Modelli Probabilistici Temporali

IALab A.A. 2018/2019

Intro

Time and uncertainty

The world changes; we need to track and predict it

Diabetes management vs vehicle diagnosis

Basic idea: copy state and evidence variables for each time step

$\mathbf{X}_t$ = set of unobservable state variables at time t
e.g., $BloodSugar_t$, $StomachContents_t$, etc.

$\mathbf{E}_t$ = set of observable evidence variables at time t
e.g., $MeasuredBloodSugar_t$, $PulseRate_t$, $FoodEaten_t$

This assumes **discrete time**; step size depends on problem

Notation: $\mathbf{X}_{a:b} = \mathbf{X}_a, \mathbf{X}_{a+1}, \ldots, \mathbf{X}_{b-1}, \mathbf{X}_b$

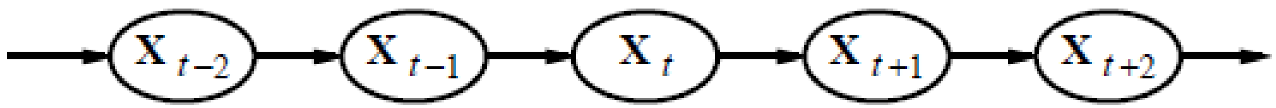
Markov processes (Markov chains)

Construct a Bayes net from these variables: parents?

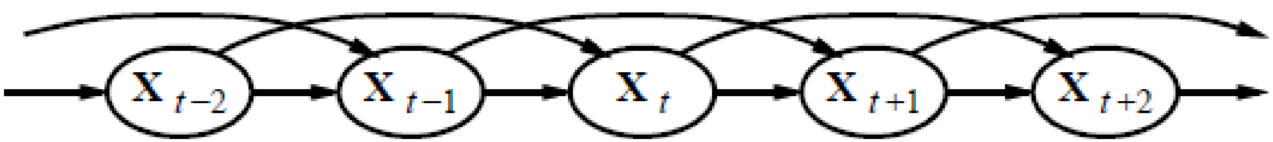
Markov assumption: $\mathbf{X}_t$ depends on bounded subset of $\mathbf{X}_{0:t-1}$

First-order Markov process: $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$
Second-order Markov process: $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-2}, \mathbf{X}_{t-1})$

First–order



Second–order

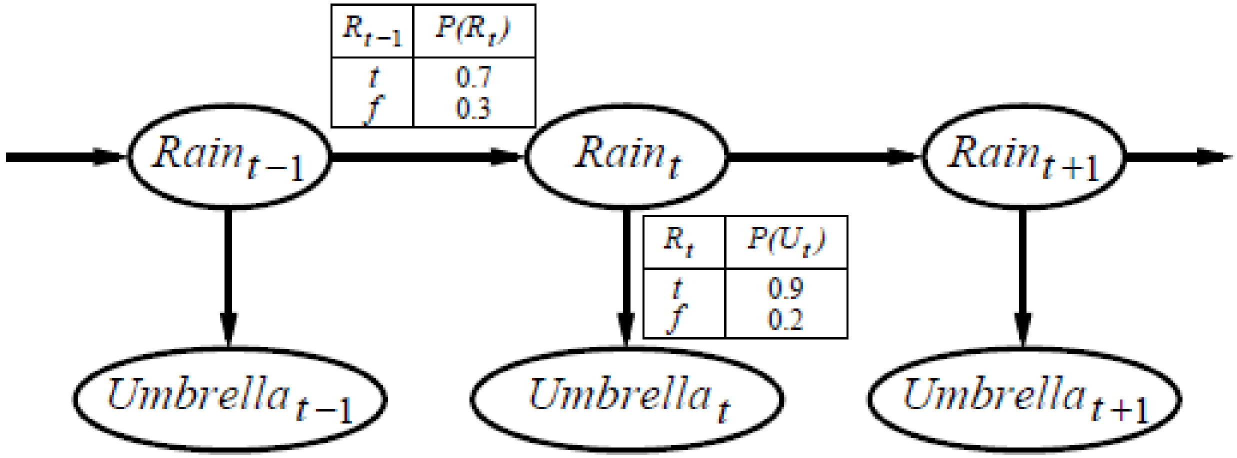


Sensor Markov assumption: $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$

Stationary process: transition model $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$ and sensor model $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$ fixed for all t

le distribuzioni condizionali non cambiano nel tempo

Example



First-order Markov assumption not exactly true in real world!

Possible fixes:

1. **Increase order** of Markov process
2. **Augment state**, e.g., add $Temp_t$, $Pressure_t$

Example: robot motion.

Augment position and velocity with $Battery_t$

Inference tasks

Filtering: $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$
belief state—input to the decision process of a rational agent

Prediction: $\mathbf{P}(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$ for $k > 0$
evaluation of possible action sequences;
like filtering without the evidence

Smoothing: $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$
better estimate of past states, essential for learning

Most likely explanation: $\arg\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}|\mathbf{e}_{1:t})$
speech recognition, decoding with a noisy channel

corrisponde alla
Simple Query

è analoga alla **MPE**
La vedrete nel corso di
Tecnologie del Linguaggio
Naturale

Quali sono utili per
risolvere un crimine?

Filtering

Aim: devise a **recursive** state estimation algorithm:

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})} = f(\mathbf{e}_{t+1}, \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}))$$

regola di Bayes condizionata a evidenza (prossima slide)

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}, \mathbf{e}_{t+1})\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})\end{aligned}$$

proprietà di Markov

I.e., prediction + estimation. Prediction by summing out $\mathbf{X}_t$:

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\Sigma_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t})P(\mathbf{x}_t|\mathbf{e}_{1:t})\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\Sigma_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t})\end{aligned}$$

$f_{1:t+1}$

$f_{1:t}$

$\mathbf{f}_{1:t+1} = \text{FORWARD}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1})$ where $\mathbf{f}_{1:t} = \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$

Time and space **constant** (independent of t)

Regola di Bayes con Evidenza

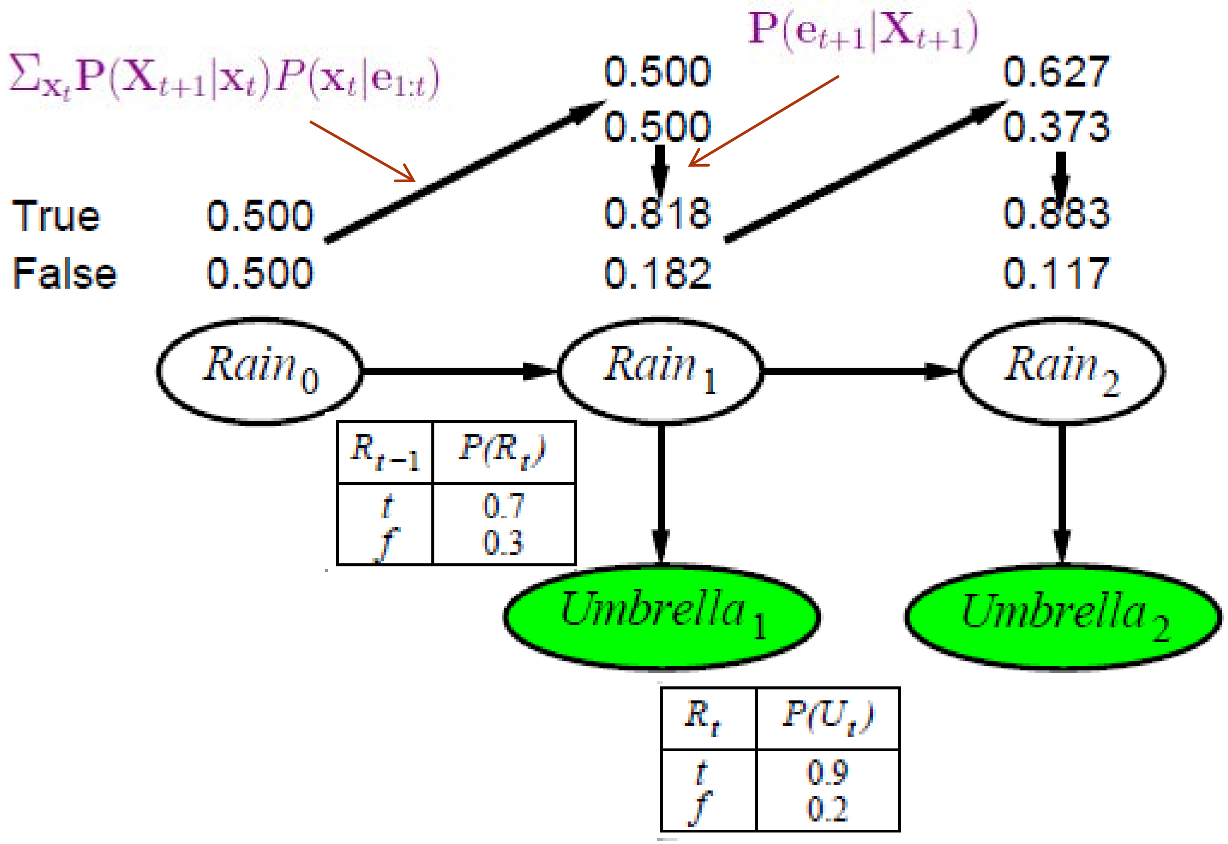
La Regola di Bayes che abbiamo visto è la seguente:

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)}$$

La sua **variante con evidenza** è la seguente:

$$\mathbf{P}(Y|X, \mathbf{e}) = \frac{\mathbf{P}(X|Y, \mathbf{e})P(Y|\mathbf{e})}{P(X|\mathbf{e})}$$

Filtering example



$P(X_1{=}T \mid e_{0:1}) = \alpha \times 0{,}9 \times 0{,}5 = 0{,}45$

$P(X_1{=}F \mid e_{0:1}) = \alpha \times 0{,}2 \times 0{,}5 = 0{,}10$

normalizzazione → <0,818; 0,182>

$P(X_2{=}T \mid x_1)P(x_1 \mid e_1) + P(X_2{=}T \mid \neg x_1)P(\neg x_1 \mid e_1) = 0{,}7 \times 0{,}818 + 0{,}3 \times 0.182 = 0{,}627$

Prediction

- la **prediction** è del tutto analoga al filtering
- nel filtering prediciamo la $\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$ a partire da $\mathbf{e}_{t+1}$ e dalla distribuzione $P(\mathbf{X}_t | \mathbf{e}_{1:t})$:

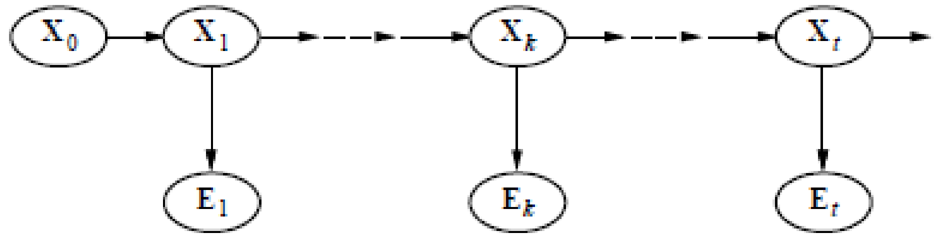
$$\boldsymbol{P}(\boldsymbol{X}_{t+1}|\boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1}|\boldsymbol{X}_{t+1}) \sum_{\boldsymbol{x}_t} \boldsymbol{P}(\boldsymbol{X}_{t+1}|\boldsymbol{x}_t) \boldsymbol{P}(\boldsymbol{x}_t|\boldsymbol{e}_{1:t})$$

- nella prediction prediciamo la $\mathbf{P}(\mathbf{X}_{t+k+1} | \mathbf{e}_{1:t})$ a partire dalla sola distribuzione $P(\mathbf{X}_{t+k} | \mathbf{e}_{1:t})$:

$$\boxed{\boldsymbol{P}(\boldsymbol{X}_{t+k+1}|\boldsymbol{e}_{1:t})} = \sum_{\boldsymbol{x}_{t+k}} \boldsymbol{P}(\boldsymbol{X}_{t+k+1}|\boldsymbol{x}_{t+k}) \boldsymbol{P}(\boldsymbol{x}_{t+k}|\boldsymbol{e}_{1:t})$$

non c'è fattore con $\mathbf{e}_{t+k+1}$

Smoothing



Divide evidence $\mathbf{e}_{1:t}$ into $\mathbf{e}_{1:k}$, $\mathbf{e}_{k+1:t}$:

$$
\begin{aligned}
\boxed{\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})} &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k},\mathbf{e}_{k+1:t}) \\
&= \alpha\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k,\mathbf{e}_{1:k}) \\
&= \alpha\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) \\
&= \alpha\mathbf{f}_{1:k}\mathbf{b}_{k+1:t}
\end{aligned}
$$

proprietà di Markov

Backward message computed by a backwards recursion:

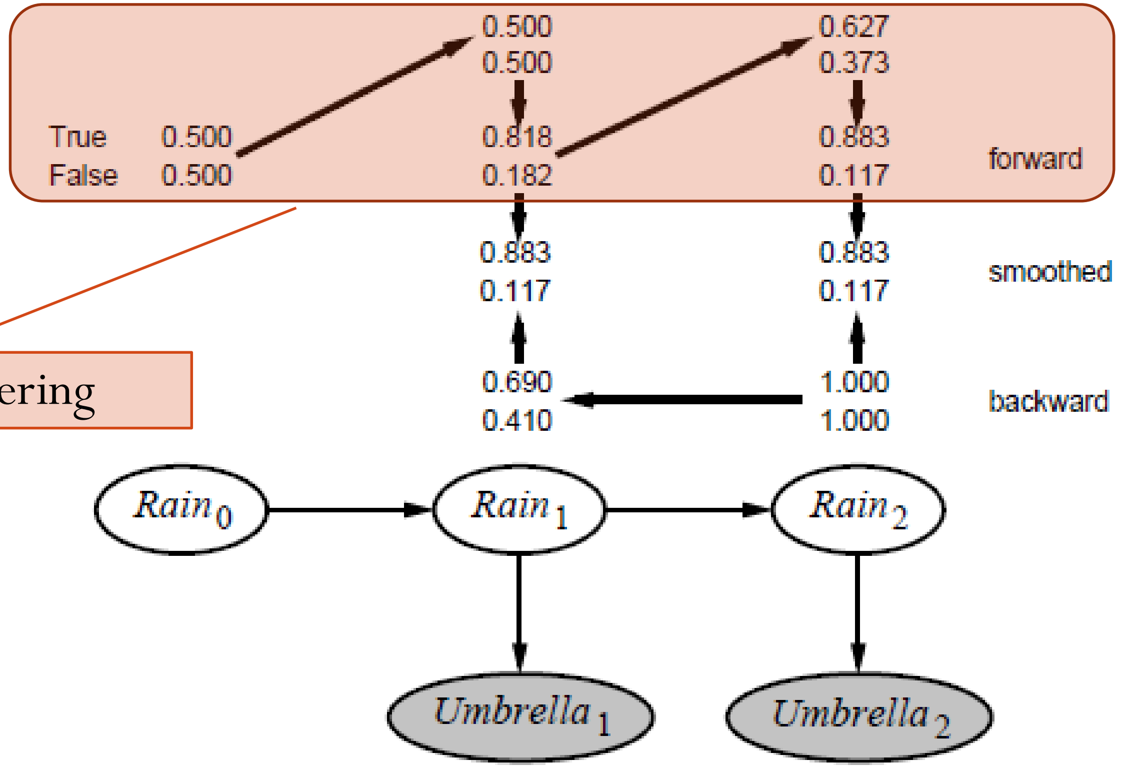
$$
\begin{aligned}
\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) &= \Sigma_{\mathbf{x}_{k+1}}\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k,\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= \Sigma_{\mathbf{x}_{k+1}}P(\mathbf{e}_{k+1:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= \Sigma_{\mathbf{x}_{k+1}}P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)
\end{aligned}
$$

$b_{k+1:t}$

$e_{k+2:t}$ c.i. e_{k+1} dato x_{k+1}

$b_{k+2:t}$

Smoothing example



vedi slide su filtering

Forward–backward algorithm: cache forward messages along the way
Time linear in t (polytree inference), space $O(t|\mathbf{f}|)$

Hidden Markov Models

Hidden Markov models

$\mathbf{X}_t$ is a single, discrete variable (usually $\mathbf{E}_t$ is too)
Domain of X_t is $\{1, \ldots, S\}$

Transition matrix $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$, e.g., $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$

Sensor matrix $\mathbf{O}_t$ for each time step, diagonal elements $P(e_t | X_t = i)$
e.g., with $U_1 = true$, $\mathbf{O}_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$

è già una osservazione specifica

Forward and backward messages as column vectors:

$$\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^\top \mathbf{f}_{1:t}$$

$$\mathbf{b}_{k+1:t} = \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t}$$

$\alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t) P(\mathbf{x}_t|\mathbf{e}_{1:t})$

$\sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)$

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$$\mathbf{b}_{k+1:t} = \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t}$$

$$\Sigma_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)$$

Forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$

lineare in t come atteso

Country dance algorithm

Can avoid storing all forward messages in smoothing by running forward algorithm backwards:

$$
\begin{aligned}
\mathbf{f}_{1:t+1} &= \alpha \mathbf{O}_{t+1} \mathbf{T}^\top \mathbf{f}_{1:t} \\
\mathbf{O}_{t+1}^{-1} \mathbf{f}_{1:t+1} &= \alpha \mathbf{T}^\top \mathbf{f}_{1:t} \\
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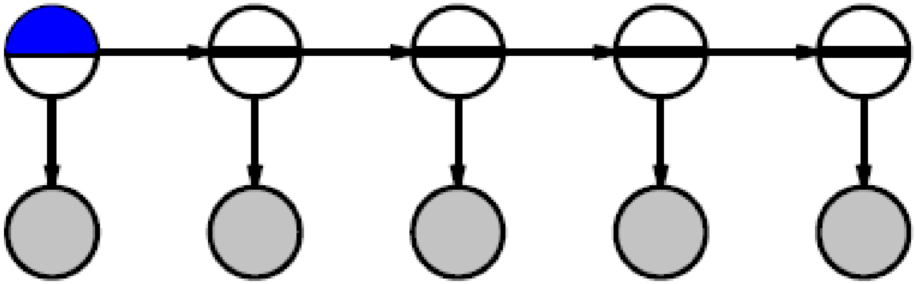
spazio di Forward-backward diventa O(S)

Algorithm: forward pass computes $\mathbf{f}_t$, backward pass does $\mathbf{f}_i$, $\mathbf{b}_i$



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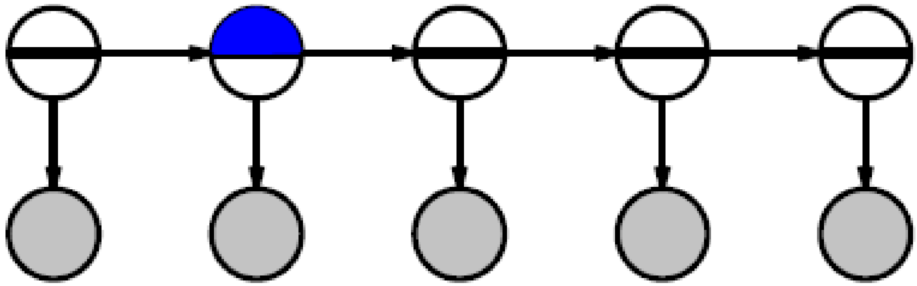

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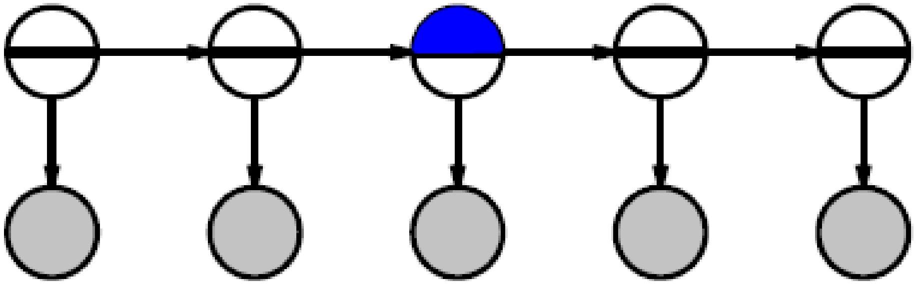

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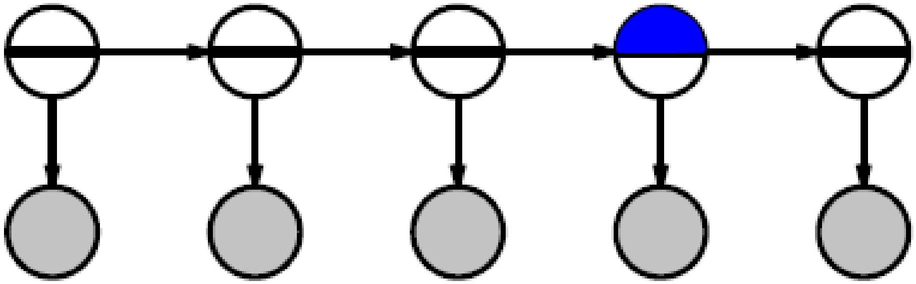

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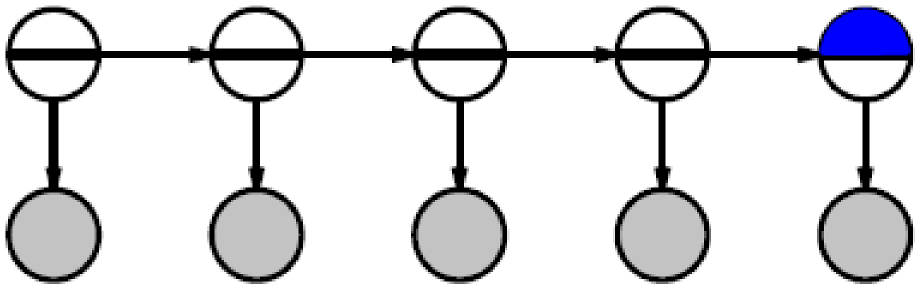

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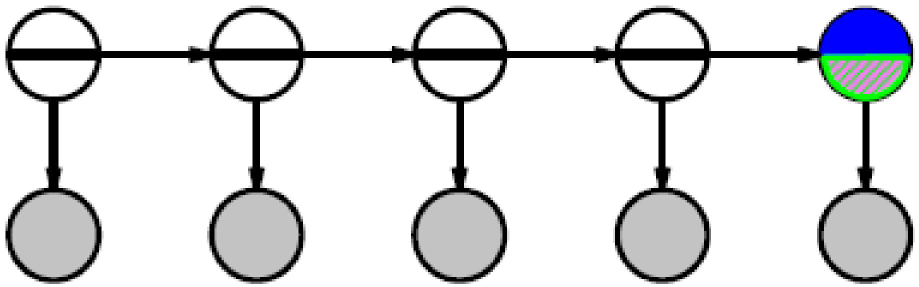

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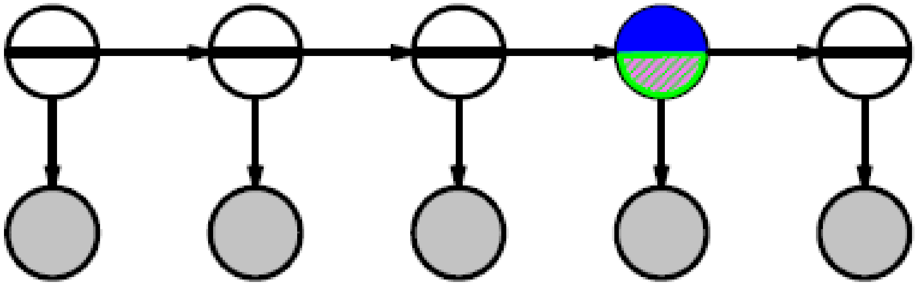

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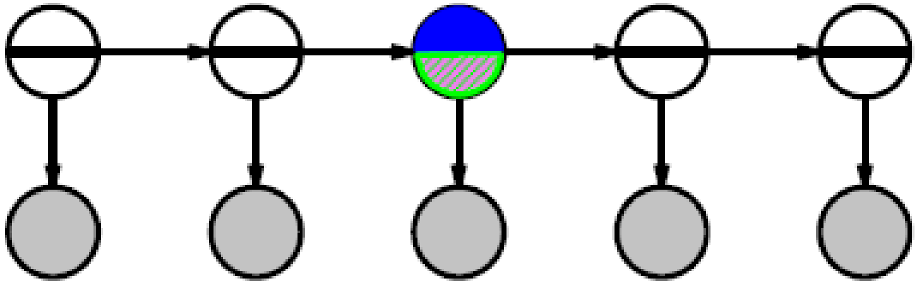

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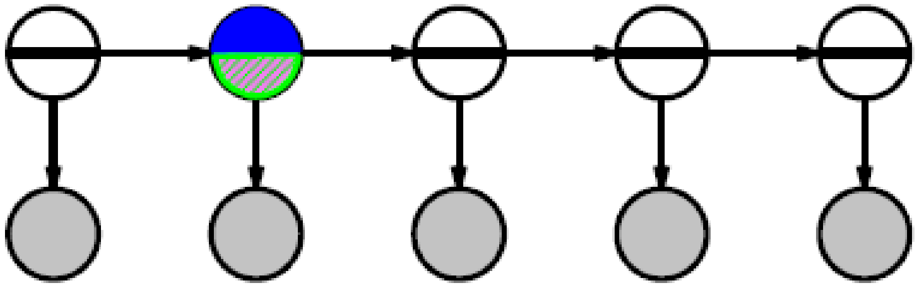

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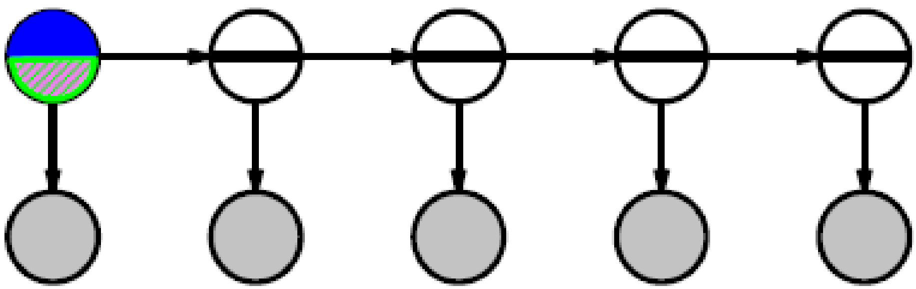


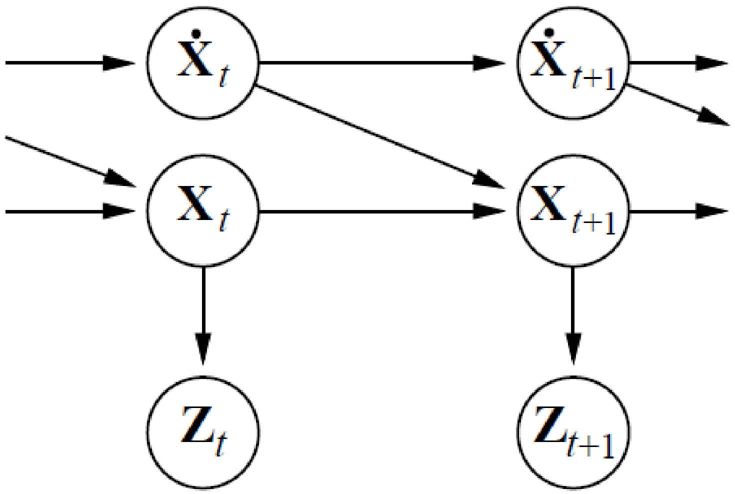
Kalman Filters

Kalman filters

Modelling systems described by a set of continuous variables,

e.g., tracking a bird flying—$\mathbf{X}_t = X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$.

Airplanes, robots, ecosystems, economies, chemical plants, planets, ...



$P(X_0) = N(\mu_0, \sigma_0)$

Gaussian prior, linear Gaussian transition model and sensor model

$P(X_{t+1} | X_t) = N(a\, X_t + b, \sigma_x)$

$P(Z_t | X_t) = N(c\, X_t + d, \sigma_z)$

Updating Gaussian distributions

$f_{1:t}$

Prediction step: if $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$ is Gaussian, then prediction

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \int_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t})\, d\mathbf{x}_t$$

is Gaussian. If $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$ is Gaussian, then the updated distribution

$f_{1:t+1}$

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

is Gaussian

Hence $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$ is multivariate Gaussian $N(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$ for all t

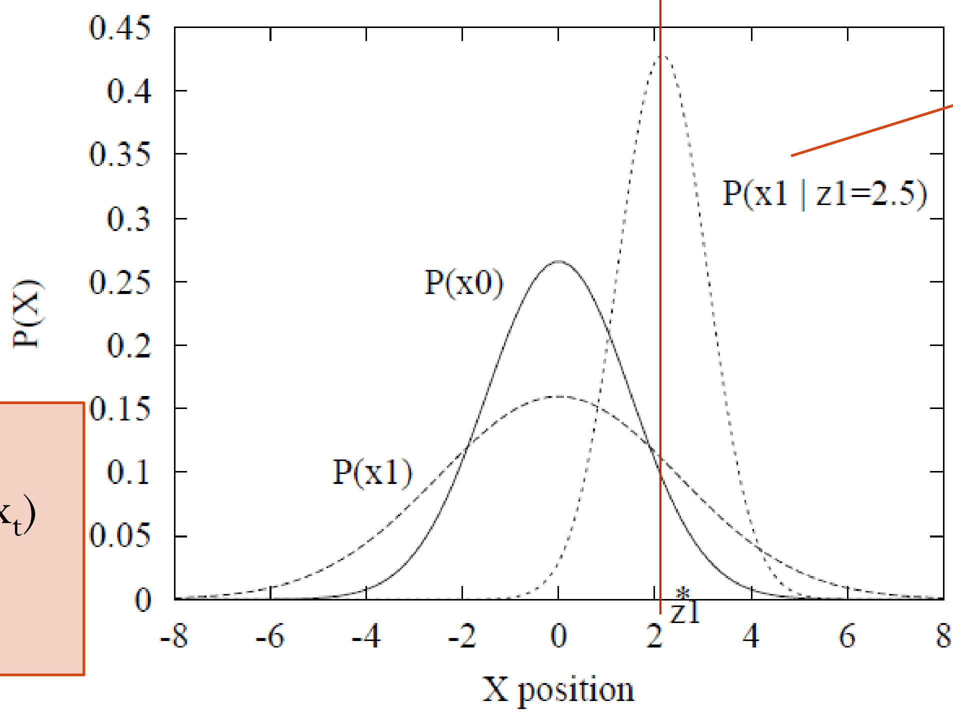
General (nonlinear, non-Gaussian) process: description of posterior grows **unboundedly** as $t \to \infty$

Simple 1-D example

Gaussian random walk on X–axis, s.d. σ_x, sensor s.d. σ_z

$f_{1:t+1} = N(\mu_{t+1}, \sigma_{t+1})$

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2) z_{t+1} + \sigma_z^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \qquad \sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$



«compromesso» tra predizione e osservazione

$P(x_0) = N(0, \sigma_0)(x_t)$

$P(x_{t+1} | x_t) = N(x_t, \sigma_x)(x_t)$

$P(z_t | x_t) = N(x_t, \sigma_z)(z_t)$

Varianza e Matrice di Covarianza

- per una singola variabile abbiamo:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

- se X = N(μ, σ) allora Var(X) = σ_X^2
- per un vettore di variabili **X**=(X_1, …, X_k) abbiamo la ***matrice di covarianza***:

$$\Sigma_{\boldsymbol{X}} = \left[cov\left[X_i, X_j\right] : i, j = 1 \ldots k\right]$$

- dove:

$$cov\left[X_i, X_j\right] = E\left[(X_i - E[X_i])(X_j - E\left[X_j\right])\right]$$

- per semplicità useremo delle $\Sigma_{\boldsymbol{X}}$ diagonali:
  - elementi sulla diagonale con Var(X_i)
  - altri elementi a 0

General Kalman update

Transition and sensor models:

«rumore» di transizione di stato

$$P(\mathbf{x}_{t+1}|\mathbf{x}_t) = N(\mathbf{F}\mathbf{x}_t, \mathbf{\Sigma}_x)(\mathbf{x}_{t+1})$$
$$P(\mathbf{z}_t|\mathbf{x}_t) = N(\mathbf{H}\mathbf{x}_t, \mathbf{\Sigma}_z)(\mathbf{z}_t)$$

«rumore» di osservazione

$\mathbf{F}$ is the matrix for the transition; $\mathbf{\Sigma}_x$ the transition noise covariance
$\mathbf{H}$ is the matrix for the sensors; $\mathbf{\Sigma}_z$ the sensor noise covariance

Filter computes the following update:

predizione x_{t+1}

predizione z_{t+1}

$$\boldsymbol{\mu}_{t+1} = \mathbf{F}\boldsymbol{\mu}_t + \mathbf{K}_{t+1}(\mathbf{z}_{t+1} - \mathbf{H}\mathbf{F}\boldsymbol{\mu}_t)$$
$$\mathbf{\Sigma}_{t+1} = (\mathbf{I} - \mathbf{K}_{t+1}\mathbf{H})(\mathbf{F}\mathbf{\Sigma}_t\mathbf{F}^\top + \mathbf{\Sigma}_x)$$

grado di «aggiustamento» della previsione

where $\mathbf{K}_{t+1} = (\mathbf{F}\mathbf{\Sigma}_t\mathbf{F}^\top + \mathbf{\Sigma}_x)\mathbf{H}^\top(\mathbf{H}(\mathbf{F}\mathbf{\Sigma}_t\mathbf{F}^\top + \mathbf{\Sigma}_x)\mathbf{H}^\top + \mathbf{\Sigma}_z)^{-1}$
is the Kalman gain matrix

$\mathbf{\Sigma}_t$ and $\mathbf{K}_t$ are independent of observation sequence, so compute offline

Caso semplice dal generale

- F=1, H=1
- $\Sigma_x = \sigma^2{}_x$, $\Sigma_z = \sigma^2{}_z$, $\Sigma_t = \sigma^2{}_t$
- $K_{t+1} = \dfrac{\sigma_t^2+\sigma_x^2}{\sigma_t^2+\sigma_x^2+\sigma_z^2}$

  quindi:
  - se $\sigma_z^2=0$ allora $K_{t+1}=1$
  - se $\sigma_t^2 + \sigma_x^2=0$ allora $K_{t+1}=0$
- $\mu_{t+1} = \mu_t + K_{t+1}(z_{t+1} - \mu_t)$

  quindi:
  - se $K_{t+1}=1$ allora $\mu_{t+1} = z_{t+1}$
  - se $K_{t+1}=0$ allora $\mu_{t+1} = \mu_t$

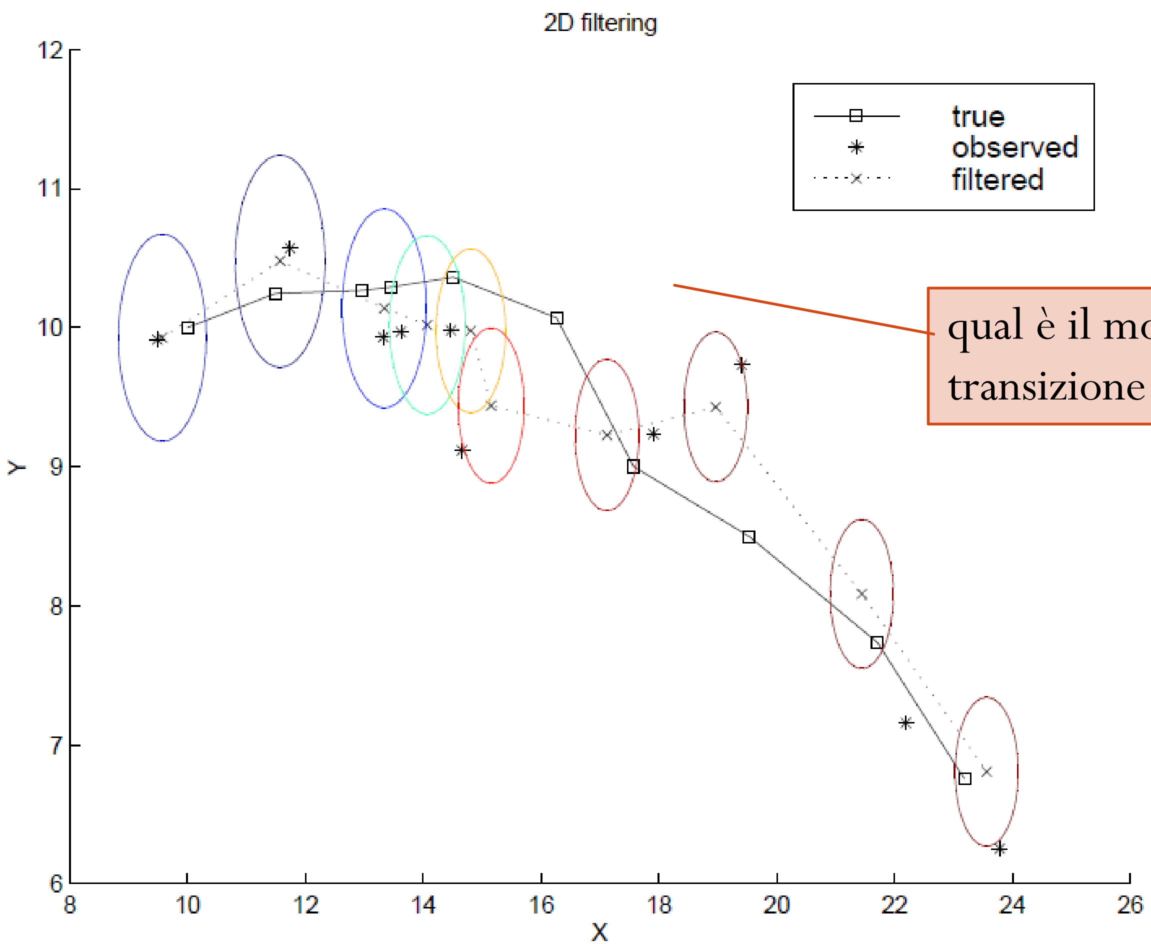
z_{t+1} totalmente affidabile

previsione precedente e previsione attuale totalmente affidabili

credo a osservazione

credo a predizione

2-D tracking example: filtering



qual è il modello di transizione di stato?