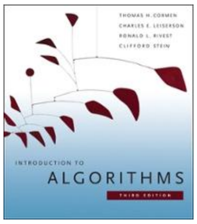
CS146 Data Structures and Algorithms



Chapter 12: Binary Search Tree

Slides made by Dr. Mike Wu of SJSU and some adopted from other universities such as Stanford and U of Virginia, etc.

BST: Dynamic Sets

- Next few lectures will focus on data structures rather than straight algorithms
- In particular, structures for *dynamic sets*
  - Elements have a *key* and *satellite data*
  - Dynamic sets support *queries* such as:
    - ***Search(S, k), Minimum(S), Maximum(S), Successor(S, x), Predecessor(S, x)***
  - They may also support *modifying operations* like:
    - ***Insert(S, x), Delete(S, x)***
- Basic operations take time proportional to the height of the tree – $O(h)$.

L12.2

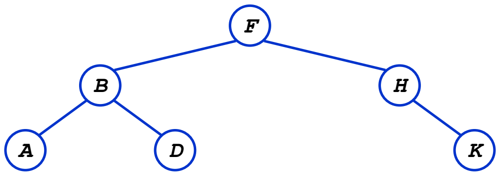
Binary Search Trees

- *Binary Search Trees* (BSTs) are an important data structure for dynamic sets
- Represented by a linked data structure of nodes.
- In addition to satellite data, elements have:
  - *key*: an identifying field inducing a total ordering
  - *left*: pointer to a left child : root of left subtree (may be NULL)
  - *right*: pointer to a right child : root of right subtree (may be NULL)
  - *p*: pointer to a parent node (NULL for root)

L12.3

Binary Search Trees

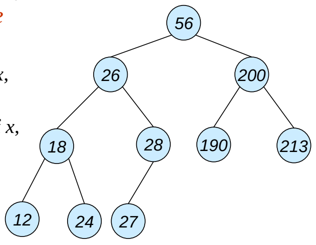
- BST property:
  key[leftSubtree(x)] ≤ key[x] ≤ key[rightSubtree(x)]
- Example:



L12.4

Binary Search Tree Property

- Stored keys must satisfy the *binary search tree* property.
  - $\forall\, y$ in left subtree of x, then $y.key \le x.key$
  - $\forall\, y$ in right subtree of x, then $y.key \ge x.key$.

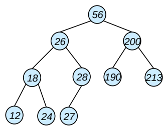


L12.5

Inorder Traversal

The binary-search-tree property allows the keys of a binary search tree to be printed, in (monotonically increasing) order, recursively.

```
Inorder-Tree-Walk (x)
1. if x ≠ NIL
2.     then Inorder-Tree-Walk(x.left)
3.          print x.key
4.          Inorder-Tree-Walk(x.right)
```



- How long does the walk take?
- Can you prove its correctness?

L12.6

Correctness of Inorder-Walk

- Must prove that it prints all elements, in order, and that it terminates.
- By induction on size of tree. Size=0: Easy.
- Size >1:
  - Prints left subtree in order by induction.
  - Prints root, which comes after all elements in left subtree (still in order).
  - Prints right subtree in order (all elements come after root, so still in order).

L12.7

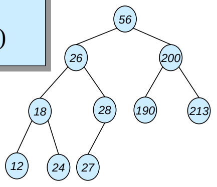
Querying a Binary Search Tree

- All dynamic-set search operations can be supported in $O(h)$ time.
- $h = \Theta(lg\ n)$ for a balanced binary tree (and for an average tree built by adding nodes in random order.)
- $h = \Theta(n)$ for an unbalanced tree that resembles a linear chain of n nodes in the worst case.
- A binary tree with n nodes (leaf nodes and internal nodes, including the root node) and height h is balanced if the following is true: $2^{h-1} \le n < 2^h$. Otherwise it is unbalanced. For example, a binary tree with height 4 can have between 8 and 15 nodes (between 1 and 8 leaf nodes) to be balanced.

L12.8

Tree Search

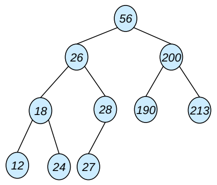
```
Tree-Search(x, k)
1.  if x == NIL or k == x.key
2.      return x
3.  if k < x.key
4.      return Tree-Search(x.left, k)
5.  else return Tree-Search(x.right, k)
```



Running time: ***O(h)***

Iterative Tree Search

```
Iterative-Tree-Search(x, k)
1.  while x ≠ NIL and k ≠ x.key
2.      if k < x.key
3.          x = x.left
4.      else x = x.right
5.  return x
```



- The iterative tree search is more efficient on most computers.
- The recursive tree search is more straightforward.

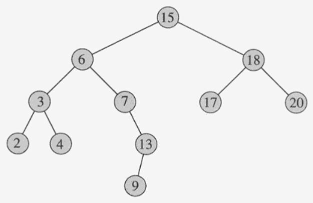


Figure 12.2 Queries on a binary search tree. To search for the key 13 in the tree, we follow the path 15 → 6 → 7 → 13 from the root. The minimum key in the tree is 2, which can be found by following *left* pointers from the root. The maximum key 20 is found by following *right* pointers from the root. The successor of the node with key 15 is the node with key 17, since it is the minimum key in the right subtree of 15. The node with key 13 has no right subtree, and thus its successor is its lowest ancestor whose left child is also an ancestor. In this case, the node with key 15 is its successor.

Finding Min & Max

- The binary-search-tree property guarantees that:
  - » The minimum is located at the left-most node.
  - » The maximum is located at the right-most node.

```
Tree-Minimum(x)
1.  while x.left ≠ NIL
2.      x = x.left
3.  return x
```

```
Tree-Maximum(x)
1.  while x.right ≠ NIL
2.      x = x.right
3.  return x
```

Q: How long do they take?

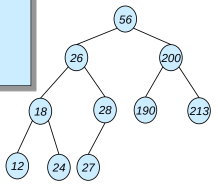
Predecessor and Successor

- Successor of node *x* is the node *y* such that *key*[*y*] is the smallest key greater than *key*[*x*].
- The successor of the largest key is NIL.
- Search consists of two cases.
  - If node *x* has a non-empty right subtree, then *x*'s successor is the minimum in the right subtree of *x*.
  - If node *x* has an empty right subtree, then:
    - As long as we move to the left up the tree (move up through right children), we are visiting smaller keys.
    - *x*'s successor *y* is the node that *x* is the predecessor of (*x* is the maximum in *y*'s left subtree).
    - In other words, *x*'s successor *y*, is the lowest ancestor of *x* whose left child is also an ancestor of *x*.

Pseudo-code for Successor

```
Tree-Successor(x)
1.  if x.right ≠ NIL
2.      return Tree-Minimum(x.right )
3.  y = x.p
4.  while y ≠ NIL and x == y.right
5.      x = y
6.      y = y.p
7.  return y
```

BST allows to determine the successor of a node without ever comparing keys.

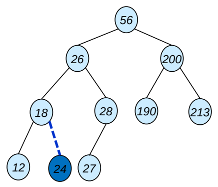


Code for **predecessor** is symmetric.

Running time: O(h)

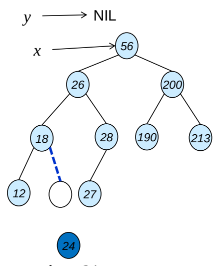
BST Insertion – Pseudocode

- Change the dynamic set represented by a BST.
- Ensure the binary-search-tree property holds after change.
- Insertion is easier than deletion.



```
Tree-Insert(T, z)
1.  y = NIL   /* trailing pointer, p of x */
2.  x = T.root
3.  while x ≠ NIL
4.      y = x
5.      if z.key < x.key
6.          x = x.left
7.      else x = x.right
8.  z.p = y
9.  if y == NIL
10.     T.root = z.  //tree T was empty
11. else if z.key < y.key
12.     y.left = z
13. else y.right = z
```

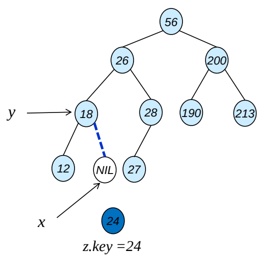
BST: Insertion (Initialization)



z.key =24

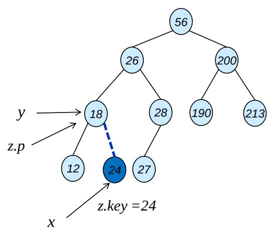
```
Tree-Insert(T, z)
1.  y = NIL   /* trailing pointer, p of x */
2.  x = T.root
3.  while x ≠ NIL
4.      y = x
5.      if z.key < x.key
6.          x = x.left
7.      else x = x.right
8.  z.p = y
9.  if y == NIL
10.     T.root = z.  //tree T was empty
11. else if z.key < y.key
12.     y.left = z
13. else y.right = z
```

BST: Insertion (After While Loop)



```
Tree-Insert(T, z)
1.  y = NIL   /* trailing pointer, p of x */
2.  x = T.root
3.  while x ≠ NIL
4.        y = x
5.        if z.key < x.key
6.              x = x.left
7.        else x = x.right
8.  z.p = y
9.  if y == NIL
10.     T.root = z.  //tree T was empty
11. else if z.key < y.key
12.       y.left = z
13. else y.right = z
```

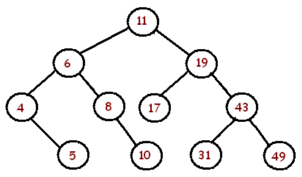
BST: Insertion (Line 9 ~ 13)



```
Tree-Insert(T, z)
1.  y = NIL   /* trailing pointer, p of x */
2.  x = T.root
3.  while x ≠ NIL
4.        y = x
5.        if z.key < x.key
6.              x = x.left
7.        else x = x.right
8.  z.p = y
9.  if y == NIL
10.     T.root = z.  //tree T was empty
11. else if z.key < y.key
12.       y.left = z
13. else y.right = z
```

BST: Insertion

- 11, 6, 8, 19, 4, 10, 5, 17, 43, 49, 31



Analysis of Insertion

- Initialization: $O(1)$
- While loop in lines 3-7 searches for place to insert z, maintaining parent y. This takes $O(h)$ time.
- Lines 8-13 insert the value: $O(1)$

$\Rightarrow$ TOTAL: $O(h)$ time to insert a node.

```
Tree-Insert(T, z)
1.  y = NIL
2.  x = T.root
3.  while x ≠ NIL
4.        y = x
5.        if z.key < x.key
6.              x = x.left
7.        else x = x.right
8.  z.p = y
9.  if y == NIL
10.     T.root = z.  //tree T was empty
11. else if z.key < y.key
12.       y.left = z
13. else y.right = z
```

Exercise: Sorting Using BSTs

```
BSTSort (A)
  for i ← 1 to n
      do Tree-Insert(A[i])
  Inorder-Tree-Walk(root)
```

- What are the worst case and best case running times?
- In practice, how would this compare to other sorting algorithms?

Sorting With Binary Search Trees

- Informal code for sorting array A of length *n*:

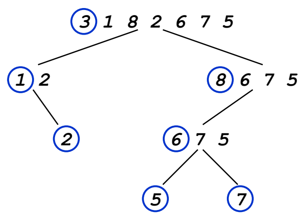
```
BSTSort(A)
    for i=1 to n
        TreeInsert(A[i]);
    InorderTreeWalk(root);
```

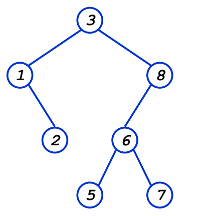
- *Argue that this is $\Omega(n \lg n)$*
- *What will be the running time in the*
  - *Worst case?*
  - *Average case? (hint: remind you of anything?)*

Sorting With BSTs

- Average case analysis
  - It's a form of quicksort!

```
for i=1 to n
    TreeInsert(A[i]);
InorderTreeWalk(root);
```





Sorting with BSTs

- Same partitions are done as with quicksort, but in a different order
  - In previous example
    - Everything was compared to 3 once
    - Then those items < 3 were compared to 1 once
    - Etc.
  - Same comparisons as quicksort, different order!
    - Example: consider inserting 5

Sorting with BSTs

- Since run time is proportional to the number of comparisons, same time as quicksort: O(n lg n)
- *Which do you think is better, quicksort or BSTsort? Why?*

L12.25

Sorting with BSTs

- Since run time is proportional to the number of comparisons, same time as quicksort: O(n lg n)
- *Which do you think is better, quicksort or BSTSort? Why?*
- A: quicksort
  - Better constants
  - Sorts in place
  - Doesn’t need to build data structure

L12.26

Tree-Delete (*T*, *x*)

- Deletion is a bit tricky
- 3 cases:

if *x* has no children ♦ case 0

then remove *x*

if *x* has one child ♦ case 1

then make *x.p* point to child

if *x* has two children (subtrees) ♦ case 2

then swap *x* with its successor

perform case 0 or case 1 to delete it

⇒ TOTAL: *O*(*h*) time to delete a node

L12.27

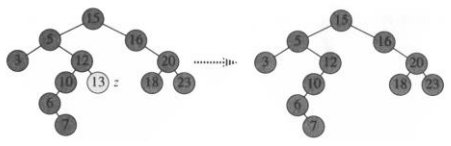
BST Operations: Delete

- *Why will case 2 always go to case 0 or case 1?*
- A: because when *x* has 2 children, its successor is the minimum in its right subtree
- *Could we swap x with predecessor instead of successor?*
- A: yes. *Would it be a good idea?*
- A: might be good to alternate

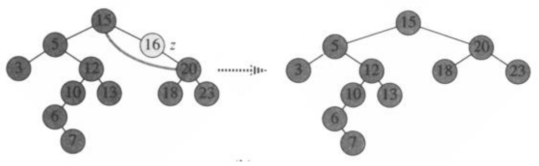
L12.28

```
Tree-Delete(T, z)
/* Determine which node to splice out: either z or z's successor. */
if z.left == NIL or z.right == NIL
        y = z
else y = Tree-Successor(z)
/* Set x to a non-NIL child of y, or to NIL if y has no children. */
if y.left ≠ NIL
        x = y.left
else x = y.right
/* y is removed from the tree by manipulating pointers of y.p and x */
if x ≠ NIL
        x.p = y.p
if y.p == NIL
        root[T] = x
else if y == y.p.left
        y.p.left = x
else y.p.right = x
/* If z's successor was spliced out, copy its data into z */
if y ≠ z
        z.key = y.key
        copy y's satellite data into z
return y
```

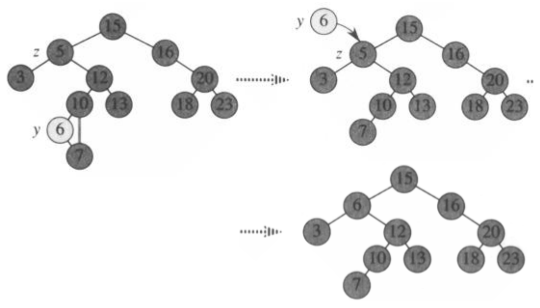
Case a: z has no children



Case b: z has only one child



Case c: z has two children



Subtree Replacement - Transplant

- To move subtree around within the subtree.
- Replace one substree rooted at node *u* as a child of its parent with another sybtree rooted at node *v*.
- Node *u*'s parent becomes node *v*'s parent, and u's parent ends up having *v* as its appropriate child.

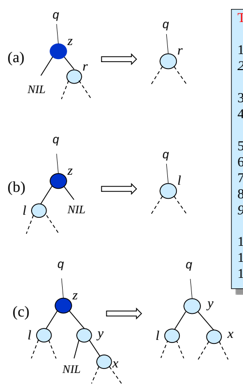
```
Transplant(T, u, v)
/* Handle u is root of T */
1.  if u.p == NIL
2.      T.root = v
/* if u is a left child */
3.  else if u == u.p.left
4.      u.p.left = v
/* if u is a right child */
5.  else u.p.right = v
/* update v.p if v is non-NIL */
6.  if v ≠ NIL
7.      v.p = u.p
```

Deletion Using Transplant(T, *u*, *v*)

```
Tree-Delete(T, z)
    /* (a) z has no left child */
1   if z.left == NIL
2.      Transplant(T, z, z.right)
    /* (b) z has a left child, but no right child */
3   else if z.right == NIL
4       Transplant(T, z, z.left)
    /* (c) z has two children */
5   else y = Tree-Minimum(z.right)  /* find z's successor */
6       if y.p ≠ z    /* y lies within y's right subtree but is not the root of this subtree */
7           Transplant(T, y, y.right)
8           y.right = z.right
9           y.right.p = y
        /* (d) if y is z's right child */
10      Transplant(T, z, y)
11      y.left = z.left     /* replace y's left child by z's left child */
12      y.left.p = y
```

TOTAL: O(h) time to delete a node

Deletion Using Transplant(T, *u*, *v*)



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5   else y = Tree-Minimum(z.right)  /* find z's successor */
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7           Transplant(T, y, y.right)
8           y.right = z.right
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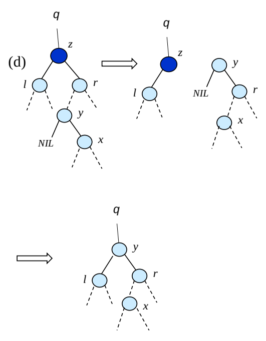
Deletion Using Transplant(T, *u*, *v*)



```
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3   else if z.right == NIL
4       Transplant(T, z, z.left)
    /* (c) z has two children */
5   else y = Tree-Minimum(z.right)  /* find z's successor */
6       if y.p ≠ z
7           Transplant(T, y, y.right)
8           y.right = z.right
9           y.right.p = y
        /* (d) if y is z's right child */
10      Transplant(T, z, y)
11      y.left = z.left     /* replace y's left child by z's left child */
12      y.left.p = y
```

Theorem 12.3

- The dynamic-set operations, INSERT and DELETE can be made to run in O(*h*) time on a binary search tree of height *h*.

The End

- Up next: guaranteeing a O(lg n) height tree