COMP533 - Maths and Statistics for AI and Data Science
Theory Assignment 1
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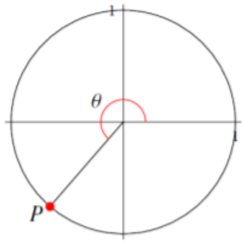
Part 1, Question 1



(Q1) Let θ be an angle defined on a unit circle, and P is the point on this circle defined by θ, see the picture to the right. Recall also that the angle determining the full circle is 360^o (360 degrees).

What are the cardinalities (sizes) of the followings 3 sets of points?

$A = \{ P : P$ is defined on all $\theta = 5i \cdot 1^o$, for all integer $i \}$ **10 points**

$B = \{ P : P$ is defined on all $\theta = \frac{1}{5} i \cdot 1^o$, for all integer $i \}$ **10 points**

$C = \{ P : P$ is defined on all $\theta = (5 + \frac{1}{5})\, i \cdot 1^o$, for all integer $i \}$ **5* points**

Provide short justification to your answers.

A: The total points of this set are $\frac{360}{5} = 72\, points$ (when i = 0, the angle is 0° and then i continues to increase to 1,2,3,... until it reaches 72 which is the same initial point at i = 0 to complete 1 cycle). So, the answer is 72.

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B: Like A, the total points of this set are $\frac{360}{0.2} = 1,800\, points$. So, the answer is 1,800.

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C: When we compute the number of points in one cycle, the number of points is $\frac{360}{5.2} = \frac{900}{13}\, points$ which are not possible because the whole points must be an integer. However, when we rotate the cycle 13 times, the total points are $\frac{900}{13}$x 13 = 900 points. So, the total unique points in this set are 900.

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Part 1, Question 2

(Q2) Identify the first 4 (four) values in each of the following sequences, decide whether they have limits when $n \to \infty$, and determine the relevant limits if they exist.

Sequence $A(n)$ such that
$A(n) = \frac{2n}{1^n} \cdot \frac{(-1)^n}{n}$, for any integer $n \geq 0$. **10 points**

Sequence $B(n)$ such that
$B(n) = \frac{(-1)^n}{n} \cdot \frac{(-1)^{3n}}{n}$, for any integer $n > 0$. **10 points**

Sequence $C(n)$ such that
$C(0) = \frac{1}{2}$ and $(n) = C(n-1) + \frac{1}{2^{n+1}}$, for any integer $n > 0$. **5* points**

Hint: Use the fact that for any integer $k \geq 1$, and $x < 1$, we get

$$(1-x) \cdot (x + x^2 + \cdots + x^k) = x - x^{k+1}.$$

Provide short justification to your answers.

Q2.1 A(n) = $\frac{2n}{n} \bullet \frac{(-1)^n}{(1)^n} = 2 \bullet \frac{(-1)^n}{(1)^n}$; n≥0

The first 4 values are A(0), A(1), A(2), and A(3)

A(0) = 2 $\bullet \frac{(-1)^0}{(1)^0} = 2$, A(1) = 2 $\bullet \frac{(-1)^1}{(1)^1} = -2$, A(2) = 2 $\bullet \frac{(-1)^2}{(1)^2} = 2$

And A(3) = 2 $\bullet \frac{(-1)^3}{(1)^3} = -2$

The limits of A(n); n → ∞ are

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$$\lim_{n \to \infty} (A(n)) = \lim_{n \to \infty} (2 \bullet \frac{(-1)^n}{(1)^n}) = either\ 2\ or\ -2$$

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So, the limit of A(n) does not exist

Q2.2 B(n) = $\frac{(-1)^{4n}}{n^2} = \frac{(1)^n}{n^2}$; n > 0

The first 4 values are A(1), A(2), A(3), and A(4)

A(1) = $\frac{(1)^1}{1^2} = 1$, A(2) = $\frac{(1)^2}{2^2} = \frac{1}{4}$, A(3) = $\frac{(1)^3}{3^2} = \frac{1}{9}$, and A(4) = $\frac{(1)^4}{4^2} = \frac{1}{16}$

The limit of B(n); n →∞ is $\lim_{n \to \infty} (B(n)) = \lim_{n \to \infty} (\frac{(1)^n}{n^2}) = \lim_{n \to \infty} (\frac{1}{n^2}) = 0$

So, B(n) has a limit when n →∞, which is 0.

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Q2.3 The first 4 values of C(n) are C(1), C(2), C(3), and C(4) because n > 0

$C(1) = C(0) + \frac{1}{2^{1+1}} = \frac{1}{2} + \frac{1}{2^2} = 0.75$

$C(2) = C(1) + \frac{1}{2^{2+1}} = 0.75 + \frac{1}{2^3} = 0.875$

$C(3) = C(2) + \frac{1}{2^{3+1}} = 0.875 + \frac{1}{2^4} = 0.9375$

$C(4) = C(3) + \frac{1}{2^{4+1}} = 0.9375 + \frac{1}{2^5} = 0.96875$

It can be concluded that, according to the pattern,

$C(n) = \frac{1}{2}^{1} + \frac{1}{2}^{2} + \frac{1}{2}^{3} + \cdots + \frac{1}{2}^{n+1}$

By giving that $(1-x)(x+x^2 + x^3 + \cdots + x^n) = x - x^{n+1}$; $n \geq 1$ and $x < 1$ 5

$(1 - \frac{1}{2})[\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots + (\frac{1}{2})^{n+1}] = \frac{1}{2} - (\frac{1}{2})^{n+1+1}$

$(1 - \frac{1}{2})C(n) = \frac{1}{2} - (\frac{1}{2})^{n+2}$

$C(n) = [\frac{1}{2} - (\frac{1}{2})^{n+2}] / (1 - \frac{1}{2})$

So $\lim_{n \to \infty} (C(n)) = (\frac{1}{2} - 0)/(\frac{1}{2}) = 1$

So, C(n) has a limit when n →∞, which is 1.

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Part 2, Questoin 1 (1)

In each group of questions **Q1**, **Q2**, **Q3**, **Q4** and **Q5** provide short answers to **only one** block **(1)** or **(2)** of 5 questions. Each individual question is worth **2 points**, which gives the total of **50 points**.

Q1 (1) Given two sets: $A = \{a, c, e\}$, and $B = \{b, d, e\}$. Compute:

(a) $A \cup B =$
(b) $A \cap B =$
(c) $A - B =$
(d) $B - A =$
(e) Do pairs (e, c) and (c, e) belong to $A \times B$?

(a) A u B = {a, b, c, d, e} 2
(b) A n B = {e} 2
(c) A - B = {a, c} 2
(d) B - A = {b, d} 2
(e) (e, c) does not belong to A x B because c is not a member of B. However, (c, e) belongs to A x B because c is a member of A and e is a member of B 2

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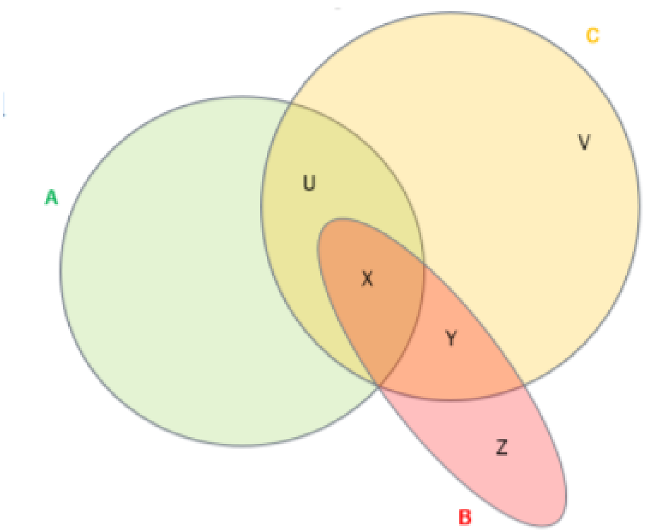
Part 2, Question 2 (1)

Q2 (1) Given three sets A (green circular region), B (red oval region), and C (orange circular region) by the Venn diagram shown on the right. Using set operators express in terms of A, B, and C the content of the 5 (five) sets represented by regions U, V, X, Y and Z.

(a) $U =$
(b) $V =$
(c) $X =$
(d) $Y =$
(e) $Z =$



(a) U = (A n C) - B 2
(b) V = (C - A) - B 2
(c) X = A n B 2
(d) Y = (B n C) - A 2
(e) Z = B - C 2

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Part 2, Question 3 (1)

Q3 (1) What are the natural domains of the following functions defined on real numbers?

(a) $f(x) = x + 5$

(b) $g(x) = \frac{1}{x-5}$

(c) $g(f(x))$

(d) $f(x) \cdot g(x)$

(e) $f(x)/g(x)$

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(a) R, because x can be any real number.

(b) R - {5} because the denominator must not be 0. So, x - 5 $\neq$ 0 which means that x $\neq$ 5 2

(c) g(f(x)) = g(x + 5) = $\frac{1}{(x+5)-5} = \frac{1}{x}$. Like (b), the denominator must not be 0, which means that x $\neq$ 0. So, the domian of this function is R - {0} 2

(d) f(x)g(x) = (x + 5)($\frac{1}{x-5}$). Like the previous example, the denominator must not be 0. So, x - 5 $\neq$ 0 which means that x $\neq$ 5. Thus, the domain of this function is R - {5} 2

(e) f(x)/g(x) = (x + 5)/($\frac{1}{x-5}$) = (x + 5)(x - 5). Since x can be any real number, the domain of this function is R

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Part 2, Question 4 (1)

Q4 (1) Compute the limits of the following sequences when n $\to \infty$.

(a) $A(n) = 2 - \frac{1}{n}$

(b) $B(n) = 3 + \frac{1}{n}$

(c) $C(n) = A(n) + B(n)$

(d) $D(n) = A(n) \cdot B(n)$

(e) $E(n) = n \cdot (A(n) + B(n))$

(a) $\lim_{n \to \infty} (A(n)) = \lim_{n \to \infty} (2) - \lim_{n \to \infty} (\frac{1}{n}) = 2 - 0 = 2$ 2

(b) $\lim_{n \to \infty} (B(n)) = \lim_{n \to \infty} (3) + \lim_{n \to \infty} (\frac{1}{n}) = 3 + 0 = 3$ 2

(c) $\lim_{n \to \infty} (C(n)) = \lim_{n \to \infty} (A(n)) + \lim_{n \to \infty} (B(n)) = 2 + 3 = 5$ 2

(d) $\lim_{n \to \infty} (D(n)) = \lim_{n \to \infty} (A(n)) \bullet \lim_{n \to \infty} (B(n)) = 2 \bullet 3 = 6$ 2

(e) $\lim_{n \to \infty} (E(n)) = \lim_{n \to \infty} (n) \bullet (\lim_{n \to \infty} (A(n)) + \lim_{n \to \infty} (B(n))) = \lim_{n \to \infty} (n) \bullet (2 + 3)$

, which means that the limit of E(n) does not exist 2

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Part 2, Question 5 (1)

Q5 (1) Answer the following questions about limits in points.

(a) Compute $\lim_{x\to 1+} \frac{x-1}{2x}$.

(b) Compute $\lim_{x\to 1-} \frac{2x+1}{2x}$.

(c) Compute $\lim_{x\to 0+} \frac{x-1}{2x}$.

(d) Compute $\lim_{x\to 0-} \frac{2x+1}{2x}$.

(e) Does the (two-sided) limit $\lim_{x\to 0}(\frac{x+1}{2x} - \frac{4x-1}{4x})$ exist?

(a) $\lim_{x\to 1+} (\frac{x-1}{2x}) = \frac{1-1}{2(1)} = 0$ 2 —

(b) $\lim_{x\to 1-} (\frac{2x+1}{2x}) = \frac{2(1)+1}{2(1)} = \frac{3}{2}$ 2

(c) $\lim_{x\to 0+} (\frac{x-1}{2x}) = \frac{0-1}{2(0)} = \frac{-1}{0}$, which means that the limit of this case does not exist. 1 — 10

(d) $\lim_{x\to 0-} (\frac{2x+1}{2x}) = \frac{2(0)+1}{2(0)} = \frac{1}{0}$, which means that the limit of this case does not exist. 1 — 10

(e) $\lim_{x\to 0} (\frac{x+1}{2x} - \frac{4x-1}{4x}) = \lim_{x\to 0} (\frac{2x+2-4x+1}{4x}) = \lim_{x\to 0} (\frac{-2x+3}{4x}) = \frac{-2(0)+3}{4(0)} = \frac{3}{0}$,
which means that the limit of this case (two-sided) does not exist.

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