Due: September 17, 2019

1. Consider the continuous-time, *linear-quadratic problem* with dynamics

$$\dot{x} = Ax + Bu, \qquad t \in [0, t_f]$$

and performance index

$$J(x_0; u(\cdot)) = \tfrac{1}{2} x^{\mathsf{T}}(t_f) S x(t_f) + \tfrac{1}{2} \int_0^{t_f} [x^{\mathsf{T}}(t) Q x(t) + u^{\mathsf{T}}(t) R u(t)] \, \mathrm{d}t$$

where $t_f = 10$, initial condition $x(0) = [5,\ 2]^{\mathsf{T}}$, and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = 1 \qquad S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Show that the optimal open-loop control for this problem is given by

$$u(t) = -R^{-1} B^{\mathsf{T}} \lambda(t)$$

Compute the optimal control history by solving the associated two-point boundary-value problem in the (x, λ) space. You may use the MATLAB command `bvpc` to solve this two-point boundary-value problem, or you can use any other code you wish, as long as you reference it properly.

(b) Compute the optimal feedback control for this problem by solving the corresponding Hamilton-Jacobi-Bellman equation. Assume that the solution of this equation is given by

$$V(t, x) = x^{\mathsf{T}} P(t) x$$

for some matrix $P(t)$. Write down the differential equation of $P(t)$ and the corresponding boundary condition. Integrate this equation to find the history of $P(t)$. Show that the optimal feedback control is given by

$$u(t, x) = -R^{-1} B^{\mathsf{T}} P(t) x$$

(c) Confirm that the open-loop and closed-loop solutions you got in (a) and (b) above agree on the optimal trajectory. Specifically, plot both control histories on the same figure. Also plot, on the same figure, the optimal co-state history $\lambda^*(t)$ and the gradient of the value function along the optimal trajectory, $V_x(t, x^*(t)) = P(t) x^*(t)$. Confirm that the two coincide, as proved in the class.

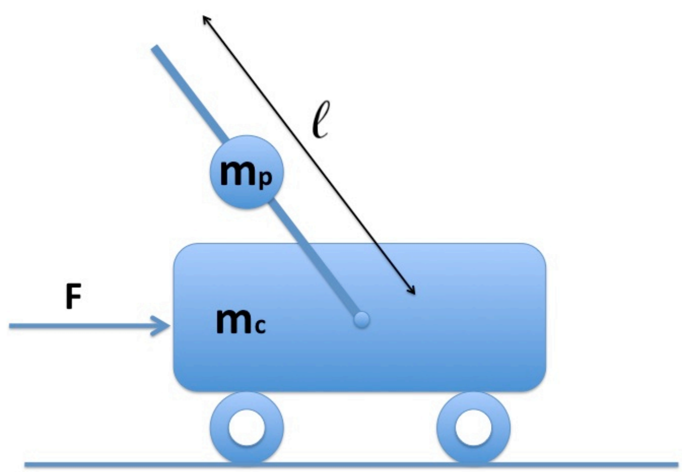
2. In the derivation of the Hamilton-Jacobi-Belman partial differential equation (for the continuous time case) and the Bellman equation (for the discrete-time case) we did in the class, we used the following *Principle of Optimality*, due to Bellman, that states:

> *Principle of Optimality (PoP): An optimal control has the property that whatever the initial control action and initial state are, the remaining control actions must constitute an optimal policy with regard to the state resulting from the first control action.*

Another way to express the PoP is to say that any part $\{x^*(t) : t' \leq t \leq t_f\}$ of an optimal trajectory $\{x^*(t) : t_0 \leq t \leq t_f\}$ starting at $x^*(t_0)$, where $t_0 < t'$, is itself optimal for the optimal control problem starting at the initial condition at that time, $x^*(t')$.

Explain in which part of the proofs given in the class (both for the continuous and discrete time cases) the PoP was used.

3. We wish to use Differential Dynamic Programming (DDP) to find the optimal control to stabilize the pendulum for the cart-pole system.



The objective of the problem is to exert a force on the cart so as to stabilize the pendulum in the up position.

A draft MATLAB code can be downloaded from `canvas` under the `code` folder.

(a) Complete the code as indicated.

(b) Plot the results.

(c) Create an animation of the stabilizing maneuver.

(d) Upload your animation (in .avi or .mp4 format) to `canvas`.