San José State University
Department of Computer Science

Ahmad Yazdankhah
ahmad.yazdankhah@sjsu.edu
www.cs.sjsu.edu/~yazdankhah

Non-Regular Languages

(Part 2)

Lecture 25

Day 29/31

CS 154

Formal Languages and Computability

Spring 2019

Agenda of Day 29

- Summary of Lecture 24
- Quiz 10
- Lecture 25: Teaching ...
  - Non-Regular Languages (Part 2)

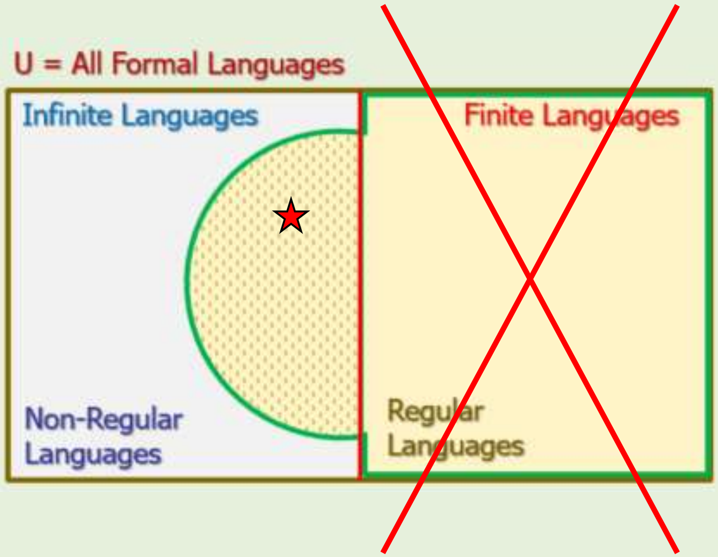
Summary of Lecture 24: We learned ...

Non-Regular Languages

- We started with this question:

How to prove a language is NONREGULAR?

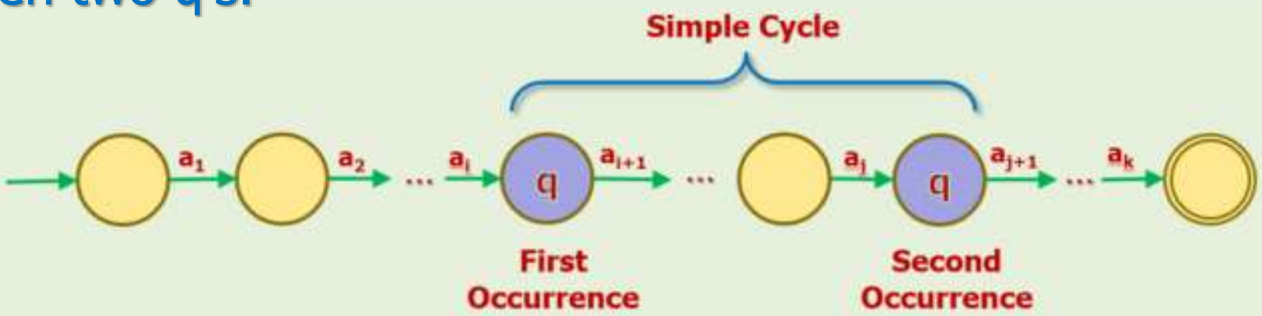
- We stated an important property of "infinite regular languages".



Summary of Lecture 24: We learned …

- We took L as a regular language.
- Since L is regular, then there is an DFA for it.
- Assume it has m states.
- Take a string $w = a_1\ a_2 \ldots a_k \in L$ whose size is $|w| \geq m$.
- Since $|w| \geq m$, based on pigeonhole principle, in the walk of w,

  at least one state is visited more than once.
- We called the first repeated-state as 'q'.
- We pick the q in such a way that there is no nested repeated-state between two q's.



Summary of Lecture 24: We learned ...

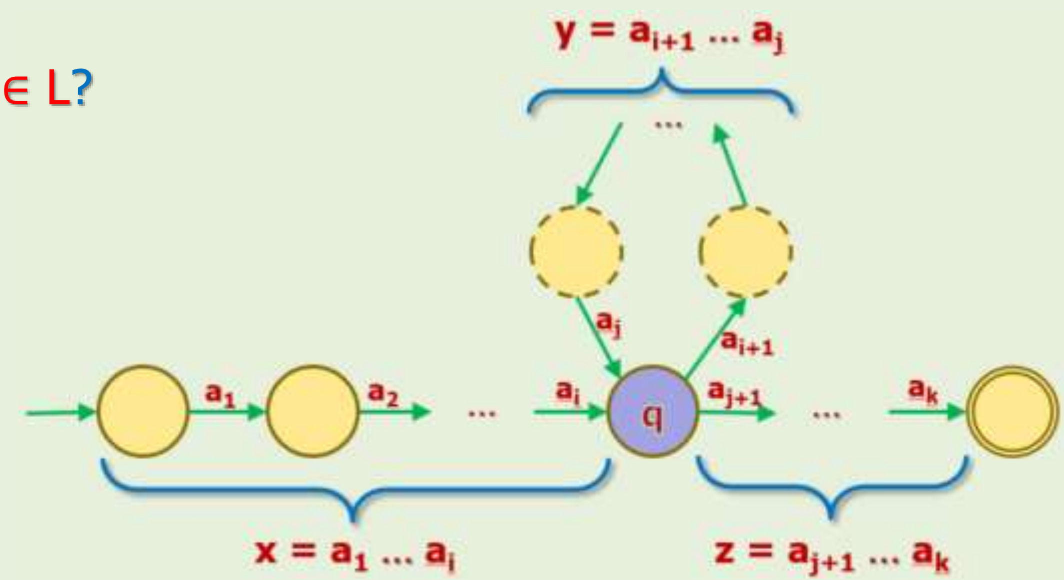
- The original DFA looks like the following figure:
- We named the 3 portions as: x y z

Questions

- $|xy| \le m$?
- $|y| \ge 1$?
- $xz = a_1\, a_2 \ldots a_i\, a_{j+1} \ldots a_k \in L$?
- How about $xyyz \in L$?
- Or, $xyyyz \in L$?
- Or in general:
  $x\, y^i\, z$, for $i = 0, 1, 2, \ldots$

- The answer is yes to all questions, so all strings $x\, y^i\, z \in L$.
- So, if some certain conditions are satisfied, we can pump any number of y's in the original string and the resulting string is still part of the language.

Any Question



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SUBJECT	CS 154	TEST NO.	10
DATE	05/02/2019	PERIOD	1 / 2 / 3

TEST RECORD	
PART 1	123
PART 2	
TOTAL	

Your list # goes here!

Quiz 10

No Scantron

Pumping Lemma

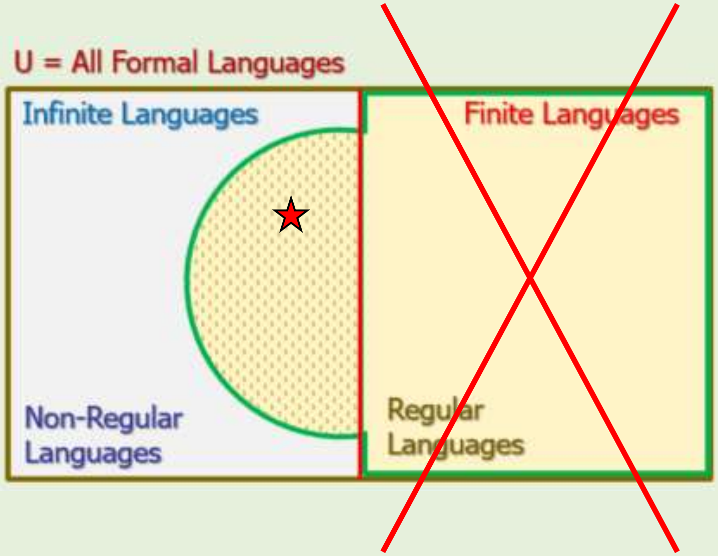
What is a Lemma?

Etymology

- "Lemma" is a smaller theorem to help proving a bigger one.

- Very occasionally lemmas can take on a life of their own.
- In computer science, "pumping lemma" is one of them.

Pumping Lemma



If L is an INFINITE REGULAR language,

Then there exists an $m \geq 1$ such that

If $w \in L$ and $|w| \geq m$

Then `//pumping lemma guarantees that ...`

We must be able to divide w into three parts xyz in such a way that all of the following conditions are satisfied:

$|xy| \leq m$, and

$|y| \geq 1$, and

$w_i = x\, y^i\, z \in L$ for i = 0, 1, 2, 3, ...

Formal Statement of Pumping Lemma

If L is an infinite regular language,

Then

(1) There exists an $m \geq 1$ such that

If (2) $w \in L$ and (3) $|w| \geq m$

Then //P. L. guarantees that ...

(4) We must be able to divide w into xyz in such a way that all of the following conditions are satisfied:

(5) $|xy| \leq m$, and

(6) $|y| \geq 1$, and

(7) $w_i = x\, y^i\, z \in L$

(8) for i = 0, 1, 2, ...

Predicate Calculus Version

FYI

This part is only for your information!

If L is an infinite regular language,

Then

$(\exists\ m \geq 1)$

$[(w \in L \text{ and } |w| \geq m\ ) \ \rightarrow$

$(\exists\ x,y,z\ )\ ($

$w = xyz\ \wedge$

$|xy| \leq m\ \wedge$

$|y| \geq 1\ \wedge$

$(\forall\ i \in \mathbb{N}\ )\ (w_i = x\, y^i\, z \in L)$

$)]$

Steps of Pumping Lemma

Step	Description	Comment
1	Take an m	Always take it as m
2	Take w	A string from the language dependent to m
3	Check $\|w\| \geq m$	
4	Find x, y, z	$w = x\ y\ z$
5	Check $\|xy\| \leq m$	
6	Check $\|y\| \geq 1$	
7	Construct $w_i = xy^iz \in L$	
8	Check various i's	For $i = 0, 1, 2, 3, \ldots$

Pumping Lemma

Example 7

- Verify the pumping lemma property on the following infinite regular language.

$$L = \{a^n b : n \geq 0\}$$

Solution

(1) Let's take the m = 2. Why not 3?

OK, **let's take it as m**.

- If we need, we'd make some boundary on m later.

(2) Let's take $w = a^m b$

- Note that m is a constant.
- It means, $a^m b$ is a string, NOT a pattern.

(3) Check w's size:
$|w| = |a^m b| = m+1 \geq m$ ✓

- Pumping lemma guarantees that:

(4) There exists x, y, z such that:

$w = a^m b = xyz = \lambda \quad a \quad a^{m-1}b$

(5) $|xy| = |a| = 1 \leq m$ ✓

(6) $|y| = 1 \geq 1$ ✓

(7) $w_i = xy^i z = \lambda\, a^i\, a^{m-1}b \in L$

(8) Check various i's:

- i=0 $w_0 = xz = a^{m-1}b \in L$ ✓
- i=1 $w_1 = xy^1 z = a^m b \in L$ ✓
- i=2 $w_2 = xy^2 z = a^{m+1}b \in L$ ✓
- i=3 $w_3 = xy^3 z = a^{m+2}b \in L$ ✓
- ...

Pumping Lemma

Example 8

- Verify the pumping lemma property on the following infinite regular language.

$$L = \{bba^n : n \geq 0\}$$

Solution

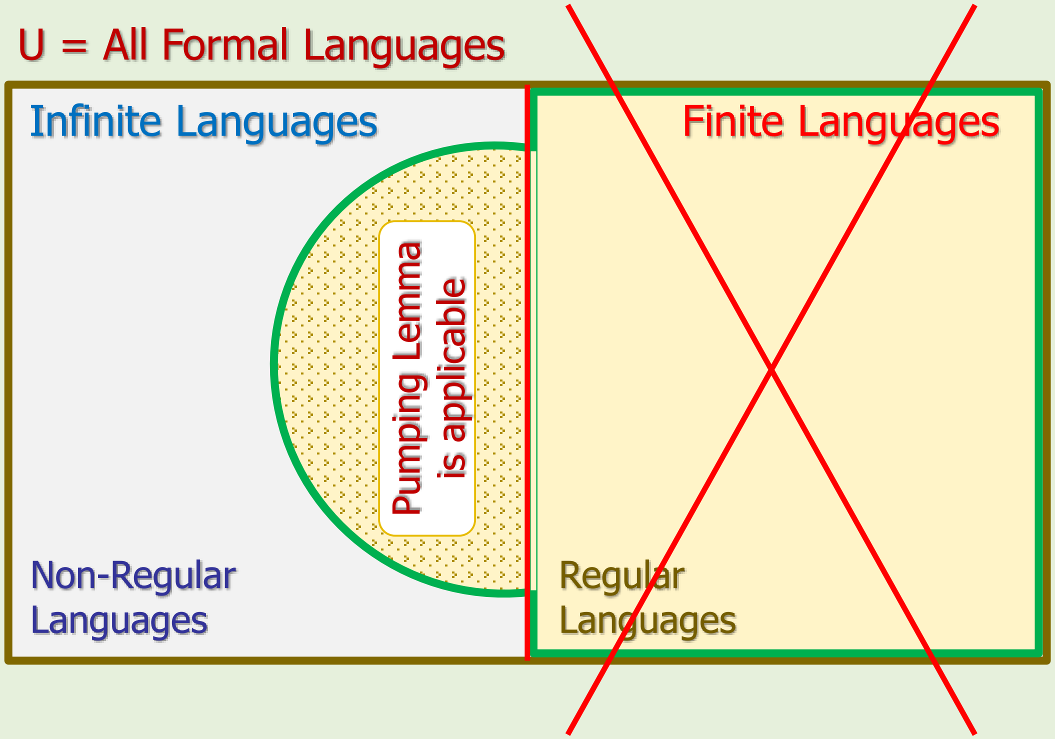
Homework

- Verify the pumping lemma property on the following infinite regular languages.

1. $L = \{a^n b^k : n \geq 0, k \geq 0\}$
2. $L = \{aaab^n (ab)^k : n \geq 0, k \geq 0\}$
3. $L = \{(ab)^n : n \geq 0\}$

Conclusion

- This is a property of INFINITE REGULAR languages.



- If an "infinite language" does not have this property, it is "non-regular".

Pumping Lemma: Notes

1. A strings such as $a^m b$ is just one string of the language, and NOT a pattern because m is a constant.
2. In the previous example (#7), one could take w something like:
   - $a^{2m}b$ or $a^{m+100}b$
   - But, try to take it as simple as possible.
3. We should always make sure that no string gets negative power.
   - For example, if, somewhere in our proof, we have something like a^{m-3}, then we should mention "we pick $m \geq 3$".
   - Recall that pumping lemma has the power of making a boundary for 'm'.
4. But if you have something like a^{m-1}, you don't need to mention it because by default $m \geq 1$.

Application of Pumping Lemma

How to Prove a Language is Non-Regular?

- Use "proof by contradiction"

1. Assume L is regular. So, the pumping lemma should hold for L.
2. Apply pumping lemma
3. Find a contradiction.
4. Then, blame your assumption and conclude that L must be non-regular.

- Recall that all non-regular languages are infinite.

- Let's take some examples!

Applications of Pumping Lemma

Example 9

- Prove $L = \{a^n b^n : n \geq 0\}$ is non-regular language.

Proof

Applications of Pumping Lemma

Example 10

- Prove $L = \{uu : u \in \{a, b\}^*\}$ is non-regular language.

Proof

Homework

- Prove that the following languages are non-regular:
  1. $L = \{uu^R : u \in \{a, b\}^*\}$
  2. $L = \{a^n b^n c^n : n \geq 0\}$
  3. $L = \{uuu : u \in \{a, b\}^*\}$
  4. $L = \{a^n b^k c^{n+k} : n \geq 0, k \geq 0\}$

More Notes About Pumping Lemma

1. Pumping lemma is difficult to understand! [Text book, P#121]

   NOT anymore! 

2. Pumping lemma is not applicable to finite languages.

   Because we need to pump infinite y's!

3. Pumping lemma cannot prove that a languages is regular.

   Because you'd need to verify infinite cases!

References

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