GB13604 - Maths for Computer Science

Lecture 1 – Introduction to Proofs

Claus Aranha

caranha@cs.tsukuba.ac.jp

College of Information Science

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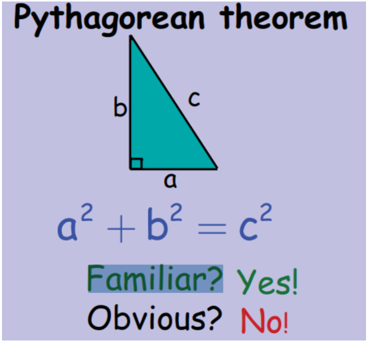
Mathematics for Computer Sciences
Part I – Proofs

What is a proof?



- Proofs are used to show how you know something
- Proofs are not obvious (more than 100 proofs for pythagoras)

Why are proofs important for Computer Sciences?

The techniques and ides of proofs can be used for debugging.

This program outputs the type of triangle

```
int triangle_type(int a, int b, int c)
  if (a == b)
    if (b == c)
      return "all sides are equal";
    else
      return "two sides are equal";
  else if (b == c)
    return "two sides are equal";
  else
    return "all sides are different";
```

- Is this program correct or incorrect?
- How can you show it with confidence?

What is a Proof?

- A proposition is a statement that can be True or False.
  - This room has 40 chairs.
  - Every intelligent being feels pain.
  - Please say your name.
  - $513 \times 435 = 223165$
  - Every even integer greater than 2 is the sum of two primes.
  - It is raining now.

- A proof is a method of proving the truth or falsehood of a proposition.
  - mathematical proofs normally use logical steps to show the truth of a mathematical proposition.

Proof Examples

- Pitagoras by pictures
- Getting rich with triangles
- 1 == -1

Morals of Proofs

- Make sure that you are applying the rules properly.
- Mindless calculation does not replace understanding.

Common Terms used in Proofs

- Proposition:
- Predicate:
- Axiom:
- Proof:
- Theorem:
- Lemma:
- Corollary:

Common Terms used in Proofs

- Proposition:
  A statement that is either true or false
- Predicate:
  A preposition that depends on variables
- Axiom:
  A preposition that is accepted to be true
- Proof:
  A sequence of axioms and proved statements that conclude with the proposition of interest
- Theorem:
  An important true proposition
- Lemma:
  A simpler proposition that is useful to prove later propositions
- Corollary:
  A proposition that follows from a theorem in a few logical steps

Our first proof method: Modus Ponens

$$\frac{P, P \text{ implies } Q}{Q} \text{ or } \frac{P, P \to Q}{Q}$$

What does “Modus Ponens” mean?

- If P is true.
- and if P being true requires that Q is true too.
- then Q is true.

How can we use Modus Ponens to prove something?

- We want to prove Q.
- Prove that when P is true, Q must be true
- Prove that P is true
- therefore, Q must be true.

Proof By Contradiction

A trivial proof:

$$\sqrt[3]{1332} \leq 11$$

Proof By Contradiction

If an assertion implies something false

Then the assertion must be false!

Better Example: $\sqrt{2}$ is irrational

Think a little bit by yourselves first.

Better Example: $\sqrt{2}$ is irrational

Let's prove by contradiction:

1. Assume that $\sqrt{2}$ is rational
2. Therefore $\sqrt{2} = \frac{m}{n}$, and m and n are integers with no common prime factors ($n \neq 0$).
3. Therefore $n\sqrt{2} = m$, $2n^2 = m^2$, and m^2 is even.
4. If m^2 is even, then m is even too. $m = 2k$ for some integer k.
5. Therefore $2n^2 = (2k)^2$, $2n^2 = 4k^2$, $n^2 = 2k^2$, and n^2 is even.
6. If n^2 is even, then n is even too. *n* and *m* are both even (contradiction).

Proof By Cases

Prove that these two code samples are the same:

Code 1

```
If (X > 0 OR (X <= 0 AND Y > 100))
  print("Hello!")
```

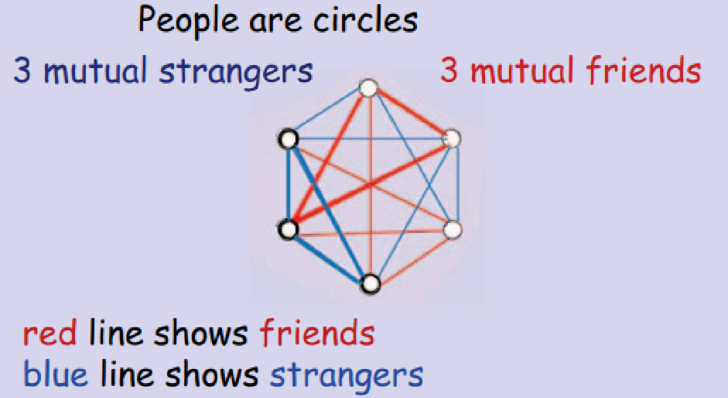
Code 2

```
If (X > 0 OR Y > 100)
  print("Hello!")
```

Proof By Cases

- “Proof by Cases” breaks a complicated problem into easier, smaller sub-problems.
- It is important to make sure that the cases cover all possibilities, or the proof is not complete.

Proof By Cases: Friends and Strangers



- Six people, every two are either friends or strangers.
- **Claim:** There is always a set of 3 mutual friends or 3 mutual strangers.

Friends and Strangers, and Ramsey's Theorem

For any k, every large enough group of people will contain k mutual friends OR k mutual strangers.

- $R(3) = 6$
- $R(4) = 18$
- $R(5) =$ unknown!

A bogus proof by cases: Prove $2a^2 > a$

1. This proof is by case analysis.
2. There are two cases:
   - Case 1: a is positive
   - Case 2: a is negative
3. One of these cases must always hold, because an integer is either positive or negative.
4. Case 1: Suppose a is positive.
5. Since a is an integer, we must have that $a \geq 1$.
6. Hence, $2a^2 = 2a \times a \geq 2a \times 1 > a$.
7. This implies the claim holds in Case 1.
8. Case 2: Suppose a is negative.
9. Since a is an integer, we must have that $a \leq -1$.
10. Hence, $2a^2 \geq 2 \times (-1 \times -1) = 2 > -1 \geq a$.
11. This implies the claim holds in Case 2.
12. The claim therefore holds in both cases.

The Well Ordering Principle

- It is a very obvious (but very useful) principle in Mathematics;
- It is so obvious that you have already used it without knowing;

The Well Ordering Principle

Every non-empty set of
Non-negative Integer Numbers
has one smallest element

The Well Ordering Principle

Obvious? yes   Trivial? no

- Every non-empty set of non-negative rational numbers has one smallest element?
- Every non-empty set of integers numbers has one smallest element?

Well Ordering Examples

- What is the smallest age of the U.Tsukuba students?
- What is the smallest number of cells in any animal?
- What is the smallest number of coins = 876 yens?

Proof $\sqrt{2}$ is irrational using well ordering

- if $\sqrt{2}$ is rational, then exist m, n so that $\sqrt{2} = \frac{m}{n}$
- We can always find $m, n > 0$ such as they have no common factors.
- Why always?

Proof $\sqrt{2}$ is irrational using well ordering

- Suppose that we choose the smallest m, n.
- Using the same idea as the previous proof, we show that both numbers must be divisible by two. ($m' = m/2, n' = n/2$)
- Now we found a number smaller than the smallest! (contradiction!)

More Proofs Using the Well Ordering Principle

- (Easy) Every integer $i > 1$ is a product of primes.
- (Medium) Every number is Postal.
    - A number n is postal if $n + 8$ can be composed of a sum of “threes” and “fives”
- (Difficult) $1 + r + r^2 + \ldots + r^n = \frac{r^n - 1}{r - 1}$

General form for a Well Ordering proof

You want to prove that $\forall n \in \mathbb{N}, P(n)$ using WOP.

1. Define a set of counter examples C, $C ::= \{n \in \mathbb{N} | \text{ not } P(n)\}$
2. Assume the minimum element of C exists, m, by WOP
3. Find a contradiction, for example:
   - Find a contradiction $c \in C, c < m$;
   - Show that $P(m)$ is actually True;

Propositions and Logic

Why Mathematical Language?

- Greeks carry swords or javelins.
- Greeks carry bronze or copper swords.

Mathematical Language

- Mathematical Language helps create non-ambiguous statements.
- We will not through all Logic operators here.
- However, it is important to understand that they are based on binary or boolean logic.

Mathematical Language / Binary Logic

Example: X XOR Y

X	Y	X XOR Y
TRUE	TRUE	FALSE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- A Truth Table is a way to understand a logic operator.
- We can use logic operators to transform ambiguous natural language sentences into clear logical propositions.
  - Greeks carry bronze or copper swords.
  - Greek carry bronze sword XOR greek carry copper sword.

Binary Logic and Truth Tables

The truth table allows us to analyze a logical formula:

- Is it always true? Is it always false?
- Is it equivalent to another logical formula?

To analyze a formula using the truth table, I need to analyse the value of each variable.

Evaluation of a Formula

Given the following variables:
P = True, Q = True, R = False

How do we evaluate the following formula?

NOT(NOT(P) OR Q) AND (R OR (P XOR Q))

Comparison of Two Formulas

We can decide whether two logical formulas are equivalent if the final column of their truth table is identical.

For example, let's prove DeMorgan's Law:

NOT(P OR Q) equiv to NOT(P) AND NOT(Q)

Satisfiability and Validity

- A logic formula is satisfiable if it is true for at least one assignment.
- A logic formula is valid if it is true for all assignments.

- Satisfiable: NOT(B)
- Not Satisfiable: B AND NOT(B)
- Valid: B OR NOT(B)

Checking for Validity and Satisfiability

Checking if a logic formula is satisfiable or not is a very importan problem in CS.

But how to do it?

Alert! If you try to use a truth table, the size of the table grows with the number of variables:

- 1 variable - 2 lines
- 2 variables - 4 lines
- 10 variables - 1024 lines
- n variables - 2^n lines...

Checking for Validity and Satisfiability

- Is there an efficient way to test for satisfiability? (SAT)
- The Efficient SAT problem is equivalent to the P=NP problem
- The validity problem is also related to the SAT problem.

Logic Quantifiers

- For all: $\forall$
- Exists: $\exists$

What is a Predicate?

A predicate is a proposition with variables in it:

$$P(X, Y) ::= [X + 2 = Y]$$

The truth value of a predicate depends on the values of the variables:

- $X = 1$, $Y = 3$, P(X,Y) is True
- $X = 2$, $Y = 2$, P(X,Y) is False

Quantifiers

- $\forall x$ – For ALL X
- $\exists y$ – There exists SOME Y

$\forall x$ works like AND. For example:

$$\forall x, x \in \{1,2,3\} | P(X) \text{ equiv } P(1) \text{ AND } P(2) \text{ AND } P(3)$$

$\exists y$ works like OR. For example:

$$\forall x, x \in \{1,2,3\} | P(X) \text{ equiv } P(1) \text{ OR } P(2) \text{ OR } P(3)$$

Quantifiers Example

For $x, y \in \mathbb{N}$ (x and y range over the integers).

$$Q(Y) ::= \exists x. x < y.$$

- Q(3) is True. ($[x < 3]$ is T for $x = 1$)
- Q(1) is True. ($[x < 1]$ is T for $x = 0$)
- Q(0) is False. ($[x < 0]$ is not T for any $x \in \mathbb{N}$)

What about a simple example for $\forall$?

Ordering Quantifiers

What is the difference when we order $\exists$ and $\forall$?

Example 1: Medicines

$\forall d \in$ diseases. $\exists m \in$ medicine.
m cures d

Example 2: Panacea

$\exists m \in$ medicine. $\forall d \in$ diseases.
m cures d

We need to be careful when writing mathematical notation!

Validity and Predicates

- Propositional Validity: A proposition is true for all truth assignments of variables.
  - Example: (P implies Q) OR (Q implies P)
- Predicate Calculus Validity: A predicate is valid when it is true for all domains.
  - Example: $\forall z.[P(z) \wedge Q(z)] \rightarrow [\forall x.P(x) \wedge \forall y.Q(y)]$

Important Ideas from Lecture I

- What are proofs?
- Proof techniques: Contradiction and Proof by Cases
- Proofs techniques can be used for debugging.
- What is the satisfiability problem.
- Why satisfiability is important to CS.

Make sure to see video 1.5.4 from OCW!

Exercise Sheet for Week 1

Please start working on the Exercise Sheet for Week 1. Deadline is Tuesday Next week (10/9).

- Each student must submit the exercise sheet separately.
- You may discuss the exercise with other students.
- You may ask the professor any questions.
- You may leave the classroom.