6.7 Static condensation

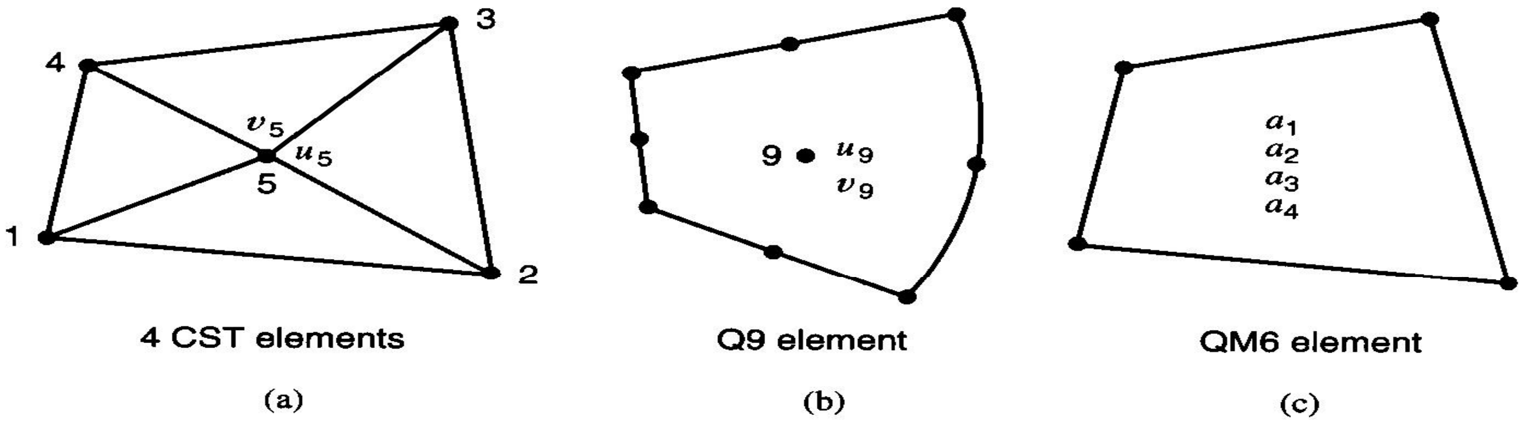


Figure 6.7-1. Examples of elements that have internal or nodeless d.o.f. The d.o.f. identified are usually treated by static condensation.

- Partitition $$\begin{bmatrix} k_{rr} & k_{rc} \\ k_{cr} & k_{cc} \end{bmatrix} \begin{Bmatrix} d_r \\ d_c \end{Bmatrix} = \begin{Bmatrix} r_r \\ r_c \end{Bmatrix}$$
- Partial solution $$\{d_c\} = -[k_{cc}]^{-1}([k_{cr}]\{d_r\} - \{r_c\})$$

$$\underbrace{\left([k_{rr}] - [k_{rc}][k_{cc}]^{-1}[k_{cr}]\right)}_{condensed\ [k]}\{d_r\} = \underbrace{\{r_r\} - [k_{rc}][k_{cc}]^{-1}\{r_c\}}_{condensed\ [r]}$$

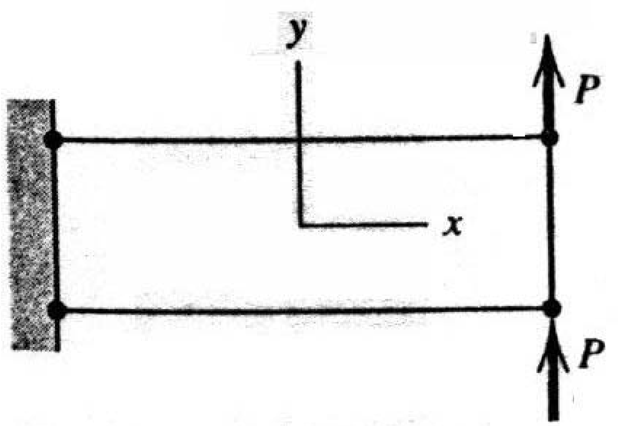
- General d.o.f reduction process called Guyan reduction

Q4: The strain fields

- Strain field:

$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$

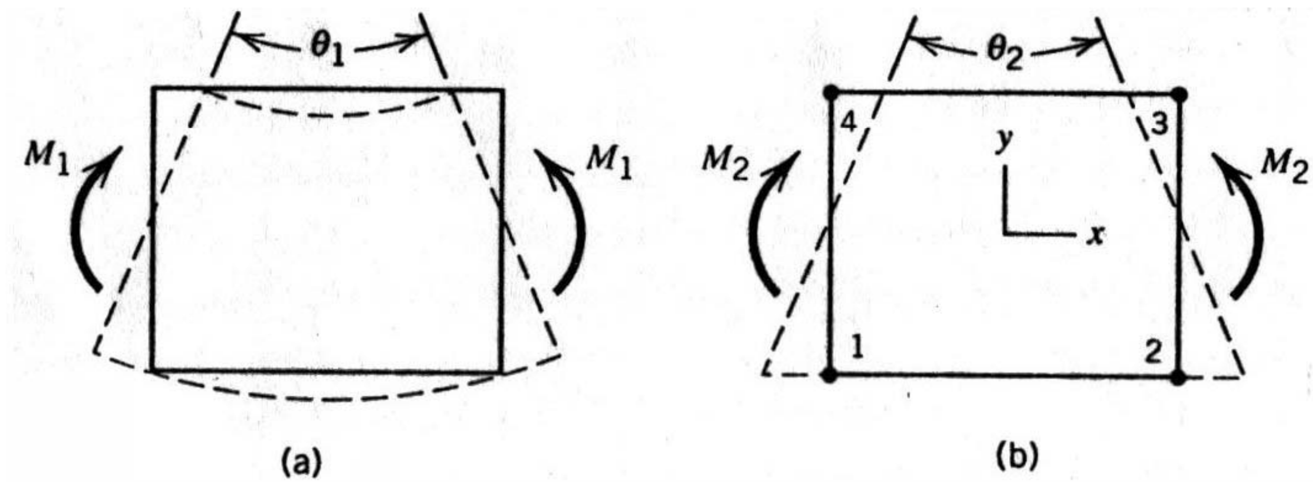
•***Observation 1:*** $\varepsilon_x \neq f(x) \Rightarrow$ Q4 cannot exactly model the beam where $\varepsilon_x \propto x$



A cantilever beam under transverse tip loading.

Q4: Behavior in Pure Bending of a Beam

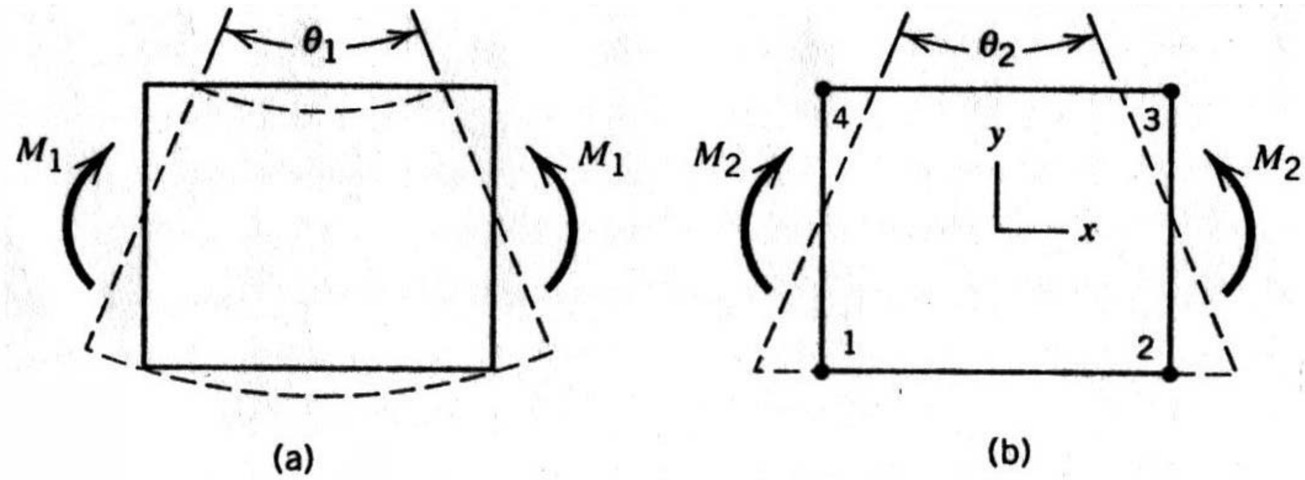
- ***Observation 2:*** When $\beta_4 \neq 0$, ε_x varies linearly in y - desirable characteristic for a beam in pure bending because normal strain varies linearly along the depth coordinate. But $\gamma_{xy} \neq 0$ is undesirable because there is no shear strain.



- Fig. (a) is the correct deformation in pure bending while (b) is the deformation of Q4 ***(sides remain straight***).
- Physical interpretation: applied moment is resisted by a spurious shear stress as well as flexural (normal) stresses.

Error in strains and stresses

- True strains $\varepsilon_x = \dfrac{-\theta_1 y}{2a}$ $\varepsilon_y = \dfrac{\nu\theta_1 y}{2a}$ $\gamma_{xy} = 0$
- Q4 strains $\varepsilon_x = \dfrac{-\theta_2 y}{2a}$ $\varepsilon_y = 0$ $\gamma_{xy} = \dfrac{-\theta_2 x}{2a}$



- Some error in σ_x larger error in σ_y very large error in shear stress

6.6 Incompatible and nodeless modes

- Additional shape functions for Q6 element

$$u = \sum_{i=1}^{4} N_i u_i + (1-\xi^2)a_1 + (1-\eta^2)a_2 \qquad \xi = \frac{x}{a}$$

$$where$$

$$v = \sum_{i=1}^{4} N_i v_i + (1-\xi^2)a_3 + (1-\eta^2)a_4 \qquad \eta = \frac{y}{b}$$

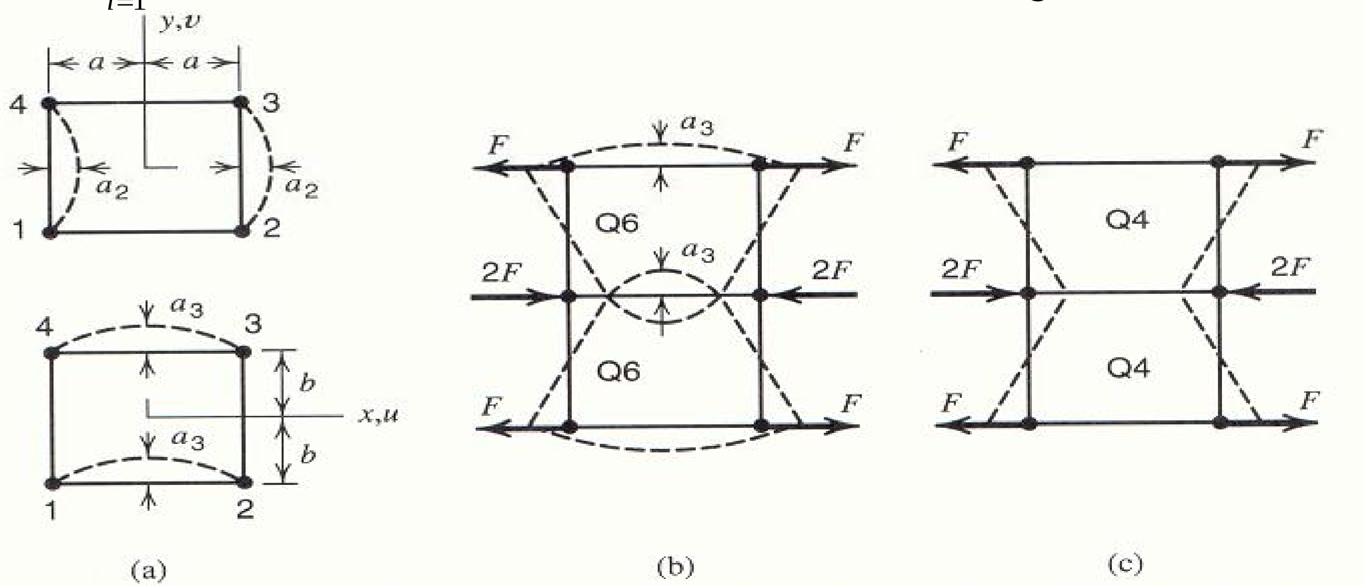


Figure 3.10-3. (a) Displacement modes $u = (1-\eta^2)a_2$ and $v = (1-\xi^2)a_3$ in the Q6 element. (b) Incompatibility between adjacent Q6 elements. (c) No incompatibility between adjacent Q4 elements.

Problems with constant strain

- Q6 fails to represent constant strain unless rectangular
- Strain matrix, $[B]=[B_d \quad B_a]$
- Solution obtained by requiring no a-load to be generated by constant stress in element

$$\int [B_a]^T \{\sigma_0\} dV = \{0\} \qquad hence \qquad \int [B_a]^T dV = \{0\}$$

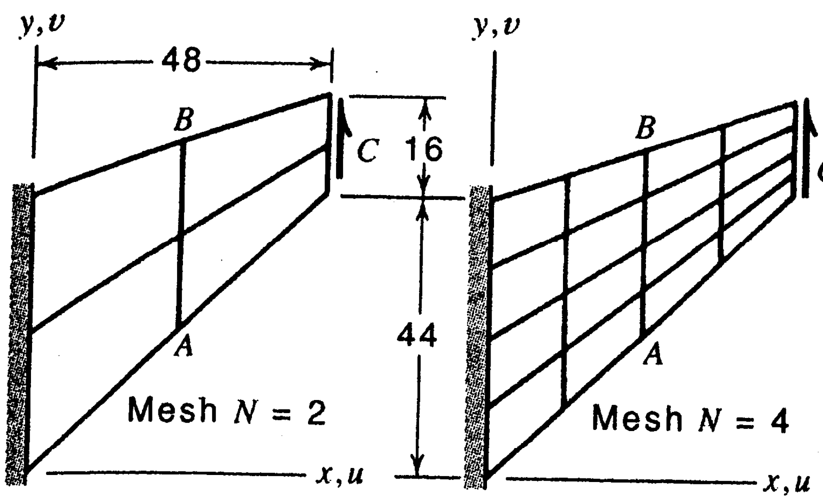
- Or

$$\int_{-1}^{1}\int_{-1}^{1} [B_a]^T tJ \; d\xi \; d\eta = [0]$$

- Accomplished by evaluating integrations involving B_a with Jacobian and its determinant evaluated at element center
- Example of reduced selective integration!
- Called QM6 element

Incompatible elements and accuracy

- Compatibility and resulting monotonicity is good, but engineers prefer accuracy



	Mesh	$N = 2$	$N = 4$
QM6	v_C	0.884	0.967
	σ_A	0.840	0.978
	σ_B	0.788	0.926
Q4	v_C	0.498	0.769
	σ_A	0.558	0.830
	σ_B	0.457	0.753

Figure 6.6-1. A swept panel with uniformly distributed load along the right side and Poisson's ratio $\nu = 0.333$. Numerical results [6.7] are y-direction deflection at the right side, maximum normal stress at A, and minimum normal stress at B (exact $= 1.000$).

6.8 Choices in numerical integration

- ”Full integration”: quadrature that is exact for rectangular geometries
- How many do you need for Q4 element?
- Reduced integration may remove resistance to some displacement modes
- For example, one point integration of Q4 element removes resistance to modes 7,8. They become spurious

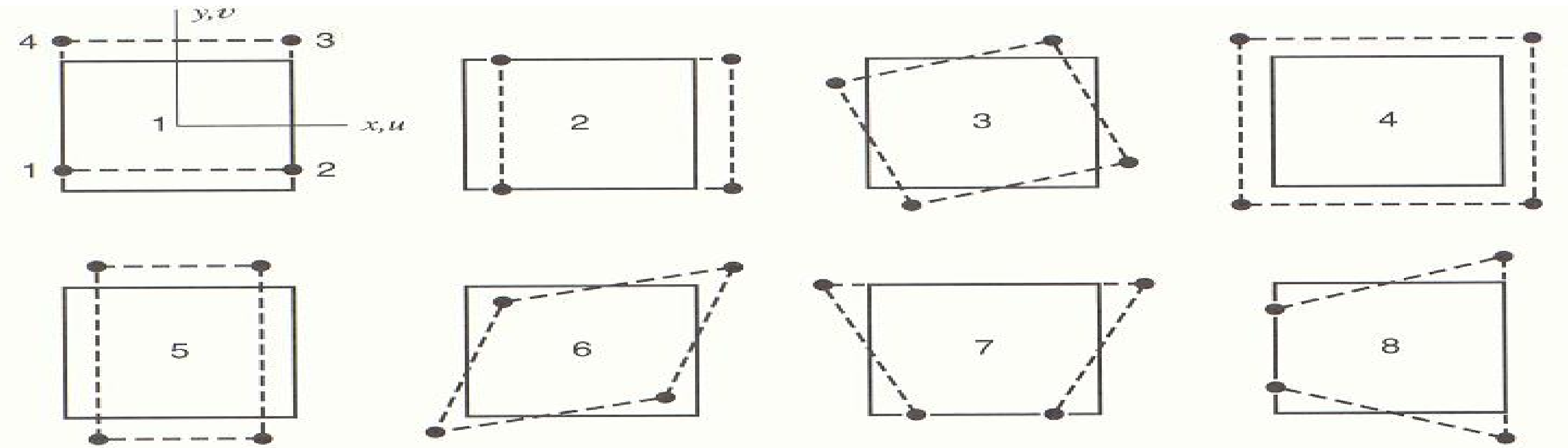


Figure 6.8-1. Independent displacement modes of a four-node plane element.

Spurious modes

- When elements are put together, spurious may be communicable or non-communicable
- In Q4-beam model below, spurious bending modes are communicable

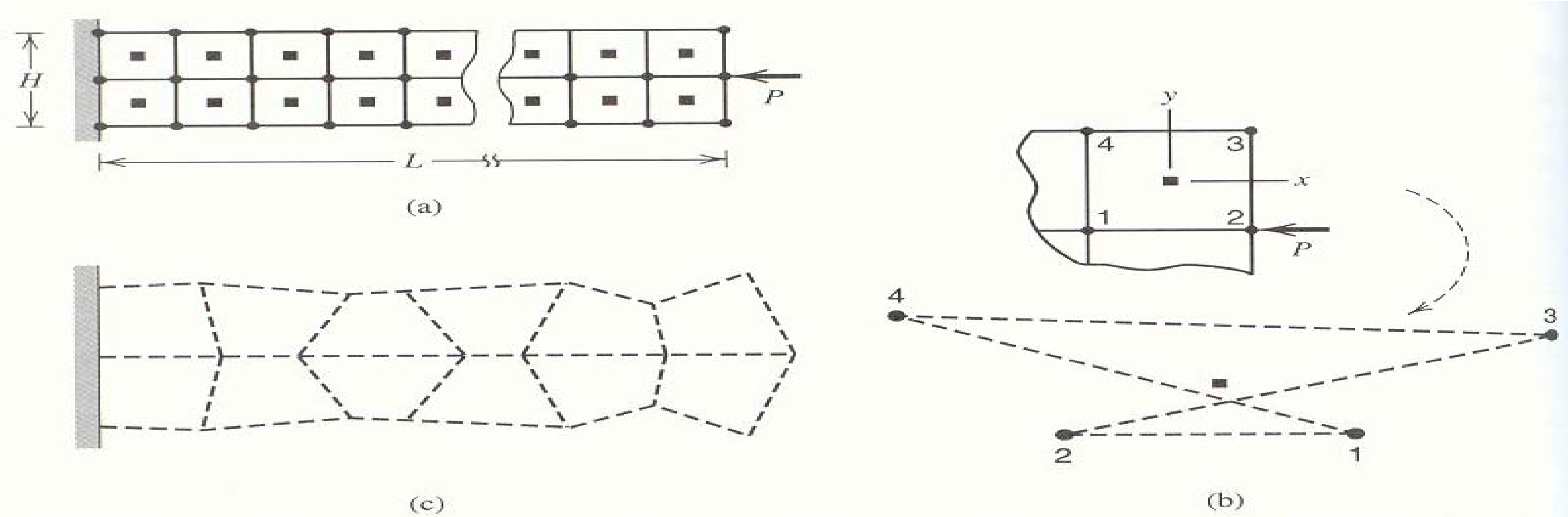


Figure 6.8-2. Four-node square elements with reduced integration. (a) Static axial load. (b) Dashed lines show deformation mode of the upper right element in a 2 by 14 mesh. (c) A computed vibration mode.

Spurious modes for quadratic element

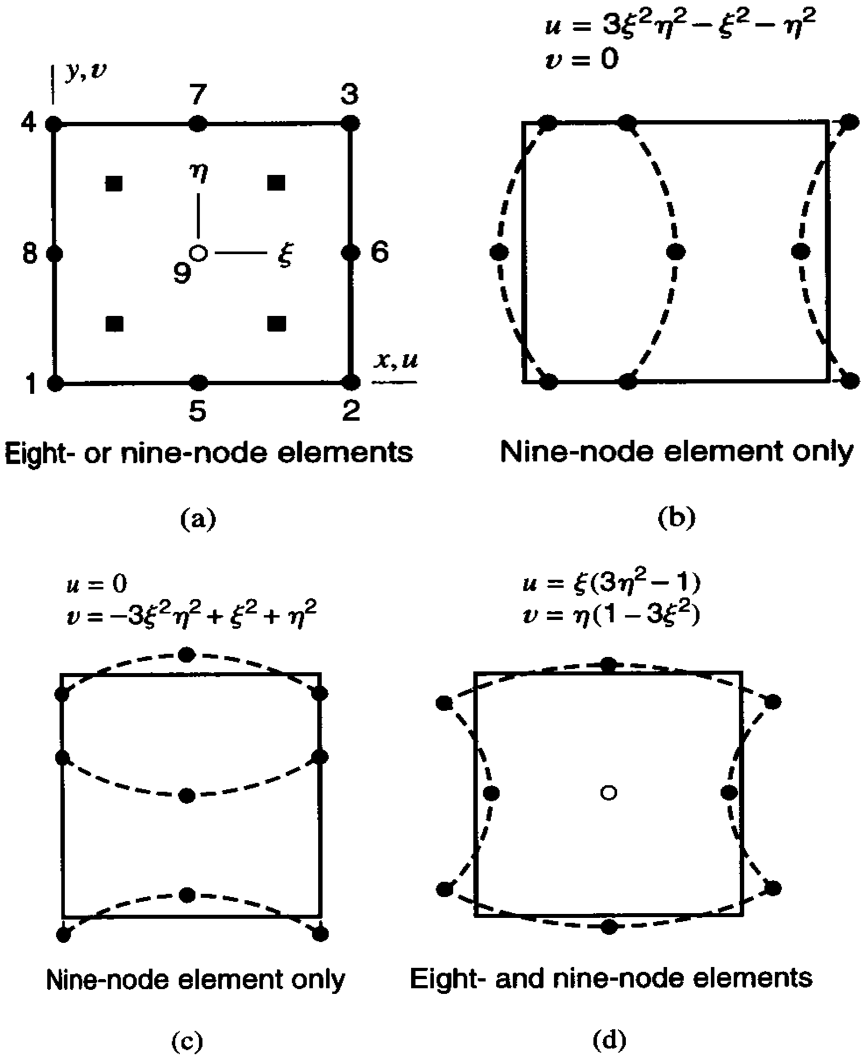


Figure 6.8-3. Spurious modes in plane quadratic elements when a 2 by 2 Gauss rule is used for stiffness matrix integration. Elements shown are initially square. Equations for u and v indicate form, not amplitude.

Why is hour glass mode (d) non-communicable?