University of Liverpool,
Department of Computer Science,
Autumn 2021

Maths and Stats for AI and Data Science COMP533

Theory Assignment 2 - part1 (Study Problems) 50/100 points

Your name:

(Q1) Consider function $f(x) = (x-a)(x+b)$, for any constant $a, b \in R$ with the natural domain R (set of all real numbers). For each task $(T1), \ldots, (T5)$ provide a short answer including justification.

$(T1)$ Identify the unique **point** $x \in R$ for which the **slope** of the tangent line at $(x, f(x))$ **is** b.

$(T2)$ Identify the unique **point** $x \in R$ for which the **slope** of the tangent line at $(x, f(x))$ **is** $-a$.

$(T3)$ Identify the unique **point** $x \in R$ for which $f(x)$ reaches the **smallest** value.

$(T4)$ Identify the **segment** of R in which all values of $f(x)$ are **negative**.

$(T5)$ Identify a **part** of the domain R in which $f(x)$ is **convex**.

A correct solution to each task is worth **5 points** which gives the total of **25 points**.

(Q2) A **permutation** of sequence $< a_1, \ldots, a_n >$ is an arbitrary arrangement of all elements of this sequence. For example, sequence $< 1,2,3 >$ has 6 different permutations $< 1,2,3 >, < 1,3,2 >, < 2,1,3>$, $< 2,3,1 >, < 3,1,2 >$, and $< 3,2,1 >$. Coincidentally 6=3! (**factorial function**), and in general any sequence of n different elements has $n!$ permutations.

Given matrix $A[n \times n]$ is called a **permutation matrix** if the product $A \times \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, where $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is interpreted as sequence $< a_1, \ldots, a_n >$, generates another vector which can be interpreted as one of the permutations of this sequence. For example, for $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, which is interpreted as sequence $< 1,2,3 >$, $A \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ which is interpreted as sequence $< 2,1,3 >$.

Note that a permutation matrix is formed of only 1s and 0s, such that each row and column contains exactly single 1.

Your first task is to create 5 permutation matrices A, B, C, D and E of size $[5 \times 5]$ which produce the following 5 permutations:

Matrix $\boldsymbol{A}$: $\boldsymbol{A} \times \begin{bmatrix} s \\ l \\ a \\ t \\ e \end{bmatrix} = \begin{bmatrix} s \\ t \\ a \\ l \\ e \end{bmatrix}$ *(slate to stale)*

Matrix **B:** $\boldsymbol{B} \times \begin{bmatrix} s \\ t \\ a \\ l \\ e \end{bmatrix} = \begin{bmatrix} t \\ a \\ l \\ e \\ s \end{bmatrix}$ *(stale to tales)*

Matrix **C:** $\boldsymbol{C} \times \begin{bmatrix} t \\ a \\ l \\ e \\ s \end{bmatrix} = \begin{bmatrix} s \\ t \\ e \\ a \\ l \end{bmatrix}$ *(tales to steal)*

Matrix **D:** $\boldsymbol{D} \times \begin{bmatrix} s \\ t \\ e \\ a \\ l \end{bmatrix} = \begin{bmatrix} t \\ e \\ a \\ l \\ s \end{bmatrix}$ *(steal to teals)*

Matrix **E:** $\boldsymbol{E} \times \begin{bmatrix} t \\ e \\ a \\ l \\ s \end{bmatrix} = \begin{bmatrix} l \\ e \\ a \\ s \\ t \end{bmatrix}$ *(teals to least)*

A correct content of each matrix is worth **4 points** which gives the total of **20 points**.

Let $\boldsymbol{F}$ be a matrix, such that $\boldsymbol{F} \times \begin{bmatrix} l \\ e \\ a \\ s \\ t \end{bmatrix} = \begin{bmatrix} s \\ l \\ a \\ t \\ e \end{bmatrix}$.

How can we express $\boldsymbol{F}$ in terms of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$?

This is an extra question for **5 points.**

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Theory Assignment 2 (coursework) 50/100

In each group of questions **Q1**,…, **Q5** provide short answers to **only one** block **(1)** or **(2)** of 5 questions. Each question is worth **2 points**, which gives the total of **50 points**.

Q1 (1) Which of the following functions and why are strictly **increasing** or **decreasing** for any $x \in R$?

(a) $f(x) = 5 - x$,
(b) $g(x) = \cos(x)$,
(c) $h(x) = \frac{1}{2021}$,
(d) $f(x) \cdot g(x)$.
(e) $(f(x)/(-h(x))$.

Q1 (2) Compute the formulae of the **first derivatives** of the following functions and compute values of the relevant derivatives in point $x = 1$.

(a) $f(x) = x^2$-3x+2,
(b) $g(x) = \frac{1}{2-x}$,
(c) $f(x) + g(x)$,
(d) $f(x) \cdot g(x)$,
(e) $f(x)/g(x)$.

Q2 (1) Compute the formulae of the **second derivatives** of the following functions.

(a) $f(x) = 3x^3 + 2x^2$,
(b) $g(x) = 2x^2 + 3x$,
(c) $f(x) \cdot g(x)$,
(d) $f(x)/g(x)$,
(e) What is the $(n-1)th$ derivative of x^n computed in point $x = 1$?

Q2 (2) Which of the following functions are **convex**, **concave** or **neither**, for $x > 1$ where $x \in R$?

(a) $f(x) = x^3 - 2x^2 + 1$,
(b) $g(x) = 1 - x^2$
(c) $g(x) - f(x)$
(d) $f(x) \cdot g(x)$
(e) $f(x)/g(x)$.

Q3 (1) For each linear function below **find two points** (x, y) defining the relevant line in R^2.

(a) $L_1: -x + y = 1$,
(b) $L_2: x - y = 1$,
(c) $L_1 + L_2$,
(d) $L_1 - L_2$,
(e) $2L_1 + L_2$.

Q3 (2) Determine whether the following systems of linear equations have **one**, **none** or **many solutions**.

(a) $x + y = -1;\ x + y = 0;$
(b) $x + y = 0;\ x - y = -1;$
(c) $x + y = 1;\ -x - y = -1;$
(d) $y = 1;\ x = 1;$
(e) $\sqrt{25}x + 5y = \frac{25}{\sqrt{25}}; x + y = 1;$

Q4 (1) Rewrite the following systems of equations to **matrix** and **vectors** representation.

(a) $\begin{cases} x + 2y = 3 \\ 3x + 2y = 1 \end{cases}$

(b) $\begin{cases} x - y = 1 \\ y - x = -1 \end{cases}$

(c) $\begin{cases} x = a \\ 2x + 3y = b \end{cases}$

(d) $\begin{cases} ax - by = c \\ bx + ay = d \end{cases}$

(e) $\begin{cases} \sqrt{2}x + \sqrt{3}y = \sqrt{5} \\ \sqrt{3}x + \sqrt{2}y = \sqrt{1} \end{cases}$

Q4 (2) Rewrite the following products as the relevant **systems** of **linear equations**.

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \times \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Q5 (1) Compute the following **products**, and **simplify** the answer if possible.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

(c) $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \times \begin{bmatrix} -\sqrt{8} \\ \sqrt{8} \end{bmatrix} =$

(d) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \times \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} =$

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$

Q5 (2) Correct one entry in each square matrix **to satisfy** the following **equations**.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \times \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ -1 \end{bmatrix}^T \times \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$

(e) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a+b \\ b-c \\ -c+a \end{bmatrix}$

Make sure the answers to all questions are submitted in PDF format via Canvas by

Friday November 12th 2021 (midnight)