GB13604 - Maths for Computer Science

Lecture 6 – Counting Part I

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This course is based on Mathematics for Computer Science, Spring 2015, by Albert Meyer and Adam Chlipala, Massachusetts Institute of Technology OpenCourseWare.

Week 6 and 7 summary

Counting

- Sums and Products
- Asymptotics
- Counting with Bijections
- Repetitions and Binomial Theorem
- Pigeonhole Principle

Sums for Children

Next two lectures: Talking about sums and counting.

Middle School Classwork:

$$89 + 102 + 115 + 128 + \ldots + 440 + 453 + 466$$

It is said that Gauss found a quick solution for this problem at 9 years old:

- 30 numbers, each 13 greater than the previous one.

Sums for Children

- $89 + (89 + 13) + \ldots + (89 + 28 \times 13) + (89 + 29 \times 13)$
- $F + (F + d) + \ldots + (L - d) + L = A$

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- $L + (L - d) + \ldots + (F + d) + F = A$

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- $F + (F + d) + \ldots + (L - d) + L = A$
- $L + (L - d) + \ldots + (F + d) + F = A$
- $(F + L) + (F + L) + \ldots + (F + L) + (F + L) = 2A$

$$A = \frac{(F + L)}{2} \times (\text{\# of terms}) \tag{1}$$

Sums for Children

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$$A = \frac{(F + L)}{2} \times (\text{\# of terms}) \tag{1}$$

$$A = \frac{(1 + n)}{2} \times (n) \tag{2}$$

Why Counting?

- Counting techniques help understand the structure of numbers:
- We will study three different sums:
  - Arithmetic Sums
  - Geometric Sums
  - Harmonic Sums

Motivation for this unit

Counting techniques can be used to estimate upper bounds and lower bounds of computer complexity in algorithms.

Arithmetic Sums and Geometric Sums

- Arithmetic Sums: Add a fixed value to each element
  $A = (a + 0d) + (a + d) + (a + 2d) + \ldots + (a + nd)$
- Geometric Sums: Multiply a fixed value to each element
  $G = (kx^0) + (kx^1) + (kx^2) + \ldots + (kx^n)$

Closed Form for the Geometric Sum

- $A = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$,
  $G = 1 + x + x^2 + \ldots + x^n = ?$

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- $G = 1 + x + x^2 + \ldots + x^n$
- $-xG = -x - x^2 - \ldots - x^{n+1}$
- $G - xG = 1 \qquad\qquad -x^{n+1}$
- $G(1-x) = 1 - x^{n+1}$, $G = \frac{1-x^{n+1}}{1-x}$

Closed Form for the Geometric Sum

- The **Proof By Induction** can show if a formula is correct, but does not show where the formula comes from
- The **Perturbation Method** can be used to find closed forms for sums.

Infinite Geometric Sums

- Geometric Sum:

$$G_n = \frac{1 - x^{n+1}}{1 - x} \tag{3}$$

- Geometric **Series**:

$$\lim_{n \to \inf} G_n = \frac{1 - \lim_{n \to \inf} x^{n+1}}{1 - x} = \frac{1}{1 - x} \tag{4}$$

(Provided that $|x| < 1$)

Application: Calculating Loans

Problem: I pay you 100$ in 1 year if you pay me X$ now

How do we calculate a fair amount for X?

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bankrate b ::= 1.03

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Problem: I pay you 100$ in 1 year if you pay me X$ now

How do we calculate a fair amount for X?

- My bank will pay me 3% interest

  bankrate b ::= 1.03

- If I deposit X$ now, I will have $b \times X$$ in 1 year.
- So I will not lose money if $b \times X \geq 100$

  $X \geq 100/1.03 = 97.09$

The future worth of money

- 1$ next year is worth $0.9709 now.
- r$ last year is worth 1 today, if $r ::= 1/b$
- n in two years is worth
- nr in one year, and it is worth
- nr^2 today.

n paid in k years is worth nr^k today, where $r ::= 1/\text{bankrate}$

Annuity

Insurance Company: I will pay you 100$ for 10 years, if you pay me $Y now. How much should Y be?

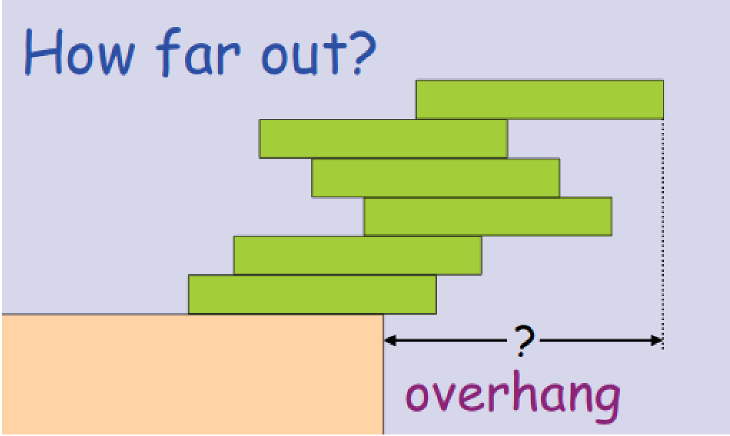
- The Insurance company needs:
- $100r + 100r^2 + 100r^3 + \ldots + 100r^{10}$

Annuity

Insurance Company: I will pay you 100$ for 10 years, if you pay me $Y now. How much should Y be?

- The Insurance company needs:
- $100r + 100r^2 + 100r^3 + \ldots + 100r^{10}$
- Insurance = $100r(1 + r + r^2 + \ldots + r^9)$
- $I = 100r \times \frac{1-r^{10}}{1-r} = 853.02$

Problem: Book Stacking

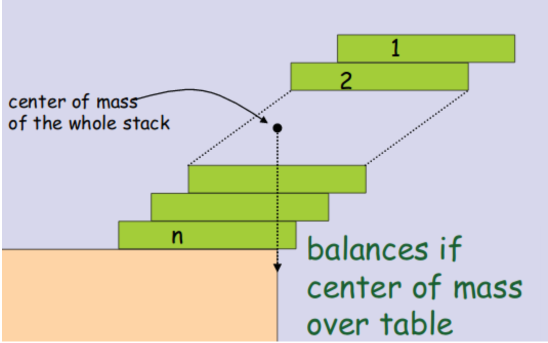


Book Stacking Problem

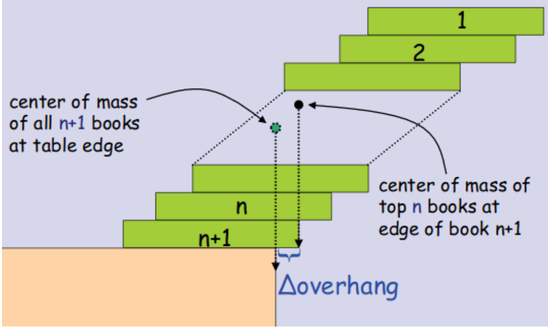
- All books have size 1.
- For 1 book: Center of Mass is 0.5

1-book overhang is 0.5.

- What about n books?



Book Stacking Problem



Δ-overhang ::= distance between CoM of n books and CoM of the $n+1$ book.

Δ-overhang

From physics, we know that:

$$\Delta = \frac{1}{2(n+1)} \tag{5}$$

- $B_n ::=$ overhang of n books
- $B_1 = 1/2$
- $B_{n+1} = B_n + \frac{1}{2(n+1)}$

- $B_n = 1/2(1 + 1/2 + 1/3 + \ldots + 1/n)$

(This is harmonic Sum!)

How big can the overhang get?

It turns out that by increasing the number of n books, we can make $1/2H_n$ as big as necessary.

So there is no upper limit for the size of the overhang, if you position the books very carefully and know integrals.

Asymptotics

Study of Asymptotics

- How fast do expressions grow?
- What is the maximum/minimum size of an expression?
- How can we compare two expressions as they get more complex?

We will look at four notations that describe the relationship between the **growth of functions**.

Asymptotic Equivalence

Def: $f(n) \sim g(n)$: (f(n) is asymptotically equal to g(n))

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \tag{6}$$

Example:

- $n^2 \sim n^2 + n$ because...
- $\lim_{n \to \infty} \frac{n^2+n}{n} = \lim_{n \to \infty} \frac{1}{n} + 1 = 1$

Asymptotic Equivalence

Lemma: $\sim$ is symmetric $(f \sim g \implies g \sim f)$

- Let $f \sim g$ be true, is $g \sim f$ true too?
- $\lim \frac{g}{f} = \lim \frac{1}{(f/g)}$
- $\lim \frac{1}{(f/g)} = \frac{1}{\lim(f/g)} = \frac{1}{1} = 1$
- $f \sim g \implies g \sim f$ ∎

Asymptotic Equivalence

Transitivity: Suppose $f \sim g$ and $g \sim h$. Prove $f \sim h$

$$1 = \lim \frac{f}{g} = \lim \frac{f/h}{g/h} = \frac{\lim(f/h)}{\lim(g/h) = 1} = \lim \frac{f}{h} \tag{7}$$

Colorary: $\sim$ is an equivalence relation

- important *sim* is a relation on functions
- $f(n) \sim g(n)$ does not care about particular values of f(n) or g(n). ($f \sim g$)

Little Oh: $o(\cdot)$ – Asymptotically Smaller

Definition: $f(n) = o(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

Example:

- $n^2 = o(n^3)$
- $\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$

Lemma: $o(\cdot)$ defines a strict partial order

Big Oh: $O(\cdot)$ – Asymptotic Order of Growth

Definition: $f(n) = O(g(n))$

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty \tag{8}$$

- The limit of f(n)/g(n) is finite.
- Could be 0, could be 1, could be something else.
- Why limsup not lim? Ignore this for now.

- **Example**: $3n^2 = O(n^2)$ because $\lim \frac{3n^2}{n^2} = \lim 3 = 3$

Why do we like $O(\cdot)$ so much?

- What O() means is that constant factors don't matter.
- Only rate of growth matters.

- When we talk about **execution time**, if the hardware changes, only a constant factor changes.

- Slow algorithms will still be slow (for bigger data!) even if the hardware changes.

Theta: $\Theta(\cdot)$ – Same Order of Growth

Definition:

$$f = \Theta(g) \iff f = O(g) \wedge g = O(f) \tag{9}$$

Lemma: Θ is an equivalence relation.

Asymptotics: Intuitive Summary

- $f \sim g$ — f and g nearly equal;
- $f = o(g)$ — f much less than g;
- $f = O(g)$ — f is about $\leq$ g;
- $f = \Theta(g)$ — f is about equal to g;

Asymptotic Properties

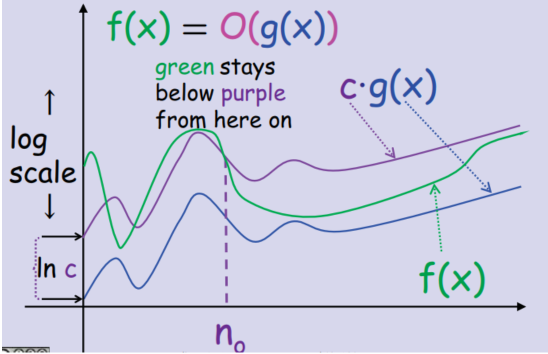
- If $f \sim g$ or $f = o(g)$ then $f = O(g)$
- If $f = o(g)$ then $g \neq O(f)$

$$(\lim f/g = 0 \implies \lim g/f = \infty)$$

Big Oh $O(\cdot)$ and limsup

Alternate Definition: $f(n) = O(g(n))$

$$\exists c, n_0, \forall n \geq n_0, f(n) \leq c \cdot g(n) \tag{10}$$



Big Oh $O(\cdot)$ and limsup

- Why is limsup necessary in the first definition?
- Suppose $f \leq 2g$ then $f = O(g)$ but f/g has no limit.
- **Example:** $f(n) = (1 + \sin^2(\frac{n\pi}{2})) \times g(n)$
- The limit of f/g in this expression alternates between 1 and 2.
- On the other hand, the limsup of f/g is 2.

A few more facts about asymptotics

- **Lemma:** $x^a = o(x^b)$ if a < b
  because $\lim \frac{x^a}{x^b} = \lim \frac{1}{x^{a-b}}$ and $a - b > 0$

- **Lemma:** $\ln(x) = o(x^\epsilon)$ for $\epsilon > 0$
  (logarithms grow slower than roots)

Proof:

- $1/y \leq y$ for $y \geq 1$, so $\int_1^z \frac{1}{y} dy \leq \int_1^z y dy$
- $\ln(z) \leq \frac{z^2}{2}$ for $z \geq 1$, so let $z = \sqrt{x^\delta}$, (for some $\delta > 0$)
- $\frac{\delta \ln(x)}{2} \leq \frac{x^\delta}{2}$, but $x^\delta = o(x^\epsilon)$ for $\delta < \epsilon$
- So $\delta \ln(x) = o(x^\epsilon)$ $\blacksquare$

Asymptotic Blunders – Be careful!

- "$\cdot = O(\cdot)$" defines a binary relation.

  Do not write O(x) = x!!

  - If $x = O(x)$ and $O(x) = x$...
  - But $2x = O(x)$ and $O(x) = x$... so $2x = x$?????

- Big Oh is not a lower bound.

  Do not write: "f is at least $O(n^2)$"

  - If you want to say that n^2 is a lower bound of f...
  - $n^2 = O(f)$

Asymptotic Blunders – Be careful!

- **False Proof:** $\sum_{i=1}^{n} i = O(n)$
  (We know that $\sum_{i=1}^{n} i = n(n+1)/2$)

- Any constant is O(1): $0 = O(1), 1 = O(1), 2 = O(1) \ldots$
- So, $i = O(1)$
- So, $\sum_{i=1}^{n} i = O(1) + O(1) + O(1) + O(1) \ldots$
- So, $\sum_{i=1}^{n} i = nO(1) = O(n)$ (???)

$O(\cdot)$ is not a quantity! Do not do arithmetic with it!

End of Class – Class Summary

- Sums: Arithmetic, Geometric, Harmonic, and Closed Forms
- Asymptotic Notation: $\sim$, o(), O(), Θ()