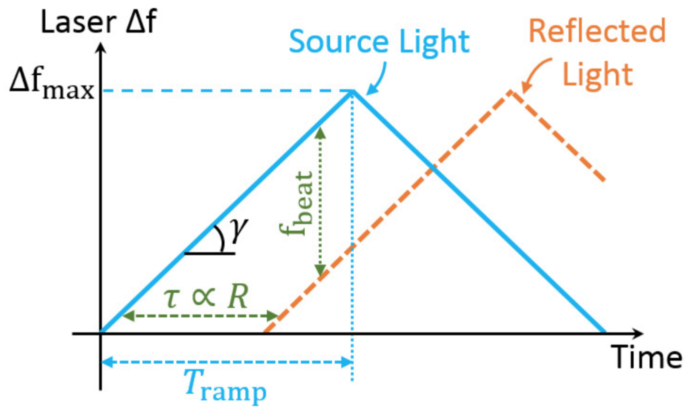


Fig. 3. Modulation profile of FMCW lidar [10]

the instantanous SNR in the time domain is $N\times$ lower than this value.

IV. Problem Simplification Leveraging Lidar System Properties

As seen from the numerical example in the previous section, solving semidefinite program for line spectral estimation is very expensive in terms of computation complexity (solving a problem of size 200 took almost a minute). Let's see what is the actual size of the sample data in typical lidar system.

Fig. 3 [10] shows the laser frequency modulation waveform for FMCW lidar (triangular pattern). Typical values of the parameters in this profile for real-world application is as follows.

- Desired maximum distance range (d_{max}): $10m$
- Chirping bandwidth (Δf_{max}): $10GHz$
- Pixel observation window (T_{ramp}): $10\mu s$
- Chirping rate ($\gamma = \Delta f_{max}/T_{ramp}$): $1PHz/s$
- Maximum beating frequency ($f_{beat,max} = 2\gamma d_{max}/c$): $66.7MHz$
- Required sampling rate ($f_{sample} = 2f_{beat,max}$): $133MHz$
- Number of samples in the time domain ($N_{sample} = T_{obs}f_{sample}$): $1,330$

This shows that the typical size of the problem is $N = 1,000 \sim 10,000$, and it is clear that full-blown semidefinite approach is not an option.

There are two potential approaches for problem simplification. First, we introduced the dual problem formulated as semidefinite program as the original problem was defined in the continuous domain with infinite dimension, and it was impossible to solve such problem numerically. We can still discretize the support space (frequency domain), and the size of that discrete grid is going to be determined by the trade-off between complexity and the resolving power beyond the resolution limit (or super-resolution factor SRF). Then the problem is now formulated on a finite field and the problem size is now $SRF \times N$. There are plenty of $l1$ minimization algorithms, and one of them could potentially show the best performance for given complexity constraint.

Secondly, we can leverage the fact that it is almost always the case that the lidar signal contains only one non-zero tone (or two non-zero tones with complement amplitudes). Multi-path propagation path from the transmitter to receiver is very unlikely since the beamforming antenna for integrated lidar system generally has extremely high directivity [11]. This introduces another constraint for the original optimization problem, and reduces the size of the optimization space.

$$\min_{\tilde{x}} \|\tilde{x}\|_1 \quad \text{subject to} \quad \|\tilde{x}\|_0 = 1 \text{ and } \|\mathcal{F}_n\tilde{x} - y\|_2 \leq \delta \tag{17}$$

There are $SRF \times N$ supports in the optimization space, and we can parallelize the search process of finding the amplitude of minimum magnitude that still satisfies $\|\mathcal{F}_n\tilde{x} - y\|_2 \leq \delta$. After that process, one can simply pick the support with smallest magnitude.

V. Conclusion

In this report, different line-spectral support recovery schemes are examined for FMCW lidar receiver with both closed-form expressions for the estimation error and numerical examples. Especially, it was confirmed that convex optimization-based super-resolution receiver achieves much superior detection performance even in rather low-SNR regime, well within the resolution limit. At the same time, we realized that the original problem formulation is not suitable for latency-critical lidar sensor due to its high complexity. I also proposed possible strategies to used super-resolution detector in the lidar system by leveraging the nature of lidar signal. In following work, I will examine the performance of proposed discretized-support, 0-norm constraint optimization-based support recovery.

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