TECHNICAL CHANGE AND THE AGGREGATE PRODUCTION FUNCTION *

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IN this day of rationally designed econometric studies and super-input-output tables, it takes something more than the usual "willing suspension of disbelief" to talk seriously of the aggregate production function. But the aggregate production function is only a little less legitimate a concept than, say, the aggregate consumption function, and for some kinds of long-run macro-models it is almost as indispensable as the latter is for the short-run. As long as we insist on practicing macro-economics we shall need aggregate relationships.

Even so, there would hardly be any justification for returning to this old-fashioned topic if I had no novelty to suggest. The new wrinkle I want to describe is an elementary way of segregating variations in output per head due to technical change from those due to changes in the availability of capital per head. Naturally, every additional bit of information has its price. In this case the price consists of one new required time series, the share of labor or property in total income, and one new assumption, that factors are paid their marginal products. Since the former is probably more respectable than the other data I shall use, and since the latter is an assumption often made, the price may not be unreasonably high.

Before going on, let me be explicit that I would not try to justify what follows by calling on fancy theorems on aggregation and index numbers.[1] Either this kind of aggregate economics appeals or it doesn't. Personally I belong to both schools. If it does, I think one can draw some crude but useful conclusions from the results.

Theoretical Basis

I will first explain what I have in mind mathematically and then give a diagrammatic exposition. In this case the mathematics seems simpler. If Q represents output and K and L represent capital and labor inputs in "physical" units, then the aggregate production function can be written as:

$$Q = F(K,L;t). \tag{1}$$

The variable t for time appears in F to allow for technical change. It will be seen that I am using the phrase "technical change" as a shorthand expression for *any kind of shift* in the production function. Thus slowdowns, speedups, improvements in the education of the labor force, and all sorts of things will appear as "technical change."

It is convenient to begin with the special case of *neutral* technical change. Shifts in the production function are defined as neutral if they leave marginal rates of substitution untouched but simply increase or decrease the output attainable from given inputs. In that case the production function takes the special form

$$Q = A(t)f(K,L) \tag{1a}$$

and the multiplicative factor $A(t)$ measures the cumulated effect of shifts over time. Differentiate (1a) totally with respect to time and divide by Q and one obtains

$$\frac{\dot{Q}}{Q} = \frac{\dot{A}}{A} + A\frac{\partial f}{\partial K}\frac{\dot{K}}{Q} + A\frac{\partial f}{\partial L}\frac{\dot{L}}{Q}$$

where dots indicate time derivatives. Now define $w_k = \frac{\partial Q}{\partial K}\frac{K}{Q}$ and $w_L = \frac{\partial Q}{\partial L}\frac{L}{Q}$ the relative shares of capital and labor, and substitute in the above equation (note that $\partial Q/\partial K = A\,\partial f/\partial K$, etc.) and there results:

$$\frac{\dot{Q}}{Q} = \frac{\dot{A}}{A} + w_K\frac{\dot{K}}{K} + w_L\frac{\dot{L}}{L}. \tag{2}$$

* I owe a debt of gratitude to Dr. Louis Lefeber for statistical and other assistance, and to Professors Fellner, Leontief, and Schultz for stimulating suggestions.

[1] Mrs. Robinson in particular has explored many of the profound difficulties that stand in the way of giving any precise meaning to the quantity of capital ("The Production Function and the Theory of Capital," *Review of Economic Studies*, Vol. 21, No. 2), and I have thrown up still further obstacles (ibid., Vol. 23, No. 2). Were the data available, it would be better to apply the analysis to some precisely defined production function with many precisely defined inputs. One can at least hope that the aggregate analysis gives some notion of the way a detailed analysis would lead.

From time series of $\dot{Q}/Q$, w_k, $\dot{K}/K$, w_L and $\dot{L}/L$ or their discrete year-to-year analogues, we could estimate $\dot{A}/A$ and thence $A(t)$ itself. Actually an amusing thing happens here. Nothing has been said so far about returns to scale. But if all factor inputs are classified either as K or L, then the available figures always show w_K and w_L adding up to one. Since we have assumed that factors are paid their marginal products, this amounts to assuming the hypotheses of Euler's theorem. The calculus being what it is, we might just as well assume the conclusion, namely that F is homogeneous of degree one. This has the advantage of making everything come out neatly in terms of intensive magnitudes. Let $Q/L = q$, $K/L = k$, $w_L = 1 - w_K$; note that $\dot{q}/q = \dot{Q}/Q - \dot{L}/L$ etc., and (2) becomes

$$\frac{\dot{q}}{q} = \frac{\dot{A}}{A} + w_K \frac{\dot{k}}{k}. \tag{2a}$$

Now all we need to disentangle the technical change index $A(t)$ are series for output per man hour, capital per man hour, and the share of capital.

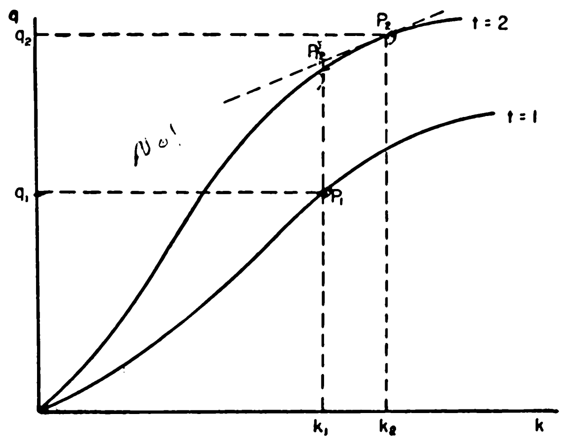
So far I have been assuming that technical change is neutral. But if we go back to (1) and carry out the same reasoning we arrive at something very like (2a), namely

$$\frac{\dot{q}}{q} = \frac{1}{F}\frac{\partial F}{\partial t} + w_k \frac{\dot{k}}{k} \tag{2b}$$

It can be shown, by integrating a partial differential equation, that if $\dot{F}/F$ is independent of K and L (actually under constant returns to scale only K/L matters) then (1) has the special form (1a) and shifts in the production function are neutral. If in addition $\dot{F}/F$ is constant in time, say equal to a, then $A(t) = e^{at}$ or in discrete approximation $A(t) = (1 + a)^t$.

The case of neutral shifts and constant returns to scale is now easily handled graphically. The production function is completely represented by a graph of q against k (analogously to the fact that if we know the unit-output isoquant, we know the whole map). The trouble is that this function is shifting in time, so that if we observe points in the (q,k) plane, their movements are compounded out of movements along the curve and shifts of the curve. In Chart 1, for instance, every ordinate on the curve for $t = 1$ has been multiplied by the same factor to give a neutral upward shift of the production function for period 2. The problem is to estimate this shift from knowledge of points P_1 and P_2. Obviously it would be quite misleading to fit a curve through raw observed points like P_1, P_2 and others. But if the shift factor for each point of time can be estimated, the observed points can be corrected for technical change, and a production function can then be found.[2]

CHART 1



The natural thing to do, for small changes, is to approximate the period 2 curve by its tangent at P_2 (or the period 1 curve by its tangent at P_1). This yields an approximately corrected point P_{12}, and an estimate for $\triangle A/A$, namely $\overline{P_{12}P_1}/q_1$. But $k_1P_{12} = q_2 - \partial q/\partial k \triangle k$ and hence $\overline{P_{12}P_1} = q_2 - q_1 - \partial q/\partial k \triangle k = \triangle q - \partial q/\partial k \triangle k$ and $\triangle A/A = \overline{P_{12}P_1}/q_1 = \triangle q/q - \partial q/\partial k\ (k/q) \triangle k/k = \triangle q/q - w_k \triangle k/k$ which is exactly the content of (2a). The not-necessarily-neutral case is a bit more complicated, but basically similar.

[2] Professors Wassily Leontief and William Fellner independently pointed out to me that this "first-order" approximation could in principle be improved. After estimating a production function corrected for technical change (see below), one could go back and use it to provide a second approximation to the shift series, and on into further iterations.

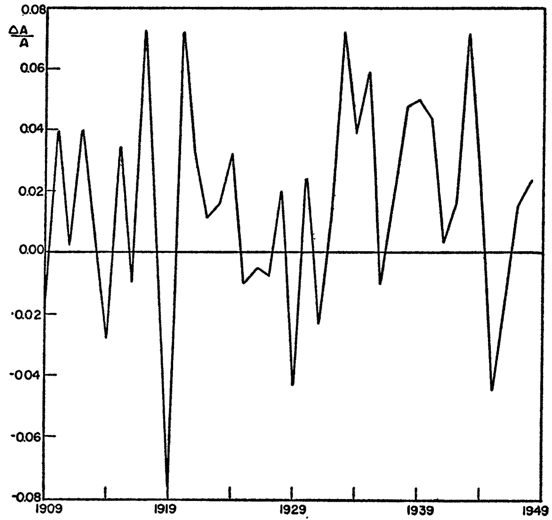
An Application to the U.S.: 1909–1949

In order to isolate shifts of the aggregate production function from movements along it, by use of (2a) or (2b), three time series are needed: output per unit of labor, capital per unit of labor, and the share of capital. Some rough and ready figures, together with the obvious computations, are given in Table 1.

The conceptually cleanest measure of aggregate output would be real net national product. But long NNP series are hard to come by, so I have used GNP instead. The only difference this makes is that the share of capital has to include depreciation. It proved possible to restrict the experiment to private non-farm economic activity. This is an advantage (a) because it skirts the problem of measuring government output and (b) because eliminating agriculture is at least a step in the direction of homogeneity. Thus my q is a time series of real private non-farm GNP per man hour, Kendrick's valuable work.

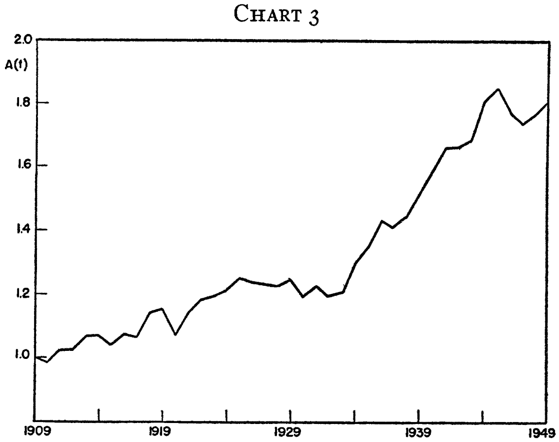
The capital time series is the one that will really drive a purist mad. For present purposes, "capital" includes land, mineral deposits, etc. Naturally I have used Goldsmith's estimates (with government, agricultural, and consumer durables eliminated). Ideally what one would like to measure is the annual flow of capital services. Instead one must be content with a less utopian estimate of the stock of capital goods in existence. All sorts of conceptual problems arise on this account. As a single example, if the capital stock consisted of a million identical machines and if each one as it wore out was replaced by a more durable machine of the same annual capacity, the stock of capital as measured would surely increase. But the maximal flow of capital services would be constant. There is nothing to be done about this, but something must be done about the fact of idle capacity. What belongs in a production function is capital in use, not capital in place. Lacking any reliable year-by-year measure of the utilization of capital I have simply reduced the Goldsmith figures by the fraction of the labor force unemployed in each year, thus assuming that labor and capital always suffer unemployment to the same percentage. This is undoubtedly wrong, but probably gets closer to the truth than making no correction at all.[3]

CHART 2



The share-of-capital series is another hodgepodge, pieced together from various sources and ad hoc assumptions (such as Gale Johnson's guess that about 35 per cent of non-farm entrepreneurial income is a return to property). Only after these computations were complete did I learn that Edward Budd of Yale University has completed a careful long-term study of factor shares which will soon be published. It seems unlikely that minor changes in this ingredient would grossly alter the final results,

CHART 3



[3] Another factor for which I have not corrected is the changing length of the work-week. As the work-week shortens, the intensity of use of existing capital decreases, and the stock figures overestimate the input of capital services.

but I have no doubt that refinement of this and the capital time-series would produce neater results.

In any case, in (2a) or (2b) one can replace the time-derivatives by year-to-year changes and calculate $\Delta q/q - w_k \Delta k/k$. The result is an estimate of $\Delta F/F$ or $\Delta A/A$, depending on whether these relative shifts appear to be neutral or not. Such a calculation is made in Table 1 and shown in Chart 2. Thence, by arbitrarily setting $A(1909) = 1$ and using the fact that $A(t+1) = A(t)\ (1 + \Delta A(t)/A(t))$ one can successively reconstruct the $A(t)$ time series, which is shown in Chart 3.

TABLE 1. — DATA FOR CALCULATION OF $A(t)$

Year	% labor force employed (1)	Capital stock ($ mill.) (2)	Col. 1 x Col. 2 (3)	Share of property in income (4)	Priv. nonfarm GNP per manhour (5)	Employed capital per manhour (6)	$\Delta A/A$ (7)	$A(t)$ (8)
1909	91.1	146,142	133,135	.335	$.623	$2.06	−.017	1.000
1910	92.8	150,038	139,235	.330	.616	2.10	.039	.983
1911	90.6	156,335	141,640	.335	.647	2.17	.002	1.021
1912	93.0	159,971	148,773	.330	.652	2.21	.040	1.023
1913	91.8	164,504	151,015	.334	.680	2.23	.007	1.064
1914	83.6	171,513	143,385	.325	.682	2.20	−.028	1.071
1915	84.5	175,371	148,188	.344	.669	2.26	.034	1.041
1916	93.7	178,351	167,115	.358	.700	2.34	−.010	1.076
1917	94.0	182,263	171,327	.370	.679	2.21	.072	1.065
1918	94.5	186,679	176,412	.342	.729	2.22	.013	1.142
1919	93.1	189,977	176,869	.354	.767	2.47	−.076	1.157
1920	92.8	194,802	180,776	.319	.721	2.58	.072	1.069
1921	76.9	201,491	154,947	.369	.770	2.55	.032	1.146
1922	81.7	204,324	166,933	.339	.788	2.49	.011	1.183
1923	92.1	209,964	193,377	.337	.809	2.61	.016	1.196
1924	88.0	222,113	195,460	.330	.836	2.74	.032	1.215
1925	91.1	231,772	211,198	.336	.872	2.81	−.010	1.254
1926	92.5	244,611	226,266	.327	.869	2.87	−.005	1.241
1927	90.0	259,142	233,228	.323	.871	2.93	−.007	1.235
1928	90.0	271,089	243,980	.338	.874	3.02	.020	1.226
1929	92.5	279,691	258,714	.332	.895	3.06	−.043	1.251
1930	88.1	289,291	254,865	.347	.880	3.30	.024	1.197
1931	78.2	289,056	226,042	.325	.904	3.33	.023	1.226
1932	67.9	282,731	191,974	.397	.879	3.28	.011	1.198
1933	66.5	270,676	180,000	.362	.869	3.10	.072	1.211
1934	70.9	262,370	186,020	.355	.921	3.00	.039	1.298
1935	73.0	257,810	188,201	.351	.943	2.87	.059	1.349
1936	77.3	254,875	197,018	.357	.982	2.72	−.010	1.429
1937	81.0	257,076	208,232	.340	.971	2.71	.021	1.415
1938	74.7	259,789	194,062	.331	1.000	2.78	.048	1.445
1939	77.2	257,314	198,646	.347	1.034	2.66	.050	1.514
1940	80.6	258,048	207,987	.357	1.082	2.63	.044	1.590
1941	86.8	262,940	228,232	.377	1.122	2.58	.003	1.660
1942	93.6	270,063	252,779	.356	1.136	2.64	.016	1.665
1943	97.4	269,761	262,747	.342	1.180	2.62	.071	1.692
1944	98.4	265,483	261,235	.332	1.265	2.63	.021	1.812
1945	96.5	261,472	252,320	.314	1.296	2.66	−.044	1.850
1946	94.8	258,051	244,632	.312	1.215	2.50	−.017	1.769
1947	95.4	268,845	256,478	.327	1.194	2.50	.016	1.739
1948	95.7	276,476	264,588	.332	1.221	2.55	.024	1.767
1949	93.0	289,360	269,105	.326	1.275	2.70	...	1.809

NOTES AND SOURCES:
Column (1): Percentage of labor force employed. 1909–26, from Douglas, *Real Wages in the United States* (Boston and New York, 1930), 460. 1929–49, calculated from *The Economic Almanac*, 1953–54 (New York, 1953), 426–28.
Column (2): Capital Stock. From Goldsmith, *A Study of Saving in the United States*, Vol. 3 (Princeton, 1956), 20–21, sum of columns 5, 6, 7, 9, 12, 17, 22, 23, 24.
Column (3): (1) x (2).
Column (4): Share of property in income. Compiled from *The Economic Almanac*, 504–505; and Jesse Burkhead, "Changes in the Functional Distribution of Income," *Journal of the American Satistical Association*, Vol. 48 (June 1953), 192–219. Depreciation estimates from Goldsmith, 427.
Column (5): Private nonfarm GNP per man hour, 1939 dollars. Kendrick's data, reproduced in *The Economic Almanac*, 490.
Column (6): Employed capital per man hour. Column (3) divided by Kendrick's man hour series, *ibid.*
Column (7): $\Delta A/A = \Delta (5)/(5) - (4) \times \Delta (6)/(6)$.
Column (8): From (7).

I was tempted to end this section with the remark that the $A(t)$ series, which is meant to be a rough profile of technical change, at least looks reasonable. But on second thought I decided that I had very little prior notion of what would be "reasonable" in this context. One notes with satisfaction that the trend is strongly upward; had it turned out otherwise I would not now be writing this paper. There are sharp dips after each of the World Wars; these, like the sharp rises that preceded them, can easily be rationalized. It is more suggestive that the curve shows a distinct levelling-off in the last half of the 1920's. A sustained rise begins again in 1930. There is an unpleasant sawtooth character to the first few years of the $\triangle A/A$ curve, which I imagine to be a statistical artifact.

The Outlines of Technical Change

The reader will note that I have already drifted into the habit of calling the curve of Chart 2 $\triangle A/A$ instead of the more general $\triangle F/F$. In fact, a scatter of $\triangle F/F$ against K/L (not shown) indicates no trace of a relationship. So I may state it as a formal conclusion that over the period 1909–49, shifts in the aggregate production function netted out to be approximately neutral. Perhaps I should recall that I have defined neutrality to mean that the shifts were pure scale changes, leaving marginal rates of substitution unchanged at given capital/labor ratios.

Not only is $\triangle A/A$ uncorrelated with K/L, but one might almost conclude from the graph that $\triangle A/A$ is essentially constant in time, exhibiting more or less random fluctuations about a fixed mean. Almost, but not quite, for there does seem to be a break at about 1930. There is some evidence that the average rate of progress in the years 1909–29 was smaller than that from 1930–49. The first 21 relative shifts average about 9/10 of one per cent per year, while the last 19 average 2¼ per cent per year. Even if the year 1929, which showed a strong downward shift, is moved from the first group to the second, there is still a contrast between an average rate of 1.2 per cent in the first half and 1.9 per cent in the second. Such *post hoc* splitting-up of a period is always dangerous. Perhaps I should leave it that there is some evidence that technical change (broadly interpreted) may have accelerated after 1929.

The over-all result for the whole 40 years is an average upward shift of about 1.5 per cent per year. This may be compared with a figure of about .75 per cent per year obtained by Stefan Valavanis-Vail by a different and rather less general method, for the period 1869–1948.[4] Another possible comparison is with the output-per-unit-of-input computations of Jacob Schmookler,[5] which show an increase of some 36 per cent in output per unit of input between the decades 1904–13 and 1929–38. Our $A(t)$ rises 36.5 per cent between 1909 and 1934. But these are not really comparable estimates, since Schmookler's figures include agriculture.

As a last general conclusion, after which I will leave the interested reader to his own impressions, over the 40 year period output per man hour approximately doubled. At the same time, according to Chart 2, the cumulative upward shift in the production function was about 80 per cent. It is possible to argue that about one-eighth of the total increase is traceable to increased capital per man hour, and the remaining seven-eighths to technical change. The reasoning is this: real GNP per man hour increased from \$.623 to \$1.275. Divide the latter figure by 1.809, which is the 1949 value for $A(t)$, and therefore the full shift factor for the 40 years. The result is a "corrected" GNP per man hour, net of technical change, of \$.705. Thus about 8 cents of the 65 cent increase can be imputed to increased capital intensity, and the remainder to increased productivity.[6]

Of course this is not meant to suggest that the observed rate of technical progress would have persisted even if the rate of investment had been much smaller or had fallen to zero. Obviously much, perhaps nearly all, innovation must be embodied in new plant and equipment to be realized at all. One could imagine this process taking place without net capital for-

[4] S. Valavanis-Vail, "An Econometric Model of Growth, U.S.A. 1869–1953," *American Economic Review, Papers and Proceedings*, XLV (May 1955), 217.

[5] J. Schmookler, "The Changing Efficiency of the American Economy, 1869–1938," this REVIEW (August 1952), 226.

[6] For the first half of the period, 1909–29, a similar computation attributes about one-third of the observed increase in GNP per man-hour to increased capital intensity.

mation as old-fashioned capital goods are replaced by the latest models, so that the capital-labor ratio need not change systematically. But this raises problems of definition and measurement even more formidable than the ones already blithely ignored. This whole area of interest has been stressed by Fellner.

For comparison, Solomon Fabricant[7] has estimated that over the period 1871–1951 about 90 per cent of the increase in output per capita is attributable to technical progress. Presumably this figure is based on the standard sort of output-per-unit-of-input calculation.

It might seem at first glance that calculations of output per unit of resource input provide a relatively assumption-free way of measuring productivity changes. Actually I think the implicit load of assumptions is quite heavy, and if anything the method proposed above is considerably more general.

Not only does the usual choice of weights for computing an aggregate resource-input involve something analogous to my assumption of competitive factor markets, but in addition the criterion output ÷ a weighted sum of inputs would seem tacitly to *assume* (a) that technical change is neutral and (b) that the aggregate production function is *strictly* linear. This explains why numerical results are so closely parallel for the two methods. We have already verified the neutrality, and as will be seen subsequently, a strictly linear production function gives an excellent fit, though clearly inferior to some alternatives.[8]

[7] S. Fabricant, "Economic Progress and Economic Change," *34th Annual Report of the National Bureau of Economic Research* (New York, 1954).

[8] For an excellent discussion of some of the problems, see M. Abramovitz "Resources and Output Trends in the U.S. since 1870," *American Economic Review, Papers and Proceedings*, XLVI (May 1956), 5–23. Some of the questions there raised could in principle be answered by the method used here. For example, the contribution of improved quality of the labor force could be handled by introducing various levels of skilled labor as separate inputs. I owe to Prof. T. W. Schultz a heightened awareness that a lot of what appears as shifts in the production function must represent improvement in the quality of the labor input, and therefore a result of real capital formation of an important kind. Nor ought it be forgotten that even straight technical progress has a cost side.

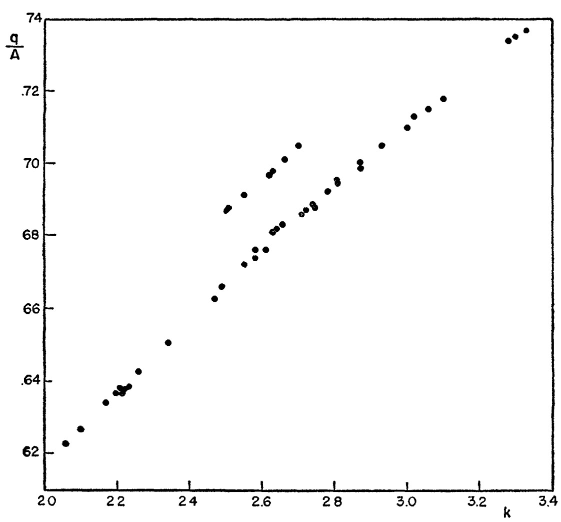
The Aggregate Production Function

Returning now to the aggregate production function, we have earned the right to write it in the form (1a). By use of the (practically unavoidable) assumption of constant returns to scale, this can be further simplified to the form

$$q = A(t)f(k,1), \tag{3}$$

which formed the basis of Chart 1. It was there noted that a simple plot of q against k would give a distorted picture because of the shift factor $A(t)$. Each point would lie on a different member of the family of production curves. But we have now provided ourselves with an estimate of the successive values of the shift factor. (Note that this estimate is quite *independent* of any hypothesis about the exact shape of the production function.) It follows from (3) that by plotting $q(t)/A(t)$ against $k(t)$ we reduce all the observed points to a *single* member of the family of curves in Chart 1, and we can then proceed to discuss the shape of $f(k,1)$ and reconstruct the aggregate production function. A scatter of q/A against k is shown in Chart 4.

CHART 4



Considering the amount of *a priori* doctoring which the raw figures have undergone, the fit is remarkably tight. Except, that is, for the layer of points which are obviously too high. These maverick observations relate to the seven last years of the period, 1943–49. From the way they lie almost exactly parallel to the main scatter, one is tempted to conclude that in 1943 the aggregate production function simply shifted. But the whole earlier procedure was designed to purify those points from shifts in the function, so that way out would seem to be closed. I suspect the explanation may lie in some systematic incomparability of the capital-in-use series. In particular during the war there was almost certainly a more intensive use of capital services through two- and three-shift operation than the stock figures would show, even with the crude correction that has been applied. It is easily seen that such an underestimate of capital inputs leads to an overestimate of productivity increase. Thus in effect each of the affected points should really lie higher and toward the right. But further analysis shows that, for the orders of magnitude involved, the net result would be to pull the observations closer to the rest of the scatter.

At best this might account for 1943–1945. There remains the postwar period. Although it is possible that multi-shift operation remained fairly widespread even after the war, it is unlikely that this could be nearly enough to explain the whole discrepancy.[9] One might guess that accelerated amortization could have resulted in an underestimate of the capital stock after 1945. Certainly other research workers, notably Kuznets and Terborgh, have produced capital stock estimates which rather exceed Goldsmith's at the end of the period. But for the present, I leave this a mystery.

In a first version of this paper, I resolutely let the recalcitrant observations stand as they were in a regression analysis of Chart 4, mainly because such casual amputation is a practice I deplore in others. But after some experimentation it seemed that to leave them in only led to noticeable distortion of the results. So, with some misgivings, in the regressions that follow I have omitted the observations for 1943–1949. It would be better if they could be otherwise explained away.

Chart 4 gives an inescapable impression of curvature, of persistent but not violent diminishing returns. As for the possibility of approaching capital-saturation, there is no trace on this gross product level, but even setting aside all other difficulties, such a scatter confers no particular license to guess about what happens at higher K/L ratios than those observed.

As for fitting a curve to the scatter, a Cobb-Douglas function comes immediately to mind, but then so do several other parametric forms, with little to choose among them.[10] I can't help feeling that little or nothing hangs on the choice of functional form, but I have experimented with several. In general I limited myself to two-parameter families of curves, linear in the parameters (for computational convenience), and at least capable of exhibiting diminishing returns (except for the straight line, which on this account proved inferior to all others).

The particular possibilities tried were the following:

$$q = \alpha + \beta k \qquad (4a)$$
$$q = \alpha + \beta \log k \qquad (4b)$$
$$q = \alpha - \beta / k \qquad (4c)$$
$$\log q = \alpha + \beta \log k \qquad (4d)$$
$$\log q = \alpha - \beta / k. \qquad (4e)$$

Of these, (4d) is the Cobb-Douglas case; (4c and e) have upper asymptotes; the semilogarithmic (4b) and the hyperbolic (4c) must cross the horizontal axis at a positive value of k and continue ever more steeply but irrelevantly downward (which means only that some positive k must be achieved before any output is forthcoming, but this is far outside the range of observation); (4e) begins at the origin with a phase of increasing returns and ends with a phase of diminishing returns — the point of inflection occurs at $k = \beta/2$ and needless to say all our observed points come well to the right of this.

The results of fitting these five curves to the scatter of Chart 4 are shown in Table 2.

The correlation coefficients are uniformly so high that one hesitates to say any more than

[9] It is cheering to note that Professor Fellner's new book voices a suspicion that the postwar has seen a substantial increase over prewar in the prevalence of multi-shift operation. See *Trends and Cycles in Economic Activity* (New York, 1956), 92.

[10] A discussion of the same problem in a different context is to be found in Prais and Houthakker, *The Analysis of Family Budgets* (Cambridge, England, 1955), 82–88. See also S. J. Prais, "Non-Linear Estimates of the Engel Curves," *Review of Economic Studies*, No. 52 (1952–53), 87–104.

TABLE 2

Curve	α	β	r
4a	.438	.091	.9982
b	.448	.239	.9996
c	.917	.618	.9964
d	−.729	.353	.9996
e	−.038	.913	.9980

that all five functions, even the linear one, are about equally good at representing the general shape of the observed points. From the correlations alone, for what they are worth, it appears that the Cobb-Douglas function (4d) and the semilogarithmic (4b) are a bit better than the others.[11]

Since all of the fitted curves are of the form $g(y) = \alpha + \beta\, h(x)$, one can view them all as linear regressions and an interesting test of goodness of fit proposed by Prais and Houthakker (*ibid.*, page 51) is available. If the residuals from each regression are arranged in order of increasing values of the independent variable, then one would like this sequence to be disposed "randomly" about the regression line. A strong "serial" correlation in the residuals, or a few long runs of positive residuals alternating with long runs of negative residuals, would be evidence of just that kind of smooth departure from linearity that one would like to catch. A test can be constructed using published tables of critical values for runs of two kinds of elements.

[11] It would be foolhardy for an outsider (or maybe even an insider) to hazard a guess about the statistical properties of the basic time series. A few general statements can be made, however. (a) The natural way to introduce an error term into the aggregate production function is multiplicatively: $Q = (1 + u)F(K,L;t)$. In the neutral case it is apparent that the error factor will be absorbed into the estimated $A(t)$. Then approximately the error in $\Delta A/A$ will be $\Delta u/1 + u$. If u has zero mean, the variance of the estimated $\Delta A/A$ will be approximately $2(1 - \rho)$ var u, where ρ is the first autocorrelation of the u series. (b) Suppose that marginal productivity distribution doesn't hold exactly, so that $K/Q \partial Q/\partial K = w_k + v$, where now v is a random deviation and w_k is the share of property income. Then the error in the estimated $\Delta A/A$ will be $v\, \Delta k/k$, with variance $(\Delta k/k)^2$ var v. Since K/L changes slowly, the multiplying factor will be very small. The effect is to bias the estimate of $\Delta A/A$ in such a way as to lead to an overestimate when property receives less than its marginal product (and k is increasing). (c) Errors in estimating $A(t)$ enter in a relatively harmless way so far as the regression analysis is concerned. Errors of observation in k will be more serious and are likely to be large. The effect will of course be to bias the estimates of β downward.

This has been done for the linear, semilogarithmic, and Cobb-Douglas functions. The results strongly confirm the visual impression of diminishing returns in Chart 4, by showing the linear function to be a systematically poor fit. As between (4b) and (4d) there is little to choose.[12]

A Note on Saturation

It has already been mentioned that the aggregate production function shows no signs of levelling off into a stage of capital-saturation. The two curves in Table 2 which have upper asymptotes (c and e) happen to locate that asymptote at about the same place. The limiting values of q are, respectively, .92 and .97. Of course these are both true asymptotes, approached but not reached for any finite value of k. It could not be otherwise: no analytic function can suddenly level off and become constant unless it has always been constant. But on the other hand, there is no reason to expect nature to be infinitely differentiable. Thus any conclusions extending beyond the range actually observed in Chart 4 are necessarily treacherous. But, tongue in cheek, if we take .95 as a guess at the saturation level of q, and use the *linear* function (4a) (which will get there first) as a lower-limit guess at the saturation level for k, it turns out to be about 5.7, more than twice its present value.

But all this is in terms of *gross output*, whereas for analytic purposes we are interested in the *net* productivity of capital. The difference between the two is depreciation, a subject about which I do not feel able to make guesses. If there were more certainty about the meaning of existing estimates of depreciation, especially over long periods of time, it would have been better to conduct the whole analysis in terms of net product.

However, one can say this. Zero net marginal productivity of capital sets in when gross marginal product falls to the "marginal rate of depreciation," i.e. when adding some capital adds only enough product to make good the depreciation on the increment of capital itself. Now in recent years NNP has run a bit over

[12] The test statistic is R, the total number of runs, with small values significant. For (4a), $R = 4$; for (4b), $R = 13$. The 1% critical value in both cases is about 9.

90 per cent of GNP, so capital consumption is a bit under 10 per cent of gross output. From Table 1 it can be read that capital per unit of output is, say, between 2 and 3. Thus annual depreciation is between 3 and 5 per cent of the capital stock. Capital-saturation would occur whenever the gross marginal product of capital falls to .03–.05. Using (4b), this would happen at K/L ratios of around 5 or higher, still well above anything ever observed.[13]

Summary

This paper has suggested a simple way of segregating shifts of the aggregate production function from movements along it. The method rests on the assumption that factors are paid their marginal products, but it could easily be extended to monopolistic factor markets. Among the conclusions which emerge from a crude application to American data, 1909–49, are the following:

1. Technical change during that period was neutral on average.
2. The upward shift in the production function was, apart from fluctuations, at a rate of about one per cent per year for the first half of the period and 2 per cent per year for the last half.
3. Gross output per man hour doubled over the interval, with 87½ per cent of the increase attributable to technical change and the remaining 12½ per cent to increased use of capital.
4. The aggregate production function, corrected for technical change, gives a distinct impression of diminishing returns, but the curvature is not violent.

[13] And this is under relatively pessimistic assumptions as to how technical change itself affects the rate of capital consumption. A warning is in order here: I have left Kendrick's GNP data in 1939 prices and Goldsmith's capital stock figures in 1929 prices. Before anyone uses the β's of Table 2 to reckon a yield on capital or any similar number, it is necessary to convert Q and K to a comparable price basis, by an easy calculation.