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Mathematical Preliminaries

(Part 1)

Lecture 02

Day 02/31

CS 154

Formal Languages and Computability

Spring 2019

Agenda of Day 02

- Waiting List Enrollment ...
- Announcement
- Summary of Lecture 01
- Lecture 02: Teaching ...
  - Mathematical Preliminaries (Part 1)

Announcement

- If you were absent last session,
  please talk to me right after the class.
- Otherwise, I might cancel your enrollment.

Summary of Lecture 01: We learned ...

Office Hours

- TR 7:15-9:15 pm
- By Appointment
- Tell me orally in this class.

OR

- Set an appointment via email.
- 24 hours before your requested time.
- I'll be in this class for my office hours.
- In some cases, I can set online office hours.

Examinations

- By default, every Thursday we'll have a short quiz!
  - So, it is not the case that I'd announce it again.
- All examinations are closed book (concepts).
- All examinations will cover everything we've covered from the beginning of the semester.
- I'll curve your final grade if it is not normal.

Summary of Lecture 01: We learned ...

Course Objective

- Dealing with the mathematical theory of computation.
- The theory of computation is divided into:
  - Formal languages
  - Automata theory
  - Computability
  - Complexity
- We'd discover the "atoms" and "molecules" of computing.

Classroom Protocol

- This is me if you use cell phone or laptop!
- For more info, please refer to the greensheet!

Any question?

Objective of This and Next Lecture

- Recap from Math 42 (discrete mathematics)
- We'll review:
  - Sets
  - Cartesian Products
  - Functions
  - Graphs

- Based on the prerequisite (Math42), we assume that:

  You are already familiar with them.

- So, we just review the most important concepts that we'd need in this course.
- There will be some questions from these topics in all tests.

Mathematical Preliminaries

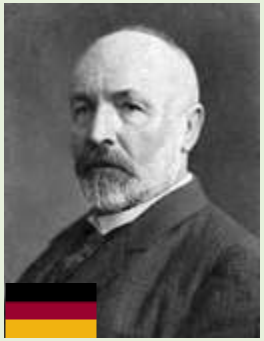
Recap from Math 42

The Basic Concepts of Set Theory

Set Theory

- "Set theory" has a great role in mathematics and consequently in other sciences.
  – In this course, we use sets tremendously.

- Created by great German mathematician, George Cantor (1845-1918).



- He is famous for the set theory and his work on the infinities.

- Specially, his method to prove that the set of real numbers is bigger than the set of natural numbers.

Sets Definition

Definition

- A set is a collection of objects (aka elements, members).

  - The definition implicitly stating that the "order" of the elements does not matter.
  - All objects in this universe are "distinct". That's why all elements of a set must be "distinct".
  - Therefore, in a set, you might repeat an element but only one of them counts as the member.

- We can also define a "list" as:

  A list is a collection of "ordered" objects.

- Do you think the "distinction" matters in lists?

Sets Representation 1: Roster Method

- One way to represent a set is enumerating its members.
- We put the elements in a pair of curly-braces, like this:

$$\{1, 2, 3, 4\}$$

Example 1

- The set of lower-case English alphabet.

$$\{a, b, c, \ldots, z\}$$

  – Sometimes we use ellipses (...) to bypass mentioning some elements if the general pattern of elements is obvious from the context.

- There are other set representations that will be covered shortly!

Sets Naming Convention

- To name a set, we usually use English capital letters such as A , B, C, etc., OR ...
- Greek capital letters such as Σ (sigma) , Γ (gamma), etc..

Example 2

- The set of lower-case English alphabet.

$$\Sigma = \{a , b , c , \ldots , z\}$$

- The set of natural numbers less than 100 and greater than 2.

$$\Gamma = \{3 , 4 , 5 , \ldots , 99\}$$

Sets Examples

Example 3

- N = {1 , 0 , -5 , 12 , 5}
- V = {train , bike , airplane , bus}
- Γ = {x , y , z}
- A = {00, 01, 10, 11}

- Is Σ a set?

  Σ = {ab , aabb, aaabbb}
- The elements are meaningless!

- Is B a set?

  B = {5 , train , apple , California}
- The elements are irrelevant!

- Is C a set?

  C = {1, 2, 3, 4}
- The elements are ordered!

- Is D a set?

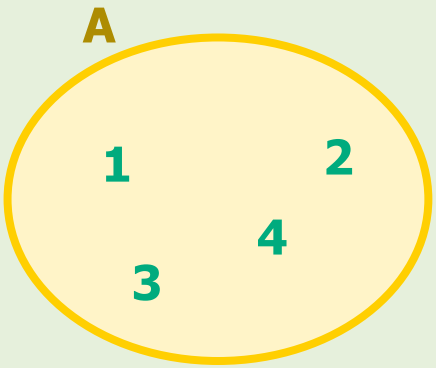
  D = {1, 2, 2, 3}
- The elements are repeated!

Sets Representation 2: Venn Diagrams

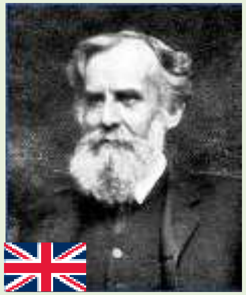
- Another way to represent a set is putting all its elements in a geometrical figure such as circle, ellipse, etc.
- These diagrams are called "Venn diagram".

Example 4

- A = {1 , 2 , 3 , 4}



- This method is named after British mathematician, John Venn (1834 – 1923).



Sets Size

- Size of a set (aka cardinality) is the number of its elements.
- The size of the set A is denoted by |A|.

Example 5

- Let A = {1 , 0 , -5 , 12 , 5} ; |A| = ?
- |A| = 5

Example 6

- Let B = {1 , 11 , 7 , -15 , 2 , 1 , 7 , 11} ; |B| = ?
- |B| = 5 (careful! Duplicate members should be eliminated.)

Sets Membership and Not Membership

- The membership of the set's elements is represented by "$\in$".
- Its negation (not membership) is represented by "$\notin$".

Example 7

- Let C = {5 , train , apple}
- train $\in$ C
  (read: train belongs to C, or train is a member of C)
- bus $\notin$ C
  (read: bus does not belong to C, or bus is not a member of C)

Membership and Not Membership Note

- A set is known when its boundary is clearly defined.

- We should be able to recognize clearly what belongs to a set and what does not.

- Thus, "not membership" is as important as "membership".

Empty Set

Definition

- Empty set is a set that has no member.
- Empty set is denoted by { } or ϕ.
  - "ϕ" is pronounced "phi".

- What is the size of ϕ?
- $|\phi| = 0$

Example 8: An Empty Set

- The set of "F-Students of this class"!!!
- The 8th day of week!

Universal Set

- We usually need to specify the "universe of our discourse".
- This universe is all possible members that affect the problem under consideration.
- We call it universal set.

Definition

- Universal set of a set is the set containing all possible elements under consideration.
- Universal set is denoted by "U".

Universal Set Examples

Example 9

- Let A = {2 , 3 , 4}.

- The universal set of A could be:
  - U = {0 , 1 , 2 , 3 , 4 , 5 , 6 , 8}, or
  - U = {1 , 2 , 3 , 4}, or
  - U = {2 , 3 , 4}, or ...

- But the universal set of A cannot be U = {2 , 3}!

Universal Set Examples

Example 10

- Let Z = {A-Students of this class}.

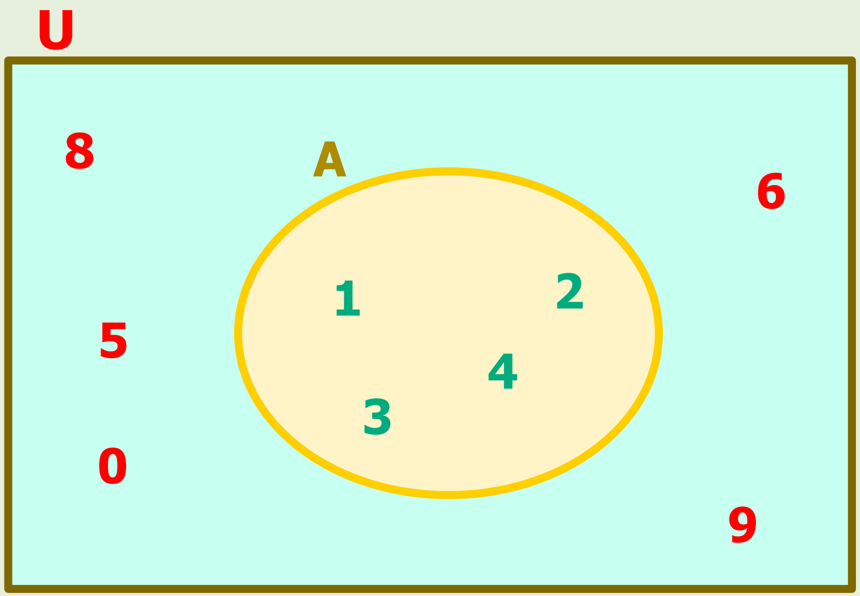
- Depends on the problem we want to solve, the universal set of Z could be:
  - U = {All students of this class}, or
  - U = {All students of SJSU}, or
  - U = {All students of the world}, or so forth.

Venn Diagram of Universal Set

- We represent a universal set by a rectangle.

Example 11

- A = {1 , 2 , 3 , 4}
- U = {0, 1, 2, 3, 4, 5, 6, 8, 9}



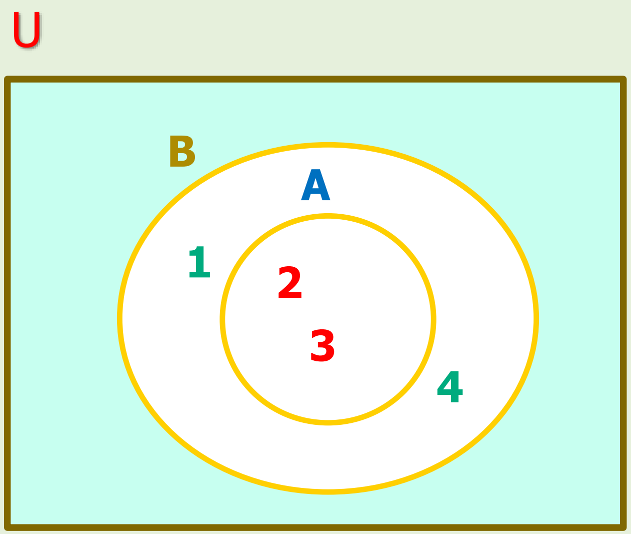
Subsets

Definition

- Set A is subset of B iff every elements of A is also an element of B.
- Subset relationship is denoted by $A \subseteq B$.

Example 12

- $B = \{1, 2, 3, 4\}$
- $A = \{2, 3\}$
- $A \subseteq B$
  - Note that A is inside B.



- We did not mention the elements of U because our focus is on the relationship between A and B.

Proper Subsets

Definition

- Set A is proper subset of B iff all elements of A belong to B, and they are not equal.
- Proper subset relationship is denoted by $A \subset B$.
- In this relationship, B is called superset of A.

Example 12 (repeated)

- $B = \{1 , 2 , 3 , 4\}$
- $A = \{2 , 3\}$
- $A \subset B$

Equality of Two Sets

Definition

- Two sets A and B are equal iff both have the same elements.
- Equality of two sets A and B is denoted by $A = B$.

- There is another way to define equality of two sets:

Equality of Two Sets by Using Subset Notation

$$A = B \text{ iff } A \subseteq B \text{ AND } B \subseteq A$$

Finite Sets

Definition

- A set is called finite if its size is a natural number.
  - The set of natural numbers is denoted by $\mathbb{N}$ and starts from 0.
    $\mathbb{N} = \{0, 1, 2, \ldots\}$
  - In some books, you might see it starts from 1
    but we prefer in this course to start it from 0.

Example 13

- Let $B = \{a, b, c, \ldots, z\}$. Is this a finite set?
- Yes, because $|B| = 26$, and 26 is a natural number.

- Is ϕ finite? Why?

Infinite Sets

Definition

- A set is infinite, if we cannot express its size by a natural number.

Example 14: Infinite Sets

- $C = \{1, 2, 3, 4, ...\}$
- $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ Integers
- $\mathbb{N} = \{0, 1, 2, ...\}$ Natural numbers

Sets Representation 3: Set Builder

- So far, we've reviewed two methods for set representation:
  1. Roster method
  2. Venn diagram method

- There are another method that is more important than these two.
- It's called set builder method.

- We use set builder method tremendously in this course.

Sets Representation 3: Set Builder

Set Builder Format

{ member variable : description of the elements' properties }

– Note that some books might use vertical bar "|" instead of colon ":".

Example 15

- Represent the following set by a set builder.

  "The set of all Natural numbers between 1 and 5 (both including)"

- $A = \{x : x \in \mathbb{N} , 1 \leq x \leq 5\}$

- This can be simulated by the following Java code:

```
for (int x = 1 ; x <= 5 ; x++) { //some code here }
```

Sets Representation 3: Set Builder

- For simplifying the representation,
  we might put the universal set description before the colon.

Example 15 (repeated)

- The set of all integers between 1 and 5 (both including)

Regular representation: $A = \{x : x \in \mathbb{N} , 1 \le x \le 5\}$

Simplified representation: $A = \{x \in \mathbb{N} : 1 \le x \le 5\}$

Sets Representation 3: Set Builder

- If the members of the set follow a pattern,
  we use the following format:

  { members pattern : description of the elements' properties }

Example 16

- Represent the following set by a set builder.

$$B = \{0 , 3 , 6 , 9 , 12 , 15 , 18\}$$

- Regular representation:

$$B = \{ x : x = 3k , k \in \mathbb{N} , 0 \le k \le 6\}$$

- Using pattern representation (preferred):

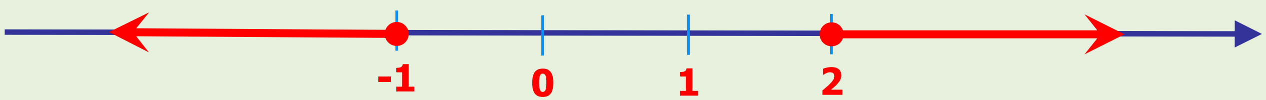
$$B = \{3k : k \in \mathbb{N} , 0 \le k \le 6\}$$

Set Builder Note

- Comma in the set builder description means "AND".
  - So, if you need "OR", you should explicitly put "OR" or "$\vee$".

Example 17

- Represent the following real numbers intervals by set builder.



$$B = \{ x \in \mathbb{R} : x \leq -1 \text{ OR } x \geq 2\}$$

- What would happen if we put comma or AND in the above set?

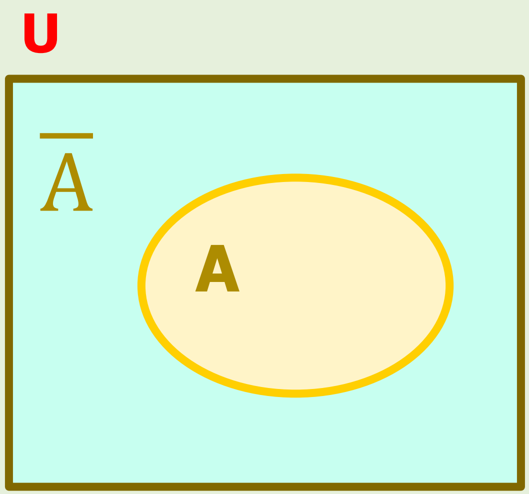
Sets Complement

Definition

- The complement of set A is called $\overline{A}$ and is defined as:

$$\overline{A} = \{x : x \notin A\}$$

- Venn diagram of $\overline{A}$:



- To find $\overline{A}$, we'd need U.

Example 20

- Let A = {3 , 6}, and U = {1, 2, 3, 4, 5, 6} ; $\overline{A}$ = ?
- $\overline{A} = \{1, 2, \cancel{3}, 4, 5, \cancel{6}\} = \{1, 2, 4, 5\}$

Exercise

Example 18

- Write all subsets of $A = \{a , b\}$.

Example 19

- Write all subsets of $B = \{1 , 2 , 3\}$

Power Set

Definition

- The set of all subsets of set A is called the power set of A.
- The power set of A is denoted by 2^A.
  - Note that 2^A is just a symbol and not an algebraic power!

- How can we define the powerset of A by set builder?

$$2^A = \{x : x \subseteq A\}$$

Example 21

- Let $A = \{a , b\}$; $2^A = ?$
- $2^A = \{\phi , \{a\} , \{b\} , \{a , b\}\}$

Power Set

Example 22

- Let $S = \{1, 2, 3\}$; $2^S = ?$
- $2^S = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Example 23

- In the previous example, what is the cardinality (size) of S and 2^S?
- $|S| = 3$
- $|2^S| = 8$

- Do you see any relation between these two cardinalities?

Size of Power Set

- If set S has n elements (i.e. $|S| = n$), then its power set has 2^n elements.

- In other words, we have the following relationship between the size of a set and the size of its power set.

$$|2^S| = 2^{|S|}$$

Example 24

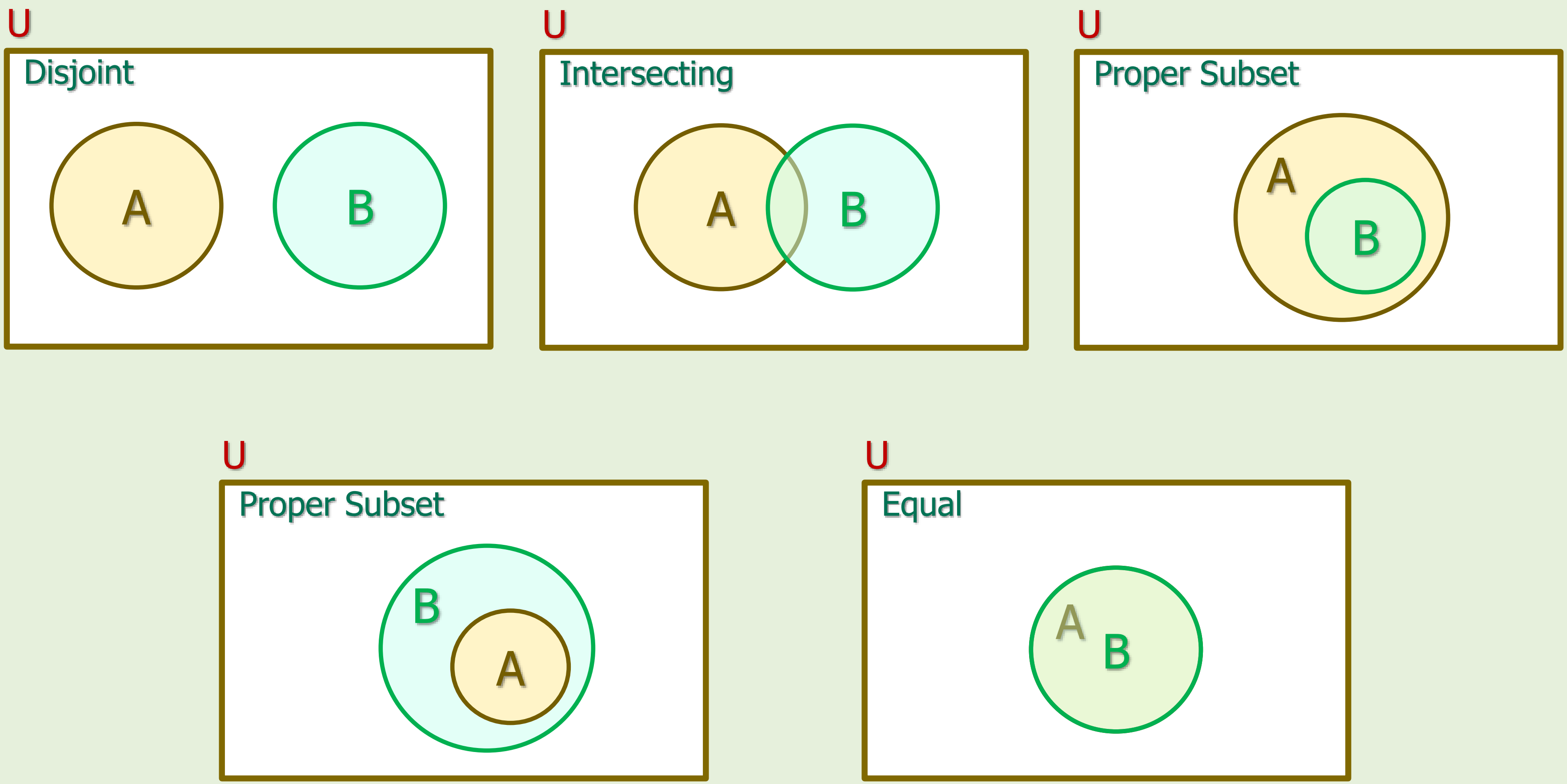
- Let $S = \{a , b , c\}$; $|2^S| = ?$
- $|2^S| = 2^{|S|} = 2^3 = 8$

Reading Assignment

Exercise

Concept	Notation
Empty set	
Universal set	
8 is member of A.	
6 is not member of B.	
A is subset of Σ.	
B is proper subset of Σ.	
Power set of A	
Size of A (aka Cardinality of A)	
Size of the power set of A	

Relationship Between Two Sets



Set Operations

Operator	Notation
Union	$A \cup B = \{x : x \in A \;\vee\; x \in B\}$
Intersection	$A \cap B = \{x : x \in A \;\wedge\; x \in B\}$
Minus	$A - B = \{x : x \in A \;\wedge\; x \notin B\}$
Complement	$\overline{A} = U - A = \{x : x \in U \;\wedge\; x \notin A\} = \{x : x \notin A\}$

Set Properties

Property	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive

Set Identities

Identity	Result	Name
$A \cup \phi =$ $A \cap U =$		Identity
$A \cup U =$ $A \cap \phi =$		Domination
$A \cup A =$ $A \cap A =$		Idempotent
$A \cup \overline{A} =$ $A \cap \overline{A} =$		Complement
$\overline{(\overline{A})} =$		Complementation
$\overline{A \cap B} =$ $\overline{A \cup B} =$		De Morgan

Empty Set Representation

- How can we represent empty set by using set builder?
- $A - A = \{ \}$
- $A - A = \{x : x \in A \text{ AND } x \notin A\}$
- $\phi = \{x : F(alse)\}$

- So, to represent empty set,
  we can put anything false in the set builder description.

- For example, the following sets represent empty sets:
- $\{x : x \text{ is the } 8^{th} \text{ day of week}\}$
- $\{x : x \notin U\}$

Homework

- Note that the homework in the lecture notes are not mandatory but STRONGLY recommended.

1. Represent the set operations by Venn diagrams.
2. Prove that $A \cup B = \overline{\overline{A} \cap \overline{B}}$
3. What is the relationship between sets A and B in the following situations:
   a) $A \cup B = A$
   b) $A - B = A$
   c) $A \cap B = A$
   d) $A - B = B - A$
   e) $A \cap B = B \cap A$

References

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