The evolution of vague categories

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1 Quantitative measures of uncertainty

1.1 Distance-based

Both sender and receiver strategies are of the form $s : C \to \Delta(A)$, *i.e.* a function s from a set of choice points C to a probability distribution over a set of actions A. We will henceforth restrict ourselves to finite C and A. For a given choice point $c \in C$, a strategy $\hat{s}$is maximally uncertain if it assigns the same probability to every action $a \in A$, *i.e.* if $\hat{s}(c)$ is a discrete uniform distribution:

Find better notation?

$$\forall a \in A : \hat{s}(c, a) = \frac{1}{|A|}$$

We say it is maximally uncertain since it has no preference whatsoever for any action in A.

We can measure how certain a strategy s is at a choice point c in terms of the distance from such a distribution $\hat{s}(c)$, for example in terms of the Kullback-Leibler divergence:

$$D_{\mathrm{KL}}(s(c) \parallel \hat{s}(c)) = \sum_{a \in A} s(c, a) \cdot \ln\left(\frac{s(c, a)}{\hat{s}(c, a)}\right) =$$

$$= \sum_{a \in A} s(c, a) \cdot \ln\left(\frac{s(c, a)}{\frac{1}{|A|}}\right) = \sum_{a \in A} s(c, a) \cdot \ln(|A| \cdot s(c, a))$$

This distance should be maximalfor any strategy $\check{s}$which puts all probability mass on some action $a' \in A$, *i.e.* a degenerate distribution localized at a':

Is it the case?

Find better notation?

$$\check{s}(c, a) = \begin{cases} 1 & a = a' \\ 0 & \text{otherwise} \end{cases}$$

For such a strategy we have:

$$D_{\mathrm{KL}}(\check{s}(c) \parallel \hat{s}(c)) = \sum_{a \in A} \check{s}(c, a) \cdot \ln(|A| \cdot \check{s}(c, a))$$

$$= \sum_{a \in A \setminus \{a'\}} \check{s}(c, a) \cdot \ln(|A| \cdot \check{s}(c, a)) + \check{s}(c, a') \cdot \ln(|A| \cdot \check{s}(c, a')) =$$

$$= \sum_{a \in A \setminus \{a'\}} 0 \cdot \ln(|A| \cdot 0) + 1 \cdot \ln(|A| \cdot 1) = \ln |A|$$

This value allows us to normalize the distance so that our measure of uncertainty ranges between 0 and 1. Finally, the measure of uncertainty of a strategy s at a choice point c thus becomes:

$$u(s, c) = 1 - \frac{1}{\ln |A|} \sum_{a \in A} s(c, a) \cdot \ln(|A| \cdot s(c, a))$$

We can then measure the overall uncertainty of a strategy s as the mean uncertainty per choice point:

$$u(s) = \frac{1}{|C|} \sum_{c \in C} u(s, c)$$

1.2 Entropy-based

Another option is to define uncertainty in terms of entropy. The most natural way is to define it in terms of mixed strategies rather than behavioral strategies. Thus, let $\vec{s} : \Delta\left(A^C\right)$ be a mixed strategy represented as a probability distribution over all possible pure strategies from choice points to actions $s : A^C$. We can define the entropy of such a strategy as:

$$E\left(\vec{s}\right) = \sum_{s \in A^C} \vec{s}\left(s\right) \cdot \ln\left(\vec{s}\left(s\right)\right)$$

This is computationally expensive to compute, since the size of the domain over which the sum is computed grows exponentially with the number of choice points. Therefore, we calculate an equivalent measure defined in terms of behavioral strategies. In general, a behavioral strategy can be converted into an infinite number of mixed strategies. However, we can define a unique mapping from a behavioral strategy $\sigma : C \to \Delta\left(A\right)$ to a mixed strategy $\vec{s}_\sigma$ where $\vec{s}_\sigma\left(s\right) = \prod_{c \in C} \sigma\left(s\left(c\right)|c\right)$, for every $s \in A^C$. Based on that, we have the following equivalences:

$$\begin{aligned}
E\left(\sigma\right) = & \quad E\left(\vec{s}_\sigma\right) \\
= & \quad -\textstyle\sum_{s \in A^C} \vec{s}_\sigma\left(s\right) \cdot \ln\left(\vec{s}_\sigma\left(s\right)\right) \\
= & \quad -\textstyle\sum_{s \in A^C} \left(\prod_{c \in C} \sigma\left(s\left(c\right)|c\right)\right) \cdot \ln\left(\prod_{c \in C} \sigma\left(s\left(c\right)|c\right)\right) \\
= & \quad -\textstyle\sum_{s \in A^C} \left(\prod_{c \in C} \sigma\left(s\left(c\right)|c\right)\right) \cdot \sum_{c \in C} \ln\left(\sigma\left(s\left(c\right)|c\right)\right) \\
= & \quad -\textstyle\sum_{s \in A^C} \sum_{c \in C} \ln\left(\sigma\left(s\left(c\right)|c\right)\right) \cdot \prod_{c' \in C} \sigma\left(s\left(c'\right)|c'\right) \\
= & \quad -\textstyle\sum_{c \in C} \sum_{s \in A^C} \ln\left(\sigma\left(s\left(c\right)|c\right)\right) \cdot \sigma\left(s\left(c\right)|c\right) \cdot \prod_{c' \in C \setminus \{c\}} \sigma\left(s\left(c'\right)|c'\right) && (1) \\
= & \quad -\textstyle\sum_{c \in C} \sum_{a \in A} \ln\left(\sigma\left(a|c\right)\right) \cdot \sigma\left(a|c\right) \cdot \sum_{s \in A^{\{c \mapsto a\} \cup C \setminus \{c\}}} \prod_{c' \in C \setminus \{c\}} \sigma\left(s\left(c'\right)|c'\right) \\
= & \quad -\textstyle\sum_{c \in C} \sum_{a \in A} \ln\left(\sigma\left(a|c\right)\right) \cdot \sigma\left(a|c\right) \cdot \prod_{c' \in C \setminus \{c\}} \sum_{a \in A} \sigma\left(a|c'\right) \\
= & \quad -\textstyle\sum_{c \in C} \sum_{a \in A} \ln\left(\sigma\left(a|c\right)\right) \cdot \sigma\left(a|c\right) \cdot \prod_{c' \in C \setminus \{c\}} 1 \\
= & \quad -\textstyle\sum_{c \in C} \sum_{a \in A} \ln\left(\sigma\left(a|c\right)\right) \cdot \sigma\left(a|c\right) && (2)
\end{aligned}$$

To illustrate the transitions from (1) to (2), let us look at an example. Let $C = \{c_1, c_2, c_3\}$, $A = \{a_1, a_2\}$, $\sigma_{xy} = \sigma\left(a_y|c_x\right)$, and $s_{xyz} = \{c_1 \mapsto a_x, c_2 \mapsto a_y, c_3 \mapsto a_z\}$. Take a given $c \in C$, for example $c = c_1$. The inner sum in (1) expands to:

$$\begin{array}{cll}
 & \ln\left(\sigma\left(s_{111}\left(c_1\right)|c_1\right)\right) \cdot \sigma\left(s_{111}\left(c_1\right)|c_1\right) \cdot \sigma\left(s_{111}\left(c_2\right)|c_2\right) \cdot \sigma\left(s_{111}\left(c_3\right)|c_3\right) & + \\
+ & \ln\left(\sigma\left(s_{112}\left(c_1\right)|c_1\right)\right) \cdot \sigma\left(s_{112}\left(c_1\right)|c_1\right) \cdot \sigma\left(s_{112}\left(c_2\right)|c_2\right) \cdot \sigma\left(s_{112}\left(c_3\right)|c_3\right) & + \\
+ & \vdots & + \\
+ & \ln\left(\sigma\left(s_{221}\left(c_1\right)|c_1\right)\right) \cdot \sigma\left(s_{221}\left(c_1\right)|c_1\right) \cdot \sigma\left(s_{221}\left(c_2\right)|c_2\right) \cdot \sigma\left(s_{221}\left(c_3\right)|c_3\right) & + \\
+ & \ln\left(\sigma\left(s_{222}\left(c_1\right)|c_1\right)\right) \cdot \sigma\left(s_{222}\left(c_1\right)|c_1\right) \cdot \sigma\left(s_{222}\left(c_2\right)|c_2\right) \cdot \sigma\left(s_{222}\left(c_3\right)|c_3\right) &
\end{array}$$

or:

$$\begin{array}{cll}
 & \ln\left(\sigma_{11}\right) \cdot \sigma_{11} \cdot \sigma_{21} \cdot \sigma_{31} & + \\
+ & \ln\left(\sigma_{11}\right) \cdot \sigma_{11} \cdot \sigma_{21} \cdot \sigma_{32} & + \\
+ & \ln\left(\sigma_{11}\right) \cdot \sigma_{11} \cdot \sigma_{22} \cdot \sigma_{31} & + \\
+ & \ln\left(\sigma_{11}\right) \cdot \sigma_{11} \cdot \sigma_{22} \cdot \sigma_{32} & + \\
+ & \ln\left(\sigma_{12}\right) \cdot \sigma_{12} \cdot \sigma_{21} \cdot \sigma_{31} & + \\
+ & \ln\left(\sigma_{12}\right) \cdot \sigma_{12} \cdot \sigma_{21} \cdot \sigma_{32} & + \\
+ & \ln\left(\sigma_{12}\right) \cdot \sigma_{12} \cdot \sigma_{22} \cdot \sigma_{31} & + \\
+ & \ln\left(\sigma_{12}\right) \cdot \sigma_{12} \cdot \sigma_{22} \cdot \sigma_{32} &
\end{array}$$

We can split this sum into one part per action, namely:

$$\begin{array}{cll}
 & \ln\left(\sigma_{11}\right) \cdot \sigma_{11} \cdot \left(\sigma_{21} \cdot \sigma_{31} + \sigma_{21} \cdot \sigma_{32} + \sigma_{22} \cdot \sigma_{31} + \sigma_{22} \cdot \sigma_{32}\right) & + \\
+ & \ln\left(\sigma_{12}\right) \cdot \sigma_{12} \cdot \left(\sigma_{21} \cdot \sigma_{31} + \sigma_{21} \cdot \sigma_{32} + \sigma_{22} \cdot \sigma_{31} + \sigma_{22} \cdot \sigma_{32}\right) &
\end{array}$$

The sums in parenthesis can be further rewritten as:

$$\begin{aligned} & \ln\left(\sigma_{11}\right) \cdot \sigma_{11} \cdot \left(\sigma_{21}+\sigma_{22}\right) \cdot \left(\sigma_{31}+\sigma_{32}\right) \quad + \\ + \quad & \ln\left(\sigma_{12}\right) \cdot \sigma_{12} \cdot \left(\sigma_{21}+\sigma_{22}\right) \cdot \left(\sigma_{31}+\sigma_{32}\right) \end{aligned}$$

Given that σ is row stochastic, each of the terms is equal to 1, thus reducing the whole expression to:

$$\ln\left(\sigma_{11}\right) \cdot \sigma_{11} + \ln\left(\sigma_{12}\right) \cdot \sigma_{12}$$

The measure of entropy is lower bounded at 0 and upper bounded at $\ln\left(\left|A^{C}\right|\right) = \ln\left(|A|^{|C|}\right) = |C| \cdot \ln\left(|A|\right)$, which means we can normalize $E\left(\sigma\right)$ between 0 and 1 by dividing by $1/|C|\cdot\ln(|A|)$.