GB13604 - Maths for Computer Science

Lecture 8 – Probability, Part I

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2018-11-21

Last updated December 5, 2018

This course is based on Mathematics for Computer Science, Spring 2015, by Albert Meyer and Adam Chlipala, Massachusetts Institute of Technology OpenCourseWare.

Last Topic: Probability

Probability is absolutely essential to the Engineering, Sciences, and Social studies. And it is also very important to understand in daily life as well.

- Probability as the study of gambling (Lottos, Casinos, Gatchas).
- Probability for Extrapolating information about society (Chances of Death and Sickness, Insurance, Average behavior of populations)
- Probability for the analysis of Noisy Data and Noisy Processes. (Stochastic Algorithms, Experiments, Resilience Engineering)

Unit Goals and Outline

- Discrete Probability
- Conditional Probability
- Independence and Causality
- Random Variables and Density Functions
- Expectation
- Deviation
- Sampling and Confidence
- Random Walks

Poker Example: Probability of Two Jacks

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- All outcomes: $\binom{52}{5}$ sets of 5 cards
- Desired outcomes: $\binom{4}{2}\binom{52-4}{5-2}$ two ace hands.
- Probability: Desired outcome/All outcomes = $\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$
- (About 0.04)

Probability as a counting problem

Basic Idea: "What fraction of the time do I get what I want?"

$$\Pr(\text{event}) = \frac{\text{Outcomes we want}}{\text{All possible outcomes}}$$

Probability as a counting problem: nomenclature

- A set of experimental **outcomes**
- A subset of outcomes is an **event**
- Probability of an event:

$$\Pr(\text{event}) = \frac{\text{\# outcomes in the event}}{\text{total \# of outcomes}}$$

Applies to a lot of cases (but not all of them)

The Monty Hall Problem

- 1970's American TV sholl "Let's make a Deal" (Hosted by Monty Hall)

- One door has a good prize (a car?) and two doors have a bad prize (a goat?)

Rules of the Monty Hall Problem

- Goats behind two doors, car behind one door;
- Contestant chooses a door;
- Monty reveals a door with a goat behind it;
- Contestant choose to stick or switch doors;

Mary Savant published a column in a science magazine about the game that sparked a debate on two positions:

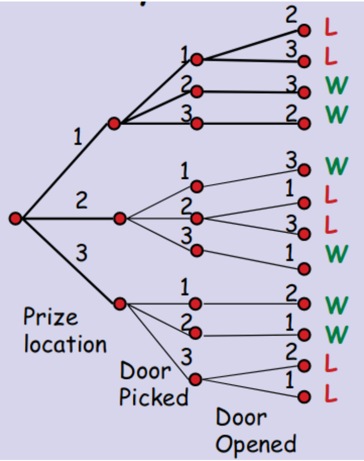
1. stick and switch are equally good
2. switch is much better than stick

Analysing Monty Hall

- What are the outcomes? What is the event?
- We will use a probability tree to analyse the game step by step and define these sets.

Analysis: Switch Strategy

Switch: Pick a door, Reveal Goat, Switch Door



- For each step, we break down the possible outcomes.
- Each branch of the tree is labeled W/L

- Winning outcomes: 6
- Losing outcomes: 6

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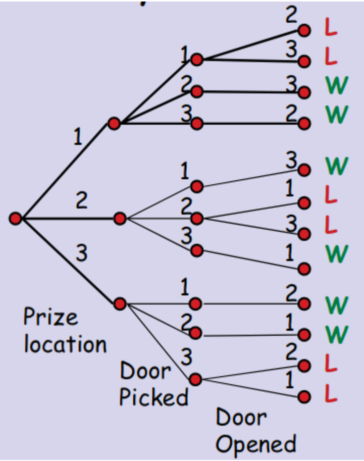

- For each step, we break down the possible outcomes.
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- Winning outcomes: 6
- Losing outcomes: 6

- Since the # of winning outcomes and losing outcomes is the same, both strategies are the same.
  (This is a **Bad Conclusion**)

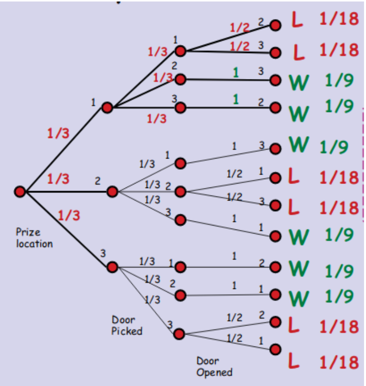
Analysis: Bad Conclusion

“Since the number of winning outcomes (6) and losing outcomes (6) is the same, then stick strategy and switch strategy are the same”

Another way: “After door opening, one goat and one prize are left. The probability of being at the goat door or prize door is the same”

Problem: Outcomes do not have the same probability

Outcomes with variable probability



- We can assign probabilities for each of the events in the probability tree.
- By counting the probabilities of the branchs, we can figure out the probability of the leaves.

- Although there are six wins, and six losses, if we compare the total probabilities:
- Total Win probability: 6/9
- Total Loss probability: 3/9

Conclusion

Switch is better than Stick

4-part method for calculating probabilities:

1. Identify the outcomes (tree helps)
2. Identify the event (winning)
3. Assign the outcome probabilities
4. Compute the probability of the event (add it up)

Avoid intuition in Probabilities!

Probability Spaces

Probability Spaces

- Sample Spaces: a **countable** set S where the elements are outcomes;
- Probability Function: $\Pr : S \to \{0, 1\}$ so that:

$$\sum_{\omega \in S} \Pr(\omega) = 1$$

- The probability function defines the probability of each outcome in the Sample Space S.

Probability Spaces and the Tree Model

The Tree Model presented in the previous example serves to help transforming the problem description into the probability space.

- **Outcomes**: Leaves of the tree
- **Outcome Probabilities**: Calculated from Branch probabilities

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- Probability Function: $\Pr : S \rightarrow \{0, 1\}$ so that:

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- Event: A subset $E \subseteq S$

$$\Pr(E) = \sum_{\omega \in E} \Pr(\omega)$$

- **Corolary:** The sum rule

The Sum Rule

For pairwise disjoint events $A_0, A_1, \ldots$,
$\Pr(A_0 \cup A_1 \cup \ldots) = \Pr(A_0) + \Pr(A_1) + \ldots$

$$\Pr(\cup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \Pr(A_i)$$

Discrete spaces = Countable spaces
Allows the use of Sums instead of Integrals

Generalized Probability Rules

- Difference Rule: $\Pr(A - B) = \Pr(A) - \Pr(A \cap B)$

Note how similar these rules are to the **Rules of set size**

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- Boole's Inequality: $\Pr(\cup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \Pr(A_i)$

Note how similar these rules are to the **Rules of set size**

QUIZ: A Coin Game

GAME: Two players flip the same, fair coin.

- What is the probability that the first player wins?
- What is the probability that no player wins?

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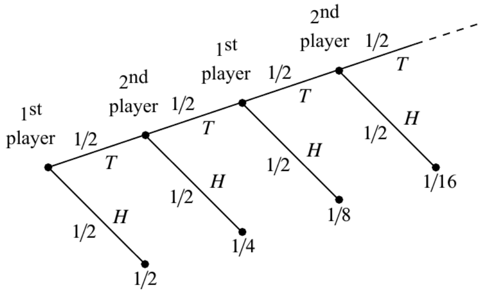

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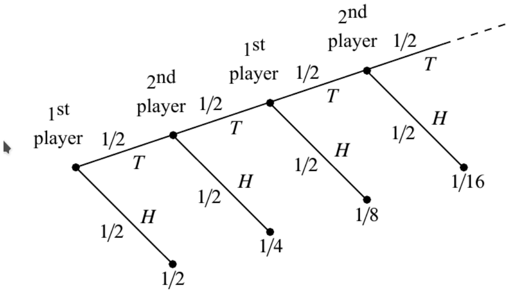

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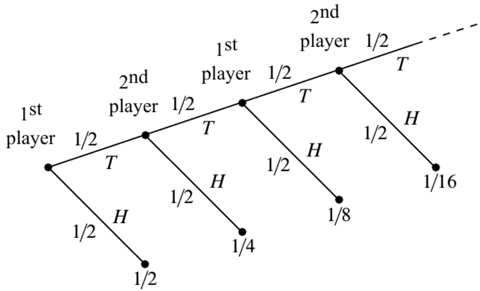


- Pr(First Player Wins) = $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots$
- Pr(First Player Wins) = $\frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{4})^n = \frac{1}{2} \cdot \frac{1}{1-1/4} = \frac{2}{3}$

Strange Dice

- Dice A: {2,6,7}
- Dice B: {1,5,9}
- Dice C: {3,4,8}

- Game 1: You roll one dice, I roll another. Which dice wins?
- Game 2: You roll one dice 2 times and add. I roll one dice 2 times and add. Which dice wins?

Conditional Probability: Definitions

Probability that one event occurs, given that another event occurred.

- What is the probability that a person will have a health problem, given their health history?
- What is the probability that a stock will rise, given past price?
- What is the probability that a server will overload, given the number of requests?

Conditional Probability: A fair dice example

Probability of rolling a 1 in a D6.

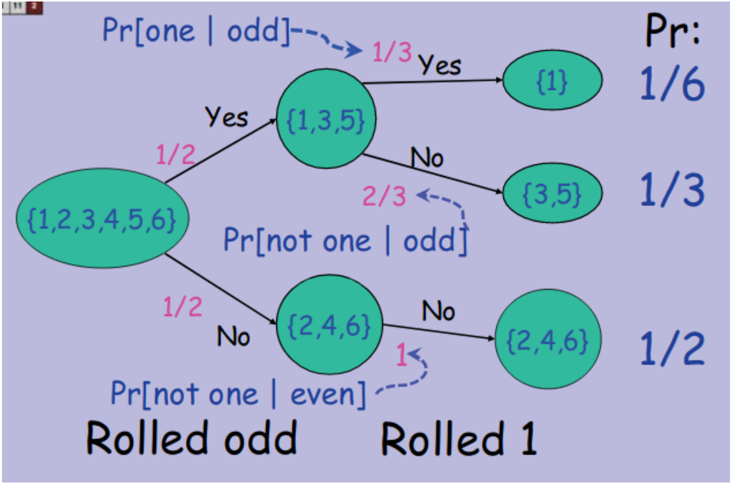
$$\text{Pr(Roll 1)} = \frac{|\{1\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{6}$$

"Knowledge" changes probabilities

Pr(Roll 1, knowing that the result was odd):

$$\frac{|\{1\}|}{|\{1,3,5\}|} = \frac{1}{3}$$

Fair dice and the Tree Model



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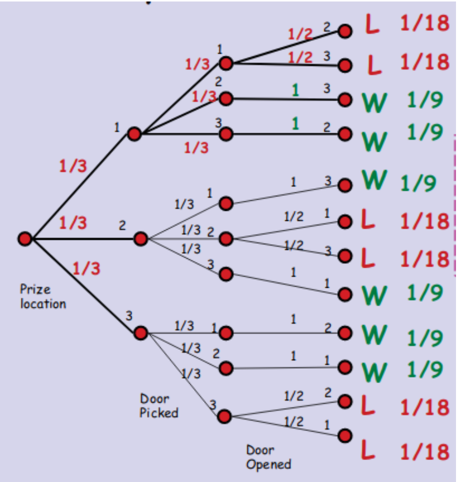

The tree's edges are the conditional probabilities!

- Pr(pick 1|prize 1) = 1/3
- Pr(pick 2|prize 3) = 1/3
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  We can compose conditional probs!

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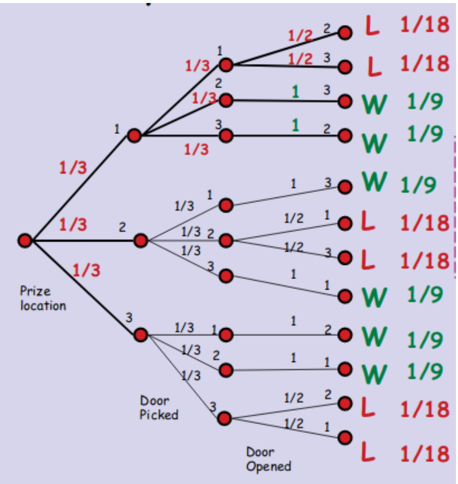

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The tree model was using conditional probabilities implicitely!

Probability: Product Rules

What is the probability that both A & B happen?

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B|A) \tag{1}$$

This implies a definition of conditional Prob:

$$\Pr(B|A) ::= \frac{\Pr(A \cap B)}{\Pr(A)} \tag{2}$$

Conditional Probability as a Probability Space

Another way to think about conditional probability is that, when we define a probability conditional to A, we create a new probability space where all events not following A have a probability of 0.

Let $\Pr_A$ be the probability function conditional to A, then:

$$\Pr_A(\omega) = 0,\ \text{if } \omega \notin A$$

$$\Pr_A(\omega) = \frac{\Pr(\omega)}{\Pr(A)},\ \text{otherwise.}$$

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- Events when dice = 3 & sum = 4: (1,3), (3,1)

- Probability: 2/3

A 99% accurate TB testing

Imagine a great looking test for Tuberculosis:

- If you have TB, the test is always correct
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- Recently, many antibiotics don't worry anymore!

- Should you worry?

99% diagnostic test and conditional probability

What is the probability that you have TB, if the %99 accuracy test says Yes?

$$\Pr(\text{TB} \mid \text{test positive}) = ?$$

What do we know about this test?

99% diagnostic test and conditional probability

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- Pr(Positive | TB) = 1
- Pr(Positive | Not TB) = 1/100

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- False Positive Rate = 1%

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- $\Pr(\text{Positive}) = ?$
- By the Total Probability formula:
- $\Pr(\text{Positive}) = \Pr(\text{Positive} \mid \text{TB}) \cdot \Pr(\text{TB}) + \Pr(\text{Positive} \mid \text{Not TB}) \cdot \Pr(\text{Not TB})$

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Do you have TB or Not?

- Conditional Probability:
- Pr(TB | Positive) = $\frac{\text{Pr(TB)}}{\text{Pr(Positive)}}$
- Pr(Positive) = 1·Pr(TB) + $\frac{1}{100}$·(1 - Pr(TB)) = 99/100 Pr(TB) + 1/100
- Pr(TB | Positive) = $\frac{\text{Pr(TB)}}{\text{Pr(Positive)}}$ = 100 · $\frac{100\ \text{Pr(TB)}}{99Pr(TB)+1}$
- Now we need to know "Pr(TB)". In the United States, it is about 11.000 per 100 million people, or 0.001.

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- Pr(TB | positive) = $\frac{100/10000}{99/10000+1} = 1/100$

Conditional Probability and Tricky Base Rates

- Even if you have the positive diagnostic, your chance of having TB is 1%.
- This is because the **TB Rate (0.01%)** is much lower than the **False Positive Rate (1%)**
- So the 99% accurate test was not so accurate here...
- This is a good example of **Bayes' Rule**

$$\Pr(B|A) = \frac{\Pr(A|B) \cdot \Pr(B)}{\Pr(A)}$$

Probability in Computer Sciences: The Birthday Paradox

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- Event space is sequences of n numbers with no repetitions.

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- In a class with 95 people, what is the probability that **two people have a birthday in the same day**?

- $n = 95, d = 365$.
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- The Birthday Principle: When $n = \sqrt{2d}$, the probability of two people having the same birthday is about 0.632
- This same principle is applied to hashing collisions.

Class Summary

- We can see probability problems as counting or combinatory problems.
  Probability = Events of Interest / Total Events
- We can use the Tree Model to make problems easier to understand.
- Conditional Probability, and Bayes' Rule

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