GB13604 - Maths for Computer Science

Lecture 7 – Counting Part II

Claus Aranha

caranha@cs.tsukuba.ac.jp

College of Information Science

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This course is based on Mathematics for Computer Science, Spring 2015, by Albert Meyer and Adam Chlipala, Massachusetts Institute of Technology OpenCourseWare.

Week 6 and 7 summary

Counting

- Sums and Products
- Asymptotics
- Counting with Bijections
- Repetitions and Binomial Theorem
- Pigeonhole Principle

Counting Rules

How do you count things and Why is it important?

- Count the number of people by counting heads;
- Count the number of people by counting tables;
- Count the number of cards in a deck;
- Count the number of possible card combinations;
- Count the number of steps in an algorithm;
- count the number of passwords;
- count the number of **problem configurations**

The Bijection Rule

The **Bijection Rule** states that if there is a bijection $A \rightarrow B$, then $|A| = |B|$.

This means that if there is a set A that we don't know the size, but we make a bijection to a set B that we know the size, then we can discover the size of A.

We will use this rule a lot in this class.

The Bijection Rule: Example

A card game has five colors of cards: Blue, Green, Red, Black and White.

How many different decks can you build of 12 cards?

000	(none)	0000	00	000
blue	green	red	black	white

Bijection Rule: Example

- **Set** A: Number of 12-card deck with 5 colors.
- **Set** B: Number of 16-bit strings with 4 “1”.

000	1		1	0000	1	00	1	000
blue		green		red		black		white

If we can count set B, we know the size of set A.

Strategy: Get really good at counting a few things, and use bijections to count everything else!

Counting Sequences

Our strategy is to create rules for couting sequences.

We define a sequence as “choosing something from set A” then “choosing something from set B” then “choosing something from set C”, etc.

If we can define rules for counting sequences, and define rules for representing other objects as sequences, we can solve many complex counting problems!

Starting from the basics: Sum rule and product rule

The Sum Rule: If Two sets (A,B) are disjoint, the size of the set that combines A and B is:

$$|A \cup B| = |A| + |B| \tag{1}$$

The Product Rule: If $|A| = m$ and $|B| = n$. The size of the sequence of choosing one thing from A and one thing from B is:

$$|A \times B| = |A| \times |B| \tag{2}$$

Sum Rule/Product Rule Example

You have 3 blue shirts, 5 black shirts, 4 pants and 3 skirts. If you wear one top and one bottom, **how amany different cloth sets do you have**?

- **TOP**: $|A| = |\text{Blu} \cup \text{Bla}| = 3 + 5 = 8$
- **BOTTOM**: $|B| = |P \cup S| = 4 + 3 = 7$
- **TxB Set**: $|A \times B| = 7 \times 8 = 56$

You have 56 combinations of tops and bottoms.

Product Rule and Bit Strings

How many bit strings exist of size n?

We can think of this problem as a sequence of choices:

- We choose 1st bit from $\{0, 1\}$
- We choose 2nd bit from $\{0, 1\}$
- ...
- We choose nth bit from $\{0, 1\}$

$$\{0, 1\}^n ::= \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \ldots \times \{0, 1\} \tag{3}$$

So by the product rule:

$$|\{0, 1\}^n| = |\{0, 1\}|^n = 2^n \tag{4}$$

Password Counting

How many different passwords exist with these rules?

- Digits and letters are acceptable;
- From 6 to 8 characters;
- Starts with a letter;
- Case sensitive;

Password Counting

- $L ::= \{a, b, c, \ldots, y, z, A, B, C, \ldots, Y, Z\}, |L| = 52$
- $D ::= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, |D| = 10$
- P_n ::= Passwords with n characters.
- $P_n ::= L \times (L \cup D)^{n-1}$
- We want to know: $P_6 \cup P_7 \cup P_8$

Password Counting

- Size of the password space:
  $|L \times (L \cup D)^5 \cup L \times (L \cup D)^6 \cup L \times (L \cup D)^7|$
- Sum Rule:
  $|L \times (L + D)^5| + |L \times (L + D)^6| + |L \times (L + D)^7|$
- Product Rule:
  $|L| \cdot (|L| + |D|)^5 + |L| \cdot (|L| + |D|)^6 + |L| \cdot (|L| + |D|)^7$
- Replacing sizes:
  $52 \times 62^5 + 52 \times 62^6 + 52 \times 62^7$
  $= 1.8 \times 10^{14}$ different passwords.

Counting Sevens

How many numbers with 4 digits have **at least one 7**?

Counting Sevens

How many numbers with 4 digits have **at least one 7**?

Count cases based on first occurence of seven

o: digit with 7 (10 choices) x: digit $\neq$ 7 (9 choices)

So we define four possible sequences:

7ooo	x7oo	xx7o	xxx7
10^3	$9 \cdot 10^2$	$9^2 \cdot 10$	9^3

3439

Generalized Product Rule

How can I choose a group of 5 students from a class of 25?

Thinking with a sequence of choices:

- Select 1 student from 25
- Select 1 student from 24
- Select 1 student from 23
- Select 1 student from 22
- Select 1 student from 21

$|G| = 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21$

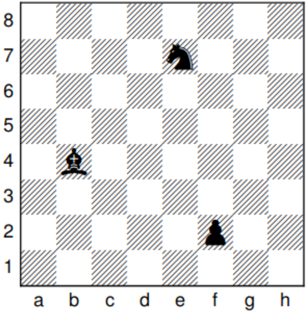
Generalized Product Rule

If we have to choose k elements from a set of size n, without repetition:

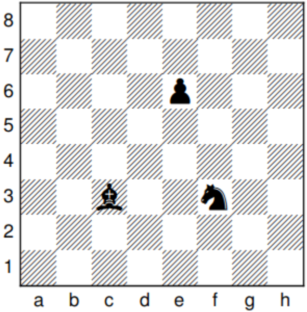
$$n \cdot n - 1 \cdot \ldots \cdot n - k \text{ OR } \frac{n!}{(n-k)!} \tag{5}$$

Generalized Product Rule: Chess Example

How many different ways can we put a bishop, a knight and a pawn in a chessboard in different rows and columns?



(a) valid



(b) invalid

Generalized Product Rule: Chess Example

Let's represent this problem as a Sequence of Choices:

- Column for Knight: c_n (from 8)
- Column for Bishop: c_b (from 7)
- Column for Pawn: c_p (from 6)
- Row for Knight: r_n (from 8)
- Row for Bishop: r_b (from 7)
- Row for Pawn: r_p (from 6)

Total Positions: $c_n c_b c_p r_n r_b r_p = 8 \cdot 7 \cdot 6 \cdot 8 \cdot 7 \cdot 6$

The Division Rule

I could count the number of students in this room by counting the number of **hands** in the room.

- A - number of students in the room
- B - number of hands in the room

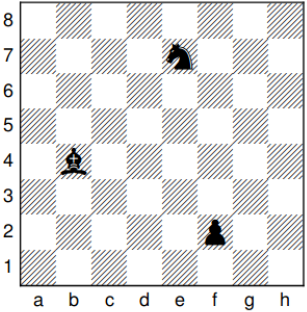
Since every student maps to two hands, then:

$$|B| = 2|A| \tag{6}$$

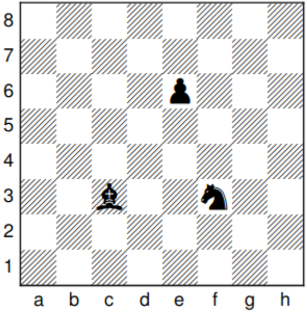
The Division Rule says that if there is a k-to-1 relationship between A and B, then $|B| = k|A|$.

Chess Example 2

How many different ways can we put two towers in a chessboard in different rows and columns?



(a) valid



(b) invalid

Chess Example 2

Let's represent this problem as a Sequence of Choices:

- Column for Tower 1: c_1 (from 8)
- Column for Tower 2: c_2 (from 7)
- Row for Tower 1: r_1 (from 8)
- Row for Tower 2: r_2 (from 7)

Total Positions: $c_1 c_2 r_1 r_2 = (8 \cdot 7)^2$ WRONG!

Chess Example 2

Total Positions: $c_1 c_2 r_1 r_2 = (8 \cdot 7)^2$ WRONG!

(2,7,3,4) is the same as (7,2,4,3) – The two towers are equal!

Every position can be "doubled" by switching the towers, so we have a 2-1 relationship.

Therefore, the number of positions is $\frac{(8 \cdot 7)^2}{2}$

Counting Subsets

How many size 4 subsets of 1 to 13 can we make?

- Set A ::= total permutations of 1 to 13
- Set B ::= size 4 subsets

Map: $a_1 a_2 a_3 a_4 a_5 \ldots a_{11} a_{12} a_{13} \in A$
To: $\{a_1 a_2 a_3 a_4\} \in B$

Counting Subsets

Map: $a_1 a_2 a_3 a_4 a_5 \dots a_{11} a_{12} a_{13} \in A$
To: $\{a_1 a_2 a_3 a_4\} \in B$

- $a_1 a_3 a_4 a_2 a_5 \dots a_{11} a_{12} a_{13}$ also maps.
- $a_1 a_2 a_3 a_4 a_{12} \dots a_7 a_9 a_5$ also maps.

For one subset in B, we can map $4!$ permutations of the first four elements, and $9!$ permutations of the others.

So the relation between B and A is $4! \cdot 9!$-to-1!

Counting Subsets: Binomial Coefficient

So to choose a 4 subset out of 13 elements:

$$13! = |A| = 9!4!|B|, |B| = \frac{13!}{9!4!} \quad (7)$$

More generally, to choose k subset from n elements:

$$\frac{n!}{n!(n-k)!} = \binom{n}{k} \quad (8)$$

(n choose k)

Everything Together: 2-pair poker hand

In the game of Poker, you draw **5 cards** of a 52-card deck with 13 ranks and 4 suits.

K♦, K♥, A♦, A♠, 3♣

A 2 pair hand has

- 2 cards of one rank;
- 2 cards of another rank;
- 1 card of a third rank;

2-pair poker hand

What is the probability of a 2-pair hand?

Total number of hands: $\binom{52}{5}$

Total number of 2-pair:

- Select first rank (1 of 13)
- Select first suits (2 of 4)
- Select second rank (1 of 12)
- Select second suits (2 of 4)
- Select third rank (1 of 11)
- Select third suit (1 of 4)

2-pair Poker Hand

1st Rank - 1st Suit - 2nd Rank - 2nd Suit - 3rd rank - 3rd suit

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

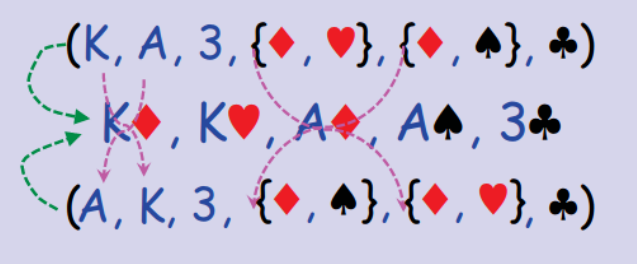
Problem!

2-pair Poker Hand

1st Rank - 1st Suit - 2nd Rank - 2nd Suit - 3rd rank - 3rd suit

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

Problem!



2-pair poker hand – Problem

Total number of 2-pair:

- Select first rank (1 of 13)
- Select first suits (2 of 4)
- Select second rank (1 of 12)
- Select second suits (2 of 4)
- Select third rank (1 of 11)
- Select third suit (1 of 4)

First and Second rank may be switched: 2-to-1 relationship!

2-pair Poker Hand – Fixed!

1st Rank - 1st Suit - 2nd Rank - 2nd Suit - 3rd rank - 3rd suit

2 Sequences to 1 hand: 2-to-1 relationship: $|A| = 2|B|$

$$\frac{1}{2} \cdot 13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

The BOOKKEEPER Principle

How many permutations has the word BOOKKEEPER?

- Total permutations: $bo_1 o_2 k_1 k_2 e_1 e_2 p e_3 r = 10!$
- But how many mappings do we have?
  $po_1 k_1 e_1 r o_2 e_2 k_2 e_3 b$ and
  $po_2 k_2 e_2 r o_1 e_1 k_1 e_3 b$ and many others...
- 2! permutations of $o_1 o_2$, 2! of $k_1 k_2$ and 3! of $e_1 e_2 e_3$
- 2!2!3!-to-1 mapping of permutations.

Total Permutations:

$$\frac{10!}{2!2!3!}$$

Generalized Multinomial Coefficient

Permutation of length n word with n_1 a's, n_2 b's, n_3 c's...

$$\binom{n}{n_1, n_2, n_3, \ldots, n_k} = \frac{n!}{n_1!n_2!n_3!\ldots n_k!} \tag{9}$$

What is the number of ways to rearrange the word SYSTEMS?

$$\binom{6}{1, 1, 3, 1, 1} = \frac{6!}{3!1!1!1!1!} = 6 \cdot 5 \cdot 4$$

The Binomial Theorem

What is the value of $(a+b)^n$?

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{10}$$

The expansion of $(a+b)^n$ is the sum of all permutations of "n" a's, "n-1" a's and 1 b's, "n-2" a's and "2" b's, ... etc.

The Binomial Theorem: Examples

Example 1: The coefficient of a^3b^5 from $(a+b)^8$ is the number of permutations of $a_1a_2a_3b_1b_2b_3b_4b_5 = \frac{8!}{3!5!}$

Example 2: The coefficient of bn^2a^3 from $(b+n+a)^6$ is the number of permutations of banana = $\frac{6!}{1!2!3!}$

Example 3: The coefficient of $x_1^{k_1}x_2^{k_2}x_3^{k_3}\dots x_n^{k_n}$ from $(x_1+x_2+\ldots+x_n)^{k_1+k_2+\ldots+k_n}$ is

$$\binom{k_1+k_2+\ldots+k_n}{k_1,k_2,\ldots,k_n} \tag{11}$$

The Pigeonhole Principle

Puzzle: A drawer has red socks, blue socks, and **black socks**. How many socks do you need to pick in the dark to be sure that you have at least one matching pair?

The Pigeonhole Principle

If there are more pigeons then pigeonholes, there is at least one hole with 2 pigeons.

Pigeonhole Example: Same number of hairs in Tokyo!

Claim: There is a group of at least 40 people with the exact number of hairs in Tokyo.

- The number of hairs in a person is between 0 and 200.000 (Set B)
- The number of people in Tokyo is 9.000.000 (Set A)
- By the Pigeonhole Principle, there are at least k ($|A| = k|B|$) people in the same group.

Therefore, there is at least one group of 40 people with the exact same number of hairs in Tokyo.

Super Useful!

Pigeonhole Principle: Pitfalls

Correct Example: If you draw 5 cards from a normal deck, then at least 2 cards have the same suit.

Incorrect Example: If you draw 5 cards from a normal deck then at least 2 cards have the **Hearts** suit.

(WRONG!)

(Example: Clubs, Clubs, Clubs, Clubs, Clubs)

The Pigeonhole principle says that **a group** will have size k, but it does not say which group will have size k.

Pigeonhole Principle: Subset Sums

- The image to the right has 90 numbers of 25 digits each;
- Are there 2 subsets with the exact same sum?

0020480135385502964448038	3171004832173501394113017
5763257331083479647409398	8247331000042995311646021
0489445991866915676240992	3208234421597368647019265
5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113
6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365
6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149
6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246
6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815
6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100
6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920
6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	4235996831123777788211249
6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220
7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427
7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190
7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530
7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910
7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856
7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348
7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372
7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947
7858918664240262356610010	9631217114906129219461111
8149436716871371161932035	3157693105325111284321993
3111474985252793452860017	5439211712248901995423441
7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458
8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044
5692168374637019617423712	8176063831682536571306791

Pigeonhole Principle: Subset Sums

- The image to the right has 90 numbers of 25 digits each;
- Are there 2 subsets with the exact same sum?

- Maximum sum: 90×10^{25}
- Maximum subsets: $2^{90} > 1.237 \times 10^{27}$

- By the pigeonhole principle, at leas two subsets have the same sum!

0020480135385502964448038	3171004832173501394113017
5763257331083479647409398	8247331000042995311646021
0489445991866915676240992	3208234421597368647019265
5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113
6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365
6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149
6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246
6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815
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1578271047286257499433886	4167283461025702348124920
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1638243921852176243192354	4235996831123777788211249
6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220
7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427
7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190
7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530
7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910
7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856
7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348
7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372
7712154432211912882310511	9602413424619187112552264
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7858918664240262356610010	9631217114906129219461111
8149436716871371161932035	3157693105325111284321993
3111474985252793452860017	5439211712248901995423441
7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458
8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044
5692168374637019617423712	8176063831682536571306791

Summary

- How to count permutations
- Division Principle: k-to-1 relations
- Binomial Theorem
- Pigeonhole Principle